

LINEAR VOLUME BOUNDS FOR DRILLING AND FILLING

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ABSTRACT. We prove uniform linear bounds on the volume variation under drilling and filling operations on finite volume hyperbolic 3-manifolds.

1. INTRODUCTION

There are two important operations that one can perform on a finite volume hyperbolic 3-manifold. One can drill out a simple closed geodesic and one can Dehn fill a cusp. It is a well-known phenomenon that all drillings and most fillings give back a hyperbolizable 3-manifold whose volume is larger for the first operation and smaller for the second. The goal of this note is to prove two results that give a *linear* quantitative control on how much the volume can change.

Setting and context. To describe what we mean by a linear quantitative control, let us introduce the relevant parameters of the two constructions.

- *Drilling.* Consider a closed hyperbolic 3-manifold M and a simple closed geodesic $\gamma \subset M$. We denote by ℓ its length and by R the radius of an embedded tubular neighborhood of γ .

We will bound $\Delta V := \text{vol}(M - \gamma) - \text{vol}(M)$ linearly in terms of ℓ/R .

- *Filling.* Consider a complete finite volume hyperbolic 3-manifold N with one cusp and the data of a cusp neighborhood $C \subset N$ bounded by the flat torus ∂C and a simple closed (flat) geodesic $\mu \subset \partial C$. We denote by A the area of ∂C and by $L = \text{length}(\mu)/\sqrt{A}$ the so-called normalized length of μ . We write N_μ for the Dehn filling of $N - C$ with slope μ .

We will bound $\Delta V := \text{vol}(N) - \text{vol}(N_\mu)$ linearly in terms of $1/L^2$.

The problems of understanding how much the volume *decreases* under filling and *increases* under drilling are central for a fine analysis of the set of volumes of all hyperbolic 3-manifolds. The former is better understood compared to the latter.

To illustrate the current state of knowledge on volume changes under filling, we review several key milestones. Thurston [Thu80, Chapter 6] showed that $\text{vol}(N_\mu) < \text{vol}(N)$ and that for every *fixed* N we have $\Delta V := \text{vol}(N) - \text{vol}(N_\mu) \rightarrow 0$ as $L \rightarrow \infty$. Neumann and Zagier [NZ85] then clarified, for every *fixed* N , that the asymptotic behavior is $\Delta V = \pi^2/L^2 + O(1/L^4)$. Later, Hodgson and Kerckhoff [HK05] greatly strengthened the above by showing that $|\Delta V - \pi^2/L^2|$ can be bounded by an explicit function provided that L is larger than some *universal* constant independent of N . The functions arising from [HK05], despite being explicit, are not so simple to

evaluate. Given the importance of getting sharp volume bounds, several efforts were made to simplify them. For example, the work of Haraway [Har19] gives $8.95/L^2 \leq \Delta V \leq 13.4/L^2$ provided that $L \geq 8.5$.

Our contribution to the filling case is a completely different perspective from [HK05] to prove bounds of the form $\Delta V \geq c/L^2$ under the *weaker* assumption that $\text{length}(\mu)$ is large enough (see Theorem C).

As for drillings, the situation is much more mysterious, but it has already been considered by Hodgson and Kerckhoff [HK05], Bridgeman [Bri98], Agol [Ago02], Agol, Storm, Thurston, and Dunfield [ASTD07, Theorem 10.1]. While very good estimates are available when the drilled geodesics are short, it is still unclear what happens for long geodesics. We quickly summarize some key advances. In the case ℓ is smaller than some universal constant, Hodgson and Kerckhoff [HK05, Theorem 6.5 and Remark 6.6] prove that $|\Delta V - \pi\ell/2|$ is bounded by an explicit function of ℓ which vanishes at 0 to at least the second order (and this can be effectivized combining the estimates of [Har19] with [Mag12, Theorem 1.2] or [FPS22, Corollary 6.13]). This gives an exact asymptotic formula for ΔV when $\ell \rightarrow 0$. If ℓ is arbitrary, the situation is more complicated. The works [Ago02] and [ASTD07, Theorem 10.1] provide the existence of explicit functions $f, g > 1$ such that $\text{vol}(M - \gamma) \leq f(R)\text{vol}(M) + g(R)\ell$.

Our main achievement for drillings is to get rid of the multiplicative constant in front of $\text{vol}(M)$ in the above inequality (see Theorem A). This shows that the increase in volume only depends on the *local* geometry around γ .

Having discussed the global picture, we now state our new results.

Drilling. Let us start with drillings. We prove the following.

Theorem A. *There exists a constant $c > 0$ such that the following holds. Let M be a closed hyperbolic 3-manifold and let $\gamma \subset M$ be a simple closed geodesic. Suppose that γ has length ℓ and has an embedded tubular neighborhood of radius at least $R \in (0, 1)$. Let $\text{vol}(M - \gamma)$ be the volume of the hyperbolic metric on $M - \gamma$. Then*

$$\Delta V := \text{vol}(M - \gamma) - \text{vol}(M) \leq c \frac{\ell}{R}.$$

Corollary B. *There exists a constant $c > 0$ such that the following holds. Let M be a closed hyperbolic 3-manifold and let $\gamma \subset M$ be a shortest non-trivial closed geodesic in M (which has length $\ell = 2 \cdot \text{inj}(M)$ where $\text{inj}(M)$ is the injectivity radius of M). Let $\text{vol}(M - \gamma)$ be the volume of the hyperbolic metric on $M - \gamma$. Then*

$$\Delta V := \text{vol}(M - \gamma) - \text{vol}(M) \leq c\ell.$$

Theorem A goes in the direction of a conjecture of Bridgeman [Bri98], which asks whether there is a linear upper bound on the volume increase under drilling operations, and, more precisely, whether one always has $\Delta V \leq \pi\ell$. Bridgeman [Bri98] proves that this is the case for certain special families of drillings. While the existence of a uniform linear upper bound is still open, Agol [Ago02, Section 3] gave experimental evidence that $\Delta V \leq \pi\ell$ does not always hold, so the potential constant in a linear upper bound should be larger than π . Note that as long as

we can uniformly bound R from below the increase in volume in Theorem A is at most linear. This leads immediately to Corollary B. In fact, it is well-known that a shortest geodesic in a closed hyperbolic 3-manifold has a tubular neighborhood of radius at least $\ell/4$, where ℓ is the length of the geodesic (see [GMT03, Proposition 1.11]). Combined with standard consequences of the Margulis Lemma, this provides a universal constant $R_0 > 0$ such that a shortest geodesic in a closed hyperbolic 3-manifold M always has a tubular neighborhood of radius $R \geq R_0$ (for much more refined estimates see [GMT03, Theorem 4.1]).

Let us mention the fact that the restriction to simple geodesics in closed hyperbolic 3-manifolds, rather than dealing with geodesic links in arbitrary finite volume ones, is not important. A straightforward generalization of Theorem A holds in this larger setting with exactly the same arguments and more robust bookkeeping. We chose to restrict ourselves to the simple case to streamline the exposition.

Filling. Now we move to fillings. We prove the following.

Theorem C. *There exists a constant $c > 0$ such that the following holds. Let N be a complete finite volume hyperbolic 3-manifold with one cusp. Let $C \subset N$ be a cusp neighborhood bounded by the flat torus ∂C and let $\mu \subset \partial C$ be a simple closed (flat) geodesic. Denote by A the area of ∂C and by $L := \text{length}(\mu)/\sqrt{A}$ the normalized length of μ . If $\text{length}(\mu) > \sqrt{c}\pi$ then the Dehn filling of $N - C$ with slope μ , denoted by N_μ , has a hyperbolic metric of volume $\text{vol}(N_\mu)$ and we have*

$$\frac{\pi^2}{2L^2} \left(1 - c \frac{\pi^2}{AL^2} \right) \leq \Delta V := \text{vol}(N) - \text{vol}(N_\mu).$$

The conclusion is similar to the lower bound due to Hodgson and Kerckhoff [HK05] and Haraway [Har19] in that the bound on ΔV is essentially linear in $1/L^2$. However, our assumptions are *weaker* from the ones of [HK05, Har19] as we ask that $\text{length}(\mu) \geq \sqrt{c}\pi$ whereas [HK05, Har19] both require that $L = \text{length}(\mu)/\sqrt{A}$ is bigger than a universal constant. Note that our condition *depends* on the choice of a cusp neighborhood C while the value of the normalized length is independent of it. We remark that, by work of Adams [Ada02], it is always possible to choose a cusp neighborhood C such that ∂C has area $A \geq \sqrt{3}$. For that choice we have $\text{length}(\mu) \geq \sqrt[4]{3}L$.

