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ANTI-DE SITTER GEOMETRY: CONVEX DOMAINS, FOLIATIONS AND VOLUME

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Abstract

We study various aspects of the geometry of globally hyperbolic anti-de Sitter 3-manifolds.
For manifolds with convex space-like boundaries, homeomorphic to the product of a closed, connected and oriented surface of genus at least two with an interval, we prove that every couple of metrics with curvature less than -1 on the surface can be realised on the two boundary components.
For globally hyperbolic maximal compact (GHMC) anti-de Sitter manifolds, we study various geometric quantities, such as the volume, the Hausdorff dimension of the limit set, the width of the convex core and the Hölder exponent of the manifold, in terms of the parameters that describe the deformation space of GHMC anti-de Sitter structures.
Moreover, we prove existence and uniqueness of a foliation by constant mean curvature surfaces of the domain of dependence of any quasi-circle in the boundary at infinity of anti-de Sitter space.

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Finally, I dedicate this work to my family: although they cannot be present physically at the defence, I know they will be thinking of me. I would not have achieved this without their support.

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Introduction

This thesis studies various aspects of anti-de Sitter geometry and its relation with Teichmüller theory. The strong link between the two subjects was first discovered in 1990 by Geoffrey Mess, who described, in his pioneering work [Mes07], the deformation space of $(2+1)$-space-times with compact space-like slices, and found a new, and somehow simpler, proof of the celebrated Thurston's Earthquake Theorem, exploiting the geometry of globally hyperbolic anti-de Sitter manifolds. Since then, this has become a very active area of research: namely, anti-de Sitter geometry turned out to be a convenient setting where to study earthquakes between hyperbolic surfaces ([BS12]), possibly with boundary ([BKS11], [Ros17]) and with conical singularities ([BS09]), quasi-conformal extensions of quasi-symmetric homeomorphisms of the circle ([BS16], [Sep17]) and polyhedra inscribed in quadrics ([DMS14]). This thesis fits into this framework and tries to enrich the existing literature about the description of the geometry of anti-de Sitter manifolds.

Three-dimensional anti-de Sitter space can be thought of as the analog of hyperbolic space in Lorentzian geometry. It can be defined as the set of time-like vectors of $\mathbb{R}^4$ endowed with a bilinear form of signature $(2,2)$, and it is the local model for Lorentzian manifolds with constant sectional curvature -1. In this thesis, we are interested in a special class of manifolds, locally isometric to anti-de Sitter space, called globally hyperbolic. Those are characterised by the existence of a space-like surface S, named Cauchy surface, that intersects any causal curve in exactly one point. This property puts strong restrictions on the topology of these manifolds, being them necessarily diffeomorphic to a product $S\times\mathbb{R}$ ([Ger70]); nonetheless their geometry is very rich. Once he fixed the topological type of the surface S, Mess studied the possible anti-de Sitter structures that can be defined on $S\times\mathbb{R}$, up to diffeomorphisms isotopic to the identity, that are maximal in the sense of isometric inclusions. If S is supposed to be closed, connected and oriented, of genus $\tau \geq 2$, Mess parameterised the deformation space of globally hyperbolic maximal anti-de Sitter structures on $S\times\mathbb{R}$ by two copies of the Teichmüller space of S. This result can be interpreted as the analog of Bers' double Uniformisation Theorem ([Ber74]) for hyperbolic quasi-Fuchsian three-manifolds. The similarity between hyperbolic quasi-Fuchsian manifolds and globally hyperbolic maximal anti-de Sitter manifolds goes further. In both we can find a convex core, that is the smallest convex subset onto which the manifold retracts, which has a very interesting geometry. If the manifold is Fuchsian, namely the two parameters of Bers' and Mess' parameterisations coincide, the convex core is a totally geodesic hyperbolic surface; otherwise it is a three dimensional domain, homeomorphic to $S \times I$, the two boundary components being naturally endowed with hyperbolic structures and pleated along measured laminations. Some aspects about the geometry of the convex core are still to be understood. In particular, two main conjectures by Thurston and Mess remain open:

Conjecture. (Thurston) *The space of quasi-Fuchsian three-manifolds can be parameterised either by the induced metrics on the boundary of the convex core or by the two geodesic measured laminations.*

Conjecture. (Mess) *The space of globally hyperbolic maximal anti-de Sitter structures on $S \times \mathbb{R}$ can be parameterised either by the induced metrics on the boundary of the convex core or by the two geodesic measured laminations.*

In both setting it is known that every couple of hyperbolic metrics ([Lab92a], [Dia13]) and every couple of filling measured laminations ([BO04], [BS12]) can be realised, but uniqueness is still unknown.

However, one can ask similar questions for other compact, convex domains that contain the convex core. In fact, it is possible to talk about the induced metrics on the two boundary components and the role of the measured geodesic laminations is replaced in a natural sense by the third fundamental forms. In the hyperbolic setting, this has been first studied by Labourie ([Lab92a]) who proved that any smooth metrics on S with curvature at least -1 can be realised on the boundary of a compact, convex hyperbolic three-manifold, and later by Schlenker ([Sch06]), who proved the uniqess part of this question and extended this result to the third fundamental form, as well.

In Chapter 2, we address the existence part of this problem in the anti-de Sitter setting. More precisely, we prove the following:

Theorem A. ([Tam18]) *Let $g_\pm$ be two smooth metrics with curvature less than -1 on a closed, connected, oriented surface S of genus $\tau \geq 2$. Then there exists a compact, convex, globally hyperbolic anti-de Sitter manifold with convex boundary $M \cong S \times I$ such that the metrics induced on the two boundary components are $g_\pm$. A similar result holds also for the third fundamental forms.*

The proof is based on a deformation argument. First, we observe that the above theorem is equivalent to the existence of a compact domain K with convex boundary, embedded in a globally hyperbolic maximal anti-de Sitter manifold

$M \cong S \times \mathbb{R}$, with induced metrics $g_\pm$ on the two boundary components. Then, using Mess' parameterisation, for every smooth metric g on S with curvature less than -1, we construct a smooth map ϕ_g from the space of equivariant isometric embeddings of (S, g) into anti-de Sitter space to $\text{Teich}(S) \times \text{Teich}(S)$, which associates to an isometric embedding the holonomy representation of the globally hyperbolic maximal anti-de Sitter manifold in which (S, g) is contained. Therefore, the proof of Theorem A follows by showing that the images of ϕ_{g_+} and ϕ_{g_-} are never disjoint, if $g_\pm$ are any two smooth metrics with curvature less than -1. This is accomplished by proving it directly for a specific couple of metrics $g_\pm$, and by then verifying that the intersection persists when deforming one of the two metrics.

The geometry of globally hyperbolic maximal anti-de Sitter manifolds can also be understood using special foliations by space-like surfaces. In case of closed Cauchy surfaces, this theory was developed by Barbot, Béguin and Zeghib, who proved that every such manifold can be foliated uniquely by constant mean curvature surfaces ([BBZ07]) and constant Gauss curvature surfaces ([BBZ11]). These results have been recently generalised in different directions: when conical singularities of angle less than π along time-like geodesics are allowed ([CS16], [QT17]), and when there is no co-compact action of a surface group ([BS16],[Tam16]). Chapter 3 focuses on the latter problem for constant mean curvature surfaces. If we identify the universal cover of a globally hyperbolic maximal anti-de Sitter manifold $M \cong S \times \mathbb{R}$ with a domain of dependence in anti-de Sitter space, the foliation by constant mean curvature surfaces is lifted to a foliation by discs of constant mean curvature of the domain of dependence. These discs intersect the boundary at infinity of anti-de Sitter space in a curve, called quasi-circle, that can be interpreted as the graph of the quasi-symmetric homeomorphism of the circle that conjugates the two Fuchsian representations in Mess' parameterisation. It is thus natural to ask if a foliation by constant mean curvature surfaces exists for more general domains of dependence, whose closure intersect the boundary at infinity of anti-de Sitter space in a general quasi-circle.

Theorem B. ([Tam16]) *Let $\phi : S^1 \to S^1$ be a quasi-symmetric homeomoprhism of the circle and let c_ϕ be the corresponding quasi-circle in the boundary at infinity of anti-de Sitter space. Then there exists a unique foliation by constant mean curvature surfaces of the domain of dependence of c_ϕ.*

The proof relies on an approximation argument. Namely, every quasi-circle c_ϕ can be seen as a limit in the Hausdorff topology of quasi-circles c_n that are graphs of quasi-symmetric homeomorphisms that conjugate two Fuchsian representations. We show that the sequence of constant mean curvature surfaces with boundary at infinity c_n converges to a constant mean curvature surface asymptotic to c_ϕ and that the surfaces obtained in this way provide the desired foliation. As an application, we exploit techniques introduced by Krasnov and Schlenker ([KS07]) to construct a family of quasi-conformal extensions of ϕ: a surface with constant mean curvature H and boundary at infinity c_ϕ provides a quasi-conformal extension Φ_H of ϕ with the following property. The map Φ_H can be decomposed uniquely as $\Phi_H = f_2 \circ f_1^{-1}$ where f_1 and f_2 are harmonic maps of the hyperbolic plane with Hopf differential $\mathrm{Hopf}(f_1) = e^{2i\theta}\mathrm{Hopf}(f_2)$, and $\theta = -\arctan(H) + \frac{\pi}{2}$.

In the second part of the thesis, we address the general question of describing the geometry of a globally hyperbolic maximal anti-de Sitter manifold with compact Cauchy surface in terms of the two points in Teichmüller space provided by Mess' parameterisation. The first interesting geometric quantity that we study in Chapter 4 is the volume of the convex core. An analogous question for hyperbolic quasi-Fuchsian manifolds was investigated by Brock ([Bro03]), who showed that the volume of the convex core is roughly equivalent to the Weil-Petersson distance between the two corresponding points in Bers' parameterisation. It turns out that a similar result does not hold in this Lorentzian setting, as we are able to construct a sequence of globally hyperbolic maximal anti-de Sitter manifolds such that the volume of their convex core diverges, but the Weil-Petersson distance between the two Mess' parameters remains bounded. However, we find a quantity that approximates the volume of the convex core up to multiplicative and additive constants:

Theorem C. ([BST17]) *Let $M_{h,h'}$ be the globally hyperbolic maximal anti-de Sitter manifold corresponding to $(h, h') \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ in Mess' parameterisation. Then the volume of the convex core of $M_{h,h'}$ is coarsely equivalent to the L^1-energy between the hyperbolic surfaces (S, h) and (S, h').*

The L^1-energy between hyperbolic surfaces is defined as the infimum, over all C^1 maps f isotopic to the identity, of the L^1-norm of the differential of f. Very few is known about this quantity: in contrast with the more studied L^2-energy that is realised by the L^2-norm of the differential of the unique harmonic map isotopic to the identity, we do not know, for instance, if the infimum is attained. As a consequence of Theorem C, we shed some light about the behaviour of the L^1-energy:

Corollary D. ([BST17]) *Let h, h' be two hyperbolic metrics on S and suppose that h' is obtained from h by an earthquake along a measured geodesic lamination λ. Then, the L^1-energy between (S, h') and (S, h') is roughly equivalent to the length of λ.*

By Thurston's Earthquake Theorem, we have two possible choices for λ, depending on whether we perfom a left or right earthquake. Our techniques show that the lengths of these two laminations are comparable, being their difference bounded by an explicit constant that depends only on the topology of the surface.

In Chapter 5, we turn our attention to other two interesting geometric quantities associated to globally hyperbolic maximal anti-de Sitter manifolds: the

Lorentzian Hausdorff dimension of the limit set and the Hölder exponent. As we discussed above, the limit set of a globally hyperbolic maximal anti-de Sitter manifold M with compact Cauchy surface can be identified with the graph c_ϕ of a quasi-symmetric homeomorphism ϕ of the circle. This is the Lorentzian analog of the quasi-circle that appears as limit set of a quasi-Fuchsian group acting on the three-dimensional hyperbolic space. However, while the Hausdorff dimension of the limit set of a quasi-Fuchsian group varies between 1 and 2 ([Sul84]) and is equal to 1 if and only if the group is Fuchsian ([Bow79]), the Hausdorff dimension of c_ϕ is always 1. In particular, it does not distinguish if a representation is Fuchsian. Glorieux and Monclair introduced a notion of Lorentzian Hausdorff dimension that fits this issue ([GM16]): roughly speaking, they replaced Euclidean balls with Lorentzian ones in the classical definition of Hausdorff dimension, obtaining thus a quantity that is always bounded by 1 and is equal to 1 if and only if the manifold is Fuchsian. The Hölder exponent of M is also related to the homeomorphism ϕ: it is the minimum between the best Hölder exponent of ϕ and ϕ^{-1}. We provide an explicit formula for this that depends only on the holonomy representation of M.

The main results of Chapter 5 concern the asymptotic behaviour of these quantities. More precisely, we use the parameterisation of the deformation space of globally hyperbolic maximal anti-de Sitter structures on $S \times \mathbb{R}$ by the cotangent bundle to the Teichmüller space of S ([KS07]) and study the asymptotic behaviour along rays of quadratic differentials:

Theorem E. ([Tam17]) *Let M_t be the family of globally hyperbolic maximal anti-de Sitter manifolds associated to the ray $(h, tq) \in T^*\mathrm{Teich}(S)$. Then the Lorentzian Hausdorff dimension of the limit set and the Hölder exponent of M_t tend to 0 when t goes to $+\infty$.*

In order to explain this result, let us first recall how the parameterisation by $T^*\mathrm{Teich}(S)$ works: to a point $(h, q) \in T^*\mathrm{Teich}(S)$ one associates the globally hyperbolic maximal anti-de Sitter manifold $M \cong S \times \mathbb{R}$ which has an embedded maximal surface with induced metric conformal to h and with second fundamental form determined by the real part of q. The existence of such a manifold is obtained by solving a quasi-linear PDE. By studying carefully this differential equation, we are able to provide estimates for the induced metric on the maximal surface along rays of quadratic differentials, and prove that its volume entropy converges to 0 when t goes to $+\infty$. The proof of the first part of Theorem E then follows from the fact that the Lorentzian Hausdorff dimension is bounded from above by the entropy of the maximal surface.

On the other hand, the asymptotic behaviour of the Hölder exponent is proved by comparing the two parameterisations. Namely, harmonic maps between hyperbolic surfaces provide a bridge between the two points of view and combining Wolf's compactification of Teichmüller space ([Wol89]) with our explicit formula for the Hölder exponent, we deduce the second part of Theorem E.

Outline of the thesis

The thesis is organised as follows. Chapter 1 introduces anti-de Sitter geometry and reviews the main classical results in this field. In Chapter 2 we study globally hyperbolic anti-de Sitter manifolds with convex space-like boundary and the problem of finding a manifold with prescribed metric on the boundary. The material of this chapter can be found in:

[Tam18] Tamburelli, A. "Prescribing metrics on the boundary of anti-de Sitter 3-manifolds". *International Mathematics Research Notices, Volume 2018, Issue 5, pp. 1281-1313, 2018.*

In Chapter 3 we prove the existence and uniqueness of a foliation by constant mean curvature surfaces of the domain of dependence of a quasi-circle in the boundary at infinity of anti-de Sitter space. The content of this chapter has been published in:

[Tam16] Tamburelli, A. "Constant mean curvature foliation of domains of dependence in anti-de Sitter space". *To appear in Transactions of the AMS.*

Chapter 4 deals with the volume of globally hyperbolic maximal anti-de Sitter manifolds: we compare the volume of the convex core and the volume of the entire manifold and find coarse estimates in terms of the L^1 energy, the Weil-Petersson distance and Thurston's asymmetric distance between the two points in Teichmüller space given by Mess' parameterisation. These results can be found in:

[BST17] Bonsante, F., Seppi, A., Tamburelli A. "On the volume of anti-de Sitter maximal globally hyperbolic three-manifolds". *Geometric and Functional Analysis, Volume 27, Issue 5, pp. 1106-1160, 2017.*

In Chapter 5, we use the parameterisation of globally hyperbolic anti-de Sitter structures by the cotangent of the Teichmüller space to describe the behaviour of the entropy of the maximal surface and the Lorentzian Hausdorff dimension of the limit set along rays of quadratic differentials. The material covered here has appeared in the preprint:

[Tam17] Tamburelli, A. "Entropy degeneration of globally hyperbolic maximal compact anti-de Sitter structures". *arXiv:1710.05827.*

Chapter 1

Anti-de Sitter geometry

In this chapter we introduce the protagonist of the thesis, i.e the three-dimensional anti-de Sitter space. The material covered here is classical, the main objective being fixing the notation and recalling the well-established results in the field.

1.1 The Klein model

Let us denote with $\mathbb{R}^{2,2}$ the vector space $\mathbb{R}^4$ endowed with the bilinear form of signature $(2,2)$:

$$\langle x, y \rangle_{2,2} = x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3 \ .$$

We define

$$\widehat{AdS}_3 = \{x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle_{2,2} = -1\} \ .$$

The restriction of the bilinear form $\langle \cdot, \cdot \rangle_{2,2}$ to the tangent space of $\widehat{AdS}_3$ induces a Lorentzian metric on $\widehat{AdS}_3$ with constant sectional curvature -1. Given a point $p \in \widehat{AdS}_3$ and a tangent vector $v \in T_p\widehat{AdS}_3$, we will say that

- v is space-like, if $\langle v, v \rangle_{2,2} > 0$;
- v is light-like, if $\langle v, v \rangle_{2,2} = 0$;
- v is time-like, if $\langle v, v \rangle_{2,2} < 0$.

Similarly, we say that a geodesic γ in $\widehat{AdS}_3$ is space-like (resp. light-like or time-like) if $\dot{\gamma}$ is space-like (resp. light-like or time-like). It is straightforward to verify that geodesics are obtained by intersecting planes through the origin of $\mathbb{R}^{2,2}$ with $\widehat{AdS}_3$. The causal type of the geodesic can be understood from the signature of the restriction of the bilinear form $\langle \cdot, \cdot \rangle_{2,2}$ to the plane:

- if it has signature $(1,1)$ we obtain a space-like geodesic;

- if it is degenerate and the intersection with $\widehat{AdS}_3$ is non-empty, we obtain a light-like geodesic;
- if it has signature $(0,2)$ we obtain a time-like geodesic.

Analogously, totally geodesic planes are obtained by intersecting $\widehat{AdS}_3$ with hyperplanes of $\mathbb{R}^{2,2}$. Given a totally geodesic plane P, we say that

- P is space-like if the induced metric on P is positive definite;
- P is light-like if the induced metric on P is degenerate;
- P is time-like if the induced metric on P is Lorentzian.

Again, the induced metric on P can be easily deduced by studying the signature of the restriction of the bilinear form $\langle \cdot, \cdot \rangle_{2,2}$ on the hyperplane that defines P.

We endow $\widehat{AdS}_3$ with the orientation induced by the standard orientation of $\mathbb{R}^4$. A time-orientation is the choice of a never-vanishing time-like vector field X on $\widehat{AdS}_3$. The isometry group of orientation and time-orientation preserving isometries of $\widehat{AdS}_3$ is the connected component of $SO(2,2)$ containing the identity.

We define anti-de Sitter space AdS_3 as the image of the projection of $\widehat{AdS}_3$ into $\mathbb{RP}^3$. More precisely, if we denote with $\pi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{RP}^3$ the canonical projection, anti-de Sitter space is

$$AdS_3 = \pi(\{x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle_{2,2} < 0\}) \ .$$

It can be easily verified that $\pi : \widehat{AdS}_3 \to AdS_3$ is a double cover, hence we can endow AdS_3 with the unique Lorentzian structure that makes π a local isometry. This is called the Klein model of anti-de Sitter space, in analogy with the more familiar Klein model of hyperbolic geometry. It follows from the definition and the above discussion that geodesics and totally geodesics planes are obtained by intersecting AdS_3 with projective lines and planes.

In order to better visualise anti-de Sitter space, it is convenient to consider the intersection with an affine chart. Let $U_3 = \{[x_0, x_1, x_2, x_3] \in \mathbb{RP}^3 \mid x_3 \neq 0\}$. The map

$$\begin{aligned} \varphi_3 : U_3 &\to \mathbb{R}^3 \\ [x] &\mapsto \left(\frac{x_0}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right) \end{aligned}$$

induces a homeomorphism between $AdS_3 \cap U_3$ and the open set $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 < 1\}$. It is clear from the construcion that in this affine chart, geodesics and totally geodesics planes are the intersection between affine lines and planes in $\mathbb{R}^3$ with Ω.

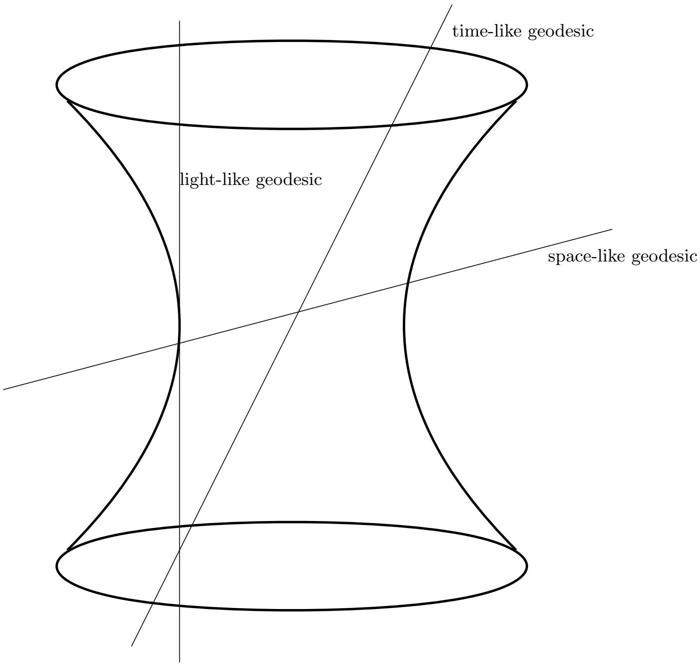

Figure 1.1: Geodesics in AdS_3.

It is natural to define the boundary at infinity of anti-de Sitter space as

$$\partial_\infty AdS_3 = \pi(\{x \in \mathbb{R}^4 \mid \langle x, x\rangle_{2,2} = 0\}).$$

It can be easily verified that $\partial_\infty AdS_3$ coincides with the image of the Segre embedding

$$s : \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^3 ,$$

hence the boundary at infinity of anti-de Sitter space is a double-ruled quadric homeomorphic to a torus. We will talk about left and right ruling in order to distinguish the two rulings. This homeomorphism can be also be described geometrically in the following way. Fix a totally geodesic plane P_0 in AdS_3. The boundary at infinity of P_0 is a circle. Let $\xi \in \partial_\infty AdS_3$. There exists a unique line of the left ruling l_ξ and a unique line of the right ruling r_ξ passing through ξ. The identification between $\partial_\infty AdS_3$ and $S^1 \times S^1$ induced by P_0 associates to ξ the intersection points $\pi_l(\xi)$ and $\pi_r(\xi)$ between l_ξ and r_ξ and the boundary at infinity of P_0. These two maps

$$\pi_l : \partial_\infty AdS_3 \to S^1 \quad \pi_r : \partial_\infty AdS_3 \to S^1$$

are called left and right projections, respectively. In the affine chart U_3 the boundary at infinity of AdS_3 coincides with the quadric of equation $x^2 + y^2 - z^2 = 1$.

The action of orientation and time-orientation preserving isometries of AdS_3 extends continuously to the boundary at infinity and it is projective on the two rulings, thus giving an identification between $SO_0(2, 2)$ and $\mathbb{P}SL(2, \mathbb{R}) \times \mathbb{P}SL(2, \mathbb{R})$.

Given a map $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$, the identification between $\partial_\infty AdS_3$ and $\mathbb{RP}^1 \times \mathbb{RP}^1$ described above allows us to represent the graph of ϕ as a curve c_ϕ on the boundary at infinity of anti-de Sitter space. If ϕ is an orientation-preserving homeomorphism, the curve c_ϕ turns out to be weakly space-like, i.e for every $\xi \in c_\phi$ the curve c_ϕ is contained in the region bounded by the lines through ξ in the left and right ruling which is connected to ξ by space-like paths. Viceversa, every weakly space-like curve $\Gamma \subset \partial_\infty AdS_3$ can be obtained as a graph of an orientation-preserving homeomorphism of the circle.
Given a weakly space-like curve Γ on the boundary at infinity of anti-de Sitter space, we define two objects that will play a fudamental role in the theory of globally hyperbolic manifolds outlined in Section 1.5:

- the convex hull of Γ is the smallest closed convex subset of AdS_3 with boundary at infinity Γ and it will be denoted with $\mathcal{C}(\Gamma)$;
- the domain of dependence $\mathcal{D}(\Gamma)$ of Γ is the set of points $p \in AdS_3 \subset \mathbb{RP}^3$ such that the plane p^*, which is the projective dual of p, is disjoint from Γ. Domains of dependence are always contained in an affine chart and admit only light-like support planes.

1.2 Anti-de Sitter space as Lie group

Let $\mathfrak{gl}(2,\mathbb{R})$ be the vector space of 2-by-2 matrices with real coefficients. The quadratic form

$$q(A) = -\det(A)$$

induces, by polarisation, a scalar product η on $\mathfrak{gl}(2,\mathbb{R})$, which in the basis consisting of elementary matrices can be represented by

$$\eta(X,Y) = X^t \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix} Y \ .$$

It follows easily that η has signature $(2,2)$.
Let us now consider the submanifold $\mathrm{SL}(2,\mathbb{R}) = \{A \in \mathfrak{gl}(2,\mathbb{R}) \mid q(A) = -1\}$. We claim that $\mathrm{SL}(2,\mathbb{R})$ endowed with the restriction of η is a 3-dimensional Lorentzian manifold. Since η is invariant by left- and right- multiplication by elements of $\mathrm{SL}(2,\mathbb{R})$ (because q is), it is sufficient to check this at $\mathrm{Id} \in \mathrm{SL}(2,\mathbb{R})$. Now

$$T_{\mathrm{Id}}\mathrm{SL}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) = \{A \in \mathfrak{gl}(2,\mathbb{R}) \mid \mathrm{trace}(A) = 0\} \ .$$

In the basis of $\mathfrak{sl}(2,\mathbb{R})$ given by

$$\mathfrak{sl}(2,\mathbb{R}) = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

the scalar product η is represented by the matrix

$$\eta|_{\mathfrak{sl}(2,\mathbb{R})}(X,Y) = X^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} Y \ .$$

Hence $(\mathrm{SL}(2,\mathbb{R}), \eta)$ is a Lorentzian manifold that we denote by $\widehat{AdS}_3$. From the above computation, it follows also that

$$\eta(X,Y) = \frac{1}{2}\mathrm{trace}(XY)$$

for every $X, Y \in \mathfrak{sl}(2,\mathbb{R})$.

The group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ acts on $\widehat{AdS}_3$ as

$$(A,B)\cdot X := AXB^{-1}$$

by isometries. In particular, η induces a Lorentzian structure on $\mathbb{P}SL(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{Id}\}$, which we identify with anti-de Sitter space AdS_3.

Remark 1.2.1. *An explicit isometry between* $\widehat{AdS}_3$ *as introduced in Section 1.1 and* $(\mathrm{SL}(2,\mathbb{R}),\eta)$ *is given by the restriction of the map*

$$\begin{aligned} \mathbb{R}^4 &\to \widehat{AdS}_3 \\ (x_0,x_1,x_2,x_3) &\mapsto \begin{pmatrix} x_0+x_1 & x_3+x_2 \\ x_2-x_3 & x_0-x_1 \end{pmatrix} \end{aligned}$$

where $(x_0,x_1,x_2,x_3) \in \mathbb{R}^4$ *are coordinates with respect to a basis in which the bilinear form of signature* $(2,2)$ *is*

$$\langle x, y\rangle_{2,2} = -x_0y_0 + x_1y_1 + x_2y_2 - x_3y_3 \ .$$

If we see $AdS_3 \subset \mathbb{P}\mathfrak{gl}(2,\mathbb{R})$, we can define the boundary at infinity of AdS_3 as

$$\partial_\infty AdS_3 = \mathbb{P}(\{A \in \mathfrak{gl}(2,\mathbb{R}) \setminus \{0\} \mid q(A) = 0\}) \ ,$$

namely the projectivisation of rank 1-matrices. This can then be identified with $\mathbb{RP}^1 \times \mathbb{RP}^1$ by

$$\begin{aligned} \partial_\infty AdS_3 &\to \mathbb{RP}^1 \times \mathbb{RP}^1 \\ [M] &\mapsto ([\mathrm{Im}(M)], [\mathrm{Ker}(M)]) \ . \end{aligned}$$

It is easy to check that the action of $\mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ on AdS_3 extends to the boundary at infinity and, in the above identification, coincides with the obvious action of $\mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ on $\mathbb{RP}^1 \times \mathbb{RP}^1$.

