RAYS OF POLYNOMIAL MAXIMAL SURFACES IN AdS_3

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INTRODUCTION

A key object of study in mathematical physics is anti de-Sitter space, a class of manifolds the provides important theoretical framework for several prominent fields.

Previous research [1] has established a homeomorphism between the space of maximal surfaces in anti-de Sitter space and polynomial quadratic differentials over $\mathbb{C}$. In this project we studied the limiting behavior of this correspondence by fixing a polynomial $p(z)$, and studying the surface associated with $tp(z)dz^2$ as $t \in \mathbb{R}$ tends to infinity.

Theorem A *Let $q = p(z)dz^2$ be a polynomial quadratic differential. There exists a nonnegative integer k so that the surface associated with tq converges to the surface associated with $w^k dw^2$ as $t \in \mathbb{R}$ tends to infinity.*

ANTI-DE SITTER GEOMETRY

We begin by endowing $\mathbb{R}^4$ with the bilinear form

$$< x, y >= x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$$

for all $x, y \in \mathbb{R}^4$. Anti-de Sitter Space is then defined as the quadric

$$\widehat{AdS_3} = \{x \in \mathbb{R}^4 | < x, x >= -1\}$$

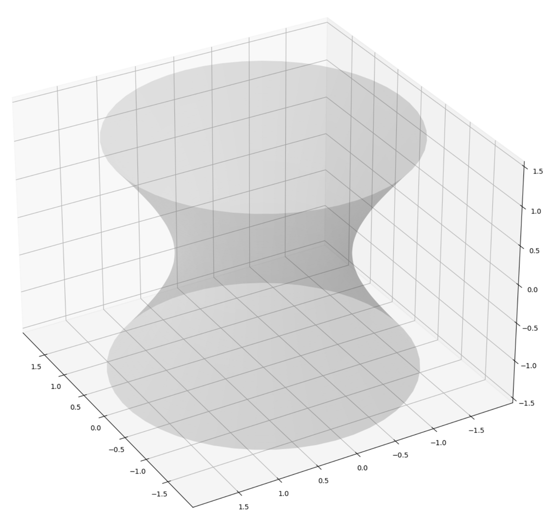
This is brought into three dimensions by being mapped to $\mathbb{RP}^3$, where we say that two vectors $v, w \in \mathbb{R}^4$ map to the same point iff there is a nonzero real scalar λ such that $v = \lambda w$. We can then define the projective mapping

$$\mathbb{P} : \mathbb{R}^4 \backslash \{0\} \longrightarrow \mathbb{RP}^3$$

where the image of the restriction of $\mathbb{P}$ to $Ad\hat{S}_3$ can be called AdS_3. If we then look at the set of points $\{(x_1, x_2, x_3, x_4) \in \mathbb{RP}^3 | \, x_4 \neq 0\}$, then there is a unique representative such that $x_4 = 1$, and thus we can identify it with $\mathbb{R}^3$ via the mapping

$$(x_1, x_2, x_3, x_4) \longrightarrow \left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right)$$

giving us a representation of AdS_3 that fills the interior of the quadric $x^2 + y^2 - z^2 = 1$ in $\mathbb{R}^3$, which looks like:



MAXIMAL SURFACES IN $\widehat{AdS_3}$

Maximal surfaces in $\widehat{AdS_3}$ are surfaces with zero mean curvature. We can define such surfaces using the first I and second fundamental forms $I\!I$. Given a parameterization f of a maximal surface S in AdS_3, we set

$$I = \begin{pmatrix} \langle f_x(p), f_x(p) \rangle & \langle f_x(p), f_y(p) \rangle \\ \langle f_x(p), f_y(p) \rangle & \langle f_y(p), f_y(p) \rangle \end{pmatrix}.$$

We will restrict our attention to conformal surfaces, i.e where I is equal to $2e^{2u}$ times the identity matrix (the notation $2e^{2u}$ will be important). We define the second fundamental form as

$$I\!I = \begin{pmatrix} \langle f_{xx}(p), N \rangle & \langle f_{xy}(p), N \rangle \\ \langle f_{yx}(p), N \rangle & \langle f_{yy}(p), N \rangle \end{pmatrix}$$

where N is defined as the unit orthogonal to f, f_x, and f_y so that the determinant of the matrix (f_x, f_y, N, f) is $2e^{2u}$. The surface is maximal when $trace(I\!I) = 0$. There, the second fundamental form is the real part of a holomorphic quadratic differential $q = h(z)dz^2$:

$$I\!I = 2Re(q) = \begin{pmatrix} 2Re(h) & -2Im(h) \\ -2Im(h) & -2Re(h) \end{pmatrix}$$

The functions u and h are related by the PDE

$$\Delta u = -\frac{1}{2}e^{2u}(I\!I_{1,1}^2 + I\!I_{1,2}^2) + 2e^{2u}$$

for u, which arises as integrability condition of the ODEs

$$\partial_x \mathcal{F} = \mathcal{F} \begin{pmatrix} 0 & u_y & (e^{-u}/\sqrt{2})I\!I_{1,1} & e^u\sqrt{2} \\ -u_y & 0 & (e^{-u}/\sqrt{2})I\!I_{1,2} & 0 \\ (e^{-u}/\sqrt{2})I\!I_{1,1} & (e^{-u}/\sqrt{2})I\!I_{1,2} & 0 & 0 \\ e^u\sqrt{2} & 0 & 0 & 0 \end{pmatrix}$$

$$\partial_y \mathcal{F} = \mathcal{F} \begin{pmatrix} 0 & -u_x & (e^{-u}/\sqrt{2})I\!I_{1,2} & 0 \\ u_x & 0 & -(e^{-u}/\sqrt{2})I\!I_{1,1} & e^u\sqrt{2} \\ (e^{-u}/\sqrt{2})I\!I_{1,2} & (e^{-u}/\sqrt{2})I\!I_{1,1} & 0 & 0 \\ 0 & e^u\sqrt{2} & 0 & 0 \end{pmatrix}$$

Therefore, we can construct a surface with given quadratic differential by solving the PDE first, then the ODE, and then taking the last column of the solution of the ODE.