The restriction to Dehn fillings of hyperbolic 3-manifolds of finite volume with one cusp, rather than allowing multiple Dehn fillings of a selection of cusps of an arbitrary finite volume hyperbolic 3-manifold, is not really important. A straightforward generalization of Theorem C holds in that larger setting with the same proof but more notation. In order to keep the exposition as linear and short as possible, we restricted ourselves to the simple setup.

On the proofs. Lastly, a few words on the proofs.

To prove Theorem A, we follow the approach of [ASTD07, Theorem 10.1] with a difference. We first change the metric of M inside a tubular neighborhood T of γ , interpolating between the metric on the tube and the metric on a suitable hyperbolic

cusps. We do so by keeping under control the scalar curvature (see Lemma 3.1). Then we compare the volume of the modified metric with that of the hyperbolic structure on $M - \gamma$. Instead of relying on Perelman's monotonicity formula, we exploit some other consequences of Perelman's work [Per02, Per03], namely, the exact computation of the sigma invariant of closed hyperbolic 3-manifolds as achieved by Anderson [And06] and Kleiner and Lott [KL08]. This allows us to keep track of the fact that we *did not change* the metric outside the tubular neighborhood T and obtain an additive error rather than a multiplicative one in the volume increase.

The proof of Theorem C follows exactly the same strategy. We change the metric of N in a cusp neighborhood C as in Gromov-Thurston's 2π -Theorem [BH96, Theorem 9] (of which we prove a quantitative version, see Proposition 5.1). This time, the interpolation between the metric on the cusp and the metric on a tube is more delicate as we want to keep under control the scalar curvature up to higher order (see Lemma 4.1). Then we use the sigma invariant comparison to conclude.

To conclude the introduction, let us comment on the constants $c > 0$ in Theorems A and C. In both cases, $c > 0$ is explicit and computable. Since we were mostly interested in the coarse behavior, we did not try to optimize their values.

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2. PRELIMINARIES

This section reviews some crucial notions and tools that will be used in the proof of the main results. It is divided in two parts. The first part introduces the so-called sigma invariant of a closed 3-manifold and gives a useful integral estimate (Proposition 2.2). In the second part, we summarize the curvature properties of a special family of metrics that can be used to describe both hyperbolic tubes and hyperbolic cusps.

2.1. The sigma invariant. The sigma invariant of a 3-manifold is the key quantity involved in the proofs of Theorems A and C.

Definition 2.1 (Sigma Invariant). Let M be a closed 3-manifold. For every Riemannian metric (M, g) define the Einstein-Hilbert functional

$$\mathcal{E}(g) := \frac{\int_M S_g \, \mathrm{dvol}_g}{\mathrm{vol}(M, g)^{1/3}}$$

where S_g is the scalar curvature of (M, g) .

Denote by $[g]$ the conformal class of (M, g) , that is, the set of Riemannian metrics on M conformally equivalent to g . The classical Yamabe constant of $(M, [g])$ is given by

$$Y(M, [g]) := \inf_{g' \in [g]} \mathcal{E}(g').$$

By considering the set $\mathcal{C}$ of all possible conformal classes of Riemannian metrics on M , one defines the sigma invariant as

$$\sigma(M) := \sup_{[g] \in \mathcal{C}} Y(M, [g]).$$

The following gives a useful lower bound.

Proposition 2.2. *Let M be a closed 3-manifold. For every Riemannian metric (M, g) we have*

$$Y(M, [g]) \geq - \left(\int_M |S_g|^{3/2} d\text{vol}_g \right)^{2/3}.$$

Proof. Consider an arbitrary conformal change of the form $g' = u^4 g$ where $u : M \rightarrow (0, \infty)$ is a smooth function. We need to control the ratio $\int_M S_{g'} d\text{vol}_{g'} / \text{vol}(M, g')^{1/3}$.

We use the following formulas for the conformal change of metrics

$$d\text{vol}_{g'} = u^6 d\text{vol}_g \quad \text{and} \quad S_{g'} = u^{-5} (8\Delta_g u + S_g u)$$

where $\Delta_g u = -\text{div}_g(\nabla_g u)$. Hence, we get $\text{vol}(M, g') = \int_M u^6 d\text{vol}_g$ and

$$\int_M S_{g'} d\text{vol}_{g'} = 8 \int_M u \Delta_g u d\text{vol}_g + \int_M S_g u^2 d\text{vol}_g.$$

By the divergence theorem, the first summand is positive

$$\int_M u \Delta_g u d\text{vol}_g = \int_M |\nabla_g u|_g^2 d\text{vol}_g.$$

Therefore $\int_M S_{g'} d\text{vol}_{g'} \geq \int_M S_g u^2 d\text{vol}_g$.

We are now ready to conclude the desired estimate. By the above discussion,

$$\mathcal{E}(g') = \frac{\int_M S_{g'} d\text{vol}_{g'}}{\text{vol}(M, g')^{1/3}} \geq \frac{\int_M S_g u^2 d\text{vol}_g}{\left(\int_M u^6 d\text{vol}_g \right)^{1/3}}.$$

By Hölder's inequality, the numerator satisfies

$$\left| \int_M S_g u^2 d\text{vol}_g \right| \leq \left(\int_M |S_g|^{3/2} d\text{vol}_g \right)^{2/3} \left(\int_M (u^2)^3 d\text{vol}_g \right)^{1/3}.$$

Thus, we can continue the chain of inequalities with

$$\geq \frac{- \left(\int_M |S_g|^{3/2} d\text{vol}_g \right)^{2/3} \left(\int_M (u^2)^3 d\text{vol}_g \right)^{1/3}}{\left(\int_M u^6 d\text{vol}_g \right)^{1/3}} = - \left(\int_M |S_g|^{3/2} d\text{vol}_g \right)^{2/3}.$$

This concludes the proof of the proposition. $\square$

In general, the sigma invariant is difficult to compute, and we only have a few examples of manifolds for which we know the exact value. Luckily, hyperbolic 3-manifolds belong to this class as proved by Anderson [And06, Equation (2.6) in Section 2] and Kleiner and Lott [KL08, Proposition 93.10] using Perelman's work [Per02, Per03].

Theorem 2.3 (see [KL08, Proposition 93.10] and [And06, Equation (2.6) in Section 2]). *Let (M, g_M) be a closed hyperbolic 3-manifold. Then*

$$\sigma(M) = \mathcal{E}(M, g_M) = -6 \cdot \text{vol}(M, g_M)^{2/3}.$$

Remark 2.4. Both [KL08, Proposition 93.10] and [And06, Equation (2.6) in Section 2] are expressed in terms of certain topological decompositions of closed 3-manifolds. We explain how to deduce Theorem 2.3 from the statement of [KL08, Proposition 93.10].

First, recall that closed hyperbolic 3-manifolds are *irreducible*, that is, every embedded 2-sphere bounds an embedded 3-ball. In particular, it is not possible to decompose them as non-trivial connected sums. Second, recall that closed hyperbolic 3-manifolds are *atoroidal*, that is, their fundamental groups do not contain $\mathbb{Z}^2$. In particular, they cannot contain π_1 -injective embedded 2-tori or Klein bottles.

Combined, these properties tell us that we cannot be in cases (a) and (b) of [KL08, Proposition 93.10] (no hyperbolic manifold is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, to an elliptic manifold, or to a graph manifold).

Hence, we are in case (c). According to [KL08, Proposition 93.10], this means that we can realize $\sigma(M)$ as $-6 \cdot \text{vol}(N, g_N)^{2/3}$ (in [KL08, Proposition 93.10] the factor is $-3/2$ not -6 , the difference is due to sectional curvature normalizations) where (N, g_N) is a complete finite volume (possibly disconnected) hyperbolic 3-manifold with the following property. The 3-manifold M can be decomposed as a connected sum of copies of $\mathbb{S}^2 \times \mathbb{S}^1$, elliptic 3-manifolds, and a 3-manifold M' containing $N \subset M'$ as an embedded submanifold and so that $M' - N$ is a (possibly empty) graph manifold with π_1 -injective boundary (what in [KL08, Proposition 93.10] is referred to as having incompressible cusps).