Geodesics of AdS_3 are obtained by intersection of projective planes with AdS_3. Therefore, if $x \in AdS_3$ and $\hat{x} \in \widehat{AdS}_3$ is a lift, then

- a space-like geodesic at x with tangent vector v is lifted to

$$\exp_x(tv) = \cosh(t)\hat{x} + \sinh(t)\hat{v} \ ;$$

- a time-like geodesic at x with tangent vector v is the projectivisation of

$$\exp_x(tv) = \cos(t)\hat{x} + \sin(t)\hat{v} \ ;$$

- a light-like geodesic at x with tangent vector v lifts to

$$\exp_x(tv) = \hat{x} + t\hat{v} \ .$$

In particular, geodesics through $[Id] \in \mathbb{P}SL(2,\mathbb{R})$ are 1-parameter subgroups.

1.3 Surfaces in anti-de Sitter manifolds

In this section we describe the theory of immersions of surfaces in anti-de Sitter space, which is a straightforward adaptation of the classical theory for Euclidean space.

Let us denote with ∇^{AdS} the Levi-Civita connection of the Lorentzian metric g_{AdS} of AdS_3. Given a smooth immersion $\sigma : \tilde{S} \to AdS_3$, the first fundamental form is the pull-back of the induced metric

$$I(V,W) = g_{AdS}(d\sigma(V), d\sigma(W)) \qquad V,W \in \Gamma(T\tilde{S}) \ .$$

We say that $\sigma(\tilde{S})$ is space-like if the first fundamental form is a Riemannian metric. From now on, we will always suppose that the immersion is space-like.
We denote with N the future-directed unit normal vector field on $\sigma(\tilde{S})$. Since the immersion is space-like, N is a time-like vector. The Levi-Civita connection ∇^I of the first fundamental form I of $\tilde{S}$ is defined from the relation:

$$\nabla^{AdS}_V W = \nabla^I_V W + II(V,W)N \ ,$$

and $II(V,W)$ is called the second fundamental form of the immersion. The shape operator $B \in End(T\tilde{S})$ of $\tilde{S}$ is defined as

$$B(V) = -\nabla^{AdS}_V N \ .$$

It turns out that B is I-self-adjoint and the second fundamental form is related to the shape operator by

$$II(V,W) = I(B(V),W) \ .$$

In particular, B is diagonalisable and its eigenvalues are called principal curvatures. The first fundamental form and the shape operator satisfy two equations:

- the anti-de Sitter version of the Guass equation:

$$\det(B) = -1 - K_I \ ,$$

where we have denoted with K_I the Gaussian curvature of the first fundamental form;

- the Codazzi equation

$$d^{\nabla^I} B = 0 \ ,$$

where $d^{\nabla^I} : \Omega^1(T\tilde{S}) \to \Omega^2(T\tilde{S})$ is the operator defined by:

$$(d^{\nabla^I} B)(V, W) = \nabla^I_V(B(W)) - \nabla^I_W(B(V)) - B([V, W]) \ .$$

As for Euclidean space, the embedding data I and B of a simply connected surface determines the immersion uniquely up to global isometries of AdS_3:

Theorem 1.3.1 (Fundamental theorem of surfaces in anti-de Sitter space)**.** *Let $\tilde{S}$ be a simply connected surface. Given a Riemannian metric I and an I-self-adjoint operator $B : T\tilde{S} \to T\tilde{S}$, satisfying the Gauss-Codazzi equations*

$$\det(B) = -1 - K_I$$
$$d^{\nabla^I} B = 0$$

there exists a smooth immersion $\sigma : \tilde{S} \to AdS_3$ such that the first fundamental form is I and the shape operator is B. Moreover, σ is uniquely determined up to post-composition with an isometry of AdS_3.

We can also define the third fundamental form of $\tilde{S}$ as

$$III(V, W) = I(B(V), B(W)) \ .$$

We notice that if $\tilde{S}$ is strictly convex, i.e. the determinant of B is strictly positive at every point, then the third fundamental form is a Riemannian metric. Notice that, by the Gauss equation, this is equivalent to say that the curvature of the induced metric is strictly smaller than -1.
The third fundamental form is linked to a duality between convex surfaces in anti-de Sitter space. More precisely, the projective duality between points and planes in $\mathbb{RP}^3$ induces a duality between convex space-like surfaces in AdS_3: given a convex space-like surface, the dual surface $\tilde{S}^*$ is defined as the set of points which are dual to the support planes of $\tilde{S}$. The relation between $\tilde{S}$ and $\tilde{S}^*$ is summarised in the following proposition:

Proposition 1.3.2 ([BBZ11])**.** *Let $\tilde{S} \subset AdS_3$ be a smooth space-like surface with curvature $\kappa < -1$. Then*

- *the dual surface $\tilde{S}^*$ is smooth and strictly convex;*

- *the pull-back of the induced metric on* $\tilde{S}^*$ *through the duality map is the third fundamental form of* $\tilde{S}$*;*
- *if* κ *is constant, the dual surface* $\tilde{S}^*$ *has curvature* $\kappa^* = -\frac{\kappa}{\kappa+1}$.

1.4 The universal cover of anti-de Sitter space

As the careful reader might have noticed from the description of the Klein model in Section 1.1, anti-de Sitter space in not simply-connected, being it diffeomorphic to a solid torus. It is sometimes convenient to work in the Universal cover, especially when dealing with space-like embeddings of surfaces into AdS_3.

Let us denote with $\mathbb{H}^2$ the hyperbolic plane. In this section we will always think of $\mathbb{H}^2$ as one connected component of the two-sheeted hyperboloid in Minkowksy space. The map

$$F : \mathbb{H}^2 \times S^1 \to \widehat{AdS}_3$$
$$(x_0, x_1, x_2, e^{i\theta}) \mapsto (x_0 \cos(\theta), x_1, x_2, x_0 \sin(\theta))$$

is a diffeomorphism, hence $\mathbb{H}^2 \times S^1$ is isometric to anti-de Sitter space, if endowed with the pull-back metric

$$(F^* g_{AdS_3})_{(x, e^{i\theta})} = (g_{\mathbb{H}^2})_x - x_0^2 d\theta^2 \ .$$

We easily deduce that the Universal cover of anti-de Sitter space can be realised as $\widetilde{AdS}_3 \cong \mathbb{H}^2 \times \mathbb{R}$ endowed with the Lorentzian metric:

$$(g_{\widetilde{AdS}})_{(x,t)} = (g_{\mathbb{H}^2})_x - x_0^2 dt^2 \ .$$

We will denote

$$\chi^2 = -\left\| \frac{\partial}{\partial t} \right\|^2$$

and

$$\mathrm{grad} t = -\frac{1}{\chi^2} \frac{\partial}{\partial t} \ .$$

The Universal cover is particularly useful to study embedded space-like surfaces. In fact, space-like surfaces in $\widetilde{AdS}_3$ are graphs of functions ([BS10, Proposition 3.2])

$$u : \mathbb{H}^2 \to \mathbb{R}$$
$$x \mapsto u(x) \ .$$

Moreover, the space-like condition provides a uniform bound on the gradient of u. For instance, let us consider the function $\hat{u}$ on $\mathbb{H} \times \mathbb{R}$ given by

$$\hat{u}(x, t) = u(x) \ .$$

The correspondent space-like surface is defined by the equation $\hat{u}(x) - t = 0$. This surface is space-like if and only if the normal vector at each point

$$\nu = -\chi^2 \mathrm{grad} t - \mathrm{grad}(\hat{u})$$

is time-like. We deduce the uniform bound

$$\|\mathrm{grad}(u)\|^2 < \frac{1}{\chi^2}$$

on the gradient of the function u. In particular, space-like surfaces are graphs of Lipschitz functions.

1.5 GHMC anti-de Sitter three manifolds

A 3-dimensional anti-de Sitter space-time is a manifold N locally isometric to AdS_3 with a fixed orientation and time-orientation. This means that N is endowed with an atlas of charts taking values on AdS_3 so that the transition functions are restrictions of elements in $\mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$.

We are actually interested in a special class of anti-de Sitter manifolds.

Definition 1.5.1. *An anti-de Sitter manifold N is Globally Hyperbolic Maximal Compact (GHMC) if it satisfies the following conditions:*

1. *Global Hyperbolicity: N contains a space-like surface, called Cauchy-surface, that intersects every inextensible causal curve in exactly one point;*
2. *Maximality: if N' is another globally hyperbolic AdS_3 space-time and $\phi : N \to N'$ is any isometric embedding sending a Cauchy surface into a Cauchy surface, then ϕ is a global isometry;*
3. *Spacial Compactness: if the Cauchy surface is compact.*

The first condition implies that N must be diffeomorphic to $S \times \mathbb{R}$ ([Ger70]), where S is homeomorphic to the Cauchy surface of N. We will always assume that S is a closed, connected, oriented surface of genus $\tau \geq 2$. We will denote with $\mathcal{GH}(S)$ the deformation space of GHMC anti-de Sitter structures on $S \times \mathbb{R}$. By the pioneering work of Mess, the deformation theory of GHMC anti-de Sitter structures is strongly related to Teichmüller theory. This becomes evident from the following result:

Theorem 1.5.2 ([Mes07])**.** *$\mathcal{GH}(S)$ is parameterised by* $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$.

The parameterisation goes as follows. First, recall that the Teichmüller space of S is identified to a certain connected component in the space of representations of $\pi_1(S)$ into $\mathbb{P}SL(2,\mathbb{R})$, considered up to conjugation. In fact, this identification is obtained by taking the conjugacy class of the holonomy represention of a hyperbolic metric on S, and the desired connected component is given by the subset of representations with maximal Euler class, called Fuchsian ([Gol80]):

$$\mathrm{Teich}(S) \cong \{\rho_0 : \pi_1(S) \to \mathbb{P}SL(2,\mathbb{R}) : \; e(\rho_0) = |\chi(S)|\}/\mathbb{P}SL(2,\mathbb{R}) \ .$$

Mess proved that for every GHMC AdS_3 manifold M, the holonomy representation

$$\rho = (\rho_l, \rho_r) : \pi_1(S) \to \mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$$

satisfies $e(\rho_l) = e(\rho_r) = |\chi(S)|$, and therefore $([\rho_l], [\rho_r])$ defines a point in $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$. The representations ρ_l and ρ_r are called left holonomy and right holonomy.

Example 1.5.3. *If h is a hyperbolic metric on S, then one can define the following metric on $M = S \times (-\pi/2, \pi/2)$, where t is the "vertical" coordinate:*

$$g_h = -dt^2 + \cos^2(t)h \ . \tag{1.1}$$

It turns out that g_h has constant sectional curvature -1, that $S \times \{0\}$ is a totally geodesic Cauchy surface, and that (M, g_h) is maximal globally hyperbolic. It can be verified that, in this case, $\rho_l = \rho_r$. The maximal globally hyperbolic manifolds for which $[\rho_l] = [\rho_r] \in \mathrm{Teich}(S)$ are called Fuchsian and correspond to the diagonal in

$$\mathcal{GH}(S) \cong \mathrm{Teich}(S) \times \mathrm{Teich}(S) \ .$$

Equivalently, they contain a totally geodesic spacelike surface isometric to $\mathbb{H}^2/\rho_0(\pi_1(S))$, where $\rho_0 := \rho_l = \rho_r$.

Going back to Theorem 1.5.2, Mess explicitly constructed an inverse of the map $\mathcal{GH}(S) \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ we have just defined. Given a couple (ρ_l, ρ_r) of Fuchsian representation, there exists a unique orientation-preserving homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that

$$\phi \circ \rho_l(\gamma) = \rho_r(\gamma) \circ \phi$$

for every $\gamma \in \pi_1(S)$. As explained in Section 1.1, we can see the graph of ϕ as a curve c_ϕ on the boundary at infinity of anti-de Sitter space. It turns out that $\rho(\pi_1(S)) = (\rho_l(\pi_1(S)), \rho_r(\pi_1(S)))$ acts properly discontiously on the domain of dependence of c_ϕ and the quotient is a GHMC anti-de Sitter manifold, with holonomy ρ. We will denote this manifold by

$$M_{h_l,h_r} := \mathcal{D}(\phi)/(\rho_l, \rho_r)(\pi_1(S)) \ ,$$

where h_l and h_r are the hyperbolic metrics of S induced by $\mathbb{H}^2/\rho_l(\pi_1(S))$ and $\mathbb{H}^2/\rho_r(\pi_1(S))$ respectively. We will often refer to h_l and h_r as the left and right metric. It follows from Mess' proof that the class of M_{h_l,h_r} in $\mathcal{GH}(S)$ only depends on the isotopy classes of h_l and h_r.

The quotient of the convex hull of c_ϕ

$$\mathcal{C}(M_{h_l,h_r}) := \mathcal{C}(\phi)/(\rho_l, \rho_r)(\pi_1(S))$$

is called the convex core of M_{h_l,h_r}, and it is the smallest compact, convex subset homotopy equivalent to M_{h_l,h_r}. The convex core has an interesting geometry: if it is not a totally geodesic space-like surface, which happens only if $h_l = h_r$, its boundary consists of two space-like surfaces homeomorphic to S, naturally endowed with hyperbolic metrics and pleated along measured laminations.

Moreover, the left and right hyperbolic metrics corresponding to the left and right representations can be constructed explicitly starting from space-like surfaces embedded in M_{h_l,h_r}. Mess gave a description in a non-smooth setting using the upper and lower boundary of the convex core of M_{h_l,h_r} as space-like surfaces. More precisely, if $m^{\pm}$ are the hyperbolic metrics on the upper and lower boundary of the convex core and $\lambda^{\pm}$ are the measured geodesic laminations along which they are pleated, the left and right metrics h_l and h_r are related to $m^{\pm}$ by an earthquake along $\lambda^{\pm}$:

$$h_l = E_l^{\lambda^+}(m_+) = E_r^{\lambda^-}(m_-) \qquad h_r = E_r^{\lambda^+}(m_+) = E_l^{\lambda^-}(m_-) \ .$$

Mess obtained in this way a new proof of Thurston's Earthquake Theorem:

Theorem 1.5.4 (Earthquake theorem)**.** *Given two hyperbolic metrics h, h' on a closed oriented surface S, there exists a unique pair of measured laminations λ_l, λ_r such that*

$$E_l^{\lambda_l}(h) = h' \qquad \textit{and} \qquad E_r^{\lambda_r}(h) = h' \ .$$

Later, this description was extended ([KS07]), thus obtaining explicit formulas for the left and right metric, in terms of the induced metric I, the complex structure J and the shape operator B of any strictly negatively curved smooth space-like surface S embedded in M_{h_l,h_r}. The construction goes as follows. We fix a totally geodesic space-like plane P_0. Let $\tilde{S} \subset AdS_3$ be the universal cover of S. Let $\tilde{S}' \subset U^1AdS_3$ be its lift into the unit tangent bundle of AdS_3 and let $p : \tilde{S}' \to \tilde{S}$ be the canonical projection. For any point $(x, v) \in \tilde{S}'$, there exists a unique space-like plane P in AdS_3 orthogonal to v and containing x. We define two natural maps $\Pi_{\infty,l}$ and $\Pi_{\infty,r}$ from $\partial_\infty P$ to $\partial_\infty P_0$, sending a point $x \in \partial_\infty P$ to the intersection between $\partial_\infty P_0$ and the unique line of the left or right foliation of $\partial_\infty AdS_3$ containing x. Since these maps are projective, they extend to hyperbolic isometries $\Pi_l, \Pi_r : P \to P_0$. Identifying P with the tangent space of $\tilde{S}$ at the point x, the pull-backs of the hyperbolic metric on P_0 by Π_l and by Π_r define two hyperbolic metrics on $\tilde{S}$

$$h_l = I((E + JB)\cdot, (E + JB)\cdot) \quad \text{and} \quad h_r = I((E - JB)\cdot, (E - JB)\cdot) \ .$$

The isotopy classes of the corresponding metrics on S do not depend on the choice of the space-like surface S and their holonomies are precisely ρ_l and ρ_r, respectively ([KS07, Lemma 3.6]).

By applying this construction to the unique maximal surface (i.e. with vanishing mean curvature) S embedded in a GHMC AdS_3 manifold, Krasnov and Schlenker deduced a correspondence between maximal surfaces and minimal Lagrangian maps between hyperbolic surfaces.

Definition 1.5.5. *An orientation-preserving diffeomorphism* $m : (S, h) \to (S, h')$ *is minimal Lagrangian if it is area-preserving and its graph is a minimal surface in* $(S \times S, h \oplus h')$.

It is known [BS10, Proposition 1.3] that minimal Lagrangian diffeomorphisms are characterized by having a decomposition $m = (f') \circ f^{-1}$, where f and f' are harmonic maps from a Riemann surface (S, X) with opposite Hopf differential.

It turns out that in this case $\Pi_{l,r}$ induce harmonic diffeomorphisms between (S, I) and $(S, h_{l,r})$, which have opposite Hopf differential. Hence we obtain a minimal Lagrangian diffeomorphism between (S, h_l) and (S, h_r) that factors through the conformal structure of the maximal surface. Moreover, all minimal Lagrangian diffeomorphisms from (S, h) to (S, h') are obtained in this way (see for instance [KS07] and [BS10] for a generalisation).

Chapter 2

Prescribing metrics on the boundary of anti-de Sitter 3-manifolds

In this chapter we prove that given two metrics g_+ and g_- with curvature $\kappa < -1$ on a closed, oriented surface S of genus $\tau \geq 2$, there exists an AdS_3 manifold N with smooth, space-like, strictly convex boundary such that the induced metrics on the two connected components of ∂N are equal to g_+ and g_-. Using the duality between convex space-like surfaces in AdS_3, we obtain an equivalent result about the prescription of the third fundamental form.

2.1 Definition of the problem and outline of the proofs

As it should be clear from Chapter 1, the 3-dimensional anti-de Sitter space AdS_3 is the Lorentzian analogue of hyperbolic space, and globally hyperbolic maximal compact AdS_3 manifolds share many similarities with hyperbolic quasi-Fuchsian manifolds. As a consequence, it is possible to formulate many classical questions of quasi-Fuchsian manifolds even in this Lorentzian setting. The question we address here is the following. Let K be a compact, convex subset with two smooth, strictly convex, space-like boundary components in a GHMC AdS_3 manifold. By the Gauss formula, the boundaries have curvature $\kappa < -1$. We can ask if it is possible to realise every couple of metrics, satisfying the condition on the curvature, on a surface S via this construction. The analogous question has a positive answer in a hyperbolic setting ([Lab92a]), where even a uniqueness result holds ([Sch06]). In this chapter, we will follow a construction inspired by the work of Labourie ([Lab92a]), in order to obtain a positive answer in the anti-de Sitter world. The main result of the chapter is thus the following:

Corollary 2.3.3. *For every couple of metrics g_+ and g_- on S with curvature less than -1, there exists a globally hyperbolic convex compact AdS_3 manifold $K \cong S \times [0,1]$, whose induced metrics on the boundary are exactly $g_\pm$.*

Using the duality between space-like surfaces in anti-de Sitter space, we obtain an analogous result about the prescription of the third fundamental form:

Corollary 2.3.4. *For every couple of metrics g_+ and g_- on S with curvature less than -1, there exists a globally hyperbolic convex compact AdS_3 manifold $K \cong S \times [0,1]$, such that the third fundamental forms on the boundary components are g_+ and g_-.*

We outline here the main steps of the proof for the convenience of the reader.
The first observation to be done is that Corollary 2.3.3 is equivalent to proving that there exists a GHMC AdS_3 manifold M containing a future-convex space-like surface isometric to (S, g_-) and a past-convex space-like surface isometric to (S, g_+). Adapting the work of Labourie ([Lab92a]) to this Lorentzian setting, we prove that the space of isometric embeddings $I(S, g_\pm)^\pm$ of $(S, g_\pm)$ into a GHMC AdS_3 manifold as a future-convex (or past-convex) space-like surface is a manifold of dimension $6\tau - 6$. On the other hand, by the work of Mess ([Mes07]), the space of GHMC AdS_3 structures is parameterised by two copies of Teichmüller space, hence a manifold of dimension $12\tau - 12$. This allows us to translate our original question into a question about the existence of an intersection between subsets in Teich$(S) \times$ Teich(S). More precisely, we will define in Section 2.3 two maps

$$\phi^\pm_{g_\pm} : I(S, g_\pm)^\pm \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$$

sending an isometric embedding of $(S, g_\pm)$ to the holonomy of the GHMC AdS_3 manifold containing it. Corollary 2.3.3 is then equivalent to the following:

Theorem 2.3.2. *For every couple of metrics g_+ and g_- on S with curvature less than -1, we have*

$$\phi^+_{g_+}(I(S, g_+)^+) \cap \phi^-_{g_-}(I(S, g_-)^-) \neq \emptyset \ .$$

In order to prove this theorem we will use tools from topological intersection theory, which we recall in Section 2.4. For instance, Theorem 2.3.2 is already known to hold under particular hypothesis on the curvatures ([BMS15]), hence we only need to check that the intersection persists when deforming one of the two metrics on the boundary, as the space of smooth metrics with curvature less than -1 is connected (see e.g. [LS00, Lemma 2.3]). More precisely, given any smooth paths of metrics $g_t^\pm$ with curvature less than -1, we will define the manifolds

$$W^\pm = \bigcup_{t \in [0,1]} I(S, g_t^\pm)^\pm$$

and the maps

$$\Phi^{\pm} : W^{\pm} \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$$

with the property that the restrictions of $\Phi^{\pm}$ to the two boundary components coincide with $\phi^{\pm}_{g_0^{\pm}}$ and $\phi^{\pm}_{g_1^{\pm}}$. We will then prove the following:

Proposition 5.1. *The maps $\Phi^{\pm}$ are smooth.*

Hence, we will have the necessary regularity to apply tools from intersection theory. In particular, we can talk about transverse maps and under this condition we can define the intersection number (mod 2) of the maps $\phi^{+}_{g_+}$ and $\phi^{-}_{g_-}$ as the cardinality (mod 2), if finite, of $(\phi^{+}_{g_+} \times \phi^{-}_{g_-})^{-1}(\Delta)$, where

$$\phi^{+}_{g_+} \times \phi^{-}_{g_-} : I(S, g_+)^+ \times I(S, g_-)^- \to (\mathrm{Teich}(S))^2 \times (\mathrm{Teich}(S))^2$$

and Δ is the diagonal in $(\mathrm{Teich}(S))^2 \times (\mathrm{Teich}(S))^2$. We will compute explicitly this intersection number (see Section 2.7) under particular hypothesis on the curvatures of g_+ and g_-: the reason for this being that the transversality condition is in general difficult to check when the metrics do not have constant curvature. It turns out that in that case the intersection number is 1.

We then start to deform one of the two metrics and check that an intersection persists. Here, one has to be careful that, since the maps are defined on non-compact manifolds, the intersection does not escape to infinity. This is probably the main technical part of the proof and requires results about the convergence of isometric embeddings (Corollary 2.5.5), estimates in anti-de Sitter geometry (Lemma 2.5.12) and results in Teichmüller theory (Lemma 2.5.11). In particular, applying these tools, we prove

Proposition 2.5.13. *For every metric g^- and for every smooth path of metrics $\{g_t^+\}_{t\in[0,1]}$ on S with curvature less than -1, the set $(\Phi^+ \times \phi^-_{g^-})^{-1}(\Delta)$ is compact*

This guarantees that when deforming one of the two metrics the variation of the intersection locus is always contained in a compact set. The proof of Theorem 2.3.2 then follows applying standard argument of topological intersection theory.

In Section 2.6, we study the map

$$p_1 \circ \Phi^+ : W^+ \to \mathrm{Teich}(S) \ ,$$

where $p_1 : \mathrm{Teich}(S) \times \mathrm{Teich}(S) \to \mathrm{Teich}(S)$ is the projection onto the left factor. The main result we obtain is the following:

Proposition 2.6.1. *Let g be a metric on S with curvature less than -1 and let h be a hyperbolic metric on S. Then there exists a GHMC AdS_3 manifold M with left metric isotopic to h containing a past-convex space-like surface isometric to (S, g).*

This is proved by showing that $p_1 \circ \phi^+_g$ is proper of degree 1 (mod 2). Again, we are able to compute explicitly the degree of the map when g has constant curvature and the general statement then follows since for any couple of metrics g and g' with curvature less than -1, the maps $p_1 \circ \phi_g$ and $p_1 \circ \phi_{g'}$ are connected by a proper cobordism.

2.2 Equivariant isometric embeddings

Let S be a connected, compact, oriented surface of genus $\tau \geq 2$ and let g be a Riemannian metric on S with curvature κ less than -1. An isometric equivariant embedding of S into AdS_3 is given by a couple (f, ρ), where $f : \tilde{S} \to AdS_3$ is an isometric embedding of the universal Riemannian cover of S into AdS_3 and ρ is a representation of the fundamental group of S into $\mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ such that

$$f(\gamma x) = \rho(\gamma) f(x) \quad \forall\, \gamma \in \pi_1(S) \quad \forall\, x \in \tilde{S} \ .$$

The group $\mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ acts on a couple (f, ρ) by post-composition on the embedding and by conjugation on the representation. We denote by $I(S, g)$ the set of equivariant isometric embeddings of S into AdS_3 modulo the action of $\mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$.

Also in an anti-de Sitter setting, an analogue of the Fundamental Theorem for surfaces in the Euclidean space holds:

Theorem 2.2.1. *There exists an isometric embedding of (S, g) into an AdS_3 manifold if and only if it is possible to define a g-self-adjoint operator $b : TS \to TS$ satisfying*

$$\begin{aligned} &\det(b) = -\kappa - 1 && \textit{Gauss equation} \\ &d^{\nabla} b = 0 && \textit{Codazzi equation} \end{aligned}$$

Moreover, the operator b determines the isometric embedding uniquely, up to global isometries.

This theorem enables us to identify $I(S, g)$ with the space of solutions of the Gauss-Codazzi equations, which can be studied using the classical techniques of elliptic operators.

Lemma 2.2.2. *The space $I(S, g)$ is a manifold of dimension $6\tau - 6$.*

Proof. We can mimic the proof of Lemma 3.1 in [Lab92a]. Consider the sub-bundle $\mathcal{F}^g \subset \mathrm{Sym}(TS)$ over S of symmetric operators $b : TS \to TS$ satisfying the Gauss equation. We prove that the operator

$$d^{\nabla} : \Gamma^{\infty}(\mathcal{F}^g) \to \Gamma^{\infty}(\Lambda^2 TS \otimes TS)$$

is elliptic of index $6\tau - 6$, equal to the dimension of the kernel of its linearization. Let J_0 be the complex structure induced by g. For every $b \in \Gamma^\infty(\mathcal{F}^g)$, the operator

$$J = \frac{J_0 b}{\sqrt{\det(b)}}$$

defines a complex structure on S. In particular we have an isomorphism

$$\begin{aligned} F : \Gamma^\infty(\mathcal{F}^g) &\to \mathcal{A} \\ b &\mapsto \frac{J_0 b}{\sqrt{\det(b)}} \end{aligned}$$

between smooth sections of the sub-bundle $\Gamma^\infty(\mathcal{F}^g)$ and the space $\mathcal{A}$ of complex structures on S, with inverse

$$\begin{aligned} F^{-1} : \mathcal{A} &\to \Gamma^\infty(\mathcal{F}^g) \\ J &\mapsto -\sqrt{-\kappa - 1} J_0 J \ . \end{aligned}$$

This allows us to identify the tangent space of $\Gamma(\mathcal{F}^g)$ at b with the tangent space of $\mathcal{A}$ at J, which is the vector space of operators $\dot{J} : TS \to TS$ such that $\dot{J}J + J\dot{J} = 0$. Under this identification the linearization of d^∇ is given by

$$L(\dot{J}) = -J_0(d^\nabla \dot{J}) \ .$$

We deduce that L has the same symbol and the same index of the operator $\overline{\partial}$, sending quadratic differentials to vector fields. Thus L is elliptic with index $6\tau - 6$.
To conclude we need to show that its cokernel is empty, or, equivalently, that its adjoint L^* is injective. If we identify $\Lambda^2 TS \otimes TS$ with TS using the metric g, the adjoint operator L^* is given by (see Lemma 3.1 in [Lab92a] for the computation)

$$(L^*\psi)(u) = -\frac{1}{2}(\nabla_{J_0 u}\psi + J\nabla_{J_0 J u}\psi) \ .$$

The kernel of L^* consists of all the vector fields ψ on S such that for every vector field u

$$J\nabla_u \psi = -\nabla_{J_0 J J_0 u}\psi \ .$$

We can interpret this equation in terms of intersection of pseudo-holomorphic curves: the Levi-Civita connection ∇ induces a decomposition of $T(TS)$ into a vertical V and a horizontal H sub-bundle. We endow V with the complex structure J, and H with the complex structure $-J_0 J J_0$. In this way, the manifold TS is endowed with an almost-complex structure and the graph of ψ is a pseudo-holomorphic curve. Since pseudo-holomorphic curves have positive intersections, if the graph of ψ did not coincide with the graph of the null section, their intersection would be positive. On the other hand, it is well-known that this intersection coincides with the Euler characteristic of S, which is negative. Hence, we conclude that ψ is identically zero and that L^* is injective. □

Similarly, we obtain the following result:

Lemma 2.2.3. *Let $\{g_t\}_{t\in[0,1]}$ be a differentiable curve of metrics with curvature less than -1. The set*

$$W = \bigcup_{t\in[0,1]} I(S, g_t)$$

is a manifold with boundary of dimension $6\tau - 5$.