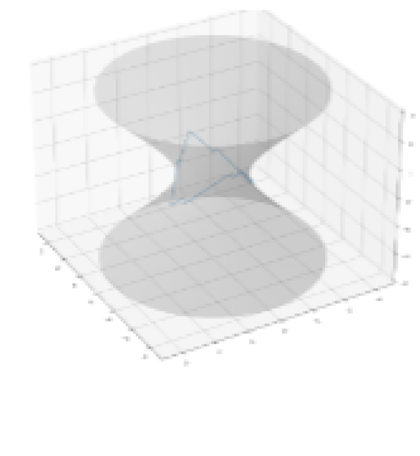
REFERENCES

[1] Andrea Tamburelli. Polynomial quadratic differentials on the complex plane and light-like polygons in the Einstein Universe. *Adv. Math.*, 352C:483–515, 2019.

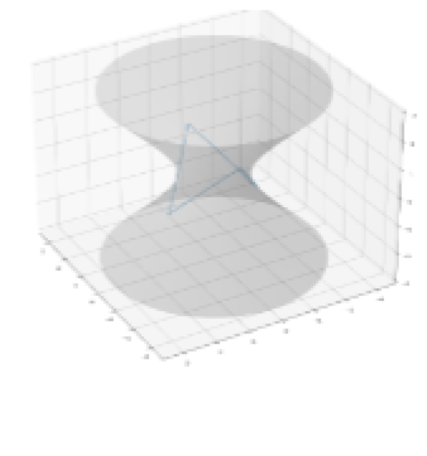
POLYGONS FROM POLYNOMIALS

As we take $t \to \infty$ and consider the maximal surface associated with the polynomial quadratic differential $q_t = tq = th(z)dz^2$, we see that the surface eventually behaves like that of a monomial. To see this, note that we can change variables to $w = az + b$ for fixed $a, b \in \mathbb{R}$, and the resulting surface is equivalent up to a transformation in $SO(2, 2)$. From this, we can show there is a unique $\alpha > 0$ such that in the coordinate $w = (a_k t)^\alpha z$ we have $q_t = \hat{q}_t(w)dw^2 \to w^k dw^2$ as $t \to \infty$.

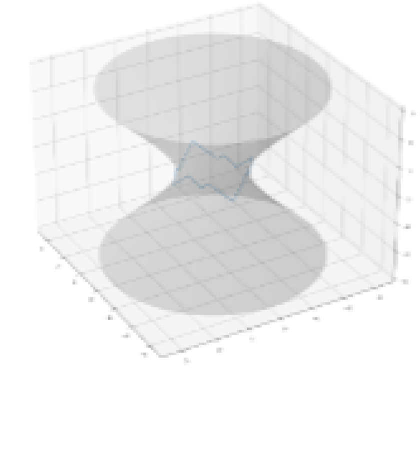
Using the sub/super solution technique for PDEs, we are able to prove that this convergence of differentials passes to a convergence of solutions to u.



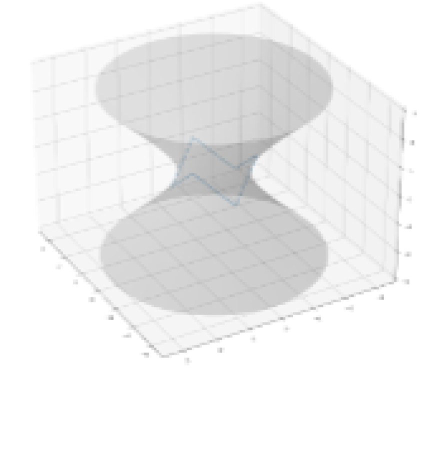
(A) Boundary for $p(z) = z^5 + 1$



(B) Boundary for $p(z) = 30000(z^5 + 1)$



(A) Boundary for $p(z) = z^5 + z$



(B) Boundary for $p(z) = 30000(z^5 + z)$

PROGRAM DETAILS

To produce these images, the program solves for the PDE

$$\Delta u_t = 2e^{2u_t} - 2e^{-2u_t}|q_t|^2$$

by perturbation of the test function $\frac{1}{4}\log(|q|^2)$. To center the polynomial at the origin, it is transformed to eliminate the z^{n-1} term, where n is the degree of q, as described in the previous section. It then solves the ODE's in the "Maximal Surfaces in $\widehat{AdS_3}$" section, where the maximal surface is extrapolated as the last column of $\mathcal{F}$ and the vertices of the polygon on the boundary are the eigenvectors with the highest associated eigenvalues along the directions $\frac{k\pi}{n+2}$.

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