As M is irreducible, such a connected sum decomposition must be trivial. More precisely, we must have $M = M'$ (recall that M , being hyperbolic, cannot be diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ or to an elliptic 3-manifold). As M does not contain π_1 -injective tori or Klein bottles, the manifold $M' - N$ must be empty. Thus $M = M' = N$. By Mostow rigidity, we can conclude that (M, g_M) and (N, g_N) are isometric. In particular, they have the same volume. Theorem 2.3 follows.

2.2. Interpolation between tubes and cusps. Our strategy to prove Theorems A and C involves a smooth interpolation between the metric on a hyperbolic tube and the metric of a hyperbolic cusp.

In order to do so, one is immediately led to study metrics of the form

$$g := dr^2 + a(r)^2 d\theta^2 + b(r)^2 dy^2.$$

on $[R_0, R_1] \times \mathbb{S}^1 \times [0, 1]$ where $a, b : [R_0, R_1] \rightarrow (0, \infty)$ are smooth functions. In fact, when we choose $R_0 = 0, R_1 = R$ and $a(r) = 2\pi \sinh(r), b(r) = \ell \cosh(r)$ we obtain the metric of a hyperbolic tube of radius R and core geodesic of length ℓ . When we choose $R_0 = -\infty, R_1 = 0$ and $a(r) = \ell_1 e^r, b(r) = \ell_2 e^r$ we obtain the metric of a hyperbolic cusp based on a flat cylinder with core curve of length ℓ_1 and height ℓ_2 .

In this section, we summarize the curvature properties of these well-studied metrics. Denote by $\partial_r, \partial_\theta, \partial_y$ the standard coordinate frame. Let $K(u, v)$ be the sectional curvature of the 2-plane generated by the vectors u, v .

Lemma 2.5 (see [Ago02, Lemma 2.3] or [BH96, Lemma 10]). *Let $g := dr^2 + a(r)^2 d\theta^2 + b(r)^2 dy^2$ be a metric on $[R_0, R_1] \times \mathbb{S}^1 \times [0, 1]$. We have the following symmetric table*

$K(\partial_i, \partial_j)$	∂_r	∂_θ	∂_y
∂_r	0	$-\frac{a''}{a}$	$-\frac{b''}{b}$
∂_θ		0	$-\frac{a'b'}{ab}$
∂_y			0

As $\partial_r, \partial_\theta, \partial_y$ is an orthogonal frame, the scalar curvature of the metric g is given by

$$S_g = 2(K(\partial_r, \partial_\theta) + K(\partial_r, \partial_y) + K(\partial_\theta, \partial_y)) = -2 \left(\frac{a''}{a} + \frac{b''}{b} + \frac{a'b'}{ab} \right).$$

In the next two sections, we produce suitable smooth transitions from tubes to cusps by carefully choosing a, b . We are going to use them for modifying the hyperbolic metrics of our initial manifold M in a tubular neighborhood of the simple closed geodesic in the drilling case and in the cuspidal end of the manifold in the filling case.

3. FROM TUBES TO CUSPS

In this section, we construct the functions $a_{\text{drill}}, b_{\text{drill}}$ used in the drilling case.

Lemma 3.1. *There exists a constant $c > 0$ such that the following holds. For every $R \in (0, 1)$ there exist functions $a_{\text{drill}}, b_{\text{drill}} : \mathbb{R} \rightarrow \mathbb{R}$ such that*

- (1) $a_{\text{drill}}, b_{\text{drill}} > 0$.
- (2) $a_{\text{drill}}(r) = \sinh(r)$ and $b_{\text{drill}}(r) = \cosh(r)$ on $(2R/3, \infty)$.
- (3) $a_{\text{drill}}(r) = \sinh(R)e^r$ and $b_{\text{drill}}(r) = \cosh(R)e^r$ on $(-\infty, R/3)$.
- (4) We have

$$\left| \frac{a''_{\text{drill}}}{a_{\text{drill}}} + \frac{b''_{\text{drill}}}{b_{\text{drill}}} + \frac{a'_{\text{drill}}b'_{\text{drill}}}{a_{\text{drill}}b_{\text{drill}}} \right| \leq c \frac{1}{R^2}.$$

- (5) We have $a_{\text{drill}}(r)b_{\text{drill}}(r) \leq \sinh(R)\cosh(R)e^{2r}$ for $r \leq R$.

Let us briefly describe the geometric meaning of the requirements for the metric that we are going to build using the functions $a_{\text{drill}}, b_{\text{drill}}$. Item (2) says that the metric is a hyperbolic tube on a region corresponding to $(2R/3, R)$. Item (3) ensures that the metric is a hyperbolic cusp on a region corresponding to $(-\infty, R/3)$. Item (4) provides a bound on the scalar curvature. Item (5) gives volume bounds.

Proof. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing smooth bump function such that

- $\eta = 0$ on $(-\infty, 1/3]$ and $\eta = 1$ on $[2/3, \infty)$.
- $|\eta'| \leq c_1$ and $|\eta''| \leq c_2$.

Set $\chi(r) := \eta(r/R)$. Clearly

- $\chi = 0$ on $(-\infty, R/3]$ and $\chi = 1$ on $[2R/3, \infty)$.
- $|\chi'| \leq c_1/R$ and $|\chi''| \leq c_2/R^2$.

Define

$$\begin{aligned}\phi(r) &:= (1 - \chi(r)) \log(\sinh(R)e^r) + \chi(r) \log(\sinh(r)) \\ \psi(r) &:= (1 - \chi(r)) \log(\cosh(R)e^r) + \chi(r) \log(\cosh(r)).\end{aligned}$$

Note that $\log(\sinh(r))$ is only well-defined for $r > 0$. However, by the definition of χ , the product $\chi(r) \log(\sinh(r))$ vanishes on $(0, R/3]$ so we extend it to be identically zero for every $r \leq 0$.

Set $a_{\text{drill}}(r) = e^{\phi(r)}$ and $b_{\text{drill}}(r) = e^{\psi(r)}$.

This automatically ensures that properties (1), (2), and (3) are satisfied.

Let us now discuss property (4). We have

$$\frac{a'_{\text{drill}}}{a_{\text{drill}}} = \phi', \quad \frac{a''_{\text{drill}}}{a_{\text{drill}}} = (\phi')^2 + \phi'' \quad \text{and} \quad \frac{b'_{\text{drill}}}{b_{\text{drill}}} = \psi', \quad \frac{b''_{\text{drill}}}{b_{\text{drill}}} = (\psi')^2 + \psi''.$$

We now compute the first derivatives

$$\begin{aligned}\phi' &= \chi'(r) \log\left(\frac{\sinh(r)}{\sinh(R)e^r}\right) + \chi(r)(\coth(r) - 1) + 1, \\ \psi' &= \chi'(r) \log\left(\frac{\cosh(r)}{\cosh(R)e^r}\right) + \chi(r)(\tanh(r) - 1) + 1.\end{aligned}$$

The terms with χ' coefficient have the following behavior. As $\sinh(x)/e^x$ is increasing, $\cosh(x)/e^x$ is decreasing, and $r \in [R/3, 2R/3]$, we have

$$\frac{\sinh(r)}{\sinh(R)e^r} \in \left(\frac{\sinh(R/3)}{\sinh(R)e^{R/3}}, 1\right) \quad \text{and} \quad \frac{\cosh(r)}{\cosh(R)e^r} \in \left(\frac{\cosh(2R/3)}{\cosh(R)e^{2R/3}}, 1\right).$$

Recall that $R \in (0, 1)$. On the interval $[0, 1]$, the functions $\frac{\sinh(y/3)}{\sinh(y)e^{y/3}}$, $\frac{\cosh(2y/3)}{\cosh(y)e^{2y/3}}$ are decreasing, so they are bounded from below by

$$c_3 = \frac{\sinh(1/3)}{\sinh(1)e^{1/3}}, \quad c_4 = \frac{\cosh(2/3)}{\cosh(1)e^{2/3}} \in (0, 1).$$

respectively. Thus,

$$\left| \log\left(\frac{\sinh(r)}{\sinh(R)e^r}\right) \right| \leq |\log(c_3)| \quad \text{and} \quad \left| \log\left(\frac{\cosh(r)}{\cosh(R)e^r}\right) \right| \leq |\log(c_4)|.$$