Proof. Again we can mimic the proof of Lemma 3.2 in [Lab92a]. Consider the subbundle $\mathcal{F} \subset \mathrm{Sym}(TS)$ over $S \times [0,1]$ of symmetric operators, whose fiber over a point (x,t) consists of the operators $b : TS \to TS$, satisfying the Gauss equation with respect to the metric g_t. The same reasoning as for the previous lemma shows that

$$d^{\nabla} : \Gamma^{\infty}(\mathcal{F}) \to \Gamma^{\infty}(\Lambda^2 TS \otimes TS)$$

is Fredholm of index $6\tau - 5$. Since $W = (d^{\nabla})^{-1}(0)$, the result follows from the implicit function theorem for Fredholm operators. □

Let N be a GHMC AdS_3 manifold endowed with a time orientation, i.e. a nowhere vanishing time-like vector field. Let S be a convex embedded surface in N. We say that S is past-convex (resp. future-convex), if its past (resp. future) is geodesically convex. We will use the convention to compute the shape operator of S using the future-directed normal. With this choice if S is past-convex (resp. future-convex) then it has strictly positive (resp. strictly negative) principal curvatures.

Definition 2.2.4. *We will denote with $I(S,g)^+$ and $I(S,g)^-$ the spaces of equivariant isometric embeddings of S as a past-convex and future-convex surface, respectively.*

2.3 Definition of the maps $\phi^{\pm}$

The parameterisation of GHMC anti-de Sitter structure described in Section 1.5 enables us to formulate our original question about the prescription of the metrics on the boundary of a compact AdS_3 manifold in terms of existence of an intersection of particular subsets of $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$.

Let K be a globally hyperbolic, convex, compact anti-de Sitter 3-manifold with strictly convex boundary. By global hyperbolicity, K is diffeomorphic to $S \times [-1,1]$, where S is a Cauchy surface of K, which we suppose to be closed, connected and oriented of genus $\tau \geq 2$. By definition of maximality, K can be embedded into a unique GHMC AdS_3 manifold N. The boundary components S^+ and S^- of K become two embedded space-like surfaces in N, the former is past-convex and the latter is future-convex. Moreover, by the Gauss equation, the metrics induced on S^+ and S^- have curvature less than -1. If we denote with g_+ and g_- the metrics induced on S^+ and S^- respectively, by lifting the embeddings $\sigma^{\pm} : (S^{\pm}, g_{\pm}) \to N$ to the Universal cover, we obtain an element of $I(S, g_+)^+$ and an element of $I(S, g_-)^-$. Viceversa, if N is a GHMC AdS_3 manifold, by cutting N along a past-convex space-like surface and a future-convex space-like surface we obtain a convex, compact, globally hyperbolic anti-de Sitter manifold with convex boundary. Thus, the question of prescribing the metrics on the boundary components of a compact, convex, globally hyperbolic anti-de Sitter manifold with strictly convex boundary is equivalent to the question of finding a future-convex and a past-convex isometric embedding into the same GHMC AdS_3 manifold.

This suggests the following construction:

Definition 2.3.1. *Let g be a metric on S with curvature $\kappa < -1$. We define the maps*

$$\phi_g^{\pm} : I(S, g)^{\pm} \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$$

$$b \mapsto (h_l(g, b), h_r(g, b)) := (g((E + Jb)\cdot, (E + Jb)\cdot), g((E - Jb)\cdot, (E - Jb)\cdot))$$

associating to every isometric embedding of (S, g) the left and right metric of the GHMC AdS_3 manifold containing it.

We recall that we use the convention to compute the shape operator using always the future-oriented normal. In this way, the above formulas hold for both future-convex and past-convex surfaces, without changing the orientation of the surface S.

We will prove (in Section 2.7) the following fact, which is the main theorem of the chapter:

Theorem 2.3.2. *For every couple of metrics g_+ and g_- on S with curvature less than -1, we have*

$$\phi_{g_+}^{+}(I(S, g_+)^+) \cap \phi_{g_-}^{-}(I(S, g_-)^-) \neq \emptyset\ .$$

Therefore, there exists a GHMC AdS_3 manifold containing a past-convex space-like surface isometric to (S, g_+) and a future-convex space-like surface isometric to (S, g_-). We deduce from this the answer to our original question:

Corollary 2.3.3. *For every couple of metrics g_+ and g_- on S with curvature less than -1, there exists a globally hyperbolic convex compact AdS_3 manifold $K \cong S \times [0, 1]$, whose induced metrics on the boundary are exactly $g_{\pm}$.*

If we apply the previous corollary to the dual surfaces, we obtain an analogous result about the prescription of the third fundamental form:

Corollary 2.3.4. *For every couple of metrics g_+ and g_- on S with curvature less than -1, there exists a compact AdS_3 manifold $K \cong S \times [0, 1]$, whose induced third fundamental forms on the boundary are exactly $g_{\pm}$.*

2.4 Topological intersection theory

As outlined in Section 2.1, the main tool used in the proof of the main theorem is the intersection theory of smooth maps between manifolds, which is developed for example in [GP74]. We recall here the basic constructions and the fundamental results.

If not otherwise stated, all manifolds considered in this section are non-compact without boundary.
Let X and Z be manifolds of dimension m and n, respectively and let A be a closed submanifold of Z of codimension k. Suppose that $m-k \geq 0$. We say that a smooth map $f : X \to Z$ is transverse to A if for every $z \in \mathrm{Im}(f) \cap A$ and for every $x \in f^{-1}(z)$ we have

$$df(T_x X) + T_z A = T_z Z \ .$$

Under this hypothesis, $f^{-1}(A)$ is a submanifold of X of codimension k.
When $k = m$ and $f^{-1}(A)$ consists of a finite number of points we define the intersection number between f and A as

$$\Im(f, A) := |f^{-1}(A)| \qquad (\mathrm{mod}\ 2)\ .$$

Remark 2.4.1. *When A is a point $p \in Z$, f is transverse to p if and only if p is a regular value for f. Moreover, if f is proper, $f^{-1}(p)$ consists of a finite number of points and the above definition coincides with the classical definition of degree (mod 2) of a smooth and proper map.*

We say that two smooth maps $f : X \to Z$ and $g : Y \to Z$ are transverse if the map

$$f \times g : X \times Y \to Z \times Z$$

is transverse to the diagonal $\Delta \subset Z \times Z$. Notice that if $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$, then f and g are transverse by definition.
Suppose now that $2 \dim X = 2 \dim Y = \dim Z$. Moreover, suppose that the maps $f : X \to Z$ and $g : Y \to Z$ are transverse and the preimage $(f \times g)^{-1}(\Delta)$ consists of a finite number of points. We define the intersection number between f and g as

$$\Im(f, g) := \Im(f \times g, \Delta) = |(f \times g)^{-1}(\Delta)| \qquad (\mathrm{mod}\ 2)\ .$$

It follows by the definition that if $\Im(f, g) \neq 0$ then $\mathrm{Im}(f) \cap \mathrm{Im}(g) \neq \emptyset$.

One important feature of the intersection number that we will use further is the invariance under cobordism. We say that two maps $f_0 : X_0 \to Z$ and $f_1 : X_1 \to Z$ are cobordant if there exists a manifold W and a smooth function $F : W \to Z$ such that $\partial W = X_0 \cup X_1$ and $F_{|X_i} = f_i$.

Proposition 2.4.2. *Let W be a non-compact manifold with boundary $\partial W = X_0 \cup X_1$. Let $H : W \to Z$ be a smooth map and denote by h_i the restriction of H to the boundary component X_i for $i = 0, 1$. Let $A \subset Z$ be a closed submanifold. Suppose that* *(i)* $\mathrm{codim} A = \dim X_i$;

(ii) *H is transverse to A;*

(iii) *$H^{-1}(A)$ is compact.*

Then $\Im(h_0, A) = \Im(h_1, A)$.

Proof. By hypothesis the pre-image $H^{-1}(A)$ is a compact, properly embedded 1-manifold, i.e. it is a finite disjoint union of circles and arcs with ending points on a boundary component of W. This implies that $h_0^{-1}(A)$ and $h_1^{-1}(A)$ have the same parity. □

In particular, we deduce the following result about the intersection number of two maps:

Corollary 2.4.3. *Let W be a non-compact manifold with boundary $\partial W = X_0 \cup X_1$. Let $F : W \to Z$ be a smooth map and denote by f_i the restriction of F to the boundary component X_i for $i = 0, 1$. Let $g : Y \to Z$ be a smooth map. Suppose that*

(i) $2 \dim X_i = 2 \dim Y = \dim Z$;

(ii) *F and g are transverse;*

(iii) *$(F \times g)^{-1}(\Delta)$ is compact.*

Then $\Im(f_0, g) = \Im(f_1, g)$.

Proof. Apply the previous proposition to the map $H = F \times g : W \times Y \to Z \times Z$ and to the submanifold $A = \Delta$, the diagonal of $Z \times Z$. □

The hypothesis of transversality in the previous propositions is not restrictive, as it is always possible to perturb the maps involved on a neighbourhood of the set on which transversality fails:

Theorem 2.4.4 (Theorem p.72 [GP74]). *Let $h : W \to Z$ be a smooth map between manifolds, where only W has boundary. Let A be a closed submanifold of Z. Suppose that h is transverse to A on a closed set $C \subset W$. Then there exists a smooth map $\tilde{h} : W \to Z$ homotopic to h such that $\tilde{h}$ is transverse to A and $\tilde{h}$ agrees with h on a neighbourhood of C.*

Now the question arises whether the intersection number depends on the particular perturbation of the map that we obtain when applying Theorem 2.4.4.

Proposition 2.4.5. *Let $h : X \to Z$ be a smooth map between manifolds. Let A be a submanifold of Z, whose codimension equals the dimension of X. Suppose that $h^{-1}(A)$ is compact. Let $\tilde{h}$ and $\tilde{h}'$ be perturbations of h, which are transverse to A*

and coincide with h *outside the interior part of a compact set* B *containing* $h^{-1}(A)$*. Then*

$$\Im(\tilde{h}, A) = \Im(\tilde{h}', A) \ .$$

Proof. Let $\tilde{H} : W = X \times [0,1] \to Y$ be an homotopy between $\tilde{h}$ and $\tilde{h}'$ such that for every $x \in (X \setminus B) \times [0,1]$ we have $\tilde{H}(x,t) = h(x)$. Notice that $\tilde{H}^{-1}(A)$ is compact. Up to applying Theorem 2.4.4 to the closed set $C = (X \setminus B) \times [0,1] \cup \partial W$, we can suppose that $\tilde{H}$ is transverse to A. By Proposition 2.4.2, we have that $\Im(\tilde{h}, A) = \Im(\tilde{h}', A)$ as claimed. □

Moreover, in particular circumstances, we can actually obtain a $1-1$ correspondence between the points of $h_0^{-1}(A)$ and $h_1^{-1}(A)$. The following proposition will not be used for the proof of the main result of the chapter, but it might be a useful tool to prove the uniqueness part of the question addressed in this chapter, as explained in Remark 2.7.3.

Proposition 2.4.6. *Under the same hypothesis as Proposition 2.4.2, suppose that the cobordism* (W, H) *between* h_0 *and* h_1 *satisfies the following additional properties:*

(i) W *fibers over the interval* $[0,1]$ *with fiber* X_t*;*

(ii) the restriction h_t *of* H *at each fiber is tranverse to* A *.*

Then $|h_0^{-1}(A)| = |h_1^{-1}(A)|$.

Proof. It is sufficient to show that in $H^{-1}(A)$ there are no arcs with ending points in the same boundary component. By contradiction, let γ be an arc with ending point in X_0. Define

$$t_0 = \sup\{t \in [0,1] \mid \gamma \cap X_t \neq \emptyset\} \ .$$

A tangent vector $\dot{\gamma}$ at a point $p \in X_{t_0} \cap \gamma$ is in the kernel of the map

$$d_pH : T_{(p,t_0)}W \to T_q(Z \times Z)/T_q(A) \ ,$$

where $q = H(p)$. The contradiction follows by noticing that on the one hand $\dot{\gamma}$ is contained in the tangent space $T_pX_{t_0}$ by construction but on the other hand $d_ph_{t_0} : T_pX_{t_0} \to T_q(Z \times Z)/T_q(A)$ is an isomorphism by transversality.
A similar reasoning works when γ has ending points in X_1. □

2.5 Some properties of the maps $\phi^{\pm}$

This section contains the most technical part of the paper. We summarise here briefly, for the convenience of the reader, what the main results of this section are.

For every metric g on S with curvature less than -1 we have defined in Section 2.3 the maps $\phi_g^{\pm}$ which associate to every isometric embedding of (S, g) into a GHMC

AdS_3 manifold M the class in Teichmüller space of the left and right metrics of M. It follows easily from Lemma 2.2.3 that for any couple of metrics g and g' with curvature less than -1 the maps $\phi^{\pm}_g$ and $\phi^{\pm}_{g'}$ are cobordant through a map $\Phi^{\pm}$. In this section we will define the maps $\Phi^{\pm}$ and will study some of its properties, which will enable us to apply the topological intersection theory described in the previous section. More precisely, the first step will consist of proving that all the maps involved are smooth. This is the content of Proposition 2.5.1 and the proof will rely on the fact that the holonomy representation of a hyperbolic metric depends smoothly on the metric. Then we will deal with the properness of the maps $\Phi^{\pm}$ (Corollary 2.5.8) that will follow from a compacteness result of isometric embeddings (Corollary 2.5.5). This will allow us also to have a control on the space where two maps ϕ_g and $\phi_{g'}$ intersect: when we deform one of the two metrics the intersection remains contained in a compact set (Proposition 2.5.13).

Recall that given a smooth path of metrics $\{g_t\}_{t\in[0,1]}$ on S with curvature less than -1, the set

$$W^{\pm} = \bigcup_{t\in[0,1]} I^{\pm}(S, g_t)$$

is a manifold with boundary $\partial W^{\pm} = I(S, g_0)^{\pm} \cup I(S, g_1)^{\pm}$ of dimension $6\tau - 5$ (Lemma 2.2.3). We define the maps

$$\Phi^{\pm} : W^{\pm} \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$$

$$b_t \mapsto (h_l(g_t, b_t), h_r(g_t, b_t)) := (g_t((E + Jb_t)\cdot, (E + Jb_t)\cdot), g_t((E - Jb_t)\cdot, (E - Jb_t)\cdot))$$

associating to an equivariant isometric embedding (identified with its Codazzi operator b_t) of (S, g_t) into AdS_3 the class in Teichmüller space of the left and right metrics of the GHMC AdS_3 manifold containing it. We remark that the restrictions of $\Phi^{\pm}$ to the boundary coincide with the maps $\phi^{\pm}_{g_0}$ and $\phi^{\pm}_{g_1}$ defined in Section 2.3.

We deal first with the regularity of the maps.

Proposition 2.5.1. *The functions $\Phi^{\pm} : W^{\pm} \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ are smooth.*

Proof. Let $\mathcal{M}_S$ be the set of hyperbolic metrics on S. We can factorise the map $\Phi^{\pm}$ as follows:

$$W^{\pm} \xrightarrow{\Phi'^{\pm}} \mathcal{M}_S \times \mathcal{M}_S \xrightarrow{\pi} \mathrm{Teich}(S) \times \mathrm{Teich}(S)$$

where $\Phi'^{\pm}$ associates to an isometric embedding of (S, g_t) (determined by an operator b_t satisfying the Gauss-Codazzi equation) the couple of hyperbolic metrics $(g_t((E+J_tb_t)\cdot, (E+J_tb_t)\cdot), g_t((E-J_tb_t)\cdot, (E-J_tb_t)\cdot))$, and π is the projection to the corresponding isotopy class, or, equivalently, the map which associates to a hyperbolic metric its holonomy representation. Since the maps $\Phi'^{\pm}$ are clearly smooth by definition, we just need to prove that the holonomy representation depends smoothly on the metric. Let h be a hyperbolic metric on S. Fix a point $p \in S$ and a unitary frame $\{v_1, v_2\}$ of the tangent space T_pS. We consider the ball model for the hyperbolic plane and we fix a unitary frame $\{w_1, w_2\}$ of $T_0\mathbb{H}^2$. We can realise every element of the fundamental group of S as a closed path passing through p. Let γ be a path passing through p and let $\{U_i\}_{i=0,\dots n}$ be a finite covering of γ such that every U_i is homeomorphic to a ball. We know that there exists a unique map $f_0 : U_0 \to B_0 \subset \mathbb{H}^2$ such that

$$\begin{cases} f_0(p) = 0 \\ d_p f_0(v_i) = w_i \\ f_0^* g_{\mathbb{H}^2} = h \ . \end{cases}$$

Then, for every $i \geq 1$ there exists a unique isometry $f_i : U_i \to B_i \subset \mathbb{H}^2$ which coincides with f_{i-1} on the intersection $U_i \cap U_{i-1}$. Let $q = f_n(p) \in \mathbb{H}^2$. The holonomy representation sends the homotopy class of the path γ to the isometry $I_q : \mathbb{H}^2 \to \mathbb{H}^2$ such that $I_q(q) = 0$. Moreover, its differential maps the frame $\{u_i = df_n(v_i)\}$ to the frame w_i. The isometry I_q depends smoothly on q and on the frame u_i, which depend smoothly on the metric because each f_i does. □

The next step is about the properness of the maps $\Phi^{\pm}$. This will involve the study of sequences of isometric embeddings of a disc into a simply-connected spacetime, which have been extensively and profitably analysed in [Sch96]. In particular, the author proved that, under reasonable hypothesis, a sequence of isometric embeddings of a disc into a simply-connected spacetime has only two possible behaviours: it converges C^∞, up to subsequences, to an isometric embedding, or it is degenerate in a precise sense:

Theorem 2.5.2 (Theorem 5.6 [Sch96]). *Let $\tilde{f}_n : D \to X$ be a sequence of uniformly elliptic* [1] *immersions of a disc D in a simply connected Lorentzian spacetime $(X, \tilde{g})$. Assume that the metrics $\tilde{f}_n^* \tilde{g}$ converge C^∞ towards a Riemannian metric $\tilde{g}_\infty$ on D and that there exists a point $x \in D$ such that the sequence of the 1-jets $j^1 \tilde{f}_n(x)$ converges. If the sequence $\tilde{f}_n$ does not converge in the C^∞ topology in a neighbourhood of x, then there exists a maximal geodesic γ of $(D, \tilde{g}_\infty)$ and a geodesic arc Γ of $(X, \tilde{g})$ such that the sequence $(\tilde{f}_n)_{|\gamma}$ converges towards an isometry $\tilde{f}_\infty : \gamma \to \Gamma$.*

We start with a straightforward application of the Maximum Principle, which we recall here in the form useful for our purposes (see e.g. [BBZ11, Proposition 4.6]).

Proposition 2.5.3 (Maximum Principle). *Let Σ_1 and Σ_2 two future-convex space-like surfaces embedded in a GHMC AdS_3 manifold M. If they intersect in a point x and Σ_1 is in the future of Σ_2 then the product of the principal curvatures of Σ_2 is smaller than the product of the principal curvatures of Σ_1.*

Proposition 2.5.4. *Let Σ be a future-convex space-like surface embedded into a GHMC AdS_3 manifold M. Suppose that the Gaussian curvature of Σ is bounded*

[1] We recall that a sequence of isometric immersions is said to be uniformly elliptic if the corresponding shape operators have uniformly positive determinant.

between $-\infty < \kappa_{min} \leq \kappa_{max} < -1$. Denote with S_{min} and S_{max} the unique future-convex space-like surfaces with constant curvature κ_{min} and κ_{max} embedded in M. Then Σ is in the past of S_{max} and in the future of S_{min}.

Proof. Consider the unique ([BBZ11, Corollary 4.7]) κ-time

$$T : I^{-}(\partial_{-}C(M)) \to (-\infty, -1) ,$$

i.e. the unique function defined on the past of the convex core of M such that the level sets $T^{-1}(\kappa)$ are future-convex space-like surfaces of constant curvature κ. The restriction of T to Σ has a maximum t_{max} and a minimum t_{min}. Consider the level sets $L_{min} = T^{-1}(t_{min})$ and $L_{max} = T^{-1}(t_{max})$. By construction Σ is in the future of L_{min} and they intersect in a point x, hence, by the Maximum Principle and the Gauss equation, we obtain the following inequality for the Gaussian curvature of Σ at the point x:

$$t_{min} \geq \kappa(x) \geq \kappa_{min} .$$

Similarly we obtain that $t_{max} \leq \kappa(y) \leq \kappa_{max}$, where y is the point of intersection between L_{max} and Σ. But this implies that Σ is in the past of the level set $T^{-1}(\kappa_{max})$ and in the future of the level set $T^{-1}(\kappa_{min})$, which correspond respectively to the surfaces S_{max} and S_{min} by uniqueness. □

Corollary 2.5.5. *Let g_n be a compact family of metrics in the C^{∞} topology with curvatures $\kappa < -1$ on a surface S. Let $f_n : (S, g_n) \to M_n = (S \times \mathbb{R}, h_n)$ be a sequence of isometric embeddings of (S, g_n) as future-convex space-like surfaces into GHMC AdS_3 manifolds. If the sequence h_n converges to an AdS metric h_{∞} in the C^{∞}-topology, then f_n converges C^{∞}, up to subsequences, to an isometric embedding into $M_{\infty} = (S \times \mathbb{R}, h_{\infty})$.*

Proof. Consider the equivariant isometric embeddings $\tilde{f}_n : (\tilde{S}, \tilde{g}_n) \to \tilde{AdS}_3$ obtained by lifting f_n to the universal cover. We denote with $\tilde{S}_n$ the images of the disc $\tilde{S}$ under the map $\tilde{f}_n$ and let $\tilde{h}_n$ be the lift of the Lorentzian metrics h_n on $\tilde{AdS}_3$. By hypothesis $\tilde{f}_n^* \tilde{h}_n = \tilde{g}_n$ admits a subsequence converging to $\tilde{g}_{\infty}$.
Fix a point $x \in \tilde{S}$. Since the isometry group of $\tilde{AdS}_3$ acts transitively on points and frames, we can suppose that $\tilde{f}_n(x) = y \in \tilde{AdS}_3$ and $j^1 \tilde{f}_n(x) = z$ for every $n \in \mathbb{N}$.
Moreover, the condition on the curvature of the metrics g_n guarantees that the sequence $\tilde{f}_n$ is uniformly elliptic.
Therefore, we are under the hypothesis of Theorem 2.5.2.
The previous proposition allows us to determine precisely in which region of M_n each surface $f_n(S)$ lies. Since the family of metrics g_n is compact, the curvatures κ_n of the surfaces $f_n(S)$ in M_n are uniformly bounded $\kappa^{min} \leq \kappa_n \leq \kappa^{max} \leq -1 - 3\epsilon$ for some $\epsilon > 0$. By the previous proposition each surface $f_n(S)$ is in the past of Σ_n^{max} and in the future of Σ_n^{min}, where Σ_n^{min} and Σ_n^{max} are the unique future-convex space-like surfaces of M_n with constant curvature κ^{min} and κ^{max}. Let Σ_{ϵ} be the unique future-convex space-like surface in M_{∞} with constant curvature $-1 - 2\epsilon$. We

think of Σ as a fixed surface embedded in $S\times\mathbb{R}$ and we change the Lorentzian metric of the ambient space. Since h_n converges to h_∞, the metrics induced on Σ_ϵ by h_n converge to the metric induced on Σ_ϵ by h_∞. In particular, for n sufficiently large the curvature of Σ_ϵ as surface embedded in $M_n=(S\times\mathbb{R},h_n)$ is bounded between $-1-3\epsilon$ and $-1-\epsilon$. Therefore, Σ_ϵ is convex in M_n and by the previous proposition Σ_ϵ is in the future of Σ_n^{max} for every n sufficiently big. This implies that each surface $f_n(S)$ is in the past of the surface Σ_ϵ.
We can now conclude that the sequence f_n must converge to an isometric embedding. Suppose by contradiction that the sequence $\tilde{f}_n$ is not convergent in the C^∞ topology in a neighbourhood of x, then there exists a maximal geodesic $\tilde{\gamma}$ of $(\tilde{S},\tilde{g}_\infty)$ and a geodesic segment $\tilde{\Gamma}$ in $\tilde{AdS}_3$ such that $(\tilde{f}_n)_{|\tilde{\gamma}}$ converges to an isometry $\tilde{f}_\infty:\tilde{\gamma}\to\tilde{\Gamma}$. This implies that $\tilde{\Gamma}$ has infinite length. The projection of $\tilde{\Gamma}$ must be contained in the past of Σ_ϵ, because each $f_n(S)$ is contained there for n sufficiently large. But the past of Σ_ϵ is disjoint from the convex core of M_∞ and this contradicts the following lemma. □

Lemma 2.5.6. *In a GHMC AdS_3-manifold every complete space-like geodesic is contained in the convex core.*

Proof. Let γ be a complete space-like geodesic in a GHMC AdS_3 manifold M. By a result of Mess (see Section 1.5), we can realise M as the quotient of the domain of dependence $\mathcal{D}(\phi)\subset AdS_3$ of a curve c_ϕ on the boundary at infinity by the action of the fundamental group of S. The lift $\bar{\gamma}$ of γ has ending points on the curve c_ϕ, hence $\bar{\gamma}$ is contained in the convex hull of c_ϕ into AdS_3 and its projection is contained in the convex core of M. □

Remark 2.5.7. *Clearly, the same result holds for equivariant isometric embeddings of past-convex space-like surfaces, as it is sufficient to reverse the time-orientation.*

Corollary 2.5.8. *The functions $\Phi^\pm: W^\pm\to \mathrm{Teich}(S)\times\mathrm{Teich}(S)$ are proper.*

Proof. We prove the claim for the function Φ^-, the other case being analogous. Let $(h_l(g_{t_n},b_{t_n}),h_r(g_{t_n},b_{t_n}))\in\mathrm{Teich}(S)\times\mathrm{Teich}(S)$ be a convergent sequence in the image of the map Φ^-. This means that the sequence of GHMC AdS_3 manifolds M_n parametrised by $(h_l(g_{t_n},b_{t_n}),h_r(g_{t_n},b_{t_n}))$ is convergent. By definition of the map Φ^-, each M_n contains an embedded future-convex, space-like surface isometric to (S,g_{t_n}), whose immersion f_n into M_n is represented by the Codazzi operator b_{t_n}. By Corollary 2.5.5, the sequence of isometric immersions f_n is convergent up to subsequences, thus Φ^- is proper. □

This allows us to show that for every metric g_- and for every smooth path of metrics $\{g_t^+\}_{t\in[0,1]}$ on S with curvature $\kappa<-1$ the intersection between $\Phi^+(W^+)$ and $\phi_{g_-}^-(I(S,g_-)^-)$ is compact. This will follow combining some technical results about the geometry of AdS_3 manifolds and length-spectrum comparisons.

Definition 2.5.9. *Let g be a metric with negative curvature on S. We define the length function*

$$\ell_g : \pi_1(S) \to \mathbb{R}^+$$

which associates to every homotopy non-trivial loop on S, the length of its g-geodesic representative.

We recall that when g is a hyperbolic metric, Thurston proved (see e.g. [FLP79]) that the length function can be extended uniquely to a function on the space of measured geodesic laminations on S, which we still denote with ℓ_g.

We will need the following technical results:

Lemma 2.5.10 (Lemma 9.6 [BMS15])**.** *Let N be a globally hyperbolic compact AdS_3 manifold foliated by future-convex space-like surfaces. Then, the sequence of metrics induced on each surface decreases when moving towards the past. In particular, if Σ_1 and Σ_2 are two future-convex space-like surfaces with Σ_1 in the future of Σ_2, then for every closed geodesic γ in Σ_1 we have*

$$\ell_{g_2}(\gamma') \leq \ell_{g_1}(\gamma) \ ,$$

where γ' is the closed geodesic on Σ_2 homotopic to γ and g_1 and g_2 are the induced metric on Σ_1 and Σ_2, respectively.

Lemma 2.5.11. *Let g_n be a compact family of smooth metrics on S with curvature less than -1. Let m_n be a family of hyperbolic metrics such that*

$$\ell_{g_n}(\gamma) \leq \ell_{m_n}(\gamma)$$

for every $\gamma \in \pi_1(S)$. Then m_n lies in a compact subset of the Teichmüller space of S.

Proof. The idea is to use Thurston asymmetric metric on Teichmüller space. To this aim, we will deduce from the hypothesis a comparison between the length spectrum of m_n and that of the hyperbolic metrics h_n in the conformal class of g_n.
Let $\kappa < -1$ be the infimum of the curvatures of the family g_n. Since g_n is a compact family, $\kappa > -\infty$. Let $\bar{g}_n = -\frac{1}{\kappa}h_n$ be the metrics of constant curvature κ in the conformal class of g_n. We claim that

$$\ell_{h_n}(\gamma) \leq \sqrt{|\kappa|}\ell_{m_n}(\gamma)$$

for every $\gamma \in \pi_1(S)$. For instance, if we write $\bar{g}_n = e^{2u_n}g_n$, the smooth function $u_n : S \to \mathbb{R}$ satisfies the differential equation

$$e^{2u_n(x)}\kappa = \kappa_{g_n}(x) + \Delta_{g_n}u_n(x) \ ,$$

where κ_{g_n} is the curvature of g_n. Since $\kappa_{g_n} \geq \kappa$, $\Delta_{g_n}u_n$ is positive at the point of maximum of u_n and $\kappa < -1$, we deduce that $e^{2u_n} \leq 1$, hence

$$\ell_{\bar{g}_n}(\gamma) \leq \ell_{g_n}(\gamma)$$

for every $\gamma \in \pi_1(S)$. It is then clear that

$$\ell_{\bar{g}_n}(\gamma) = \frac{1}{\sqrt{\kappa}}\ell_{h_n}(\gamma)$$

for every $\gamma \in \pi_1(S)$ and the claim follows.
Moreover, by the inequality

$$\ell_{h_n}(\gamma) \leq \sqrt{|\kappa|}\ell_{g_n}(\gamma) \quad \forall\, \gamma \in \pi_1(S)$$

we deduce that h_n is contained in a compact set of Teichmüller space: if that were not the case, there would exists a curve γ such that $\ell_{h_n}(\gamma) \xrightarrow{n\to\infty} +\infty$, which is impossible because g_n is a compact family.
We can conclude now using Thurston asymmetric metric: given two hyperbolic metrics h and h', Thurston asymmetric distance between h and h' is defined as

$$d_{Th}(h, h') = \sup_{\gamma\in\pi_1(S)} \log\left(\frac{\ell_h(\gamma)}{\ell_{h'}(\gamma)}\right) .$$

It is well-known ([Thu98]) that if h'_n is a divergent sequence than $d_{Th}(K, h_n) \to +\infty$, where K is any compact set in Teichmüller space. Now, by the length spectrum comparison

$$\ell_{h_n}(\gamma) \leq \sqrt{|\kappa|}\ell_{m_n}(\gamma) \quad \forall\, \gamma \in \pi_1(S) ,$$

we deduce that $d_{Th}(h_n, m_n) \leq \log(\sqrt{|\kappa|}) < +\infty$, hence m_n must be contained in a compact set. □

We will need also the following fact about the geometry of the convex core of a GHMC AdS_3 manifold.

Lemma 2.5.12 (Proposition 5 [Dia13])**.** *Let M be a GHMC AdS_3 manifold. Denote by m^+ and m^- the hyperbolic metrics on the upper and lower boundary of the convex core of M. Let λ^+ and λ^- be the measured geodesic laminations on the upper and lower boundary of the convex core of M. For all $\epsilon > 0$, there exists some $A > 0$ such that, if m^+ is contained in a compact set and $\ell_{m^+}(\lambda^+) \geq A$, then $\ell_{m^-}(\lambda^+) \leq \epsilon\ell_{m^+}(\lambda^+)$.*

Proposition 2.5.13. *For every metric g^- and for every smooth path of metrics $\{g_t^+\}_{t\in[0,1]}$ on S with curvature $\kappa < -1$, the set $(\Phi^+ \times \phi^-_{g^-})^{-1}(\Delta)$ is compact.*

Proof. We need to prove that every sequence of isometric embeddings $(b^+_{t_n}, b^-_n)$ in $(\Phi^+ \times \phi^-_{g^-})^{-1}(\Delta)$ admits a convergent subsequence. By definition, for every $n \in \mathbb{N}$, there exists a GHMC AdS_3 manifold M_n containing a past-convex surface isometric to $(S, g^+_{t_n})$ with shape operator $b^+_{t_n}$ and a future-convex surface isometric to (S, g^-) with shape operator b^-_n. By Lemma 2.5.10 and Lemma 2.5.11, the metrics m^+_n and m^-_n on the upper and lower boundary of the convex core of M_n are contained in a

compact set of Teich(S) .
We are going to prove now that the sequences of left and right metrics of M_n are contained in a compact set of Teichmüller space, as well. Suppose by contradiction that the sequence of left metric h_{l_n} of M_n is not contained in a compact set. By Mess parameterisation (see Section 2.3, or [Mes07]), the left metrics are related to the metrics m_n^+ and to the measured geodesic laminations λ_n^+ of the upper-boundary of the convex core by an earthquake:

$$h_{l_n} = E^l_{\lambda_n^+}(m_n^+) \ .$$

Since h_{l_n} is divergent, the sequence of measured laminations λ_n^+ is divergent, as well. In particular, this implies that $\ell_{m_n^+}(\lambda_n^+)$ goes to infinity. Therefore, by Lemma 2.5.12, for every $\epsilon > 0$ there exists n_0 such that the inequality $\ell_{m_n^-}(\lambda_n^+) \leq \epsilon \ell_{m_n^+}(\lambda_n^+)$ holds for $n \geq n_0$. From this we deduce a contradiction, because we prove that the inequality

$$\ell_{m_n^-}(\lambda_n^+) \leq \epsilon \ell_{m_n^+}(\lambda_n^+) \qquad \forall\ n \geq n_0$$

implies that the sequence m_n^- is divergent, which contradicts what we proved in the previous paragraph. For instance, if m_n^- were contained in a compact set of Teichmüller space, there would exist (using again Thurston's asymmetric metric) a constant $C > 1$ such that

$$\frac{\ell_{m_n^+}(\gamma)}{\ell_{m_n^-}(\gamma)} \leq C \quad \forall\ n \geq n_0 \ .$$

By density this inequality must hold also for every measured geodesic lamination on S. But we have seen that for every $\epsilon > 0$ we can find n_0 such that for every $n \geq n_0$ we have

$$\frac{\ell_{m_n^+}(\lambda_n^+)}{\ell_{m_n^-}(\lambda_n^+)} \geq \frac{1}{\epsilon} \ ,$$

thus obtaining a contradiction.
A similar argument proves that also the sequence of right metrics h_{r_n} must be contained in a compact set of Teich(S).
Since the sequences of left and right metrics of M_n converge, up to subsequence, we can concretely realise the corresponding subsequence M_n as $(S \times \mathbb{R}, h_n)$ such that h_n converges in the C^∞-topology to an anti-de Sitter metric h_∞ and each M_n contains a future-convex space-like surface with embedding data (g^-, b_n^-) and a past-convex space-like surface with embedding data $(g^+_{t_n}, b^+_{t_n})$. The proof is then completed applying Corollary 2.5.5. $\square$

2.6 Prescription of an isometric embedding and half holonomy

This section is dedicated to the proof of the following result about the existence of an AdS_3 manifold with prescribed left metric containing a convex space-like surface with prescribed induced metric:

Proposition 2.6.1. *Let g be a metric on S with curvature less than -1 and let h be a hyperbolic metric on S. There exists a GHMC AdS_3 manifold M with left metric isotopic to h containing a past-convex space-like surface isometric to (S, g).*

If we denote with

$$p_1 : \mathrm{Teich}(S) \times \mathrm{Teich}(S) \to \mathrm{Teich}(S)$$

the projection onto the left factor, Propostition 2.6.1 is equivalent to proving that the map $p_1 \circ \phi_g^+ : I(S, g)^+ \to \mathrm{Teich}(S)$ is surjective. After showing that $p_1 \circ \phi_g^+$ is proper (Corollary 2.6.4), this will follow from the fact that its degree (mod 2) is non-zero.

In order to prove properness of the map $p_1 \circ \phi_g^+$, we will need the following well-known result about the behaviour of the length function while performing an earthquake.

Lemma 2.6.2 (Lemma 7.1 [BS09])**.** *Given a geodesic lamination $\lambda \in \mathcal{ML}(S)$ and a hyperbolic metric $g \in \mathrm{Teich}(S)$, let $g' = E_l^\lambda(g)$. Then for every closed geodesic γ in S the following estimate holds*

$$\ell_g(\gamma) + \ell_{g'}(\gamma) \geq \lambda(\gamma) \ .$$

Proposition 2.6.3. *For every path of metrics $\{g_t\}_{t\in[0,1]}$ with curvature less than -1, the projection $p_1 : \Phi^+(W^+) \to \mathrm{Teich}(S)$ is proper.*

Proof. Let $h_l(g_{t_n}, b_{t_n})$ be a convergent sequence of left metrics. We need to prove that the corresponding sequence of right metrics $h_r(g_{t_n}, b_{t_n})$ is convergent, as well. By hypothesis, (S, g_{t_n}) is isometrically embedded as past-convex space-like surface in each GHMC AdS_3 manifold M_n parametrised by $(h_l(g_{t_n}, b_{t_n}), h_r(g_{t_n}, b_{t_n}))$. By Lemma 2.5.10 and Lemma 2.5.11, the metrics m_n^+ on the past-convex boundary of the convex core of M_n are contained in a compact set of $\mathrm{Teich}(S)$. Moreover, by a result of Mess ([Mes07]), the left metrics $h_l(g_{t_n}, b_{t_n})$, the metrics m_n^+ and the measured laminations on the convex core λ_n^+ are related by an earthquake

$$h_l(g_{t_n}, b_{t_n}) = E_l^{\lambda_n^+}(m_n^+) \ .$$

Since $h_l(g_{t_n}, b_{t_n})$ is convergent, by Lemma 2.6.2, the sequence of measured laminations λ_n^+ must be contained in a compact set. Therefore, by continuity of the right earthquake

$$E_r : \mathrm{Teich}(S) \times \mathcal{ML}(S) \to \mathrm{Teich}(S)$$
$$(h, \lambda) \mapsto E_r^\lambda(h)$$

the sequence

$$h_r(g_{t_n}, b_{t_n}) = E_r^{\lambda_n^+}(m_n^+)$$

is convergent, up to subsequences. □

In particular, considering a constant path of metrics, we obtain the following:

Corollary 2.6.4. *The projection* $p_1 : \phi_g^+(I(S,g)^+) \to \mathrm{Teich}(S)$ *is proper .*

Proposition 2.6.5. *For every metric* g *of curvature* $\kappa < -1$*, the map*

$$p_1 \circ \phi_g^+ : I(S,g)^+ \to \mathrm{Teich}(S)$$

is proper of degree 1 *mod* 2*.*

Proof. Consider a path of metrics $(g_t)_{t\in[0,1]}$ with curvature less than -1 connecting $g = g_0$ with a metric of constant curvature g_1. By Corollary 2.5.8 and Corollary 2.6.4, the maps $p_1 \circ \phi_{g_0}^+ : I(S, g_0)^+ \to \mathrm{Teich}(S)$ and $p_1 \circ \phi_{g_1}^+ : I(S, g_1)^+ \to \mathrm{Teich}(S)$ are proper and cobordant, hence they have the same degree (mod 2). (This follows from Remark 2.4.1, Proposition 2.4.2 and Proposition 2.6.3). Thus, we can suppose that g has constant curvature $\kappa < -1$.
We notice that there exists a unique element in $I(S,g)^+$ such that $h_l(g,b) = -\kappa g$: a direct computation shows that $b = \sqrt{-\kappa-1}E$ works and uniqueness follows by the theory of landslides developed in [BMS13]. We sketch here the argument and we invite the interested reader to consult the aforementioned paper for more details. Pick $\theta \in (0,\pi)$ such that $\kappa = -\frac{1}{\cos^2(\theta/2)}$. The landslide

$$L^1_{e^{i\theta}} : \mathrm{Teich}(S) \times \mathrm{Teich}(S) \to \mathrm{Teich}(S)$$
$$(h, h^*) \mapsto h'$$

associates to a couple of hyperbolic metrics (h,h^*), the left metric of a GHMC AdS_3 manifold containing a space-like embedded surface with induced first fundamental form $I = \cos^2(\theta/2)h$ and third fundamental form $III = \sin^2(\theta/2)h^*$. It has been proved ([BMS13, Theorem 1.14]) that for every $(h,h') \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$, there exists a unique h^* such that $L^1_{e^{i\theta}}(h,h^*) = h'$. Moreover, the shape operator b of the embedded surface can be recovered by the formula ([BMS13, Lemma 1.9])

$$b = \tan(\theta/2)B$$

where $B : TS \to TS$ is the unique h-self-adjoint operator such that $h^* = h(B\cdot, B\cdot)$. Therefore, if we choose $h = h' = -\kappa g$, the uniqueness of the operator b follows by the uniqueness of h^* and B.
Hence, the degree (mod 2) of the map is 1, provided $-\kappa g$ is a regular value. Let $\dot{b} \in T_b I(S,g)^+$ be a non-trivial tangent vector. We remark that, since elements of $I(S,g)^+$ are g-self-adjoint, Codazzi tensor of determinant $-1-\kappa$, the tangent space $T_b I(S,g)^+$ can be identified with the space of traceless, Codazzi, g-self-adjoint

tensors. We are going to prove that the deformation induced on the left metric is non-trivial, as well. Let b_t be a path in $I(S,g)^+$ such that $b_0 = b = \sqrt{-\kappa-1}E$ and $\frac{d}{dt}b_t = \dot{b}$ at $t=0$. The complex structures induced on S by the metrics $h_l(g,b_t)$ are

$$J_t = (E + Jb_t)^{-1}J(E+Jb_t)$$

where J is the complex structure induced by g. Taking the derivative of this expression at $t=0$ we get

$$\dot{J} = \frac{2}{\kappa}[E - \sqrt{-\kappa-1}J]\dot{b}$$

which is non-trivial in $T_{-\kappa g}\mathrm{Teich}(S)$ because, as explained in Theorem 1.2 of [FT84], the space of traceless and Codazzi operators in $T_J\mathcal{A}$ has trivial intersection with the kernel of the differential of the projection $\pi : \mathcal{A} \to \mathrm{Teich}(S)$, which sends a complex structure J to its isotopy class. □

In particular, for every smooth metric g on S with curvature less than -1, the map $p_1 \circ \phi_g^+ : I(S,g)^+ \to \mathrm{Teich}(S)$ is surjective (a proper, non-surjective map has vanishing degree (mod 2)) and we deduce Proposition 2.6.1.

2.7 Proof of the main result

We have now all the ingredients to prove Theorem 2.3.2. As outlined in the Introduction, the first step consists of verifying that in one particular case, i.e. when we choose the metrics $g'_+ = -\frac{1}{\kappa}h$ and $g'_- = -\frac{1}{\kappa^*}h$, where h is any hyperbolic metric and $\kappa^* = -\frac{\kappa}{\kappa+1} = \kappa = -2$, the maps $\phi^+_{g'_+}$ and $\phi^-_{g'_-}$ have a unique transverse intersection. It is a standard computation to verify that $b^+ = E$ and $b^- = -E$ are Codazzi operators corresponding to an isometric embedding of (S, g'_+) as a past-convex space-like surface and to an isometric embedding of (S, g'_-) as a future-convex space-like surface respectively into the GHMC AdS_3 manifold M parametrised by $(h,h) \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$. This manifold M is unique due to the following:

Theorem 2.7.1 (Theorem 1.15 [BMS15])**.** *Let h_+ and h'_- be hyperbolic metrics and let κ_+ and κ_- be real numbers less than -1. There exists a GHMC AdS_3 manifold M which contains an embedded future-convex space-like surface with induced metric $\frac{1}{|\kappa_-|}h_-$ and an embedded past-convex space-like surface with induced metric $\frac{1}{|\kappa_+|}h_+$. Moreover, if $\kappa_+ = -\frac{\kappa_-}{\kappa_-+1}$, then M is unique.*

We notice that M is Fuchsian, i.e. it is parametrised by a couple of isotopic metrics in Teichmüller space. A priori, there might be other isometric embeddings of (S, g'_+) as a past-convex space-like surface and of (S, g'_-) as a future-convex space-like surface into M not equivalent to the ones found before. Actually, this is not the case due to the following result about isometric embeddings of convex surfaces into Fuchsian Lorentzian manifolds:

Theorem 2.7.2 (Theorem 1.1 [LS00]). *Let (S,g) be a Riemannian surface of genus $\tau \geq 2$ with curvature strictly smaller than -1. Let $x_0 \in \tilde{AdS}_3$ be a fixed point. There exists an equivariant isometric embedding (f,ρ) of (S,g) into $\tilde{AdS}_3$ such that ρ is a representation of the fundamental group of S into the group* $\mathrm{Isom}(\tilde{AdS}_3, x_0)$ *of isometries of $\tilde{AdS}_3$ fixing x_0. Such an embedding is unique modulo* $\mathrm{Isom}(\tilde{AdS}_3, x_0)$.

As a consequence, if we denote with Δ the diagonal of $\mathrm{Teich}(S)^2 \times \mathrm{Teich}(S)^2$, we have proved that

$$(\phi^+_{g'_+} \times \phi^-_{g'_-})^{-1}(\Delta) = (E, -E) \in I(S, g'_+)^+ \times I(S, g'_-)^- \ .$$

We need to verify next that at this point the intersection

$$\phi^+_{g'_+}(I(S, g'_+)^+) \cap \phi^-_{g'_-}(I(S, g'_-)^-)$$

is transverse. Suppose by contradiction that the intersection is not transverse, then there exists a non-trivial tangent vector $\dot{b}^+ \in T_E I(S, g'_+)^+$ and a non-trivial tangent vector $\dot{b}^- \in T_{-E} I(S, g'_-)^-$ such that

$$d\phi^+_{g'_+}(\dot{b}^+) = d\phi^-_{g'_-}(\dot{b}^-) \in T_h\mathrm{Teich}(S) \times T_h\mathrm{Teich}(S) \ .$$

We recall that elements of $T_E I(S, g'_+)^+$ can be represented by traceless, g'_+-self-adjoint, Codazzi operators. With this in mind, let us compute explicitly $d\phi^+_{g'_+}(\dot{b}^+)$. Let b^+_t be a smooth path in $I(S, g'_+)^+$ such that $b^+_0 = E$ and $\frac{d}{dt}_{|t=0} b^+_t = \dot{b}^+ \neq 0$. The complex structures induced on S by the left metrics $h_l(b_t)$ are

$$J^+_l = (E + Jb^+_t)^{-1} J (E + Jb^+_t) \ ,$$

where J is the complex structure of (S, g'_+). We compute now the derivative of this expression at $t = 0$. First notice that, since the operators b_t are g'_+-self-adjoint, Jb^+_t is traceless, hence the Hamilton-Cayley equation reduces to $(Jb^+_t)^2 + \det(Jb^+_t)E = (Jb^+_t)^2 + E = 0$. We deduce that

$$(E + Jb^+_t)(E - Jb^+_t) = 2E \ .$$

Therefore, the variation of the complex structures induced by the left metrics is

$$\begin{aligned}
\dot{J}^+_l &= \frac{d}{dt}_{|t=0} J^+_l = \frac{d}{dt}_{|t=0} \frac{1}{2}(E - Jb^+_t) J (E + Jb^+_t) \\
&= \frac{1}{2}(-J\dot{b}^+) J (E + J) + \frac{1}{2}(E - J) J^2 \dot{b}^+ \\
&= -(E - J)\dot{b}^+
\end{aligned}$$

where, in the last passage we used the fact that, since $\dot{b}^+$ is traceless and symmetric, the relation $J\dot{b}^+ = -\dot{b}^+ J$ holds.

With a similar procedure we compute the variation of the complex structures of the right metrics and we obtain

$$\dot{J}_r^+ = (E+J)\dot{b}^+ \in T_J\mathcal{A} \ .$$

Noticing that $\dot{J}_l^+$ and $\dot{J}_r^+$ are both traceless Codazzi operators, the image of $\dot{b}^+$ under the differential $d\phi^+_{g'_+}$ is simply

$$d\phi^+_{g'_+}(\dot{b}^+) = (-(E-J)\dot{b}^+, (E+J)\dot{b}^+) \in T_h\mathrm{Teich}(S) \times T_h\mathrm{Teich}(S)$$

because, as explained in Theorem 1.2 of [FT84], the space of traceless and Codazzi operators in $T_J\mathcal{A}$ is in direct sum with the kernel of the differential of the projection $\pi : \mathcal{A} \to \mathrm{Teich}(S)$, which sends a complex structure J to its isotopy class and gives an isomorphism between the space of traceless, Codazzi, self-adjoint tensors and $T_h\mathrm{Teich}(S)$.
With a similar reasoning we obtain that

$$d\phi^-_{g'_-}(\dot{b}^-) = (-(E+J)\dot{b}^-, (E-J)\dot{b}^-) \in T_h\mathrm{Teich}(S) \times T_h\mathrm{Teich}(S) \ .$$

By imposing that $d\phi^+_{g'_+}(\dot{b}^+) = d\phi^-_{g'_-}(\dot{b}^-)$ we obtain the linear system

$$\begin{cases} (-E+J)\dot{b}^+ = -(E+J)\dot{b}^- \\ (E+J)\dot{b}^+ = (E-J)\dot{b}^- \end{cases}$$

which has solutions if and only if $\dot{b}^+ = \dot{b}^- = 0$. Therefore, the intersection is transverse and we can finally state that

$$\Im(\phi^+_{g'_+}, \phi^-_{g'_-}) = 1 \ .$$

Now we use the theory described in Section 2.4 to prove that an intersection persists under a deformation of one metric that fixes the other. Let g_+ and g_- be two arbitrary metrics on S with curvature less than -1. We will still denote with g'_+ and with g'_- the metrics introduced in the previous paragraph with self-dual constant curvature and in the same conformal class. Consider two paths of metrics $\{g^t_+\}_{t\in[0,1]}$ and $\{g^t_-\}_{t\in[0,1]}$ with curvature less than -1 such that $g^0_+ = g_+$, $g^1_+ = g'_+$, $g^0_- = g_-$ and $g^1_- = g'_-$. We will first prove that

$$\phi^+_{g_+}(I(S,g_+)^+) \cap \phi^-_{g'_-}(I(S,g'_-)^-) \neq \emptyset \ .$$

Suppose by contradiction that this intersection is empty. Then the map $\phi^+_{g_+} \times \phi^-_{g'_-}$ is trivially transverse to Δ. Consider the manifold

$$W^+ = \bigcup_{t\in[0,1]} I(S, g^+_t)^+$$

and the map

$$\Phi^+ \times \phi^-_{g'_-} : X = W^+ \times I(S, g'_-)^- \to (\mathrm{Teich}(S))^4 = Y$$

as defined in Section 2.5. By assumption the restriction of $\Phi^+ \times \phi^-_{g'_-}$ to the boundary is transverse to Δ and by Proposition 2.5.13, the set $D = (\Phi^+ \times \phi^-_{g'_-})^{-1}(\Delta)$ is compact. Let B be the interior of a compact set containing D and let $C = (X \setminus B) \cup \partial X$. By construction, $\Phi^+ \times \phi^-_{g_-}$ is transverse to Δ along the closed set C. Applying Theorem 2.4.4, there exists a smooth map $\Psi : X \to Y$ which is transverse to Δ and which coincides with $\Phi^+ \times \phi^-_{g'_-}$ on C. In particular, the value on the boundary remains unchanged and $\Psi^{-1}(\Delta)$ is still a compact set. By Proposition 2.4.3, the intersection number of the maps

$$\phi^+_{g'_+} : I(S, g'_+)^+ \to \mathrm{Teich}(S) \times \mathrm{Teich}(S) \quad \text{and} \quad \phi^+_{g_+} : I(S, g_+)^+ \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$$

with the map

$$\phi^-_{g'_-} : I(S, g_-)^- \to \mathrm{Teich}(S) \times \mathrm{Teich}(S) \ ,$$

as defined in Section 2.4, must be the same. This gives a contradiction, because

$$0 = \Im(\phi^+_{g_+}, \phi^-_{g'_-}) \neq \Im(\phi^+_{g'_+}, \phi^-_{g'_-}) = 1 \ .$$

So we have proved that $\phi^+_{g_+}(I(S, g_+)^+) \cap \phi^-_{g'_-}(I(S, g'_-)^-) \neq \emptyset$, but we do not know if the intersection is transverse. Repeating the above argument choosing the closed set $C = (X \setminus B) \cup I(S, g'_+)^+$, we obtain that a perturbation ψ of $\phi^+_{g_+} \times \phi^-_{g'_-}$ which is transverse to Δ and coincides with $\phi^+_{g_+} \times \phi^-_{g'_-}$ outside the interior of a compact set containing $(\phi^+_{g_+} \times \phi^-_{g'_-})^{-1}(\Delta)$ has intersection number $\Im(\psi, \Delta) = 1$. By Proposition 2.4.5, every perturbation of the map $\phi^+_{g_+} \times \phi^-_{g'_-}$ obtained in this way has intersection number with Δ equal to 1.

This enables us to deform the metric g_- without losing the intersection, by repeating a similar argument. Suppose by contradiction that

$$\phi^+_{g_+}(I(S, g_+)^+) \cap \phi^-_{g_-}(I(S, g_-)^-) = \emptyset \ .$$

Consider the manifold

$$W^- = \bigcup_{t \in [0,1]} I(S, g_t^-)^-$$

and the map

$$\Phi^- \times \phi^+_{g_+} : X = W^- \times I(S, g_+)^+ \to (\mathrm{Teich}(S))^4 = Y \ .$$

By assumption the restriction of the map $\Phi^- \times \phi^+_{g_+}$ to the first boundary component $X_0 = I(S, g_+)^+ \times I(S, g_-)^-$ is transverse to Δ and by Proposition 2.5.13, the pre-image $D = (\Phi^- \times \phi^+_{g_+})^{-1}(\Delta)$ is a compact set. Let B be the interior of a compact set containing D and let $C = (X \setminus B) \cup X_0$. By construction, $\Phi^- \times \phi^+_{g_+}$ is transverse to Δ along the closed set C. Applying Theorem 2.4.4, there exists a smooth map $\Psi : X \to Y$ which is transverse to Δ and which coincides with $\Phi^- \times \phi^+_{g_+}$ on C. In particular, the value on the boundary X_0 remains unchanged and $\Psi^{-1}(\Delta)$ is still a compact set. Moreover, the value of Ψ on the other boundary component is a perturbation of $\phi^+_{g_+} \times \phi^-_{g'_-}$ which is transverse to Δ and coincides with $\phi^+_{g_+} \times \phi^-_{g'_-}$ outside the interior of a compact set containing $(\phi^+_{g_+} \times \phi^-_{g'_-})^{-1}(\Delta)$. Hence, by Proposition 2.4.5 and by Proposition 2.4.3, the intersection number $\Im(\phi^+_{g_+}, \phi^-_{g_-})$ must be equal to 1, thus giving a contradiction.

Remark 2.7.3. *It might be possible to prove the uniqueness of this intersection by applying Proposition 2.4.6. To this aim, it would be necessary to show that for every couple of metrics g_+ and g_- with curvature strictly smaller than -1, the functions $\phi^+_{g_+} : I(S, g_+)^+ \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ and $\phi^-_{g_-} : I(S, g_-)^- \to \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ are transverse.*

Chapter 3

Constant mean curvature foliation of domains of dependence

We prove that, given an acausal curve Γ in the boundary at infinity of AdS_3 which is the graph of a quasi-symmetric homeomorphism ϕ, there exists a unique foliation of its domain of dependence $\mathcal{D}(\Gamma)$ by constant mean curvature surfaces with bounded second fundamental form. Moreover, these surfaces provide a family of quasi-conformal extensions of ϕ.

3.1 Definition of the problem and outline of the proofs

Recently, after the work of Bonsante and Schlenker [BS10] and of Bonsante and Seppi [BS16], Anti-de Sitter geometry has turned out to be a useful tool to construct quasi-conformal extensions of quasi-symmetric homeomorphisms of the unit disc. Indeed, the graph of a quasi-symmetric map $\phi : S^1 \to S^1$ describes a curve (called quasi-circle) c_ϕ on the boundary at infinity of the 3-dimensional anti-de Sitter space AdS_3. Bonsante and Schlenker proved that a smooth surface S (satisfying some technical conditions) with asymptotic boundary c_ϕ defines a quasi-conformal extension of ϕ. Moreover, some remarkable properties of the quasi-conformal extension can be deduced from the geometry of the surface itself: for example, when the surface S is maximal (i.e. it has vanishing mean curvature) the quasi-conformal extension induced by S is minimal Lagrangian ([BS10]); when S is a smooth convex κ-surface, the corresponding quasi-conformal extension is a landslide ([BS16]); when the surface S is the past-convex boundary of the convex-hull of c_ϕ, the quasi-symmetric homeomorphism ϕ is exactly the boundary map of the earthquake $E^{2\lambda} : \mathbb{H}^2 \to \mathbb{H}^2$, where λ is the pleating locus of S ([Mes07]).

In this chapter we study quasi-conformal extensions induced by constant mean curvature surfaces (in brief H-surfaces). The first problem we address is the existence of an H-surface with a given quasi-circle Γ as asymptotic boundary, thus generalizing the work of Bonsante and Schlenker ([BS10]) for maximal surfaces. We will prove that for each $H \in \mathbb{R}$, there exists an H-surface with asymptotic boundary Γ and bounded principal curvatures. Although the technical part of the proof is based on the same apriori estimates as in [BS10], the starting point for the construction of this H-surface is different, thus obtaining a somehow new proof also in the case when $H = 0$. Namely, we construct this H-surface S_H as a limit of H-surfaces $(S_H)_n$, with asymptotic boundary Γ_n, with the property that Γ_n is the graph of a quasi-symmetric homeomorphism conjugating two cocompact Fuchsian groups and Γ_n converges to Γ in the Hausdorff topology. The existence of this approximating sequence $(S_H)_n$ is a consequence of some results in [BBZ07] and [BS16].