Consider then the terms with χ coefficient. On $[0, 1]$ we have

$$\coth(r) - 1 = \frac{2}{1 - e^{-2r}} \in \left(0, \frac{1}{r}\right) \quad \text{and} \quad \tanh(r) - 1 = -\frac{2}{1 + e^{-2r}} \in (-2, -1).$$

Therefore, recalling that $R \in (0, 1)$, for $r \in [R/3, 2R/3]$, we get

$$|\coth(r) - 1| \leq \frac{3}{R} \quad \text{and} \quad |\tanh(r) - 1| \leq 2.$$

Using the fact $|\chi'| \leq c_1/R$ we get

$$\begin{aligned} |\phi'(r)| &\leq \frac{c_1}{R} |\log(c_3)| + \frac{3}{R} + 1 = \frac{1}{R} (c_1 |\log(c_3)| + 3 + R) \leq \frac{c_5}{R}, \\ |\psi'(r)| &\leq \frac{c_1}{R} |\log(c_4)| + 2 + 1 = \frac{1}{R} (c_1 |\log(c_4)| + 3R) \leq \frac{c_6}{R} \end{aligned}$$

where $c_5 = c_1 |\log(c_3)| + 4$ and $c_6 = c_1 |\log(c_4)| + 3$.

As for the second derivatives, we have

$$\begin{aligned} \phi'' &= \chi''(r) \log \left(\frac{\sinh(r)}{\sinh(R)e^r} \right) + 2\chi'(r)(\coth(r) - 1) - \chi(r) \frac{1}{\sinh(r)^2} \\ \psi'' &= \chi''(r) \log \left(\frac{\cosh(r)}{\cosh(R)e^r} \right) + 2\chi'(r)(\tanh(r) - 1) + \chi(r) \frac{1}{\cosh(r)^2}. \end{aligned}$$

Using the fact $|\chi'| \leq c_1/R$, $|\chi''| \leq c_2/R^2$ and that

$$\frac{1}{\sinh(r)^2} \leq \frac{1}{r^2} \quad \text{and} \quad \frac{1}{\cosh(r)^2} \leq 1,$$

we get

$$\begin{aligned} |\phi''(r)| &\leq \frac{c_2}{R^2} |\log(c_3)| + 6 \frac{c_1}{R^2} + \frac{9}{R^2} = \frac{1}{R^2} (c_2 |\log(c_3)| + 6c_1 R + 9R^2) \leq \frac{c_7}{R^2}, \\ |\psi''(r)| &\leq \frac{c_2}{R^2} |\log(c_4)| + 4 \frac{c_1}{R} + 1 = \frac{1}{R^2} (c_2 |\log(c_4)| + 4c_1 R + R^2) \leq \frac{c_8}{R^2} \end{aligned}$$

where $c_7 = c_2 |\log(c_3)| + 6c_1 + 9$ and $c_8 = c_2 |\log(c_4)| + 4c_1 + 1$.

In conclusion

$$\left| \frac{a''_{\text{drill}}}{a_{\text{drill}}} + \frac{b''_{\text{drill}}}{b_{\text{drill}}} + \frac{a'_{\text{drill}} b'_{\text{drill}}}{a_{\text{drill}} b_{\text{drill}}} \right| = (\phi')^2 + (\psi')^2 + |\phi' \psi'| + |\phi''| + |\psi''| \leq \frac{c}{R^2}$$

where $c = c_5^2 + c_6^2 + c_5 c_6 + c_7 + c_8$.

We need to check (5). Recall that we are assuming $r \leq R$. We have

$$\begin{aligned} \phi(r) + \psi(r) &= (1 - \chi(r)) \log(\sinh(R) \cosh(R) e^{2r}) + \chi(r) \log(\sinh(r) \cosh(r)) \\ &= \log(\sinh(R) \cosh(R) e^{2r}) + \chi(r) \log \left(\frac{\sinh(r) \cosh(r)}{\sinh(R) \cosh(R) e^{2r}} \right) \\ &\leq \log(\sinh(R) \cosh(R) e^{2r}) \end{aligned}$$

as $\chi(r) = 0$ for $r \leq 0$ and $\sinh(r) \cosh(r) / \sinh(R) \cosh(R) e^{2r} < 1$ (for $r \leq R$).

This concludes the proof of the lemma. $\square$

4. FROM CUSPS TO TUBES

In this section, we construct the functions $a_{\text{fill}}, b_{\text{fill}}$ used in the filling case.

Lemma 4.1. *There exists a constant $c > 0$ such that the following holds. Consider $\ell_1 > 2\pi, \ell_2 > 0$. Define $R := \log(\ell_1/\pi) \geq -\log(2)$ and $\kappa := 2\pi\ell_2/\ell_1$. There exist functions $a_{\text{fill}}, b_{\text{fill}} : (-R, \infty) \rightarrow \mathbb{R}$ that satisfy the following properties.*

- (1) $a_{\text{fill}}, b_{\text{fill}} > 0$.
- (2) $a_{\text{fill}}(r) = 2\pi \sinh(r+R)$ and $b_{\text{fill}}(r+R) = \kappa \cosh(r+R)$ on $(-R, -\log(2)/2)$.
- (3) $a_{\text{fill}}(r) = \ell_1 e^r$ and $b_{\text{fill}}(r) = \ell_2 e^r$ on $(-\log(2)/4, 0)$.
- (4) We have

$$\left| \frac{a''_{\text{fill}}}{a_{\text{fill}}} + \frac{b''_{\text{fill}}}{b_{\text{fill}}} + \frac{a'_{\text{fill}} b'_{\text{fill}}}{a_{\text{fill}} b_{\text{fill}}} - 3 \right| \leq c \frac{\pi^4}{\ell_1^4}.$$

- (5) We have

$$a_{\text{fill}} b_{\text{fill}} \leq \ell_1 \ell_2 e^{2r}.$$

As before, let us explain the geometric meaning of the requirements for the metric that we will define later on using the functions $a_{\text{fill}}, b_{\text{fill}}$. Item (2) tells us that the metric is a hyperbolic tube on a region corresponding to $(-R, -\log(2)/2)$. Item (3) makes sure that the metric agrees with a (suitable) hyperbolic cusp on a region corresponding to $(-\log(2)/4, 0)$. The parameters ℓ_1, ℓ_2 describe the shape of the horospherical torus that bounds the cusp (essentially, they will be the sizes of a fundamental rectangle). Item (4) provides a bound on the scalar curvature. Item (5) gives volume bounds.

Proof. Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth non-decreasing function such that $\chi(r) = 0$ on $(-\infty, -\log(2)/2)$ and $\chi(r) = 1$ on $(-\log(2)/4, \infty)$ whose first and second derivatives are bounded by $|\chi'| \leq c_1$ and $|\chi''| \leq c_2$. Define

$$\begin{aligned} \phi(r) &:= (1 - \chi(r)) \log(2\pi \sinh(r+R)) + \chi(r) \log(\ell_1 e^r) \\ \psi(r) &:= (1 - \chi(r)) \log(\kappa \cosh(r+R)) + \chi(r) \log(\ell_2 e^r). \end{aligned}$$

Set $a_{\text{fill}}(r) = e^{\phi(r)}$ and $b_{\text{fill}}(r) = e^{\psi(r)}$.

This automatically ensures that properties (1), (2), and (3) are satisfied.