Moreover, extending the results of [BBZ07], we prove the following:

Theorem 3.3.1. *Given a quasi-circle $\Gamma \subset \partial_\infty AdS_3$, there exists a foliation by constant mean curvature surfaces S_H for $H \in (-\infty, +\infty)$ of the domain of dependence $\mathcal{D}(\Gamma)$.*

In the second part of the chapter, we estimate the principal curvatures of a constant mean curvature surface. Those results will then be used to prove the uniqueness of the foliation (Theorem 3.5.2) and to prove that each H-surface bounding a quasi-circle induces a quasi-conformal extension of a quasi-symmetric homeomorphism (Proposition 3.6.2).

3.2 Quasi-symmetric and quasi-conformal maps

In this section we recall some well-known results about quasi-symmetric homeomorphisms of S^1. The graph of a quasi-symmetric homeomorphism ϕ describes a curve c_ϕ on the boundary at infinity of AdS_3 and, under some additional conditions, smooth negatively curved surfaces bounding c_ϕ provide quasi-conformal extensions of ϕ ([BS10], [KS07]). We recall here briefly this construction.

In Chapter 1 we have seen that it is possible to identify the boundary at infinity of the 3-dimensional anti-de Sitter space with $S^1 \times S^1$. With this identification, we can represent the graph of a homeomorphism $\phi : S^1 \to S^1$ as a curve on the boundary at infinity of AdS_3, namely

$$c_\phi = \{(x, \phi(x)) \in \partial_\infty AdS_3 \mid x \in S^1\} \ .$$

A homeomorphism $\phi : S^1 \to S^1$ is quasi-symmetric if there exists a constant $C > 0$ such that

$$\sup_Q |\log |cr(\phi(Q))|| \leq C \ ,$$

where the supremum is taken over all quadruple Q of points in S^1 with cross ratio $cr(Q) = -1$, and we use the following definition of cross-ratio

$$cr(x_1, x_2, x_3, x_4) = \frac{(x_4 - x_1)(x_3 - x_2)}{(x_2 - x_1)(x_3 - x_4)} .$$

Definition 3.2.1. *An acausal curve $\Gamma \subset \partial_\infty AdS_3$ is a quasi-circle, if it is the graph of a quasi-symmetric homeomorphism.*

Remark 3.2.2. *It follows from the identification between the boundary at infinity of AdS_3 and $S^1 \times S^1$ that an acausal curve Γ is a quasi-circle if and only if $\phi = \pi_r \circ \pi_l^{-1}$ is quasi-symmetric. Moreover, Γ is the graph of ϕ.*

An orientation-preserving homeomorphism $f : D^2 \to D^2$ is quasi-conformal if f is absolutely continuous on lines and there exists a constant $k < 1$ such that

$$|\mu_f| = \left| \frac{\overline{\partial} f}{\partial f} \right| \leq k .$$

A map with this property can also be called K-quasi-conformal, where

$$K = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty} \in [1, +\infty)$$

The relation between quasi-symmetric homeomorphisms of the circle and quasi-conformal maps of the unit disc is provided by the following well-known theorem:

Theorem 3.2.3 ([Ahl38])**.** *Every quasi-conformal map $\Phi : D^2 \to D^2$ extends to a quasi-symmetric homeomorphism of S^1. Conversely, any quasi-symmetric homeomorphism $\phi : S^1 \to S^1$ admits a quasi-conformal extension to D^2.*

If we represent the graph of a quasi-symmetric homeomorphism ϕ as a curve c_ϕ on the boundary at infinity of AdS_3, in [KS07] it is explained how to obtain quasiconformal extensions of ϕ using smooth, negatively curved, space-like surfaces with boundary at infinity c_ϕ. The construction goes as follows. We fix a totally geodesic space-like plane P_0. Let S be a space-like, negatively-curved surface embedded in AdS_3. Let $\tilde{S}' \subset U^1 AdS_3$ be its lift into the unit tangent bundle of AdS_3 and let $p : \tilde{S}' \to \tilde{S}$ be the canonical projection. For any point $(x, v) \in \tilde{S}'$, there exists a unique space-like plane P in AdS_3 orthogonal to v and containing x. We define two natural maps $\Pi_{\infty,l}$ and $\Pi_{\infty,r}$ from $\partial_\infty P$ to $\partial_\infty P_0$, sending a point $x \in \partial_\infty P$ to the intersection between $\partial_\infty P_0$ and the unique line of the left or right foliation of $\partial_\infty AdS_3$ containing x. Since these maps are projective, they extend to hyperbolic isometries $\Pi_l, \Pi_r : P \to P_0$. We then define the map $\Phi = \Pi_r \circ \Pi_l^{-1}$. This map is always a local diffeomorphism of $\mathbb{H}^2$ when the surface is negatively curved, as the differentials of the maps Π_l and Π_r are given by

$$d\Pi_l = E + JB \qquad d\Pi_r = E - JB .$$

On the other hand, Φ is not always a global diffeomorphism, but the following lemma gives some sufficient conditions on the surface S which guarantee that Φ is proper (and hence a homeomorphism) and that its boundary value coincides with ϕ:

Lemma 3.2.4 (Lemma 3.18 [KS07])**.** *Let S be a space-like, negatively-curved surface in AdS_3 whose boundary at infinity Γ does not contain any light-like segment. Suppose that there is no sequence of points x_n on S such that the totally geodesic planes P_n tangent to S at x_n converge to a light-like plane P whose past end-point and future end-point are not in Γ. Then for any sequence of points $x_n \in S$ converging to $x \in \Gamma$ we have that $\Pi_l(x_n) \to \pi_l(x)$ and $\Pi_r(x_n) \to \pi_r(x)$.*

Remark 3.2.5. *As noticed in [KS07], the hypothesis of Lemma 3.2.4 are satisfied in case of a smooth, convex, space-like surface bounding a quasi-circle.*

3.3 Existence of a CMC foliation

This section is devoted to the proof of the following:

Theorem 3.3.1. *Given a quasi-circle $\Gamma \subset \partial_\infty AdS_3$, there exists a foliation by constant mean curvature surfaces S_H for $H \in (-\infty, +\infty)$ of the domain of dependence $\mathcal{D}(\Gamma)$.*

As outlined in Section 3.1, the main idea to construct a constant mean curvature surface with a given quasi-circle as boundary at infinity is a process by approximation. In fact, as a consequence of the work [BBZ07], the existence (and uniqueness) of a constant mean curvature foliation is known for a particular class of quasi-circles:

Theorem 3.3.2 (Theorem 1.1 [BBZ07])**.** *Let Γ be a quasi-circle which is the graph of a quasi-symmetric homeomorphism that conjugates two cocompact Fuchsian groups. Then there exists a unique foliation by equivariant H-surfaces of the domain of dependence of Γ, where H varies in $(-\infty, +\infty)$.*

Moreover, by a recent result in [BS16, Lemma 7.2], every quasi-circle can be uniformly approximated by a sequence of quasi-circles, which are the graphs of quasi-symmetric homeomorphisms conjugating two cocompact Fuchsian groups.

Therefore, given a quasi-circle Γ, we will consider a sequence of quasi-circles Γ_n, which are the graphs of quasi-symmetric homeomorphisms conjugating two cocompact Fuchsian groups, converging in the Hausdorff topology to Γ. For each $H \in (-\infty, +\infty)$, Theorem 3.3.2 provides a sequence of H-surfaces $(S_H)_n$ with boundary at infinity Γ_n. In this section we will prove that the sequence $(S_H)_n$ converges C^∞ on compact sets to an H-surface $(S_H)_\infty$ with boundary at infinity Γ. This will give us the existence of a surface with given boundary at infinity and given constant mean curvature H for every $H \in (-\infty, +\infty)$. We will then prove that these surfaces provide a foliation of the domain of dependence of Γ.

We first recall some definitions. In the Universal cover of AdS_3, given a space-like surface M, we recall that M is the graph of a function $u : \mathbb{H}^2 \to \mathbb{R}$. We define the gradient function with respect to the vector field $T = -\chi\nabla t$ as

$$v_M = -\langle \nu, T\rangle = \frac{1}{\sqrt{1-\chi^2|\nabla u|^2}}$$

where ν is the unit future-oriented normal vector field. The shape operator of M is defined by

$$B(X) = -\nabla_X \nu$$

for every vector field X on M. The mean curvature of M is

$$H = \frac{\text{trace}(B)}{2} \ .$$

We can write explicitely a formula for the mean curvature of M, in terms of u and T (see e.g. [Bar88]):

$$H = \frac{1}{2v_M}(\text{div}_M(\chi \text{grad}_M u) + \text{div}_M T) \ . \tag{3.1}$$

We will need the following a-priori estimate for the gradient function v_M, which is a consequence of the work of Bartnik [Bar88]. Given a point $p \in \widetilde{AdS_3}$, we denote with $I^+(p)$ the set of points in the future of p, and similarly with $I^-(p)$ the set of points in the past of p. We will indicate with $I^+_\epsilon(p)$ the set of points in the future of p at distance at least ϵ. We have the following:

Lemma 3.3.3. *Let $p \in \widetilde{AdS_3}$ and $\epsilon > 0$. Let K be a compact domain contained on a region where the covering map $\pi : \widetilde{AdS_3} \to AdS_3$ is injective. Let $H \in \mathbb{R}$ be fixed. There exists a constant $C = C(p, \epsilon, K)$ such that for every H-surface M that verifies*

- *$\partial M \cap I^+(p) = \emptyset$;*
- *$M \cap I^+(p) \subset K$,*

we have that

$$\sup_{M\cap I^+_\epsilon(p)} v_M < C \ .$$

Proof. Consider the time function

$$\tau(x) = d_{AdS}(x, p) - \frac{\epsilon}{2} \ ,$$

where $d_{AdS}(x, p)$ is the Lorentzian distance between x and p. This function is smooth on $V = K \cap I^+(p)$. By assumption on M, the region $M \cap V$ contains the set $\{\tau \geq 0\} \cap M$ and $M \cap I^+_\epsilon(p)$ is contained in V. We can thus apply Theorem 3.1 in [Bar88] and conclude that

$$\sup_{M\cap I^+_\epsilon(p)} v_M < C \ ,$$

where the constant C depends on the C^2 norms of t and T and on the C^0 norm of the Ricci tensor on the domain $V \cap \{\tau \geq 0\}$ with respect to a reference Riemannian metric. □

We will also need the following result that provides some barriers for constant mean curvature surfaces in anti-de Sitter manifolds:

Proposition 3.3.4. *Let Σ be a space-like surface with constant mean curvature $H \in \mathbb{R}$ embedded in AdS_3 with boundary at infinity a quasi-circle Λ. Suppose that Σ the lift of a compact surface embedded in a GHMC anti-de Sitter manifold. Then there exists $\kappa \leq -1$ such that Σ is in the past of the past-convex surface S_κ^+ and in the future of the future-convex surface S_κ^- with constant Gauss curvature κ and asymptotic boundary Λ.*

Proof. If $H = 0$, the statement holds, since a maximal surface is contained in the convex hull of Λ. For the other values of H we choose $\kappa < -1$ such that $\sqrt{-1-\kappa} > |H|$. We claim that the past-convex space-like surface S_κ^+ with constant curvature κ, whose existence is proved in [BS16], must be in the future of Σ. If not, the surfaces Σ and S_κ^+ would intersect transversely, but, since constant curvature surfaces provide a foliation of $\mathcal{D}(\Lambda) \setminus \mathcal{C}(\Lambda)$, there would exist a $\kappa' < \kappa$ such that the surface $S_{\kappa'}$ with constant Gauss curvature κ' is tangent to Σ at a point x. By the Maximum Principle, the mean curvature of $S_{\kappa'}$ at x must be smaller than the mean curvature of Σ at x, but this is impossible for our choice of κ'.
With a similar reasoning we obtain that the future-convex space-like surface S_κ must be in the past of Σ. □

Let us fix $H \in \mathbb{R}$. We have now all the ingredients to prove the existence of an H-surface with given asymptotic boundary. Let Γ be a quasi-circle on AdS_3 and let Γ_n be a sequence of quasi-circles converging to Γ in the Hausdorff topology that are the graphs of quasi-symmetric homeomorphisms that conjugate two cocompact Fuchsian groups. Let $(S_H)_n$ be the H-surface with asymptotic boundary Γ_n provided by Theorem 3.3.2.

Theorem 3.3.5. *The sequence of H-surfaces $(S_H)_n$ converges C^∞ on compact sets to an H-surface $(S_H)_\infty$ with boundary at infinity Γ.*

Proof. We consider their lifts $(\tilde{S}_H)_n$ to the Universal cover $\widetilde{AdS}_3$. We denote with $(\tilde{\Sigma}_\kappa^\pm)_n$ the lifts of the two constant curvature surfaces provided by Proposition 3.3.4. In general, we will use the notation with a tilda to indicate the lift of an object to the Universal cover. By Theorem 7.8 in [BS16], the sequence $(\tilde{\Sigma}_\kappa^\pm)_n$ converges to constant curvature surfaces $\tilde{\Sigma}_\kappa^\pm$ with boundary at infinity $\tilde{\Gamma}$. We denote with K' the domain

$$K' = I^-(\tilde{\Sigma}_\kappa^+) \cap I^+(\tilde{\Sigma}_\kappa^-) \ .$$

For any point $\tilde{p} \in \widetilde{\mathcal{D}(\Gamma)} \cap I^-(\tilde{\Sigma}_\kappa^-)$, we choose $\epsilon(\tilde{p})$ such that the family $\{I^+_{\epsilon(\tilde{p})}(\tilde{p}) \cap K'\}$ is an open covering of K'. Since

$$K'_n = I^-((\tilde{\Sigma}_\kappa^+)_n) \cap I^+((\tilde{\Sigma}_\kappa^-)_n)$$

converges to K', there exists an n_0 such that for every $n \geq n_0$ the closed set

$$K = \overline{\bigcup_{n \geq n_0} K'_n}$$

is contained in the open covering $\cup\{I^+_{\epsilon(\tilde{p})}(\tilde{p})\}$ constructed above.
Given a number $R > 0$ we denote with B_R the ball of radius R in $\mathbb{H}^2$ centered at the origin in the Poincaré model. The intersection $(B_R \times \mathbb{R}) \cap K$ is compact, so there is a finite number of points $\tilde{p}_1, \dots, \tilde{p}_m$ such that

$$(B_R \times \mathbb{R}) \cap K \subset \bigcup_{j=1}^m I^+_{\epsilon(\tilde{p}_j)}(\tilde{p}_j) \ .$$

We notice that, since $\tilde{p}_j \in \widetilde{\mathcal{D}(\Gamma)}$, the intersection $I^+(\tilde{p}_j) \cap \widetilde{\mathcal{D}(\Gamma)}$ is compact. Moreover, since the plane dual to $\tilde{p}_j$ is disjoint from $\tilde{\Gamma}$ for every $j = 1, \dots, m$, if we choose n_0 big enough, the same is true for Γ_n for every $n \geq n_0$, because Γ_n converges to Γ in the Hausdorff topology. In this way we can ensure that the set $K_j = \overline{I^+(\tilde{p}_j)} \cap K$ is compact and contained on a region where the covering map $\pi : \widetilde{AdS_3} \to AdS_3$ is injective. By Lemma 3.3.3, there is a constant C_j such that

$$\sup_{M \cap I^+_{\epsilon(\tilde{p}_j)}(\tilde{p}_j)} v_M < C_j$$

for every constant mean curvature surface M satisfying

(i) $\partial M \cap I^+(\tilde{p}_j) = \emptyset$;

(ii) $M \cap I^+(\tilde{p}_j)$ is contained in K_j .

Condition (i) is clearly satisfied for $n \geq n_0$ by definition of the set K. As for Condition (ii), the boundary of $(\tilde{S}_H)_n$ is disjoint from the future of $\tilde{p}_j$ for every $j = 1, \dots m$ due to our choice of n_0.
If we denote with v_n the gradient function associated to the surface $(\tilde{S}_H)_n$, it follows that

$$\sup_{(\tilde{S}_H)_n \cap (B_R \times \mathbb{R})} v_n \leq \max\{C_1, \dots, C_m\} \tag{3.2}$$

for every $n \geq n_1$.
We deduce that for every R there is a constant $C(R)$ such that the gradient function v_n is bounded by $C(R)$ for n sufficiently large.
Let $u_n : \mathbb{H}^2 \to \mathbb{R}$ such that $(\tilde{S}_H)_n$ are the graph of the function u_n. By comparing

Equation (3.1) with estimate (3.2), we see that the restriction of u_n on B_R is the solution of a uniformly elliptic quasi-linear PDE with bounded coefficients. Since u_n and the gradient ∇u_n are uniformly bounded on B_R (see Section 1.4), by elliptic regularity the norms of u_n in $C^{2,\alpha}(B_{R-1})$ are uniformly bounded. We can thus extract a subsequence u_{n_k} which converges C^2 to some function u_∞ on compact sets. Since u_∞ is a C^2-limit of solutions of Equation (3.1), it is still a solution and its graph S_H has constant mean curvature H.
The boundary at infinity of S_H coincides with Γ because it is the Hausdorff limit of the curves Γ_n, which converge to Γ, by construction. □

Moreover, we can deduce that the principal curvatures of the surface $(S_H)_\infty$ are uniformly bounded, due to the following:

Lemma 3.3.6. *Let S be an H-surface embedded in AdS_3, which is the lift of a space-like compact surface embedded into a GHMC AdS_3 manifold. Then the principal curvatures μ and μ' of S are bounded by some constant depending only on H. More precisely,*

$$H \leq \lambda_1 \leq \sqrt{H^2+1}+H \qquad \textit{and} \qquad -\sqrt{H^2+1}+H \leq \lambda_2 \leq H \ .$$

Proof. Let B be the shape operator of S. We consider $B_0 = B - HE$, the traceless part of B (here E is the identity operator). Since H is constant, the operator B_0 is Codazzi. Let e_1 and e_2 be tangent vectors in a orthonormal frame of S that diagonalises B_0. Since B_0 is traceless, the eigenvalues are opposite, and we will denote with $\lambda \geq 0$ the eigenvalue of e_1. Let ω be 1-form connection of the Levi-Civita connection ∇ for the induced metric on S, defined by the relation

$$\nabla_x e_1 = \omega(x) e_2 \ .$$

The Codazzi equation for B_0 can be read as follows,

$$\begin{cases} \lambda\omega(e_1) = d\lambda(e_2) \\ \lambda\omega(e_2) = -d\lambda(e_1) \ . \end{cases}$$

If we define $\beta = \log(\lambda)$ we obtain

$$\begin{cases} \omega(e_1) = d\beta(e_2) \\ \omega(e_2) = -d\beta(e_1) \end{cases}$$

Moreover, if we denote with κ the Gaussian curvature of S, we have

$$-K = d\omega(e_1,e_2) = e_1(\omega(e_2)) - e_2(\omega(e_1)) - \omega([e_1,e_2]) = \Delta\beta \ ,$$

where Δ is the Laplacian that is positive at the points of local maximum. On the other hand by the Gauss equation,

$$-K = \det(B)+1 = \det(B_0+HE)+1 = \det(B_0)+H^2+1 = -e^{2\beta}+H^2+1 \ .$$

Since the surface S is the lift of a compact surface, the function β has maximum at a point x_0. By the fact that $\Delta\beta(x_0) \geq 0$, we deduce that

$$\lambda = e^{\beta} \leq \sqrt{H^2+1} \ .$$

Since the eigenvalues of B are $\mu = \lambda + H$ and $\mu' = -\lambda + H$, we obtain the claim. □

We have thus found for every value of $H \in \mathbb{R}$ a constant mean curvature surface S_H, with bounded principal curvatures, bounding a given quasi-circle Γ at infinity. We conclude this section by showing that these surfaces provide a foliation of the domain of dependence $\mathcal{D}(\Gamma)$.

Proposition 3.3.7. *Let $\{S_H\}_{H\in\mathbb{R}}$ be the family of H-surfaces provided by Theorem 3.3.5 with boundary at infinity Γ. Then $\{S_H\}_{H\in\mathbb{R}}$ foliates the domain of dependence $\mathcal{D}(\Gamma)$.*

Proof. We first show that if $H_1 < H_2 \in \mathbb{Q}$, then S_{H_1} and S_{H_2} are disjoint and S_{H_1} is in the past of S_{H_2}. By construction S_{H_1} and S_{H_2} are C^∞ limits of the sequences $(S_{H_1})_n$ and $(S_{H_2})_n$ of constant mean curvature surfaces with boundary at infinity Γ_n which is a graph of a quasi-symmetric homeomorphism that conjugates two cocompact Fuchsian groups. By Theorem 3.3.2, they are leaves of the constant mean curvature foliation of $\mathcal{D}(\Gamma_n)$ and, in particular, $(S_{H_1})_n$ is in the past of $(S_{H_2})_n$ for every n. Hence, the same holds for S_{H_1} and S_{H_2}. This shows that they cannot intersect transversely. Moreover, it is not possible that S_{H_1} and S_{H_2} are tangent at one point, since the trace of the shape operator of S_{H_2} is bigger than the trace of the shape operator of S_{H_1} and this would contradict the Maximum Principle (see Lemma 3.5.1).
We now show that if we take two sequences of rational numbers H'_k converging increasingly to $H \in \mathbb{R}$ and H''_k converging decreasingly to $H \in \mathbb{R}$, then $S_{H'_k}$ and $S_{H''_k}$ converge to the same limit S_H. We first notice that the limits

$$S'_H = \lim_{H'_k \to H} S_{H'_k} \qquad \text{and} \qquad S''_H = \lim_{H''_k \to H} S_{H''_k}$$

exist by a similar argument as in the proof of Theorem 3.3.5. Moreover, S'_H must be in the past of S''_H. Suppose by contradiction that S'_H and S''_H are distinct. Let U be an open set contained in the past of S''_H and in the future of S'_H. Since we know that the domain of dependence of Γ_n is foliated by constant mean curvature surfaces, for n large enough there exists a surface S_{h_n} with constant mean curvature $h_n \in \mathbb{Q}$ and boundary at infinity Γ_n. By the uniform convergence on compact sets, for n larger than some n_0, we must have $h_n < H''_k$ and $h_n > H'_k$ for every $k \in \mathbb{N}$, which gives the contradiction.
So far we have proved that the H-surfaces $\{S_H\}_{H\in\mathbb{R}}$ provide a foliation of a subset of $\mathcal{D}(\Gamma)$. We need to prove that

$$\bigcup_{H\in\mathbb{R}} S_H = \mathcal{D}(\Gamma) \ .$$

Suppose by contradiction that there exists a point $p \in \mathcal{D}(\Gamma)$ which does not lie in any of the surfaces S_H. Since the domain of dependence of Γ_n converges to the domain of dependence of Γ, there exists a sequence of points $p_n \in \mathcal{D}(\Gamma_n)$ converging to p. Since $\mathcal{D}(\Gamma_n)$ is foliated by constant mean curvature surfaces, there exists a sequence $H_n \in \mathbb{R}$ such that $p_n \in S_n^{H_n} \subset \mathcal{D}(\Gamma_n)$. We claim that the sequence H_n is bounded. We can assume that H_n is positive for n big enough. Since $p \in \mathcal{D}(\Gamma)$, the boundary at infinity of the dual plane p^* is disjoint from Γ. We choose a space-like plane P in the future of p^* with the following properties: the boundary at infinity of P is disjoint from Γ and $p \in \mathcal{D}(P) \cap \mathcal{D}(\Gamma)$. Since Γ_n converges to Γ in the Hausdorff topology, the asymptotic boundary of P is disjoint from Γ_n for n big enough. Moreover, the surfaces $S_n^{H_n}$ converge to a nowhere time-like surface S_∞ passing at p, because they are graphs of uniformly Lipschitz functions (see Section 1.4). We notice that there exists a surface Σ_{H_0} with constant mean curvature H_0 and boundary at infinity $\partial_\infty P$, such that Σ_{H_0} intersects S_∞ in a compact set and $p \in I^-(\Sigma_{H_0})$. This surface Σ_{H_0} is obtained by taking equidistant surfaces from the space-like plane P_0. Let $F_P : \mathcal{D}(P) \to \mathbb{R}$ be the time function defined on the domain of dependence of P such that each level set $F^{-1}(H) = \Sigma_H$ is a constant mean curvature surface with asymptotic boundary $\partial_\infty P$. It follows by construction that $F_P^{-1}((-\infty, H_0)) \cap S_\infty$ is compact. Since the sequence $S_n^{H_n}$ converges uniformly on compact set to S_∞, the same is true for $F_P^{-1}((-\infty, H_0)) \cap S_n^{H_n}$, for n sufficiently big. We define

$$H_n^- = \inf_{x \in S_n^{H_n}} F_P(x) \ .$$

By the previous remarks $H_n^- \leq H_0$ for every n sufficiently large, and it is assumed at some point $x_n \in S_n^{H_n}$. By construction, $S_n^{H_n}$ is tangent to $\Sigma_{H_n^-}$ at the point x_n and $S_n^{H_n}$ is contained in the future of $\Sigma_{H_n^-}$. Therefore, by the Maximum Principle, we deduce that

$$H_n \leq H_n^- \leq H_0$$

as claimed. Therefore, there exist two real numbers H^+ and H^- such that $S_n^{H_n}$ is in the past of $S_n^{H^+}$ and in the future of $S_n^{H^-}$ for every n. But the sequences $S_n^{H^+}$ and $S_n^{H^-}$ converge, by Theorem 3.3.5, to constant mean curvature surfaces S_{H^+} and S_{H^-} with boundary at infinity Γ. But this implies that p is contained in the subset of $\mathcal{D}(\Gamma)$ foliated by the surfaces S_H and this gives the contradiction. □

This completes the proof of Theorem 3.3.1.

3.4 Study of the principal curvatures of an H-surface

The aim of this section is to give precise estimates for the principal curvatures of an H-surface with bounded second fundamental form. These results will then be used in Section 3.6 in order to associate to each surface in the foliation of the domain of dependence of a quasi-circle a quasi-conformal extension of the corresponding quasi-symmetric homeomorphism.

The main tool used to estimate the principal curvatures of an H-surface with bounded second fundamental form is the following compactness result for sequences of H-surfaces with bounded second fundamental form. This is a straightforward generalization of Lemma 5.1 in [BS10].