Let us now discuss property (4). We have

$$\frac{a'_{\text{fill}}}{a_{\text{fill}}} = \phi', \quad \frac{a''_{\text{fill}}}{a_{\text{fill}}} = (\phi')^2 + \phi'' \quad \text{and} \quad \frac{b'_{\text{fill}}}{b_{\text{fill}}} = \psi', \quad \frac{b''_{\text{fill}}}{b_{\text{fill}}} = (\psi')^2 + \psi''.$$

So we need to control three terms. The first is $(\phi')^2 + (\psi')^2$ which we conveniently write as

$$\begin{aligned} (\phi')^2 + (\psi')^2 &= (1 + (\phi' - 1))^2 + (1 + (\psi' - 1))^2 \\ &= 2 + 2[(\phi' - 1) + (\psi' - 1)] + (\phi' - 1)^2 + (\psi' - 1)^2. \end{aligned}$$

The second is $\phi'\psi'$, which we write as

$$\begin{aligned}\phi'\psi' &= (1 + (\phi' - 1))(1 + (\psi' - 1)) \\ &= 1 + [(\phi' - 1) + (\psi' - 1)] + (\phi' - 1)(\psi' - 1).\end{aligned}$$

The third is $\phi'' + \psi''$. Let us begin by computing the first derivative.

$$\begin{aligned}\phi'(r) - 1 &= -\chi'(r) \log \left(\frac{2\pi \sinh(r+R)}{\ell_1 e^r} \right) + (1 - \chi(r))(\coth(r+R) - 1) \\ \psi'(r) - 1 &= -\chi'(r) \log \left(\frac{\kappa \cosh(r+R)}{\ell_2 e^r} \right) + (1 - \chi(r))(\tanh(r+R) - 1).\end{aligned}$$

The terms with $\chi'(r)$ coefficient can be simplified using the definition of $R = \log(\ell_1/\pi)$ and $\kappa = 2\pi\ell_2/\ell_1$ as follows.

$$\frac{2\pi \sinh(r+R)}{\ell_1 e^r} = \frac{\pi e^R}{\ell_1} \left(1 - e^{-2(r+R)} \right) = 1 - e^{-2(r+R)}$$

and

$$\frac{\kappa \cosh(r+R)}{\ell_2 e^r} = \frac{\kappa e^R}{2\ell_2} \left(1 + e^{-2(r+R)} \right) = 1 + e^{-2(r+R)}.$$

The terms with the $1 - \chi(r)$ coefficient can be better understood by writing

$$\coth(r+R) - 1 = \frac{2e^{-2(r+R)}}{1 - e^{-2(r+R)}}$$

and

$$\tanh(r+R) - 1 = -\frac{2e^{-2(r+R)}}{1 + e^{-2(r+R)}}.$$

Putting together the above, we get

$$\begin{aligned}\phi'(r) - 1 &= -\chi'(r) \log \left(1 - e^{-2(r+R)} \right) + (1 - \chi(r)) \frac{2e^{-2(r+R)}}{1 - e^{-2(r+R)}}, \\ \psi'(r) - 1 &= -\chi'(r) \log \left(1 + e^{-2(r+R)} \right) - (1 - \chi(r)) \frac{2e^{-2(r+R)}}{1 + e^{-2(r+R)}}.\end{aligned}$$

Recall that $r \in [-\log(2)/2, -\log(2)/4]$ and observe that

$$e^{-2(r+R)} = \frac{\pi^2}{\ell^2} e^{-2r} < \frac{1}{4} e^{-2r} < \frac{1}{2}.$$

As $|\log(1 \pm x)| \leq x$ for $x \in (0, 1)$ and $1/(1 \pm x) \leq 3/2 \mp 1/2$ for $x \in (0, 1/2)$, we have $|\phi'(r) - 1| \leq c_3 e^{-2R}$ and $|\psi'(r) - 1| \leq c_4 e^{-2R}$ where $c_3 = 2(c_1 + 4)$, $c_4 = 2(c_1 + 2)$ only depends on χ .

Summing the expressions for $\phi'(r) - 1$ and $\psi'(r) - 1$, we get

$$(\phi'(r) - 1) + (\psi'(r) - 1) = -\chi'(r) \log \left(1 - e^{-4(r+R)} \right) + (1 - \chi(r)) \frac{4e^{-4(r+R)}}{1 - e^{-4(r+R)}}.$$

Using the same controls on $\log(1 \pm x)$ and $1/(1 \pm x)$ as above (note that $e^{-4(r+R)} < e^{-2(r+R)}$) we deduce that $|(\phi'(r) - 1) + (\psi'(r) - 1)| \leq c_5 e^{-4R}$ where $c_5 = 4(c_1 + 8)$ only depends on χ .

As for the second derivatives, we have

$$\begin{aligned}\phi''(r) &= -\chi''(r) \log\left(\frac{2\pi \sinh(r+R)}{\ell_1 e^r}\right) - 2\chi'(r)(\coth(r+R) - 1) - (1 - \chi(r)) \frac{1}{\sinh(r+R)^2}, \\ \psi''(r) &= -\chi''(r) \log\left(\frac{\kappa \cosh(r+R)}{\ell_2 e^r}\right) - 2\chi'(r)(\tanh(r+R) - 1) + (1 - \chi(r)) \frac{1}{\cosh(r+R)^2}.\end{aligned}$$

Simplifying the expressions as above, we obtain

$$\begin{aligned}\phi''(r) &= -\chi''(r) \log\left(1 - e^{-2(r+R)}\right) - 2\chi'(r) \frac{2e^{-2(r+R)}}{1 - e^{-2(r+R)}} - (1 - \chi(r)) \left(\frac{2e^{-(r+R)}}{1 - e^{-2(r+R)}}\right)^2, \\ \psi''(r) &= -\chi''(r) \log\left(1 + e^{-2(r+R)}\right) + 2\chi'(r) \frac{2e^{-2(r+R)}}{1 + e^{-2(r+R)}} + (1 - \chi(r)) \left(\frac{2e^{-(r+R)}}{1 + e^{-2(r+R)}}\right)^2.\end{aligned}$$

Summing the two terms, we get

$$\phi''(r) + \psi''(r) = -\chi''(r) \log\left(1 - e^{-4(r+R)}\right) - 2\chi'(r) \frac{4e^{-4(r+R)}}{1 - e^{-4(r+R)}} - (1 - \chi(r)) \frac{16e^{-4(r+R)}}{(1 - e^{-4(r+R)})^2}.$$

Again, using $|\log(1 \pm x)| \leq x$ and $1/(1 \pm x) \leq 2$ we deduce that there is a constant $c_6 = 4(c_2 + 16c_1 + 32)$ such that $|\phi''(r) + \psi''(r)| \leq c_6 e^{-4R}$. Putting all the terms together, we conclude the following inequality

$$\begin{aligned}&|(\phi')^2 + (\psi')^2 + \phi'\psi' + \phi'' + \psi'' - 3| \\ &= |2[(\phi' - 1) + (\psi' - 1)] + (\phi' - 1)^2 + (\psi' - 1)^2 + [(\phi' - 1) + (\psi' - 1)] + (\phi' - 1)(\psi' - 1) + \phi'' + \psi''| \\ &\leq 3|(\phi' - 1) + (\psi' - 1)| + |\phi' - 1||\psi' - 1| + |\phi' - 1|^2 + |\psi' - 1|^2 + |\phi'' + \psi''| \\ &\leq (3c_5 + c_3c_4 + c_3^2 + c_4^2 + c_6)e^{-4R} = c\pi^4/\ell^4\end{aligned}$$

where $c = 3c_5 + c_3c_4 + c_3^2 + c_4^2 + c_6$ only depends on χ .

Lastly, we need to check (5). We have

$$\begin{aligned}\phi(r) + \psi(r) &= (1 - \chi(r)) \log(2\pi\kappa \sinh(r+R) \cosh(r+R)) + \chi(r) \log(\ell_1 \ell_2 e^{2r}) \\ &= (1 - \chi(r)) \log\left(\frac{2\pi\kappa e^{2R}}{4} e^{2r} (1 - e^{-4(r+R)})\right) + \chi(r) \log(\ell_1 \ell_2 e^{2r}).\end{aligned}$$

Using $\frac{2\pi\kappa e^{2R}}{4} = \ell_1 \ell_2$ we continue the chain of equalities as

$$\begin{aligned}&= (1 - \chi(r)) \log\left(\ell_1 \ell_2 e^{2r} (1 - e^{-4(r+R)})\right) + \chi(r) \log(\ell_1 \ell_2 e^{2r}) \\ &= \log(\ell_1 \ell_2 e^{2r}) + (1 - \chi(r)) \log\left(1 - e^{-4(r+R)}\right) \leq \log(\ell_1 \ell_2 e^{2r}).\end{aligned}$$

This concludes the proof of the lemma. $\square$

5. METRICS ON TUBES

Having defined the functions $a_{\text{fill}}, b_{\text{fill}}$, we use them to prove the following extension result (which might be of independent interest).

Proposition 5.1. *There exists a constant $c_{\text{scal}} > 0$ such that the following holds. Let V be a solid torus. Suppose that the boundary ∂V is equipped with a flat metric of area A . Assume that the length of the flat geodesic representative on ∂V of the meridian $\mu \subset \partial V$ is $\ell > 2\pi$. Then there exists a smooth Riemannian metric (V, g) with the following properties.*

- *The metric g is a hyperbolic cusp metric in a collar of ∂V , the boundary ∂V is a horospherical section of the cusp, and the restriction of g to the boundary agrees with the prescribed flat metric on ∂V .*
- *Denote by S_g the scalar curvature of the metric g . We have*

$$\int_V \left(\frac{|S_g|}{6} \right)^{3/2} \text{dvol}_g \leq \frac{A}{2} \left(1 - \frac{\pi^2}{\ell^2} \right) \left(1 + c_{\text{scal}} \frac{\pi^4}{\ell^4} \right).$$

Proposition 5.1 can be seen as a quantitative version of the 2π -Theorem (compare with [BH96, Theorem 9], [CL98, Proposition 3.1], and [FKP08, Theorem 2.1]). It will be used in the proofs of both Theorem A and Theorem C.

Proof. Let $R := \log(\ell/\pi)$.

Consider the normal covering $W \rightarrow \partial V$ corresponding to $\ker(\pi_1(\partial V) \rightarrow \pi_1(V)) = \mathbb{Z}\mu$ (the loop $\mu \subset \partial V$ is a flat representative of the meridian). By covering theory, the meridian μ lifts to a simple closed curve on W . We denote by $\delta : W \rightarrow W$ a generator of the deck group $\pi_1(\partial V)/\mathbb{Z}\mu$, so that $\partial V = W/\langle \delta \rangle$.

Equipped with the induced flat metric, W is a flat cylinder which we can isometrically identify with

$$(\mathbb{S}^1 \times \mathbb{R}, \ell_1^2 d\theta^2 + dy^2)$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Since μ lifts homeomorphically to a simple closed geodesic on W of length ℓ , we must have $\ell_1 = \ell$. Under the identification $W \simeq \mathbb{S}^1 \times \mathbb{R}$ the deck transformation δ translates into the isometry

$$\tau : (\theta, y) \rightarrow (\theta + \alpha, y + s)$$

and we have $\partial V = W/\langle \delta \rangle = (\mathbb{S}^1 \times \mathbb{R})/\langle \tau \rangle$. In order to compute s , we observe the following. Every fundamental domain of τ has area equal to A , the area of ∂V . By the explicit formula for τ , a fundamental domain is given by $\mathbb{S}^1 \times [0, s]$ which has area $\ell_1 s$. Thus $s = A/\ell_1 = A/\ell$.

By the above discussion, the isometric diffeomorphism

$$f : (\mathbb{S}^1 \times \mathbb{R})/\langle \tau \rangle \rightarrow \partial V$$

maps $\mathbb{S}^1 \times \{0\}$ to a parallel copy of μ , the geodesic representative on ∂V of the meridian. We now extend this diffeomorphism to a diffeomorphism of a suitable

solid torus with boundary $(\mathbb{S}^1 \times \mathbb{R})/\langle \tau \rangle$ and V . In order to do so, observe that τ extends to a diffeomorphism of

$$U := (\mathbb{S}^1 \times \mathbb{R} \times [-R, 0]) / (\theta, y, -R) \sim (\theta', y, -R)$$

which we keep denoting by τ with a slight abuse of notation, by the formula

$$\tau(\theta, y, r) = (\theta + \alpha, y + s, r).$$

The group $\langle \tau \rangle$ generated by this extension acts freely and properly discontinuously on U . The quotient $U/\langle \tau \rangle$ is a solid torus with boundary $(\mathbb{S}^1 \times \mathbb{R} \times \{0\})/\langle \tau \rangle$ and meridian given by $\mathbb{S}^1 \times \{0\} \times \{0\}$. As $f : \partial U/\langle \tau \rangle \rightarrow \partial V$ maps the meridian of the source to the meridian of the target, it can be extended to a diffeomorphism $U/\langle \tau \rangle \rightarrow V$.

From now on we work on U whose boundary $\mathbb{S}^1 \times \mathbb{R} \times \{0\}$ is equipped with the metric $\ell_1^2 d\theta^2 + dy^2$. We will consider metrics of the form $g = dr^2 + a(r)^2 d\theta^2 + b(r)^2 dy^2$. All these metrics are τ -invariant. Hence, they descend to Riemannian metrics on $U/\langle \tau \rangle$. Note that g is potentially singular at the $r = -R$ locus (which is a line). It is well-known that g is smooth everywhere if $a'(-R) = 2\pi$ (see [BH96] or [FKP08]). We choose $a(r) = a_{\text{fill}}(r)$ and $b(r) = b_{\text{fill}}(r)$ as given by Lemma 4.1 with input $\ell_1 = \ell$ and $\ell_2 = 1$. Note that property (2) of Lemma 4.1 guarantees that $a'_{\text{fill}}(-R) = 2\pi$. Note also that property (4) of Lemma 4.1 together with Lemma 2.5 implies that the scalar curvature S_g of the resulting metric satisfies

$$\left| \left(\frac{|S_g|}{6} \right)^{3/2} - 1 \right| \leq \left| \left(1 + 2c \frac{\pi^4}{\ell^4} \right)^{3/2} - 1 \right|.$$

We bound the right-hand side as follows. By the mean value theorem, for $x \geq 0$ we have $(1+x)^{3/2} - 1 = \frac{3}{2}(\sqrt{1+x_0})x$ for some $0 \leq x_0 \leq x$. Thus, recalling that $\ell > 2\pi$,

$$\left| \left(1 + 2c \frac{\pi^4}{\ell^4} \right)^{3/2} - 1 \right| \leq \frac{3}{2} \left(\sqrt{1 + 2c \frac{\pi^4}{\ell^4}} \right) 2c \frac{\pi^4}{\ell^4} \leq 3c \sqrt{1 + c/8} \frac{\pi^4}{\ell^4}.$$

Set $c_{\text{scal}} := 3c\sqrt{1 + c/8}$. We now compute

$$\int_V \left| \frac{S_g}{6} \right|^{3/2} d\text{vol}_g.$$

As $|S_g|/6$ is close to 1, we first compare the integral with $\text{vol}(V, g)$ controlling the difference. Then we show that $\text{vol}(V, g)$ is bounded by $(1 - \pi^2/\ell^2)A/2$. In order to do these computations, we work on the fundamental domain $F := \mathbb{S}^1 \times [0, s] \times [-R, 0]$ for the action of τ on $\mathbb{S}^1 \times \mathbb{R} \times [-R, 0]$. We have

$$\left| \int_V \left(\frac{|S_g|}{6} \right)^{3/2} d\text{vol}_g - \text{vol}(V, g) \right| \leq \int_V \left| \left(\frac{|S_g|}{6} \right)^{3/2} - 1 \right| d\text{vol}_g \leq c' \frac{\pi^4}{\ell^4} \text{vol}(V, g).$$

Let us compute the volume. By Lemma 4.1 (5), we have $a_{\text{fill}} b_{\text{fill}} \leq \ell e^{2r}$. Thus

$$\begin{aligned} \text{vol}(V, g) &= \text{vol}(F, g) \\ &= \int_0^{A/\ell} \int_0^1 \int_{-R}^0 a_{\text{fill}} b_{\text{fill}} \, dr \, d\theta \, dy \\ &= \frac{A}{\ell} \int_{-R}^0 a_{\text{fill}} b_{\text{fill}} \, dr \\ &\leq \frac{A}{\ell} \int_{-R}^0 \ell e^{2r} \, dr = \frac{A}{2} \left(1 - \frac{\pi^2}{\ell^2} \right). \end{aligned}$$

In conclusion

$$\begin{aligned} \int_V \left(\frac{|S_g|}{6} \right)^{3/2} \, d\text{vol}_g &= \left(\int_V \left(\frac{|S_g|}{6} \right)^{3/2} \, d\text{vol}_g - \text{vol}(V, g) \right) + \text{vol}(V, g) \\ &\leq \left(1 + c_{\text{scal}} \frac{\pi^4}{\ell^2} \right) \text{vol}(V, g) \leq \left(1 + c_{\text{scal}} \frac{\pi^4}{\ell^2} \right) \frac{A}{2} \left(1 - \frac{\pi^2}{\ell^2} \right) \end{aligned}$$

as desired. $\square$

6. PROOF OF THEOREM A

We begin by recalling the setup. We consider (M, g_M) a closed hyperbolic 3-manifold and a simple closed geodesic $\gamma \subset M$ in it. Under the assumption that γ has an embedded tubular neighborhood T of radius at least $R \in (0, 1)$ we are going to show that the unique finite volume hyperbolic metric on $M - \gamma$ has volume at most $\text{vol}(M, g_M) + c_{\text{drill}} \ell / R$ where ℓ is the length of γ and c_{drill} is a universal constant.