Lemma 3.4.1. *Let $C > 0$ be a fixed constant. Choose a point $x_0 \in AdS_3$ and a future-oriented unit time-like vector $n_0 \in T_{x_0}AdS_3$. There exists $r > 0$ as follows. Let P_0 be the totally geodesic space-like plane orthogonal to n_0 at x_0. Let D_0 be the disk of radius r centered at x_0 in P_0. Let $H \in \mathbb{R}$ be fixed and let S_n be a sequence of H-surfaces containing x_0 and orthogonal to n_0 with second fundamental form bounded by C. After extracting a sub-sequence, the restrictions of S_n to the cylinders above D_0 converge C^∞ to an H-surface with boundary contained in the cylinder over ∂D_0.*

Proof. For all n, the surface S_n is the graph of a function f_n over a totally geodesic plane P_n. The bound on the second form of S_n, along with the fact that S_n is orthogonal to n_0 indicates that for some $r > 0$, there exists $\epsilon > 0$ such that

$$\|\nabla f_n\| \leq \frac{1-\epsilon}{\chi}$$

on the disk of center x_0 and radius r.
Moreover, since the second fundamental form of S_n are uniformly bounded, also the Hessian of f_n are uniformly bounded by a constant depending on r.
Therefore, we can extract a sub-sequence, still denoted with f_n, which converges $C^{1,1}$ to a function f_∞ on the disk of center x_0 and radius r. We notice that, since the gradient of f_∞ is uniformly bounded, the graph of f_∞ is a space-like surface.
By definition, the fact that S_n are H-surfaces translates to the fact that f_n is solution of Equation (3.1). Since f_∞ is a $C^{1,1}$-limit of the sequence f_n, it is itself a weak solution of the same equation. By elliptic regularity, it follows that f_∞ is C^∞ and f_n is actually converging to f_∞ in the C^∞ sense. Therefore, the graph of f_∞ over the disk of radius r is an H-surface, which is the C^∞ limit of the restriction of the H-surfaces S_n to the cylinder above the disk of radius r. □

We will need also a stability result for sequences of quasi-circles in $\partial_\infty AdS_3$. This will be a consequence of the following compactness property of quasi-symmetric homeomorphisms:

Proposition 3.4.2 ([BZ06])**.** *Let $\phi_n : S^1 \to S^1$ be a family of uniformly quasi-symmetric homeomorphisms of S^1, i.e. there exists a constant M such that*

$$\sup_n \sup_Q |\log|cr(\phi_n(Q))|| \leq M \ ,$$

where the supremum is taken over all possible quadruples Q of points in S^1 with $cr(Q) = -1$. Then there exists a subsequence ϕ_{n_k} for which one of the following holds:

- *the homeomorphisms ϕ_{n_k} converge uniformly to a quasi-symmetric homeomorphism ϕ;*
- *the homeomorphisms ϕ_{n_k} converge uniformly on the complement of any open neighborhood of a point of S^1 to a constant map.*

In terms of anti-de Sitter geometry, the above proposition can be translated as follows:

Proposition 3.4.3. *Let Γ_n be a sequence of uniformly quasi-circles, i.e. Γ_n are graphs of a family of unifomly quasi-symmetric homeomorphisms. Then, there exists a subsequence Γ_{n_k} which converges in the Hausdorff topology either to the boundary of a light-like plane or to a quasi-circle Γ_∞.*

Under particular assumptions, we can guarantee that the limit of a sequence of quasicircle is never the boundary of a light-like plane:

Lemma 3.4.4. *Let S be a space-like surface whose asymptotic boundary is a quasi-circle Γ. Suppose that there exists $\epsilon \in (-\pi/2, \pi/2)$ such that the surface S_ϵ at time-like distance ϵ from S is convex. Fix a point $x_0 \in AdS_3$ and a future-directed unit vector $n_0 \in T_{x_0}AdS_3$. Let x_n be a sequence of points in S and let ϕ_n be a sequence of isometries of AdS_3 such that $\phi_n(x_n) = x_0$ and the future-directed normal vector to S at x_n is sent to n_0. Then no subsequences of $\phi_n(\Gamma)$ converge to the boundary of a light-like plane.*

Proof. We can choose an affine chart such that $x_0 = (0,0,0) \in \mathbb{R}^3$ and $n_0 = (1,0,0)$. Let $x'_n \in S_\epsilon$ be the point corresponding to x_n under the normal flow. In particular $d(x_n, x'_n) = \epsilon$ for every n. Since ϕ_n is an isometry, we deduce that $\phi_n(x'_n) = x'_0 = (\arcsin(\epsilon), 0, 0)$. Moreover, if we denote with P_n the totally geodesic space-like plane tangent to S_ϵ at $x_{n'}$, by construction $\phi_n(P_n) = P_0$, where P_0 is the totally geodesic space-like plane through x'_0 orthogonal to n_0. Notice that, since S_ϵ is equidistant from S, they have the same asymptotic boundary Γ and S_ϵ is contained in the domain of dependence of Γ
Suppose by contradiction that there exists a subsequence, still denoted with $\phi_n(\Gamma)$, which converges to the boundary of a light-like plane. Let $\xi \in \partial_\infty AdS_3$ be the self-intersection point of the boundary at infinity of this light-like plane. Since S_ϵ is convex, P_n is a support plane, hence its boundary at infinity is disjoint from Γ. After applying the isometry ϕ_n, this implies that P_0 is disjoint from $\phi_n(\Gamma)$ for every n. We deduce that ξ lies in P_0. But this implies that for n big enough, the point x'_0 is not contained in the domain of dependence of $\phi_n(\Gamma)$, which is impossible because each x'_n is contained in the domain of dependence of Γ for every n. □

We have now all the tools to study the principal curvature of an H-surface with bounded second fundamental form. In Section 3.3, we have seen that if we express the principal curvatures of an H-surface as $\pm\lambda + H$, and we define $\mu = \log(\lambda)$, then μ satisfies the differential equation

$$\Delta\mu = e^{2\mu} - H^2 - 1 \ . \tag{3.3}$$

The main result of this section is the following:

Proposition 3.4.5. *Let S be an H-surface with bounded principal curvatures. If its boundary at infinity is a quasi-circle then S is uniformly negatively curved.*

We will first show that an H-surface bounding a quasi-circle cannot be flat. In case $H = 0$, a flat maximal surface in AdS_3 was described in [BS10] as a maximal horosphere in AdS_3. It turns out that for general H, flat constant mean curvature surfaces are equidistant surfaces from this maximal horosphere.

Let us first recall the construction of the maximal horosphere. Consider a space-like line l in AdS_3 and its dual line $l^{\perp}$, which is obtained as the intersection of totally geodesic planes dual to points of l. We recall that $l^{\perp}$ can also be described as the set of points at distance $\pi/2$ from l. The maximal horosphere S_0 is defined as the set of points at distance $\pi/4$ from l. It can be easily checked that S_0 is flat. Moreover, for every point $x \in S_0$, the surface S_0 has an orientation-reversing and time-reversing isometry obtained by reflection along a plane P tangent to a point $x \in S_0$, followed by a rotation of angle $\pi/2$ around the time-like geodesic orthogonal to P at x. This shows that the principal curvatures of S_0 must be opposite to each other, hence S_0 is a maximal surface. We then deduce by the Gauss formula that the principal curvatures are necessarily ± 1 at every point. We notice that the boundary at infinity of S_0 consists of four light-like segments, hence S_0 does not bound a quasi-circle.

We are now going to prove that flat H-surfaces can be obtained as surfaces at constant distance from the maximal horosphere. We will need the following well-known formulas for the variation of the induced metric and the shape operator in a foliation by equidistant surfaces.

Proposition 3.4.6 (Lemma 1.14 [Sep17])**.** *Let S be a space-like surface in AdS_3 with induced metric I and shape operator B. Let S_ρ be the surface at time-like distance ρ from S, obtained by following the normal flow. Then the induced metric on the surface S_ρ is given by*

$$I_\rho = I((\cos(\rho)E + \sin(\rho)B)\cdot, (\cos(\rho)E + \sin(\rho)B)\cdot) \ .$$

Moreover, the shape operator of S_ρ is given by

$$B_\rho = (\cos(\rho)E + \sin(\rho)B)^{-1}(-\sin(\rho)E + \cos(\rho)B) \ .$$

If we apply the previous proposition to the maximal horosphere S_0, we obtain that, choosing a local orthogonal frame that diagonalises B, the induced metrics and the shape operator of the surface at time-like distance ρ from S_0 can be written as

$$I_\rho = \begin{vmatrix} \cos(\rho)+\sin(\rho) & 0 \\ 0 & \cos(\rho)-\sin(\rho) \end{vmatrix} \quad B_\rho = \begin{vmatrix} \frac{-\sin(\rho)+\cos(\rho)}{\cos(\rho)+\sin(\rho)} & 0 \\ 0 & \frac{-\sin(\rho)-\cos(\rho)}{\cos(\rho)-\sin(\rho)} \end{vmatrix} .$$

We deduce that equidistance surfaces form S_0 are smooth for $\rho \in (-\pi/4, \pi/4)$ and, for every value of ρ in this interval, the surface S_ρ is flat. Namely, by the Gauss formula,

$$\kappa_{S_\rho} = -1 - \det(B_\rho) = -1 + 1 = 0 .$$

Moreover, since the shape operator B_ρ has constant trace

$$\text{trace}(B_\rho) = \frac{-\sin(\rho)+\cos(\rho)}{\cos(\rho)+\sin(\rho)} + \frac{-\sin(\rho)-\cos(\rho)}{\cos(\rho)-\sin(\rho)} = -2\tan(2\rho)$$

we deduce that a flat surface with constant mean curvature H is the surface at time-like distance $\rho = \frac{1}{2}\arctan(-H)$ from the maximal horosphere. Notice that, since these surfaces are at bounded distance from the maximal horosphere, they have the same boundary at infinity, which, we recall, consists of four light-like segments.

The proof of Proposition 3.4.5 will then follow from the above description of flat H-surfaces by applying the Maximum Principle "at infinity":

Proof of Proposition 3.4.5. Since S has bounded second fundamental form, its Gaussian curvature is bounded. We will denote with κ_{sup} the upper bound of the Gaussian curvature of S.
If κ_{sup} is attained, then the maximum principle applied to Equation (3.3) implies that $\kappa_{\text{sup}} \leq 0$ and if $\kappa_{\text{sup}} = 0$, then the surface is flat. This latter case cannot happen, since by hypothesis S bounds a quasi-circle, hence, if the upper bound of the Gaussian curvature is attained, the surface S is uniformly negatively curved.
We will know apply the Maximum Principle "at infinity" to get the same conclusion when the upper bound is not attained. Consider a sequence of points $x_n \in S$ such that the Gaussian curvature of S at x_n satisfies

$$\kappa_{\text{sup}} - \frac{1}{n} \leq \kappa(x_n) \leq \kappa_{\text{sup}} .$$

Let ϕ_n be a sequence of isometries of AdS_3 which sends x_n to a fixed point x_0 and the future-directed unit normal vector to S at x_n to a fixed vector $n_0 \in T_{x_0}AdS_3$. Since S has bounded second fundamental form, Lemma 3.4.1 shows that $\phi_n(S)$ converges, up to subsequences, to an H-surface S_∞ in a neighborhood of x_0. By construction, the curvature of S_∞ has a local maximum at x_0 equal to κ_{sup}, hence the maximum principle applied again to Equation (3.3) shows that $\kappa_{\text{sup}} \leq 0$.

We are left to show that $\kappa_{\sup} \neq 0$. For this we will need to use that the boundary at infinity of S is a quasi-circle and the description of flat constant mean curvature surfaces. Suppose by contradiction that $\kappa_{\sup} = 0$. Then the sequence $\phi_n(S)$ converges in a neighborhood of x_0 to the flat H-surface S_∞ described above. Moreover, Lemma 3.4.1 implies that $\phi_n(S)$ converges, up to subsequences, to S_∞ uniformly on compact set. In particular, the boundary at infinity of $\phi_n(S)$ converges to the boundary at infinity of S_∞, which consists of four light-like segments. But this is not possible by Proposition 3.4.3. □

Remark 3.4.7. *Proposition 3.4.5 and the Gauss formula imply that the principal curvatures of an H-surface with bounded second fundamental form and asymptotic boundary a quasi-circle can be written as $\pm\lambda + H$, where $\lambda \in [0, \sqrt{H^2+1} - \epsilon]$ for some $\epsilon > 0$.*

We conclude this section with the following result about the existence of a convex surface equidistant from a negatively curved H-surface. This is a crucial step, together with Lemma 3.4.4, to prove the uniqueness of the constant mean curvature foliation of the domain of dependence of a quasi-circle.

Proposition 3.4.8. *Let S be a negatively curved H-surface with bounded second fundamental form. Then there exists a convex surface equidistant from S.*

Proof. We will do the proof for $H \geq 0$, the other case being analogous. Let us write the principal curvatures of S as $\mu = \lambda + H$ and $\mu' = -\lambda + H$. By Remark 3.4.7, we know that $\lambda \in [0, \sqrt{1+H^2} - \epsilon]$ for some $\epsilon > 0$. We can clearly suppose $\epsilon < 1$. By Proposition 3.4.6, we know that the principal curvatures of the surface S_ρ at time-like distance ρ from S can be expressed as

$$\mu_\rho = \frac{\mu - \tan(\rho)}{1 + \mu\tan(\rho)} = \tan(\rho_0 - \rho)$$

$$\mu'_\rho = \frac{\mu' - \tan(\rho)}{1 + \mu'\tan(\rho)} = \tan(\rho_1 - \rho)$$

where $\rho_0 = \arctan(\mu)$ and $\rho_1 = \arctan(\mu')$. We will denote

$$\alpha = \arctan\left(\sqrt{1+H^2} + H - \epsilon\right) \quad \beta = \arctan(H) \quad \gamma = \arctan\left(H - \sqrt{1+H^2} + \epsilon\right) .$$

By definition, we have

$$0 < \beta < \rho_0 < \alpha < \frac{\pi}{2} \qquad \gamma < \rho_1 < \beta < \frac{\pi}{2} .$$

We deduce that for every $\rho \in (\alpha - \pi/2, 0]$ the surface S_ρ is smooth, since the principal curvatures are non-degenerate. Moreover, for every $\rho \in (\alpha - \pi/2, \gamma)$, the surface S_ρ has positive principal curvatures. Here, we should be careful that $\alpha - \pi/2 < \gamma$, but this can easily be verified to be true under the assumption $0 < \epsilon \leq 1$. Namely, $\alpha - \pi/2 < \gamma$ if and only if

$$\arctan\left(H - \sqrt{1+H^2} + \epsilon\right) > \arctan\left(\sqrt{1+H^2} + H - \epsilon\right) - \frac{\pi}{2}$$
$$= \arctan\left(\frac{-1}{H + \sqrt{1+H^2} - \epsilon}\right)$$

and it is sufficient to verify that

$$\epsilon + H - \sqrt{1+H^2} > \frac{-1}{H + \sqrt{1+H^2} - \epsilon} ,$$

which is true, under the hypothesis that $0 < \epsilon \leq 1$, since

$$-1 < -(1-\epsilon)^2 \leq -\epsilon^2 + 2\epsilon\sqrt{1+H^2} - 1 = (\epsilon + H - \sqrt{1+H^2})(H - \epsilon + \sqrt{1+H^2}) .$$

□

3.5 Uniqueness of the CMC foliation

In Section 3.3 we have proved the existence of a foliation by constant mean curvature surfaces of the domain of dependence of a quasi-circle. In this section we prove the uniqueness of such a foliation.

In order to prove the uniqueness of the foliation provided by Theorem 3.3.1, the main idea consists of proving that a constant mean curvature surface with bounded second fundamental form and with boundary at infinity a quasi-circle must coincide with a leaf of the foliation. The main tool we will use is the Maximum Principle for constant mean curvature surfaces in Lorentzian manifold, which we are going to recall.

Lemma 3.5.1 (Maximum Principle [BBZ07])**.** *Let Σ and Σ' be smooth space-like surfaces in a time-oriented Lorentzian manifold M. Assume that Σ and Σ' are tangent at some point p, and assume that Σ' is contained in the future of Σ. Then, the mean curvature of Σ' at p is smaller or equal than the mean curvature of Σ at p.*

We have now all the instruments to prove the uniqueness of the foliation by constant mean curvature surfaces of the domain of dependence of a quasi-circle.

Theorem 3.5.2. *Let Γ be a quasi-circle and let $\mathcal{D}(\Gamma)$ be its domain of dependence. Then there exists a unique foliation of $\mathcal{D}(\Gamma)$ by constant mean curvature surfaces with bounded second fundamental form.*

Proof. The existence is provided by Theorem 3.3.1. In particular, we can define a time function $F : \mathcal{D}(\Gamma) \to \mathbb{R}$ with the property that the level sets $F^{-1}(H)$ are H-surfaces for each $H \in \mathbb{R}$.

To prove the uniqueness of this foliation, we are going to show that every other surface S with constant mean curvature H, with bounded second fundamental form and with asymptotic boundary Γ must coincide with the level set $F^{-1}(H)$.
Consider the restriction of F to S. Suppose that F admits maximum h_{max} at a point $x \in S$ and minimum h_{min} at a point $y \in S$. Notice that $h_{max} < +\infty$ and $h_{min} > -\infty$, otherwise S would touch the domain of dependence. By construction, the surfaces $F^{-1}(h_{max}) = S_{max}$ and $F^{-1}(h_{min}) = S_{min}$ are tangent to S at x and y, respectively. Moreover, S_{max} is in the future of S and S_{min} is in the past of S. Hence, by the Maximum Principle we obtain that

$$h_{max} \leq H \leq h_{min} \ .$$

Therefore, $h_{min} = h_{max} = H$ and S coincides with a level set of the function F as claimed.
In the general case, F does not admit maximum and minimum on S, but we can still apply a similar reasoning "at infinity". We define

$$h^+ = \sup_{x \in S} F(x) \quad \text{and} \quad h^- = \inf_{x \in S} F(x) \ .$$

Fix a point x_0 in AdS_3 and a future-directed unit normal vector n_0 at x_0. Let x_n be a sequence of points in S such that $F(x_n)$ tends to h^+ for $n \to \infty$. Let ϕ_n be a sequence of isometries of AdS_3 such that $\phi_n(x_n) = x_0$ and the surface $\phi_n(S)$ is orthogonal to n_0 at x_0. By Lemma 3.4.1, the sequence of surfaces $S_n = \phi_n(S)$ converges C^∞ to an H-surface S_∞ in a neighborhood of x_0. Let Γ_∞ be the boundary at infinity of S_∞. Since S_n converges uniformly on compact sets to S_∞, the curve Γ_∞ is the limit of the sequence $\phi_n(\Gamma)$. By Lemma 3.4.4 and Proposition 3.4.8, the limit curve Γ_∞ is a quasi-circle, hence Theorem 3.3.1 provides a foliation of the domain of dependence of Γ_∞ by constant mean curvature surfaces. Let $F_\infty : \mathcal{D}(\Gamma_\infty) \to \mathbb{R}$ be the time function such that its level sets are leaves of the foliation. By construction, the function F_∞ restricted to S_∞ admits maximum at x_0, so, by the Maximum Principle, $h^+ \leq H$. Repeating a similar procedure for h_-, we obtain the inequalities

$$h^+ \leq H \leq h^-$$

from which we deduce that $h^+ = h^- = H$ and that S must coincide with the level set $F^{-1}(H)$. □

3.6 Application

In this section we will use the theory described in Section 3.2 in order to associate to every constant mean curvature surface with asymptotic boundary the graph of a quasi-symmetric homeomorphism ϕ and with bounded second fundamental form a quasi-conformal extension of ϕ.

In Section 3.2, we have recalled that, given a negatively curved space-like surface embedded in AdS_3, we can construct two local diffeomorphisms $\Pi_{l,r} : S \to \mathbb{H}^2$.

Proposition 3.6.1. *Let S be an H-surface with bounded second fundamental form. Suppose that the boundary at infinity of S is a quasi-circle Γ. Then the mappings $\Pi_{l,r} : S \to \mathbb{H}^2$ are global diffeomorphisms and they extend the maps $\pi_{l,r} : \Gamma \to S^1$*

Proof. We will do the proof for the map Π_l, the other case being analogous. By Proposition 3.4.8, there exists a convex surface S_ϵ equidistant from S. We remark that S and S_ϵ has the same boundary at infinity. Let $\Pi_{l,\epsilon}$ be the map defined in Section 3.2 associated to the surface S_ϵ. By Lemma 3.2.4 and Remark 3.2.5, the map $\Pi_{l,\epsilon}$ is a global diffeomorphism and extends π_l.
We claim now that, if we denote with $\eta : S \to S_\epsilon$ the diffeomorphism induced by the normal flow, we have $\Pi_l = \Pi_{l,\epsilon} \circ \eta$. This is sufficient to conclude the proof, as the map η can be extended to the identity on the asymptotic boundaries. We now prove the claim. Let $p \in S$. Up to isometry we can suppose that, in the affine chart $U_3 = \{x_3 \neq 0\}$, we have $p = (0, 0, 0)$ and the tangent plane to S at p is the space-like plane P of equation $x = 0$. In addition, we can suppose that the totally-geodesic space-like plane we fix in the definition of Π_l (see Section 3.2) is exactly P. With this assumption, we clearly have $\Pi_l(p) = p$. Moreover, $\eta(p) = (\epsilon, 0, 0)$ and the tangent plane to S_ϵ at $\eta(p)$ has equation $x = \epsilon$. On the other hand, since the left foliation is parametrised by

$$(x, \cos(\theta) - x\sin(\theta), \sin(\theta) + x\cos(\theta)) = (x, \sqrt{1+x^2}\cos(\theta+\alpha), \sqrt{1+x^2}\sin(\theta+\alpha)) \ ,$$

where $\tan\alpha = x$, we have that

$$\Pi_{l,\epsilon}(\epsilon, t\sqrt{1+\epsilon^2}\cos(\theta), t\sqrt{1+\epsilon^2}\sin(\theta)) = \frac{1}{\sqrt{1+\epsilon^2}}(0, t\cos(\theta-\beta), t\sin(\theta-\beta)) \ ,$$

where $\tan\beta = \epsilon$. Therefore, $\Pi_{l,\epsilon}(\eta(p)) = \Pi_{l,\epsilon}(\epsilon, 0, 0) = (0, 0, 0) = p = \Pi_l(p)$, as claimed. □

As a consequence the map $\Phi = \Pi_r \circ \Pi_l^{-1}$ is a global diffeomorphism and extends the quasi-symmetric map ϕ, whose graph is the quasi-circle Γ. We now use the estimates on the principal curvatures proved in Section 3.4 to show that Φ is quasi-conformal.

Proposition 3.6.2. *Let S be an H-surface with bounded second fundamental form whose boundary at infinity is the graph of a quasi-symmetric homeomorphism ϕ. Then the associated map Φ is a quasi-conformal extension of ϕ.*

Proof. It is sufficient to prove that the map $\Pi_l : S \to \mathbb{H}^2$ is quasi-conformal from the induced metric on S to the hyperbolic metric on $\mathbb{H}^2$. As seen in Section 3.2, the differential of Π_l is $E + JB$, where J is the complex structure on S and B is its shape operator. We thus need to bound the module of the complex dilatation of the map

$A = (E + JB)^t(E + JB)$. In a suitable orthogonal frame for S, we can suppose that B is diagonal, hence

$$E + JB = \begin{vmatrix} 1 & \lambda - H \\ \lambda + H & 1 \end{vmatrix} ,$$

where we wrote the principal curvatures of S as $\pm\lambda + H$ with $\lambda \in [0, \sqrt{1 + H^2} - \epsilon]$. Therefore,

$$A = \begin{vmatrix} 1 + (H - \lambda)^2 & 2\lambda \\ 2\lambda & (\lambda + H)^2 + 1 \end{vmatrix}$$

and its complex dilatation is

$$\mu = -\frac{\lambda(H + i)}{1 + H^2} .$$

Thus,

$$|\mu|^2 = \frac{\lambda^2}{1 + H^2} = 1 - \frac{1 + H^2 - \lambda^2}{1 + H^2} < 1$$

as wanted. □

It turns out that the quasi-conformal homeomorphism Φ is a landslide. We recall that an area-preserving homeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ is a θ-landslide if it can be decomposed as

$$\Phi = f_2 \circ f_1^{-1}$$

where $f_i : \mathbb{H}^2 \to \mathbb{H}^2$ are harmonic maps whose Hopf differentials satisfy

$$\mathrm{Hopf}(f_1) = e^{2i\theta}\mathrm{Hopf}(f_2) .$$

Proposition 3.6.3. *Let $S_H \subset AdS_3$ be a space-like H-surface bounding a quasi-circle at infinity. Then the map $\Pi_r \circ \Pi_l^{-1}$ is a θ-landslide, where*

$$\theta = -\arctan(H) + \frac{\pi}{2} .$$

Proof. Since, by definition, $\Phi = \Pi_r \circ \Pi_l^{-1}$, it is sufficient to prove that $\Pi_{l,r} : (S_H, I) \to \mathbb{H}^2$ are harmonic with

$$\mathrm{Hopf}(\Pi_l) = e^{2i\theta}\mathrm{Hopf}(\Pi_r) .$$

In order to prove that Π_r is harmonic, it is sufficient to prove that if we write

$$\Pi_r^* g_{\mathbb{H}^2} = I(\cdot, b\cdot) ,$$

then the traceless part of b is Codazzi for I. By definition of the map Π_r, we know that

$$\Pi_r^* g_{\mathbb{H}^2} = I((E - JB)\cdot, (E - JB)\cdot) = I(\cdot, (E - JB)^*(E - JB))$$

where $(E - JB)^*$ denotes the adjoint for the metric I. Since B is I-self-adjoint and J is skew-symmetric for I, we deduce that $(E - JB)^* = (E + JB)$, thus $b =$

$(E+JB)(E-JB)$. Let us now decompose the operator B as $B = B_0 + HE$, where B_0 is traceless. Then

$$\begin{aligned}(E+JB)(E-JB) &= E + BJ - JB + B^2 \\ &= E + B_0 J + HJ - JB_0 - HJ + B_0^2 + H^2 E + 2HB_0 \\ &= (1+H^2)E + 2(HE - J)B_0 + B_0^2 \\ &= (1+\lambda^2+H^2)E + 2(HE-J)B_0\end{aligned}$$

where, in the last passage, we have used the fact that $B_0^2 = \lambda^2 E$, λ being the positive eigenvalue of B_0. We deduce that the traceless part b_0 of b is given by $b_0 = 2(HE - J)B_0$, which is Codazzi because H is constant, B is Codazzi and J is integrable and compatible with the metric I (hence $d^{\nabla^I} J = 0$). Therefore, $I(\cdot, b_0 \cdot)$ is the real part of a holomorphic quadratic differential Φ_r and the metric $\Pi_r^* g_{\mathbb{H}^2}$ can be written as

$$\Pi_r^* g_{\mathbb{H}^2} = I(\cdot, b\cdot) = e_r I + \Phi_r + \overline{\Phi_r}$$

where $e_r : S \to \mathbb{R}^+$ is the energy density of the map Π_r. Since Φ_r is holomorphic for the complex structure J, this shows that Π_r is harmonic with Hopf differential

$$\mathrm{Hopf}(\Pi_r) = \Phi_r = I(HE - J)B_0 + iIJ(HE - J)B_0 \ .$$

A similar computation for Π_l shows that

$$\mathrm{Hopf}(\Pi_l) = \Phi_l = I(HE + J)B_0 + iIJ(HE + J)B_0 \ .$$

By using conformal coordinates, we deduce that

$$\frac{\mathrm{Hopf}(\Pi_l)}{\mathrm{Hopf}(\Pi_r)} = \frac{H+i}{H-i} = e^{2i\theta}$$

with

$$\theta = -\arctan(H) + \frac{\pi}{2}$$

as claimed. □

Chapter 4

The volume of GHMC anti-de Sitter 3-manifolds

In this chapter, we study the volume of globally hyperbolic maximal compact anti-de Sitter manifolds, in relation to some geometric invariants depending only on the two points h and h' in the Teichmüller space of S provided by Mess' parameterisation. The main result of the chapter is that the volume coarsely behaves like the minima of the L^1-energy of maps from (S, h) to (S, h'). As a corollary, we show that the volume of GHMC AdS_3 manifolds is bounded from above by the exponential of (any of the two) Thurston's Lipschitz asymmetric distances, up to some explicit constants, and bounded from below by the exponential of the Weil-Petersson distance.

4.1 Definition of the problem and outline of the proofs

In the celebrated paper [Bro03], Brock proved that the volume of the convex core of a quasi-Fuchsian manifold M behaves coarsely like the Weil-Petersson distance between the two components in $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$ provided by Bers' parameterisation ([Ber60]). The main purpose of this chapter is to study the analogous question for GHMC anti-de Sitter manifolds.

Our first result concerns the relation between the volume of the convex core and Thurston's distances on Teichmüller space. Recall that Thurston's distance $d_{\mathrm{Th}}(h, h')$ is essentially the logarithm of the best Lipschitz constant of a diffeomorphism from (S, h) to (S, h'). This definition satisfies the properties of a distance on $\mathrm{Teich}(S)$, except the symmetry. We prove the following:

Theorem 4.5.2. *Let $M_{h,h'}$ be a globally hyperbolic maximal compact AdS_3 manifold.*

Then

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \pi|\chi(S)|\exp(\min\{d_{\mathrm{Th}}(h,h'), d_{\mathrm{Th}}(h',h)\}) \ .$$

One may thus try to see if the volume of the convex core is coarsely equivalent to the minimum of Thurston asymmetric distances. This turns out to be false in general, as we can produce examples of manifolds M_{h_n,h'_n} in which the minimum $\min\{d_{\mathrm{Th}}(h_n,h'_n), d_{\mathrm{Th}}(h'_n,h_n)\}$ goes to infinity while $\mathrm{Vol}(\mathcal{C}(M_{h_n,h'_n}))$ stays bounded, thus showing that there cannot be a bound from below on the volume using any of Thurston's asymmetric distances. However, in these examples both sequences h_n and h'_n diverge in Teichmüller space. We prove that this condition is necessary for this to happen. More precisely, we prove that if K is any compact subset of $\mathrm{Teich}(S)$, then the volume of the convex core of a GHMC anti-de Sitter 3-manifold parameterised by points in $K \times \mathrm{Teich}(S)$ (or $\mathrm{Teich}(S) \times K$) is coarsely equivalent to the minimum of Thurston asymmetric distances.

On the other hand, we obtain a coarse bound from below on the volume of the convex core of $M_{h,h'}$ by using the Weil-Petersson distance $d_{\mathrm{WP}}(h,h')$.

Theorem 4.6.2. *Let $M_{h,h'}$ be a GHMC AdS_3 manifold. Then there exist some positive constants C and C' such that*

$$\exp(C d_{\mathrm{WP}}(h,h')) - C' \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \ .$$

It follows easily from Theorem 4.5.2 that there are examples in which $d_{\mathrm{WP}}(h_n,h'_n)$ remains bounded, but $\mathrm{Vol}(\mathcal{C}(M_{h_n,h'_n}))$ diverges, thus the volume of the convex core of $M_{h,h'}$ cannot be bounded from above by the Weil-Petersson distance between h and h'.