Observe that the tubular neighborhood T of γ can be parametrized by the so-called Fermi coordinates, that is, on T the metric of M can be written as

$$g_M = dr^2 + 4\pi^2 \sinh(r)^2 d\theta^2 + \ell^2 \cosh(r)^2 dy^2$$

where $y \in [0, 1]$ is a length parameter along γ while $r \in [0, R], \theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ are respectively the radial and angular parameters around γ .

Heuristically speaking, the basic picture is as follows. Ideally, one should have $\sigma(M - \gamma) = -6^{3/2} \text{vol}(M - \gamma)^{2/3}$ (similar to Theorem 2.3) and would like to exploit the inequality $\sigma(M - \gamma) \geq -(\int_{M-\gamma} |S_g|^{3/2} d\text{vol}_g)^{2/3}$ (similar to Proposition 2.2) to compare $\text{vol}(M - \gamma)$ to the integral of the scalar curvature for a Riemannian metric we obtain by modifying the hyperbolic metric g_M on $T - \gamma$ (replacing with one of those defined using Lemma 3.1). This strategy cannot be carried out exactly as stated since Theorem 2.3 and Proposition 2.2 only work for closed manifolds ($M - \gamma$ has a cusp). However, this is just a technical obstacle that we will bypass using a Dehn filling trick (as in [ASTD07]).

Let us start with the proof. It is convenient to define the following. We will call $[r_0, r_1]$ -shell the submanifold of T that in Fermi coordinates corresponds to the region where the radial coordinate r varies in the interval $[r_0, r_1]$.

Fix $\epsilon > 0$ arbitrary. According to our strategy, we replace the metric g_M on the $[\epsilon, R]$ -shell S_ϵ with the metric

$$g_{S_\epsilon} := dr^2 + 4\pi^2 a_{\text{drill}}(r)^2 d\theta^2 + \ell^2 b_{\text{drill}}(r)^2 dy^2$$

where $a_{\text{drill}}, b_{\text{drill}}$ are the functions provided by Lemma 3.1 with input R . Note that g_{S_ϵ} coincides with g_M on the $[2R/3, R]$ -shell and with

$$g_C := dr^2 + 4\pi^2 \sinh(R)^2 e^{2r} d\theta^2 + \ell^2 \cosh(R)^2 e^{2r} dy^2$$

in the $[\epsilon, R/3]$ -shell. The metric g_C naturally extends to every $r \leq \epsilon$ with the same formula. The extension is isometric to a hyperbolic cusp where the slices $r = \text{constant}$ are horospherical tori.

The next step is just a trick to remedy the fact that the sigma invariant computation only works for closed hyperbolic manifolds. What we do is to realize $M - \gamma$ as a geometric limit of closed hyperbolic manifolds using long Dehn fillings. By standard properties of geometric convergence, the volume of $M - \gamma$ can be computed as the limit of the volumes of the approximating hyperbolic manifolds.

Consider the smaller tube $T_\epsilon := T - S_\epsilon$. Choose an infinite sequence of pairwise distinct essential simple closed curves μ_n on the torus ∂T_ϵ and define M_n to be the Dehn filling of $M - T_\epsilon$ with filling slope μ_n , that is

$$M_n = (M - T_\epsilon) \cup_{\mu_n} \mathbb{D}^2 \times \mathbb{S}^1$$

where the gluing map $\mathbb{D}^2 \times \mathbb{S}^1 \rightarrow \partial T_\epsilon$ is completely determined (up to isotopy) by the requirement that a meridian of $\mathbb{D}^2 \times \mathbb{S}^1$ is mapped to μ_n . By Thurston's Hyperbolic Dehn Filling Theorem (see for example [Mar22, Chapter 15]), for large enough n we have that M_n admits a hyperbolic metric g_{M_n} whose volume $\text{vol}(M_n, g_{M_n})$ converges to the volume of $(M - \gamma, g_{M-\gamma})$ where $g_{M-\gamma}$ denotes the hyperbolic metric on $M - \gamma$.

By Proposition 5.1, for every n large enough we can smoothly extend the metric g_C to a metric $g_{\mathbb{D}^2 \times \mathbb{S}^1, n}$ on the added solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ so that

$$\limsup \int_{\mathbb{D}^2 \times \mathbb{S}^1} |S_{g_n}|^{3/2} d\text{vol}_{g_n} \leq 6^{3/2} \cdot \frac{\text{Area}(\partial T_\epsilon, g_C)}{2} = e^{2\epsilon} \sinh(R) \cosh(R) \pi \ell.$$

By Theorem 2.3, we have $-6 \cdot \text{vol}(M_n, g_{M_n})^{2/3} = \sigma(M_n)$.

By Proposition 2.2 we have

$$\sigma(M_n) \geq - \left(\int_{M_n} |S_{g_n}|^{3/2} d\text{vol}_{g_n} \right)^{2/3}$$

for any Riemannian metric g_n on M_n . In particular, this holds for the Riemannian metric defined by

$$g_n := \begin{cases} g_M & \text{on } M - T, \\ g_{S_\epsilon} & \text{on } S_\epsilon, \\ g_{\mathbb{D}^2 \times \mathbb{S}^1, n} & \text{on } \mathbb{D}^2 \times \mathbb{S}^1. \end{cases}$$

Thus,

$$|\sigma(M_n)|^{3/2} = 6^{3/2} \cdot \text{vol}(M_n, g_{M_n}) \leq \int_{M_n} |S_{g_n}|^{3/2} d\text{vol}_{g_n}.$$

We split the integral as

$$\begin{aligned} \int_{M_n} |S_{g_n}|^{3/2} d\text{vol}_{g_n} &= \int_{M-T} |S_{g_M}|^{3/2} d\text{vol}_{g_M} + \int_{S_\epsilon} |S_{g_S}|^{3/2} d\text{vol}_{g_S} \\ &\quad + \int_{\mathbb{D}^2 \times \mathbb{S}^1} |S_{g_{\mathbb{D}^2 \times \mathbb{S}^1, n}}|^{3/2} d\text{vol}_{g_{\mathbb{D}^2 \times \mathbb{S}^1, n}}. \end{aligned}$$

As the metric g_n coincides with the hyperbolic metric on $M - T$, the first term is

$$\int_{M-T} |S_g|^{3/2} d\text{vol}_g = 6^{3/2} \cdot \text{vol}(M - T) \leq 6^{3/2} \text{vol}(M).$$

On S_ϵ , we compute

$$\begin{aligned} &\int_0^\ell \int_0^1 \int_\epsilon^R |S_g|^{3/2} a_{\text{drill}} b_{\text{drill}} dr d\theta dy \\ &\leq 2\pi\ell(2c)^{3/2} \frac{1}{R^3} \int_\epsilon^R a_{\text{drill}} b_{\text{drill}} dr && \text{by Lemma 3.1 (4)} \\ &\leq 2\pi\ell(2c)^{3/2} \frac{1}{R^3} \int_\epsilon^R \sinh(R) \cosh(R) e^{2r} dr && \text{by Lemma 3.1 (5)} \\ &= (2c)^{3/2} \pi\ell \frac{\sinh(R) \cosh(R)}{R^3} (e^{2R} - e^{2\epsilon}). \end{aligned}$$

Lastly, on $\mathbb{D}^2 \times \mathbb{S}^1$ we have

$$\limsup \int_{\mathbb{D}^2 \times \mathbb{S}^1} |S_{g_{\mathbb{D}^2 \times \mathbb{S}^1, n}}|^{3/2} d\text{vol}_{g_{\mathbb{D}^2 \times \mathbb{S}^1, n}} \leq 6^{3/2} e^{2\epsilon} \sinh(R) \cosh(R) \pi\ell.$$

Putting together the three terms and dividing by $6^{3/2}$, we obtain

$$\text{vol}(M_n, g_{M_n}) \leq \text{vol}(M) + \frac{(2c)^{3/2}}{6^{3/2}} \pi\ell \frac{\sinh(R) \cosh(R)}{R^3} (e^{2R} - e^{2\epsilon}) + e^{2\epsilon} \sinh(R) \cosh(R) \pi\ell$$

Passing to the limit as $n \rightarrow \infty$ and then as $\epsilon \rightarrow 0$, we get

$$\text{vol}(M - \gamma) \leq \text{vol}(M) + \pi \frac{\ell}{R} \left(\frac{(2c)^{3/2}}{6^{3/2}} \frac{\sinh(R) \cosh(R) (e^{2R} - 1)}{R^2} + \frac{\sinh(R) \cosh(R)}{R} \right).$$

In order to conclude, we just observe that $\sinh(R) \cosh(R) (e^{2R} - 1)/R^2 \leq c_9$ and that $\sinh(R) \cosh(R)/R \leq c_{10}$ for $R \in (0, 1)$ (where c_9 can be chosen to be roughly $c_9 = 11.59$ and $c_{10} = 1.81$). This concludes the proof of Theorem A. $\square$

7. THE PROOF OF THEOREM C

Let us start with the setup. We consider a complete finite volume hyperbolic 3-manifold (N, g_N) with one cusp. We also fix a cusp neighborhood C bounded by the flat torus ∂C and a simple closed (flat) geodesic $\mu \subset \partial C$. We are going to show that if $\ell := \text{length}(\mu) > \sqrt{c_{\text{fill}}} \pi$, where $c_{\text{fill}} > 0$ is a universal constant, then the

Dehn filling N_μ of $N - C$ with slope μ has a unique hyperbolic metric (N_μ, g_{N_μ}) of volume at most

$$\text{vol}(N, g_N) - \frac{\pi^2}{2L^2} \left(1 - c_{\text{fill}} \frac{\pi^2}{AL^2} \right).$$

Consider the cusp neighborhood C bounded by the horospherical torus ∂C . By Proposition 5.1, we can find a metric g_μ on the Dehn filling

$$N_\mu = (N - C) \cup_\mu \mathbb{D}^2 \times \mathbb{S}^1$$

that agrees with the hyperbolic metric of M on $M - C$ and satisfies

$$\int_{\mathbb{D}^2 \times \mathbb{S}^1} |S_g|^{3/2} d\text{vol}_g \leq 6^{3/2} \frac{A}{2} \left(1 - \frac{\pi^2}{\ell^2} \right) \left(1 + c_{\text{scal}} \frac{\pi^4}{\ell^4} \right).$$

By Theorem 2.3, we have $-6 \cdot \text{vol}(N_\mu, g_{N_\mu})^{2/3} = \sigma(N_\mu)$.

By Proposition 2.2 we have

$$\sigma(N_\mu) \geq - \left(\int_{N_\mu} |S_g|^{3/2} d\text{vol}_g \right)^{2/3}.$$

for every Riemannian metric g on N_μ . In particular

$$|\sigma(N_\mu)|^{3/2} = 6^{3/2} \cdot \text{vol}(N_\mu) \leq \int_{N_\mu} |S_g|^{3/2} d\text{vol}_g.$$

We use the metric $g = g_\mu$ and compute the integral

$$\int_{N_\mu} |S_{g_\mu}|^{3/2} d\text{vol}_{g_\mu} = \int_{N-C} |S_{g_N}|^{3/2} d\text{vol}_{g_N} + \int_{\mathbb{D}^2 \times \mathbb{S}^1} |S_{g_\mu}|^{3/2} d\text{vol}_{g_\mu}.$$

As the metric g_μ coincides with the hyperbolic metric on $N - C$, the first term is

$$\int_{N-C} |S_g|^{3/2} d\text{vol}_g = 6^{3/2} \cdot \text{vol}(N - C) = 6^{3/2} \left(\text{vol}(N) - \frac{A}{2} \right).$$

By Proposition 5.1, on $\mathbb{D}^2 \times \mathbb{S}^1$ we have

$$\int_{\mathbb{D}^2 \times \mathbb{S}^1} |S_g|^{3/2} d\text{vol}_g \leq 6^{3/2} \frac{A}{2} \left(1 - \frac{\pi^2}{\ell^2} \right) \left(1 + c_{\text{scal}} \frac{\pi^4}{\ell^4} \right).$$

Putting together the terms and dividing by $6^{3/2}$, we obtain

$$\text{vol}(N_\mu) + \frac{A}{2} \left(\frac{\pi^2}{\ell^2} - c_{\text{scal}} \frac{\pi^4}{\ell^4} + c_{\text{scal}} \frac{\pi^6}{\ell^6} \right) \leq \text{vol}(N).$$

This concludes the proof of Theorem C. $\square$

REFERENCES

- [Ada02] Colin C. Adams, *Waist size for cusps in hyperbolic 3-manifolds*, Topology **41** (2002), no. 2, 257–270 (English).
- [Ago02] Ian Agol, *Volume change under drilling*, Geom. Topol. **6** (2002), 905–916.
- [And06] Michael T. Anderson, *Canonical metrics on 3-manifolds and 4-manifolds*, Asian J. Math. **10** (2006), no. 1, 127–164.
- [ASTD07] Ian Agol, Peter A. Storm, William P. Thurston, and Nathan Dunfield, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, J. Am. Math. Soc. **20** (2007), no. 4, 1053–1077.
- [BH96] Steven A. Bleiler and Craig D. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996), no. 3, 809–833.
- [Bri98] Martin Bridgeman, *Bounds on volume increase under Dehn drilling operations*, Proc. Lond. Math. Soc. (3) **77** (1998), no. 2, 415–436.
- [CL98] Daryl Cooper and Marc Lackenby, *Dehn surgery and negatively curved 3-manifolds*, J. Differ. Geom. **50** (1998), no. 3, 591–624.
- [FKP08] David Futer, Efstratia Kalfagianni, and Jessica S. Purcell, *Dehn filling, volume, and the Jones polynomial*, J. Differ. Geom. **78** (2008), no. 3, 429–464.
- [FPS22] David Futer, Jessica S. Purcell, and Saul Schleimer, *Effective bilipschitz bounds on drilling and filling*, Geom. Topol. **26** (2022), no. 3, 1077–1188.
- [GMT03] David Gabai, Robert Meyerhoff, and Nathaniel Thurston, *Homotopy hyperbolic 3-manifolds are hyperbolic*, Ann. Math. (2) **157** (2003), no. 2, 335–431.
- [Har19] Robert C. III Haraway, *Practical bounds for a Dehn parental test*, Proc. Am. Math. Soc. **147** (2019), no. 1, 427–442.
- [HK05] Craig D. Hodgson and Steven P. Kerckhoff, *Universal bounds for hyperbolic Dehn surgery*, Ann. Math. (2) **162** (2005), no. 1, 367–421.
- [KL08] Bruce Kleiner and John Lott, *Notes on Perelman’s papers*, Geom. Topol. **12** (2008), no. 5, 2587–2855.
- [Mag12] Aaron D. Magid, *Deformation spaces of Kleinian surface groups are not locally connected*, Geom. Topol. **16** (2012), no. 3, 1247–1320.
- [Mar22] Bruno Martelli, *An introduction to geometric topology*, Independently published, 2022.
- [NZ85] Walter D. Neumann and Don Zagier, *Volumes of hyperbolic three-manifolds*, Topology **24** (1985), 307–332.
- [Per02] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, arXiv:math/0211159 [math.DG] (2002)., 2002, Id/No 0211159.
- [Per03] ———, *Ricci flow with surgery on three-manifolds*, Preprint, arXiv:math/0303109 [math.DG] (2003)., 2003, Id/No 0303109.
- [Thu80] William P. Thurston, *The geometry and topology of three-manifolds*, <http://library.msri.org/books/gt3m/>, 1980, Princeton University lecture notes.

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