We consider also a form of holomorphic 1-energy, which was already introduced in [TV95]. Given two hyperbolic surfaces (S,h) and (S,h'), this is the functional $E_{\partial}(\cdot,h,h') : \mathrm{Diffeo}_0(S) \to \mathbb{R}$ defined by

$$E_{\partial}(f,h,h') = \int_S ||\partial f|| d\mathrm{A}_h \ ,$$

where $||\partial f||$ is the norm of the $(1,0)$-part of the differential of f, computed with respect to the metrics h and h'. In [TV95], Trapani and Valli proved that the functional $E_{\partial}(\cdot,h,h')$ admits a unique minimum, which coincides with the unique minimal Lagrangian diffeomorphism $m : (S,h) \to (S,h')$ isotopic to the identity. Using the known construction ([BS10], [KS07]) which associates a minimal Lagrangian diffeomorphism from (S,h) to (S,h'), isotopic to the identity, to the unique maximal surface in $M_{h,h'}$, we obtain the following theorem which gives a precise description of the coarse behaviour of the volume of the convex core in terms of the holomorphic 1-energy:

Theorem 4.3.8. *Let $M_{h,h'}$ a GHMC AdS$_3$ manifold. Then*

$$\frac{\sqrt{2}}{4}E_{\partial}(m,h,h') - \pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \frac{\sqrt{2}}{4}E_{\partial}(m,h,h') \ ,$$

where $m : (S,h) \to (S,h')$ is the minimal Lagrangian map isotopic to the identity, that is, the unique minimum of the 1-holomorphic energy functional $E_{\partial}(\cdot,h,h')$.

A direct corollary involves the L^1-energy, considered as the following functional:

$$E_d(f,h,h') = \int_S ||df||d\mathrm{A}_h \ .$$

Theorem 4.4.4. *Let $M_{h,h'}$ be a GHMC AdS$_3$ manifold. Then*

$$\frac{1}{4}\inf E_d(\cdot,h,h') - \frac{\sqrt{2}}{2}\pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \frac{\sqrt{2}}{2}\inf E_d(\cdot,h,h') \ .$$

A more combinatorial version of the relation between maximal surfaces and minimal Lagrangian maps is the association, already discovered by Mess, of (left and right) earthquake maps between the two pleated surfaces which form the boundary of the convex core of $M_{h,h'}$. Roughly speaking, the role of the L^1-energy between hyperbolic surfaces is played by the length of the two measured geodesic laminations on (S,h) which provide the earthquake maps of Thurston's Earthquake Theorem.

If we denote by $E^\lambda : \mathrm{Teich}(S) \to \mathrm{Teich}(S)$ the transformation which associates to $h \in \mathrm{Teich}(S)$ the metric $h' = E^\lambda(h)$ obtained by a (left or right) earthquake along λ, we get the following inequalities involving the volume and the length of (both) earthquake laminations:

Theorem 4.2.7. *Given a GHMC AdS$_3$ manifold $M_{h,h'}$, let λ be the (left or right) earthquake lamination such that $E^\lambda(h) = h'$. Then*

$$\frac{1}{4}\ell_\lambda(h) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{1}{4}\ell_\lambda(h) + \frac{\pi^2}{2}|\chi(S)| \ .$$

A direct consequence is the fact that the length of the left and right earthquake laminations is comparable, that is, their difference is bounded only in terms of the topology of S:

Corollary 4.2.6. *Given two hyperbolic metrics h and h' on S, if λ_l and λ_r are the measured laminations such that $E_l^{\lambda_l}(h) = h'$ and $E_r^{\lambda_r}(h) = h'$, then*

$$|\ell_{\lambda_l}(h) - \ell_{\lambda_r}(h)| \leq 2\pi^2|\chi(S)| \ .$$

Let us outline here, for the convenience of the reader, the main steps for the proof of the above results. A first main difference between the quasi-Fuchsian and the anti-de Sitter setting consists in the fact that the volume of the whole manifold $M_{h,h'}$ is finite. By considering the foliation by constant curvature surfaces ([BBZ07]) of the complement of the convex core, we will show that the volume of the whole manifold and the volume of the convex core are coarsely equivalent. More precisely we prove the following:

Proposition 4.2.1. *Given a GHMC AdS_3 manifold M, let M_- and M_+ be the two connected components of the complement of $\mathcal{C}(M)$. Then*

$$\mathrm{Vol}(M_-) \leq \frac{\pi^2}{2}|\chi(S)| \qquad \textit{and} \qquad \mathrm{Vol}(M_+) \leq \frac{\pi^2}{2}|\chi(S)| \ ,$$

with equality if and only M is Fuchsian.

Then, using a foliation by equidistant surfaces from the boundary of the convex core, we prove the following formula (already mentioned in [BBD+12]) which connects the volume of the convex core, the volume of the whole manifold and the length of the left and right earthquake laminations,

$$\mathrm{Vol}(M) + \mathrm{Vol}(\mathcal{C}(M)) = \frac{1}{4}(\ell_{\lambda_l}(h) + \ell_{\lambda_r}(h)) + \pi^2|\chi(S)| \ . \tag{4.1}$$

As a consequence, we obtain that the difference between the length of the right earthquake lamination λ_r and the length of the left earthquake lamination λ_l is uniformly bounded, which seems to be a non-trivial result to obtain using only techniques from hyperbolic geometry. Theorem 4.2.7 will then follow by combining Equation (4.1) with Proposition 4.2.1.

Another main consequence of Proposition 4.2.1 is the fact that the volume of the convex core of $M_{h,h'}$ is coarsely equivalent to the volume of every submanifold in which it is contained. Starting from the unique maximal surface embedded in $M_{h,h'}$ ([BBZ07]), we construct a submanifold with smooth boundary $\Omega_{h,h'}$ which contains the convex core and whose volume can be computed explicitly in terms of the function (already introduced in [BMS15])

$$\begin{aligned} F : \mathrm{Teich}(S) \times \mathrm{Teich}(S) &\to \mathbb{R}^+ \\ (h,h') &\mapsto \int_S \mathrm{trace}(b) d\mathrm{A}_h \ , \end{aligned}$$

where $b : TS \to TS$ is the Codazzi, h-self-adjoint operator such that $h' = h(b\cdot, b\cdot)$ associated to the minimal Lagrangian diffeomorphism from (S,h) to (S,h'). In particular, using explicit formulas that relate the embedding data of the maximal surface in $M_{h,h'}$ with the operator b, we can prove that

$$\mathrm{Vol}(\Omega_{h,h'}) = \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}\int_S \mathrm{trace}(b) d\mathrm{A}_h$$

which, combined again with Proposition 4.2.1 implies that the volume of the convex core is coarsely equivalent to the functional F defined above. Moreover, it turns out that the integral of the trace of b coincides, up to a multiplicative constant, with the holomorphic L^1-energy

$$E_\partial(m, h, h') = \int_S \|\partial f\| d\mathrm{A}_h$$

when $m : (S, h) \to (S, h')$ is the minimal Lagrangian diffeomorphism isotopic to the identity. This will lead to Theorem 4.3.8 and Corollary 4.4.3. In addition, since the trace of b is bounded from above by twice the Lipschitz constant of m, we will deduce Theorem 4.5.2.

As for the relation between the volume of the convex core of $M_{h,h'}$ and the Weil-Petersson distance between h and h', the main technical tool consists in the following estimate:

Theorem 4.6.3. *There exists a universal constant $a > 0$ such that for every $\lambda \in \mathcal{ML}(S)$ and for every $h \in \mathrm{Teich}(S)$, we have*

$$\|\mathrm{grad}\ell_\lambda(h)\|_{\mathrm{WP}} \geq \frac{a}{|\chi(S)|}\ell_\lambda(h) \ .$$

This will be obtained by a careful analysis of Riera's formula [Rie05] for the norm of the Weil-Petersson gradient of the length function. With this in hand, the proof of Theorem 4.6.2 then goes as follows. By a result of Bers', we can fix a pants decomposition P for h such that the length of all curves α_j in P are smaller than a universal constant $L > 0$. If the metric h' is obtained by performing a (left or right) earthquake along a lamination λ, then the length of the curves α_j increases at most by

$$\ell_{\alpha_j}(h') \leq L + \frac{\ell_\lambda(h)}{d(L)} \ ,$$

where $d(L)$ is a constant depending only on L. We can thus say that the point $h' \in \mathrm{Teich}(S)$ belongs to the set

$$V_m(S) = \{h \in \mathrm{Teich}(S) \mid \ell_{\alpha_j}(h) < m\}$$

if we put $m = L + \ell_\lambda(h)/d(L)$. As a consequence of Theorem 4.6.3, the integral curve of the vector field $X = -\mathrm{grad}\ell_{\lambda_P}/\|\mathrm{grad}\ell_{\lambda_P}\|_{\mathrm{WP}}$, where we denoted with λ_P the measured geodesic lamination consisting of the simple closed curves α_j with unit weight, starting at h' will intersect the set $V_L(S)$ in a finite time t_0, which we are able to express explicitely in terms of $\ell_\lambda(h)$ and the constants L and $d(L)$. Theorem 4.6.2 will then follow from the fact that the set $V_L(S)$ has bounded diameter for the Weil-Petersson metric and from Theorem 4.2.7.

4.2 Volume and length of earthquake laminations

In this section, we will discuss an explicit relation between the volume of a maximal globally hyperbolic manifold (or the volume of its convex core) and the length of the (left and right) earthquake laminations. Before that, we will prove that the volume of the complement of the convex hull is bounded by the volume of a Fuchsian manifold. That is, the volume of $M \setminus \mathcal{C}(M)$ is maximal in the Fuchsian case. Hence, from a coarse point of view, the volume of the manifold M is essentially the same as the volume of its convex core.

Let us first notice that, by the description of Fuchsian AdS_3 manifolds, given in Example 1.5.3, the volume of any Fuchsian GHMC AdS_3 manifold M_F homeomorphic to $S \times \mathbb{R}$ is:

$$\mathrm{Vol}(M_F) = \pi^2 |\chi(S)| \ . \tag{4.2}$$

4.2.1 Volume of the complement of the convex hull.

Given a globally hyperbolic maximal compact AdS_3 manifold M, we will denote

$$M \setminus \mathcal{C}(M) = M_+ \sqcup M_- \ ,$$

where M_+ is the connected component adjacent to $\partial_+\mathcal{C}(M)$, and M_- the other connected component. The following proposition estimates the volume of the complement of the convex core.

Proposition 4.2.1. *In the above setting, we have*

$$\mathrm{Vol}(M_-) \leq \frac{\pi^2}{2}|\chi(S)| \qquad \textit{and} \qquad \mathrm{Vol}(M_+) \leq \frac{\pi^2}{2}|\chi(S)| \ ,$$

with equality if and only M is Fuchsian.

It will then obviously follow that

$$\mathrm{Vol}(M \setminus \mathcal{C}(M)) \leq \pi^2 |\chi(S)| \ , \tag{4.3}$$

that is, the volume of $\mathrm{Vol}(M \setminus \mathcal{C}(M))$ is at most the volume of a Fuchsian manifold.

Proof. We will give the proof for M_+. By [BBZ11], there exists a function

$$F : M_+ \to [0, \infty)$$

such that $S_\kappa = F^{-1}(\kappa)$ is the surface of constant curvature $K = -1 - \kappa$. By the Gauss equation in the AdS_3 setting, if B is the shape operator of S_κ, then $\kappa = \det B_x$ for every point $x \in S_\kappa$. Let φ_t be the flow of the vector field $\mathrm{grad}F/||\mathrm{grad}F||^2$. By definition $\varphi_t(S_\kappa) = S_{\kappa+t}$ and following this flow, one obtains the following expression for the volume of M_+:

$$\mathrm{Vol}(M_+) = \int_0^\infty \int_{S_\kappa} \frac{d\mathrm{Area}_{S_\kappa}}{||\mathrm{grad}F||} d\kappa \ . \tag{4.4}$$

Let us now fix some $\kappa \in (0,\infty)$. Let $x(\rho)$ be the normal flow of S_κ, which is well-defined for a small ρ. We adopt the convention that the unit normal of S_κ is pointing towards the concave side of S_κ. Let $S_\kappa(\rho)$ be the parallel surface of S_κ at distance ρ, in the concave side. Using the formula for the shape operator of $S_\kappa(\rho)$, see [KS07] or [Sep17, Lemma 1.14], we get:

$$B_\rho = (\cos(\rho)E + \sin(\rho)B)^{-1}(-\sin(\rho)E + \cos(\rho)B)$$

where B_ρ is the shape operator of $S_\kappa(\rho)$. Hence

$$\det B_\rho = \frac{\sin^2\rho + \cos^2\rho \det B - (\sin\rho\cos\rho)\mathrm{trace}B}{\cos^2\rho + \sin^2\rho\det B + (\sin\rho\cos\rho)\mathrm{trace}B} \ .$$

With our convention, $\mathrm{trace}B < 0$ since S_κ is concave, and thus, using the inequality $(\mathrm{trace}B)^2 \geq 4\det B = 4\kappa$, we have

$$\det B_\rho \geq \frac{\sin^2\rho + \cos^2\rho\,\kappa + 2\sin\rho\cos\rho\sqrt{\kappa}}{\cos^2\rho + \sin^2\rho\,\kappa - 2\sin\rho\cos\rho\sqrt{\kappa}} \ ,$$

which implies that $\inf\det B_\rho \geq f(\rho)$, where

$$f(\rho) = \frac{\sin^2\rho + \cos^2\rho\,\kappa + 2\sin\rho\cos\rho\sqrt{\kappa}}{\cos^2\rho + \sin^2\rho\,\kappa - 2\sin\rho\cos\rho\sqrt{\kappa}} \ .$$

Hence by an application of the maximum principle (compare for instance [BBZ11]) one has that $S_\kappa(\rho)$ lies entirely in the concave side of $S_{f(\rho)}$. In other words, $F(x(\rho)) \geq f(\rho)$. Observe that the timelike vector fields $\dot{x}(\varrho)$ and $\mathrm{grad}F(x(\varrho))$ are collinear when $\varrho = 0$. Hence for every $\epsilon > 0$, there exists $\rho_0 > 0$ such that for every $\varrho < \rho_0$ we have

$$\langle \mathrm{grad}F(x(\varrho)), \dot{x}(\varrho)\rangle \leq ||\mathrm{grad}F(x(\varrho))||(1+\epsilon) \ ,$$

hence

$$F(x(\rho)) - F(x(0)) = \int_0^\rho \langle \mathrm{grad}F(x(\varrho)), \dot{x}(\varrho)\rangle d\varrho \leq (1+\epsilon)\int_0^\rho ||\mathrm{grad}F(x(\varrho))||d\varrho \ ,$$

for $\rho < \rho_0$. On the other hand

$$F(x(\rho)) - F(x(0)) \geq f(\rho) - \kappa \ ,$$

and thus by differentiating at $\rho = 0$:

$$||\mathrm{grad}F(x(0))||(1+\epsilon) \geq \left.\frac{d}{d\rho}\right|_{\rho=0} f(\rho) = 2\sqrt{\kappa}(\kappa+1) \ .$$

We can finally conclude the computation. From Equation (4.4), we have

$$\begin{aligned} \mathrm{Vol}(M_+) &\leq \int_0^{+\infty} \frac{(1+\epsilon)}{2\sqrt{\kappa}(\kappa+1)} \int_{S_\kappa} d\mathrm{Area}_{S_\kappa} d\kappa \\ &= (1+\epsilon)\int_0^{+\infty} \frac{2\pi|\chi(S)|d\kappa}{2\sqrt{\kappa}(\kappa+1)^2} = \frac{\pi^2}{2}|\chi(S)|(1+\epsilon) \ , \end{aligned}$$

where we have used the Gauss-Bonnet formula, the fact that the Gaussian curvature of S_κ is $-1-\kappa$, and that

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}(1+x)^2} = \frac{\pi}{2} .$$

Finally, let us observe that equality holds if and only if $(\mathrm{trace}B)^2 = 4\det B$ at every point, which is the case in which all the surfaces S_κ are umbilical. This implies that the boundary of the convex core is totally geodesic, and thus M is a Fuchsian manifold. □

A direct consequence of Proposition 4.2.1, using Equation (4.3), is that the volume of M and of the convex core of M are roughly comparable:

Corollary 4.2.2. *Given a maximal globally hyperbolic manifold M,*

$$\mathrm{Vol}(\mathcal{C}(M)) \leq \mathrm{Vol}(M) \leq \mathrm{Vol}(\mathcal{C}(M)) + \pi^2|\chi(S)| .$$

4.2.2 Length of earthquake laminations

In this subsection we will prove a coarse relation between the volume of a maximal globally hyperbolic manifold $M_{h,h'}$ and the length of the earthquake laminations of the (both left and right) earthquake maps from (S,h) to (S,h'), provided by the Earthquake Theorem (Theorem 1.5.4).

Before stating the main results of this subsection, we finally need to recall the definition of length of a measured geodesic lamination. Let us denote by $\mathcal{ML}(S)$ the set of measured laminations on S, up to isotopy. The set of weighted multicurves

$$(\mathbf{c},\mathbf{a}) = ((c_1,a_1),\ldots,(c_n,a_n)) ,$$

where c_i are essential simple closed curves on S and a_i are positive weights, is dense in $\mathcal{ML}(S)$. The well-posedness of the following definition then follows from [Bon86].

Definition 4.2.3. *Given a closed orientable surface S of genus $g \geq 2$, we denote*

$$\ell : \mathcal{ML}(S) \times \mathrm{Teich}(S) \to [0,+\infty)$$

the unique continuous function such that, for every weighted multicurve $(\mathbf{c},\mathbf{a})$,

$$\ell((\mathbf{c},\mathbf{a}),[h]) = \sum_{i=0}^{n} a_i \mathrm{length}_h(c_i) ,$$

where $\mathrm{length}_h(c)$ *denotes the length of the h-geodesic representative in the isotopy class of c. Then we define the* length function *associated to a measured lamination λ as the function*

$$\ell_\lambda : \mathrm{Teich}(S) \to [0,+\infty)$$

defined by $\ell_\lambda([h]) = \ell(\lambda,[h])$.

Similarly, we also recall the definition of topological intersection for measured geodesic laminations:

Definition 4.2.4. *Given a closed orientable surface S of genus $g \geq 2$, we denote*

$$\iota : \mathcal{ML}(S) \times \mathcal{ML}(S) \to [0, +\infty)$$

the unique continuous function such that, for every pair of simple closed curves $\lambda = (c, w)$ and $\lambda' = (c', w')$,

$$\iota(\lambda, \lambda') = w \cdot w' \cdot \#(\gamma \cap \gamma') \ ,$$

where γ and γ' are geodesic representatives of c and c' for any hyperbolic metric on S.

The following is the first step towards a relation between the volume of a maximal globally hyperbolic manifold and the length of the left and right earthquake laminations of the Earthquake Theorem (Theorem 1.5.4).

Lemma 4.2.5. *Given a GHMC AdS_3 manifold $M = M_{h,h'}$, let λ_l and λ_r be the measured laminations such that $E_l^{\lambda_l}(h) = h'$ and $E_r^{\lambda_r}(h) = h'$. Then*

$$\mathrm{Vol}(\mathcal{C}(M)) + \mathrm{Vol}(M_+) = \frac{1}{4}\ell_{\lambda_r}(h) + \frac{\pi^2}{2}|\chi(S)| \ , \tag{4.5}$$

and

$$\mathrm{Vol}(\mathcal{C}(M)) + \mathrm{Vol}(M_-) = \frac{1}{4}\ell_{\lambda_l}(h) + \frac{\pi^2}{2}|\chi(S)| \ . \tag{4.6}$$

The proof follows from the arguments in [BB09, Section 8.2.3].

Corollary 4.2.6. *Given two hyperbolic metrics h and h' on S, if λ_l and λ_r are the measured laminations such that $E_l^{\lambda_l}(h) = h'$ and $E_r^{\lambda_r}(h) = h'$, then*

$$|\ell_{\lambda_l}(h) - \ell_{\lambda_r}(h)| \leq 2\pi^2|\chi(S)| \ .$$

Proof. From Equations (4.5) and (4.6), it follows that

$$\frac{1}{4}|\ell_{\lambda_l}(h) - \ell_{\lambda_r}(h)| = |\mathrm{Vol}(M_+) - \mathrm{Vol}(M_-)| \leq \max\{\mathrm{Vol}(M_-), \mathrm{Vol}(M_+)\} \leq \frac{\pi^2}{2}|\chi(S)| \ ,$$

where the last inequality is the content of Proposition 4.2.1. □

Theorem 4.2.7. *Given a GHMC AdS_3 manifold $M_{h,h'}$, let λ_l and λ_r be the measured laminations such that $E_l^{\lambda_l}(h) = h'$ and $E_r^{\lambda_r}(h) = h'$. Then*

$$\frac{1}{4}\ell_{\lambda_l}(h) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{1}{4}\ell_{\lambda_l}(h) + \frac{\pi^2}{2}|\chi(S)| \ , \tag{4.7}$$

and analogously

$$\frac{1}{4}\ell_{\lambda_r}(h) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{1}{4}\ell_{\lambda_r}(h) + \frac{\pi^2}{2}|\chi(S)| \ . \tag{4.8}$$

Proof. From Lemma 4.2.5, we have

$$\mathrm{Vol}(\mathcal{C}(M)) = \frac{1}{4}\ell_{\lambda_l}(h) + \frac{\pi^2}{2}|\chi(S)| - \mathrm{Vol}(M_-) \ ,$$

and thus the claim follows, using that

$$0 \leq \mathrm{Vol}(M_-) \leq \frac{\pi^2}{2}|\chi(S)|$$

by Proposition 4.2.1. The other inequality holds analogously. □

4.3 Holomorphic energy

In this section we will discuss the relation between the volume of a maximal globally hyperbolic anti-de Sitter manifold and several types of 1-energy, that is, the holomorphic 1-energy obtained by integrating the norm $||\partial f||$ of the $(1,0)$-part of the differential of a diffeomorphism f between Riemannian surfaces, and the integral of the 1-Schatten norm of the differential of f.

4.3.1 Volume of a convex set bounded by K-surfaces

As a consequence of Proposition 4.2.1, the volume of the convex core of a GHMC AdS_3 manifold $M_{h,h'}$ is coarsely equivalent to the volume of every domain of $M_{h,h'}$ in which it is contained. Using this fact, we will be able to compare the volume of $M_{h,h'}$ with the minima of certain functionals which depend on $(h,h') \in \mathrm{Teich}(S)\times\mathrm{Teich}(S)$.

As explained in Subsection 1.5, in $M_{h,h'}$, there exists a unique embedded maximal surface $\Sigma_0 = \Sigma_{h,h'}$ (i.e with vanishing mean curvature) with principal curvatures in $(-1,1)$. By an application of the maximum principle, Σ_0 is contained in the convex core of $M_{h,h'}$. Moreover, using the formulas for the shape operator B_ρ of equidistant surfaces (see [KS07] or [Sep17, Lemma 1.14]), it is straightforward to verify that a foliation by equidistant surfaces Σ_ρ from Σ_0 is defined at least for $\rho \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ and the surfaces $\Sigma_{-\frac{\pi}{4}}$ and $\Sigma_{\frac{\pi}{4}}$ are convex resp. concave, with constant Gaussian curvature -2. Therefore, the domain with boundary

$$\Omega_{h,h'} = \bigcup_{\rho\in[-\frac{\pi}{4},\frac{\pi}{4}]} \Sigma_\rho$$

contains the convex core and by definition

$$\mathrm{Vol}(\Omega_{h,h'}) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{Area}(\Sigma_\rho)d\rho \ .$$

By exploiting the analytic relation between maximal surfaces in AdS_3-manifolds and minimal Lagrangian diffeomorphisms between hyperbolic surfaces, we can express explicitly this volume as a functional of h and h'.

In fact (recalling Definition 1.5.5), the minimal Lagrangian map $m : (S, h) \to (S, h')$ can be characterised in the following way, see [Lab92b]:

Lemma 4.3.1. *Given two hyperbolic metrics h and h' on S, an orientation-preserving diffeomorphism $m : (S, h) \to (S, h')$ is minimal Lagrangian if and only if there exists a bundle morphism $b \in \Gamma(\mathrm{End}(TS))$ such that*

(1) $m^*h' = h(b\cdot, b\cdot)$

(2) $\det(b) = 1$

(3) b is h-self-adjoint

(4) b satisfies the Codazzi equation $d^\nabla b = 0$ for the Levi-Civita connection ∇ of h.

Moreover, if we denote with I_0 the induced metric on Σ_0 and with B_0 its shape operator, and we identify (Σ_0, I_0) and (S, h) using the left projection, the following relations hold (see [KS07, BS10, BS16]):

$$I_0 = \frac{1}{4}h((E+b)\cdot, (E+b)\cdot) \qquad \text{and} \qquad B_0 = -(E+b)^{-1}J_h(E-b)\ . \tag{4.9}$$

Here J_h is the complex structure on S compatible with the metric h. In particular, it can be checked directly that the surface Σ_0 is maximal precisely when conditions $(1) - (4)$ of Lemma 4.3.1 hold, that is, when the associated map is minimal Lagrangian.

Therefore, by using the above formulas and the fact that the metric on the parallel surface Σ_ρ at distance ρ from Σ_0 is given by

$$I_\rho = I_0((\cos(\rho)E + \sin(\rho)B_0)\cdot, (\cos(\rho)E + \sin(\rho)B_0)\cdot)\ ,$$

the area form of (Σ_ρ, I_ρ) is

$$d\mathrm{A}_{\Sigma_\rho} = \det(\cos(\rho)E + \sin(\rho)B_0)d\mathrm{A}_{\Sigma_0} = (\cos^2(\rho) + \sin^2(\rho)(\det B_0))d\mathrm{A}_{\Sigma_0}\ .$$

Moreover, from Equation (4.9), we have

$$d\mathrm{A}_{\Sigma_0} = \frac{1}{4}\det(E+b)d\mathrm{A}_h \qquad \text{and} \qquad \det B_0 = \frac{\det(E-b)}{\det(E+b)} = \frac{2-\mathrm{trace}(b)}{2+\mathrm{trace}(b)}\ .$$

Therefore we get:

$$\begin{aligned}
\mathrm{Area}(\Sigma_\rho) &= \int_{\Sigma_0} (\cos^2(\rho) + \sin^2(\rho)(\det B_0))d\mathrm{A}_{\Sigma_0} \\
&= \cos^2(\rho)\int_{\Sigma_0} d\mathrm{A}_{\Sigma_0} + \sin^2(\rho)\int_{\Sigma_0} \det(B_0)d\mathrm{A}_{\Sigma_0} \\
&= \frac{\cos^2(\rho)}{4}\int_{\Sigma_0} \det(E+b)d\mathrm{A}_h + \frac{\sin^2(\rho)}{4}\int_{\Sigma_0} \det(E-b)d\mathrm{A}_h \\
&= \frac{\cos^2(\rho)}{4}\int_{\Sigma_0} (2+\mathrm{trace}(b))d\mathrm{A}_h + \frac{\sin^2(\rho)}{4}\int_{\Sigma_0} (2-\mathrm{trace}(b))d\mathrm{A}_h \\
&= \pi|\chi(S)| + \frac{1}{4}(\cos^2(\rho) - \sin^2(\rho))\int_{\Sigma_0} \mathrm{trace}(b)d\mathrm{A}_h \ ,
\end{aligned}$$

where in the last step we used the Gauss-Bonnet equation for the hyperbolic metric h. Integrating for $\rho \in [-\pi/4, \pi/4]$ we have

$$\mathrm{Vol}(\Omega_{h,h'}) = \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}\int_{\Sigma_0} \mathrm{trace}(b)d\mathrm{A}_h \ . \tag{4.10}$$

Recall that from [Lab92b], there exists a unique minimal Lagrangian map $m : (S,h) \to (S,h')$ isotopic to the identity between any two hyperbolic surfaces (S,h) and (S,h'). Hence we can now prove:

Corollary 4.3.2. *Let $M_{h,h'}$ be a GHMC AdS$_3$ manifold. Let $b : TS \to TS$ be the unique h-self-adjoint Codazzi operator such that $m^*h' = h(b\cdot, b\cdot)$, where $m : (S,h) \to (S,h')$ is the minimal Lagrangian diffeomorphism. Then*

$$\frac{1}{4}\int_S \mathrm{trace}(b)d\mathrm{A}_h - \pi|\chi(S)| \le \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \le \frac{1}{4}\int_S \mathrm{trace}(b)d\mathrm{A}_h + \frac{\pi^2}{2}|\chi(S)| \ .$$

Proof. By the previous computation, we have

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \le \mathrm{Vol}(\Omega_{h,h'}) = \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}\int_S \mathrm{trace}(b)d\mathrm{A}_h \ .$$

On the other hand, by an adaptation of the proof of Proposition 4.2.1, since the boundary of $\Omega_{h,h'}$ consists of the disjoint union of the two surfaces with constant curvature -2 in $M_{h,h'}$, for every $\epsilon > 0$, we have

$$\begin{aligned}
\mathrm{Vol}(\Omega_{h,h'} \setminus \mathcal{C}(M_{h,h'})) &\le 2(1+\epsilon)\int_0^1 \frac{1}{2\sqrt{\kappa}(\kappa+1)}\int_{S_\kappa} d\mathrm{Area}_{S_\kappa} d\kappa \\
&= 2\pi|\chi(S)|(1+\epsilon)\int_0^1 \frac{d\kappa}{\sqrt{\kappa}(\kappa+1)^2} = |\chi(S)|\left(\pi + \frac{\pi^2}{2}\right)(1+\epsilon) \ ,
\end{aligned}$$

and therefore

$$\mathrm{Vol}(\Omega_{h,h'} \setminus \mathcal{C}(M_{h,h'})) \le |\chi(S)|\left(\pi + \frac{\pi^2}{2}\right) \ .$$

Hence, using Equation (4.10),

$$\begin{aligned}\mathrm{Vol}(\mathcal{C}(M_{h,h'})) &= \mathrm{Vol}(\Omega_{h,h'}) - \mathrm{Vol}(\Omega_{h,h'} \setminus \mathcal{C}(M_{h,h'})) \\ &\geq \mathrm{Vol}(\Omega_{h,h'}) - |\chi(S)| \left(\pi + \frac{\pi^2}{2}\right) \geq \frac{1}{4}\int_S \mathrm{trace}(b) d\mathrm{A}_h - \pi|\chi(S)| \ ,\end{aligned}$$

as claimed. □

4.3.2 Holomorphic energy and Schatten energy

As a consequence of Corollary 4.3.2, the coarse properties of the volume of a GHMC AdS_3 manifold depend only on the function

$$\begin{aligned} F : \mathrm{Teich}(S) \times \mathrm{Teich}(S) &\to \mathbb{R}^+ \\ (h, h') &\mapsto \int_S \mathrm{trace}(b) d\mathrm{A}_h \ , \end{aligned}$$

where b is the Codazzi tensor, satisfying the conditions $(1) - (4)$ of Lemma 4.3.1 above, for the minimal Lagrangian map $m : (S, h) \to (S, h')$. The properties of F have already been introduced and studied in [BMS15]. Here we point out the relation with an L^1-energy on Teichmüller space.

Let us denote by $C^1_{\mathrm{id}}(S)$ the space of C^1 maps $f : (S, h) \to (S, h')$ homotopic to the identity. Equivalently, by identifying (S, h) with $\mathbb{H}^2/\rho(\pi_1(S))$ and (S, h') with $\mathbb{H}^2/\rho'(\pi_1(S))$ (where ρ, ρ' are the holonomy representations of $\pi_1(S)$ into $\mathrm{Isom}(\mathbb{H}^2)$), $C^1_{\mathrm{id}}(S)$ coincides with the space of (ρ, ρ')-equivariant C^1 maps of $\mathbb{H}^2$ into itself.

Definition 4.3.3. *Given two hyperbolic surfaces (S, h) and (S, h'), the* 1*-Schatten energy is the functional $E_{Sch}(\cdot, h, h') : C^1_{\mathrm{id}}(S) \to \mathbb{R}^+$*

$$E_{Sch}(f, h, h') = \int_S \mathrm{trace}(b_f) d\mathrm{A}_h$$

where b_f is the unique h-self-adjoint operator such that $f^*h' = h(b_f\cdot, b_f\cdot)$.

Remark 4.3.4. *At every point $x \in S$, the tensor b_f at x coincides with the square root of df^*df, where df^* is the h-adjoint operator of df. Hence* $\mathrm{trace}(b_f)$ *coincides with the* 1*-Schatten norm of the operator df at x, and this justifies the definition of $E_{Sch}(f)$ as the* 1*-Schatten energy.*

Remark 4.3.5. *The* 1*-Schatten energy of a C^1 map f is related to the holomorphic energy*

$$E_\partial(f, h, h') = \int_S \|\partial f\| d\mathrm{A}_h$$

studied by Trapani and Valli in [TV95]. (Here we are considering ∂f as a holomorphic 1-form with values in f^*TS and we denote with $\|\cdot\|$ the norm on $T^*S \otimes f^*TS$ induced by the metrics h and h'). A computation in local coordinates, using Remark 4.3.4, shows that the eigenvalues of df^*df are

$$\mu_1 = \frac{1}{2}(\|\partial f\| - \|\overline{\partial} f\|)^2 \quad \textit{and} \quad \mu_2 = \frac{1}{2}(\|\partial f\| + \|\overline{\partial} f\|)^2 \ .$$

Therefore one obtains:

$$\mathrm{trace}(b_f) = \sqrt{2}\max\{||\partial f||, ||\overline{\partial} f||\} \ . \tag{4.11}$$

In particular, when f is orientation-preserving (for instance if f is a minimal Lagrangian diffeomorphism), then $\|\partial f\|^2 - \|\overline{\partial} f\|^2 > 0$ and therefore

$$\mathrm{trace}(b_f) = \sqrt{2}\|\partial f\| \ .$$

In conclusion, this shows that

$$E_\partial(f,h,h') \leq \frac{\sqrt{2}}{2} E_{Sch}(f,h,h') \ , \tag{4.12}$$

with equality when f is an orientation-preserving diffeomorphism.

Let us denote by $\mathrm{Diffeo}_0(S,h,h')$ the space of orientation preserving diffeomorphisms $f:(S,h)\to(S,h')$ isotopic to the identity. Trapani and Valli proved that the holomorphic 1-energy $E_\partial(\cdot,h,h')$ is minimized on $\mathrm{Diffeo}_0(S,h,h')$ by the unique minimal Lagrangian map $m:(S,h)\to(S,h')$:

Proposition 4.3.6 (Lemma 3.3 [TV95])**.** *Given two hyperbolic metrics (S,h) and (S,h'), the functional*

$$E_\partial(\cdot,h,h') : \mathrm{Diffeo}_{\mathrm{id}}(S,h,h') \to \mathbb{R}^+$$

admits a unique minimum attained by the minimal Lagrangian map $m:(S,h)\to(S,h')$ isotopic to the identity.

We will actually need the fact that the minimal Lagrangian map $m:(S,h)\to(S,h')$ also minimizes E_{Sch} on $C^1_{\mathrm{id}}(S)$, which is an improvement of Proposition 4.3.6:

Proposition 4.3.7. *Given two hyperbolic metrics (S,h) and (S,h'), the functional*

$$E_{Sch}(\cdot,h,h') : C^1_{\mathrm{id}}(S) \to \mathbb{R}^+$$

admits a minimum attained by the minimal Lagrangian map $m:(S,h)\to(S,h')$ isotopic to the identity.

The proof follows from the convexity of the functional E_{Sch}, see [BMS17]. In fact, the space $\mathrm{Diffeo}_{\mathrm{id}}(S,h,h')$ of diffeomorphisms isotopic to the identity is open in $C^\infty_{\mathrm{id}}(S)$ (i.e. the space of C^∞ self maps of S homotopic to the identity). Moreover, by Remark

4.3.5, E_{Sch} and E_∂ coincide on $\mathrm{Diffeo}_{\mathrm{id}}(S, h, h')$, up to a factor. By Proposition 4.3.6, m is a local minimum of E_{Sch} on $C^\infty_{\mathrm{id}}(S)$, and thus a global minimum on $C^\infty_{\mathrm{id}}(S)$ by convexity. By density of $C^\infty_{\mathrm{id}}(S)$ in $C^1_{\mathrm{id}}(S)$, and the continuity of E_{Sch} on $C^1_{\mathrm{id}}(S)$, it follows that m is a global minimum of $E_{Sch}(\cdot, h, h')$ on $C^1_{\mathrm{id}}(S)$, as well.

The above results enable us to conclude the following theorem:

Theorem 4.3.8. *Let $M_{h,h'}$ be a GHMC AdS_3 manifold. Then*

$$\frac{1}{4}E_{Sch}(m, h, h') - \pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{1}{4}E_{Sch}(m, h, h') + \frac{\pi^2}{2}|\chi(S)| \ ,$$

where $m : (S, h) \to (S, h')$ is the minimal Lagrangian map isotopic to the identity, that is, the minimum of the 1-Schatten energy functional $E_{Sch}(\cdot, h, h') : C^1_{\mathrm{id}}(S) \to \mathbb{R}$.

Proof. By Proposition 4.3.7, the minimum of $E_{Sch}(\cdot, h, h')$ is achieved at the minimal Lagrangian map m, and we have by definition

$$E_{Sch}(m, h, h') = \int_S \mathrm{trace}(b) d\mathrm{A}_h \ .$$

Hence the statement follows from Corollary 4.3.2. □

4.4 L^1-energy between hyperbolic surfaces

We conclude this section by showing that the volume is also coarsely comparable to the L^1-energy on Teichmüller space. The L^1-energy, or total variation, is defined as:

Definition 4.4.1. *Given two hyperbolic surfaces (S, h) and (S, h'), the* 1-energy*, or* total variation*, of f is the functional*

$$E_d(\cdot, h, h') : C^1_{\mathrm{id}}(S) \to \mathbb{R}^+$$

defined by

$$E_d(f, h, h') = \int_S ||df|| d\mathrm{A}_h \ .$$

For the proof of the inequality

$$\frac{1}{4} \inf_{f \in C^1_{\mathrm{id}}(S)} E_d(\cdot, h, h') - \frac{\sqrt{2}}{2}\pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'}))$$

of Theorem 4.4.3, we will need the fact that

$$\inf_{f \in C^1_{\mathrm{id}}(S)} E_d(\cdot, h, h') \leq \ell_\lambda(h) + 2\sqrt{2}\pi|\chi(S)| \ ,$$

so as to apply Theorem 4.2.7. This follows from the following lemma:

Lemma 4.4.2. *Given two hyperbolic surfaces (S,h) and (S,h'), let λ be the measured lamination such that $E_l^\lambda(h) = h'$ (or $E_r^\lambda(h) = h'$). Then there exists a sequence $f_n \in C^1_{\mathrm{id}}(S)$ such that*

$$\lim_{n\to+\infty} E_d(f_n, h, h') \leq \ell_\lambda(h) + 2\sqrt{2}\pi|\chi(S)| \ .$$

Proof. We will give the proof for left earthquakes. Suppose first that λ is a weighted simple closed geodesic (γ, w). Let U_ϵ be the ϵ-neighborhood of γ on (S,h). Choose coordinates (t,r) on U_ϵ, so that the geodesic γ is parameterized by arclength by the coordinate $(t,0)$, for $t\in[0,L]$, and the point (t,r) is at signed distance r from the point $(t,0)$. Hence the metric on U_ϵ has the form $dr^2+\cosh^2(r)dt^2$. Then define $f_\epsilon(r,t) = (r, t+g_\epsilon(r))$ on U_n, where $g_\epsilon(r)$ is a smooth increasing map such that $g_\epsilon(-\epsilon)=0$ and $g_\epsilon(\epsilon)=w$. By definition of earthquake map, we can then extend f_ϵ to be an isometry on $S\setminus U_\epsilon$. By a direct computation,

$$||df_{\epsilon(r,t)}|| = \sqrt{2+g'_\epsilon(r)^2} \ ,$$

hence

$$\begin{aligned}\int_{U_\epsilon} ||df_\epsilon|| d\mathrm{A}_h &= L\int_{-\epsilon}^{\epsilon} \sqrt{2+g'_\epsilon(r)^2}\cosh(r)dr \leq L\int_{-\epsilon}^{\epsilon}(\sqrt{2}+g'_\epsilon(r))\cosh(r)dr \\ &\leq \sqrt{2}\mathrm{Area}(U_\epsilon) + L\cosh(\epsilon)(g(\epsilon)-g(-\epsilon)) \ .\end{aligned}$$

Therefore, using that $L(g(\epsilon)-g(-\epsilon)) = Lw = \ell_\lambda(h)$ and that f_ϵ is an isometry outside of U_ϵ, we get

$$\begin{aligned}\int_S ||df_\epsilon|| d\mathrm{A}_h &= \int_{S\setminus U_\epsilon} ||df_\epsilon|| d\mathrm{A}_h + \int_{U_\epsilon} ||df_\epsilon|| d\mathrm{A}_h \\ &\leq \cosh(\epsilon)\ell_\lambda(h) + \sqrt{2}\mathrm{Area}(S\setminus U_\epsilon) + \sqrt{2}\mathrm{Area}(U_\epsilon) \\ &= \cosh(\epsilon)\ell_\lambda(h) + \sqrt{2}\mathrm{Area}(S) = \cosh(\epsilon)\ell_\lambda(h) + 2\sqrt{2}\pi|\chi(S)| \ .\end{aligned}$$

As we let $\epsilon\to 0$, this concludes that

$$\lim_{\epsilon\to 0}\int_S ||df_\epsilon|| d\mathrm{A}_h \leq \ell_\lambda(h) + 2\sqrt{2}\pi|\chi(S)| \ .$$

Let us now take an arbitrary measured geodesic lamination λ. Let λ_n be a sequence of weighted multicurves converging to λ, so that:

- $|\ell_{\lambda_n}(h) - \ell_\lambda(h)| \leq 1/n$.
- The metrics $h_n = E_l^{\lambda_n}(h)$ and $h' = E_l^\lambda(h)$ are $(1+1/n)$-bi-Lipschitz.

In fact, the second step follows from the continuity of the earthquake map $E_l : \mathcal{ML}(S)\times \mathrm{Teich}(S) \to \mathrm{Teich}(S)$. Let us now take $f_n : (S,h)\to(S,h_n)$ (constructed as before) so that

$$\int_S ||df_n|| d\mathrm{A}_h \leq \ell_{\lambda_n}(h) + 2\sqrt{2}\pi|\chi(S)| + \frac{1}{n} \ .$$

Let $g_n : (S, h_n) \to (S, h')$ be the $(1 + 1/n)$-bi-Lipschitz diffeomorphisms. Since $h_n \to h'$, we can assume $g_n \to \mathrm{id}$. Then for the map $g_n \circ f_n : (S, h) \to (S, h')$, we have:

$$\begin{aligned}\int_S ||d(g_n \circ f_n)|| d\mathrm{A}_h &\leq \left(1 + \frac{1}{n}\right) \int_S ||df_n|| d\mathrm{A}_h \\ &\leq \left(1 + \frac{1}{n}\right) \left(\ell_{\lambda_n}(h) + 2\sqrt{2}\pi|\chi(S)| + \frac{1}{n}\right) \\ &\leq \left(1 + \frac{1}{n}\right) \left(\ell_{\lambda}(h) + 2\sqrt{2}\pi|\chi(S)| + \frac{2}{n}\right) .\end{aligned}$$

Hence the constructed sequence $g_n \circ f_n : (S, h) \to (S, h')$ converges to the earthquake map e_λ and satisfies:

$$\lim_{n\to+\infty} \int_S ||d(g_n \circ f_n)|| d\mathrm{A}_h \leq \ell_\lambda(h) + 2\sqrt{2}\pi|\chi(S)| ,$$

hence concluding the proof. □

We are now able to prove the main result connecting the volume of GHMC AdS_3 manifolds with the minima of the L^1-energy:

Theorem 4.4.3. *Let $M_{h,h'}$ be a GHMC AdS_3 manifold. Then*

$$\frac{1}{4} \inf_{f\in C^1_{\mathrm{id}}(S)} E_d(\cdot, h, h') - \frac{\sqrt{2}}{2}\pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\sqrt{2}}{2} \inf_{f\in C^1_{\mathrm{id}}(S)} E_d(\cdot, h, h') + \frac{\pi^2}{2}|\chi(S)| .$$

Proof. From Lemma 4.4.2 and Theorem 4.2.7, we have

$$\frac{1}{4} \inf_{f\in C^1_{\mathrm{id}}(S)} E_d(\cdot, h, h') \leq \frac{1}{4}(\ell_\lambda(h) + 2\sqrt{2}\pi|\chi(S)|) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) + \frac{\sqrt{2}}{2}\pi|\chi(S)| ,$$

hence the lower bound follows. On the other hand, using the fact that

$$||df||^2 = ||\partial f||^2 + ||\overline{\partial} f||^2 ,$$

from Equation (4.11) we have for every $f \in C^1_{\mathrm{id}}(S)$:

$$\mathrm{trace}(b_f) = \sqrt{2} \max\{||\partial f||, ||\overline{\partial} f||\} \leq \sqrt{2}||df|| .$$

Thus

$$E_{Sch}(f, h, h') = \int_S \mathrm{trace}(b_f) dA_h \leq \sqrt{2} \int_S ||df|| dA_h .$$

Hence the upper bound follows from Theorem 4.3.8 and Proposition 4.3.7:

$$\begin{aligned}
\mathrm{Vol}(\mathcal{C}(M_{h,h'})) &\leq \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}E_{Sch}(m,h,h') \\
&= \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}\inf_{f\in C^1_{\mathrm{id}}(S)} E_{Sch}(f,h,h') \\
&\leq \frac{\pi^2}{2}|\chi(S)| + \frac{\sqrt{2}}{4}\inf_{f\in C^1_{\mathrm{id}}(S)} E_d(f,h,h') \ ,
\end{aligned}$$

thus concluding the proof. □

4.5 Thurston's asymmetric distance

In this Section, we will apply Corollary 4.3.2 to compare the volume of the convex core of a GHMC AdS_3-manifold and Thurston's asymmetric distance on Teichmüller space.

4.5.1 The general upper bound

Thurston asymmetric distance on Teichmüller space is deeply related to the hyperbolic geometry of surfaces. We briefly recall here the main definitions for the convenience of the reader.

Let h and h' two hyperbolic metrics on S. Given a diffeomorphism isotopic to the identity $f:(S,h)\to(S,h')$ we define the Lipschitz constant of f as

$$L(f) = \sup_{x\neq y\in S}\frac{d_{h'}(f(x),f(y))}{d_h(x,y)} \ .$$

Definition 4.5.1. *Thurston asymmetric distance between $h,h'\in \mathrm{Teich}(S)$ is*

$$d_{\mathrm{Th}}(h,h') = \inf_{f\in \mathrm{Diff}_{\mathrm{id}}} \log(L(f))$$

where the infimum is taken over all diffeomorphisms $f:(S,h)\to(S,h')$ isotopic to the identity.

Thurston showed that the Lipschitz constant $L(f)$ can also be computed by comparing lengths of closed geodesics for the metrics h and h'. More precisely, in [Thu98] he proved that

$$L(f) = \sup_c \frac{\ell_c(h')}{\ell_c(h)} \ , \tag{4.13}$$

where c varies over all simple closed curves c in S.

An application of Theorem 4.3.8 leads to the following comparison between the volume of the convex core of a GHMC AdS_3 manifold and Thurston asymmetric distance.

Theorem 4.5.2. *Let $M_{h,h'}$ be a GHMC AdS_3 manifold. Then*

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \pi|\chi(S)|\exp(\min\{d_{\mathrm{Th}}(h,h'), d_{\mathrm{Th}}(h',h)\}) \ .$$

Proof. We will first prove that

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \pi|\chi(S)|\exp(d_{\mathrm{Th}}(h,h')) \ .$$

First of all, let us observe that the Lipschitz constant of a diffeomorphism $f : (S,h) \to (S,h')$ can be expressed as:

$$L(f) = \sup_{v \in TS} \frac{||df(v)||_{h'}}{||v||_h} = \sup_{x\in S} ||df_x||_\infty \ ,$$

Here, $||df_x||_\infty$ is the spectral norm of $df_x : (T_xS, h_x) \to (T_{f(x)}S, h'_{f(x)})$. Now, from Theorem 4.3.8, for every diffeomorphism $f : (S,h) \to (S,h')$ isotopic to the identity, we have

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}E_{Sch}(f,h,h') \ .$$

Since the spectral norm of df is the maximum eigenvalue of $\sqrt{df^*df}$, the 1-Schatten norm is bounded by twice the spectral norm, hence we get

$$E_{Sch}(f,h,h') = \int_S \mathrm{trace}(b_f) d\mathrm{A}_h \leq 2 \sup_{x\in S} ||df_x||_\infty \int_S d\mathrm{A}_h = 4\pi|\chi(S)|L(f) \ .$$

Hence we obtain:

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \pi|\chi(S)| \inf_f L(f) = \frac{\pi^2}{2}|\chi(S)| + \pi|\chi(S)|e^{d_{\mathrm{Th}}(h,h')} \ .$$

For the main statement, observe that the involution

$$\begin{aligned} \mathrm{SL}(2,\mathbb{R}) &\to \mathrm{SL}(2,\mathbb{R}) \\ A &\mapsto A^{-1} \end{aligned}$$

induces an orientation-reversing isometry of AdS_3 which swaps the left and right metric in Mess' parameterization (see Section 1.5). Therefore, the volumes of the convex cores of $M_{h,h'}$ and $M_{h',h}$ are equal. Hence it follows that

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \pi|\chi(S)|\exp(d_{\mathrm{Th}}(h',h))$$

is also true. This concludes the proof. □

4.5.2 A negative result

We are now showing that it is not possible to find a lower-bound for the volume of the convex core in terms of the Thurston asymmetric distance between the left and right metric.

Proposition 4.5.3. *There is no continuous, proper function* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$g(\min\{d_{\mathrm{Th}}(h, h'), d_{\mathrm{Th}}(h', h)\}) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \ .$$

for every couple of metrics $h, h' \in \mathrm{Teich}(S)$.

Proof. It is sufficient to show that it is possible to find a sequence of GHMC AdS_3 manifolds such that the volume of the convex core remains bounded but both Thurston's asymmetric distances between the left and right metric diverge.
Choose a simple closed curve $\mu \in P$ that disconnects the surface in such a way that one connected component S_1 is a surface of genus 1 with geodesic boundary equal to μ. Fix a pant decomposition P containing the curve μ. Let $\alpha \subset S_1$ be the curve in the pant decomposition of S contained in the interior of S_1. Fix a simple closed curve β in S_1 (see Figure 4.1) which intersects α in exactly one point. Choose then a hyperbolic metric h on S such that the geodesic representative of β intersects α orthogonally. For every $n \in \mathbb{N}$ we define an element $h_n \in \mathrm{Teich}(S)$ with the property that all Fenchel-Nielsen coordinates of h_n coincide with those of h but the length of the curve α, which we impose to be equal to $1/n$. In particular, the h_n-geodesic representative of β intersects α orthogonally for every n.

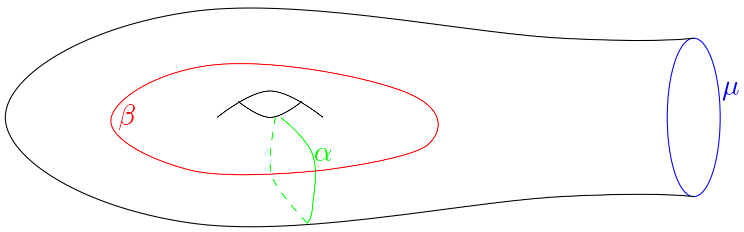

Figure 4.1: Curves described in the proof of Proposition 4.5.3

Consider the measured geodesic laminations λ_n consisting of the simple closed curve α with weight n. We define a second sequence of hyperbolic metrics h'_n as $h'_n = E_l^{\lambda_n}(h_n)$. Notice that these metrics are obtained from h_n by performing n^2 Dehn-twists along α. We are going to show that the volume of the convex core of the GHMC AdS_3 manifolds $M_n = M_{h_n,h'_n}$ remains bounded but the two Thurston's asymmetric distances between h_n and h'_n go to infinity when n tends to $+\infty$.
By Equation (4.7) in Theorem 4.2.7, the volume of the convex core of M_n is coarsely equivalent to the length of λ_n, which by definition is

$$\ell_\lambda(h_n) = \ell_\alpha(h_n) \cdot \frac{1}{n} = 1 \ ,$$

hence the volume remains bounded.
On the other hand, since the curve β intersects α orthogonally, for every metric h_n we claim that

$$\ell_\beta(h_n) = 4\text{arcsinh}\left(\frac{\cosh(\frac{\ell_\mu(h)}{4})}{\sinh(\frac{1}{2n})}\right) .$$

To prove the claim, we cut the surface S_1 along the curve α, thus obtaining a pair of pants P' with geodesic boundaries given by μ and two copies of α. If we cut again P' into two right-angled exagons (see Figure 4.2), the length of the curve β can be computed using standard hyperbolic trigonometry [Thu97]. Here we are also using the fact that the length of the curve μ does not depend on n.

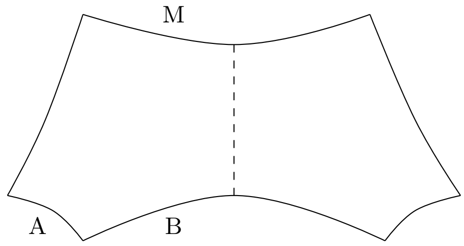

Figure 4.2: The lengths of the edges A,B and M satisfy $\sinh(A)\sinh(B/2) = \cosh(M/2)$.

Hence we obtain

$$\ell_\beta(h_n) \le C_1 |\log(n)| + C_2 \ , \tag{4.14}$$

for some constants C_1, C_2, when n is sufficiently big. Moreover, by a simple application of the triangle inequality (see [BS09, Lemma 7.1], and recall Definition 4.2.4 for the intersection of measured geodesic laminations), we can deduce that

$$\ell_\beta(h'_n) + \ell_\beta(h_n) \ge \iota(\lambda_n, \beta) = n \ ,$$

thus

$$\frac{\ell_\beta(h'_n)}{\ell_\beta(h_n)} \ge -1 + \frac{n}{\ell_\beta(h_n)} \to +\infty$$

when n tends to $+\infty$ by using Equation (4.14). Therefore, by definition $d_{\mathrm{Th}}(h_n, h'_n) \to +\infty$.
To prove that also $d_{\mathrm{Th}}(h'_n, h_n)$ is unbounded, it is sufficient to repeat the same argument for the curve $\beta' = D_\alpha^{n^2}(\beta)$ obtained from β by performing n^2 Dehn-twists along α. Namely, by construction, the curve β' intersects orthogonally the curve α for the metric h'_n, thus the same estimate as in Equation (4.14) holds for the length of the curve β' with respect to the metric h'_n. $\square$

4.5.3 Discussion of the optimality

In this subsection, we will construct some examples to show that the result of Theorem 4.5.2 is optimal, in some sense. The first situation we consider is the case of a sequence of manifolds $M_{h,h'}$ for which one metric is fixed, and the other metric diverges in Teich(S). In this case, the volume of $M_{h,h'}$ is in fact bounded also from below by the exponential of Thurston's asymmetric distance:

Proposition 4.5.4. *Let Ω be a compact set in* Teich(S) *and let $M_{h,h'}$ be a GHMC AdS_3 manifold with $h \in \Omega$. There exists a constant $C = C(\Omega) > 0$ such that*

$$C(\Omega)\exp(d_{\mathrm{Th}}(h, h')) - C(\Omega) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'}))$$

for every $h \in \Omega$ and every $h' \in$ Teich(S).

Proof. Let h' be any hyperbolic metric on S, let λ be the measured lamination such that $h' = E_l^\lambda(h)$, and let α be any simple closed curve on S. By a simple formula, we have:

$$\ell_\alpha(h') \leq \ell_\alpha(h) + \iota(\lambda, \alpha) \ .$$

In fact, it is easy to check that this formula is true when λ is a simple closed curve, since the h'-geodesic representative of α is shorter than the piecewise-geodesic curve obtained by glueing the image of the h-geodesic representative of α and subintervals of the simple closed curve λ according to the earthquake measure. The general case follows by a continuity argument. Hence we have:

$$\frac{\ell_\alpha(h')}{\ell_\alpha(h)} \leq 1 + \frac{\iota(\lambda, \alpha)}{\ell_\alpha(h)} \ .$$

We claim that there exists a constant $C = C(\Omega) > 0$ such that, for every pair of measured laminations $\mu, \lambda \in \mathcal{ML}(S)$,

$$\frac{\iota(\mu, \lambda)}{\ell_\mu(h)\ell_\lambda(h)} \leq D(\Omega) \ .$$

The proof will then follows directly from the claim, since we will then have

$$\frac{\ell_\alpha(h')}{\ell_\alpha(h)} \leq 1 + D(\Omega)\ell_\lambda(h) \leq 1 + 4D(\Omega)\mathrm{Vol}(\mathcal{C}(M_{h,h'}))$$

by Theorem 4.2.7, for every simple closed curve α. Therefore (recall Equation (4.13))),

$$C(\Omega)\exp(d_{\mathrm{Th}}(h, h')) - C(\Omega) = C(\Omega)\sup_\alpha \frac{\ell_\alpha(h')}{\ell_\alpha(h)} - C(\Omega) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \ ,$$

where $C(\Omega) = 1/4D(\Omega)$.
To prove the claim, suppose by contradiction there exists no such constant $D(\Omega)$, and therefore there exist a sequence $h_n \in \Omega$, and sequences $\mu_n, \lambda_n \in \mathcal{ML}(S)$ such that

$$\frac{\iota(\mu_n, \lambda_n)}{\ell_{\mu_n}(h_n)\ell_{\lambda_n}(h_n)} \to +\infty \ .$$

Now, up to extracting subsequences, we can assume $h_n \to h_\infty \in \Omega \subset \mathrm{Teich}(S)$. Moreover, by the compactness of the space of projective measured laminations on S, we can assume that there exist $a_n, b_n > 0$ such that $a_n\mu_n \to \mu_\infty$ and $b_n\lambda_n \to \lambda_\infty$, for $\mu_\infty, \lambda_\infty \neq 0$. This leads to a contradiction, as

$$\frac{\iota(a_n\mu_n, b_n\lambda_n)}{\ell_{a_n\mu_n}(h_n)\ell_{b_n\lambda_n}(h_n)} = \frac{\iota(\mu_n, \lambda_n)}{\ell_{\mu_n}(h_n)\ell_{\lambda_n}(h_n)} \to \frac{\iota(\mu_\infty, \lambda_\infty)}{\ell_{\mu_\infty}(h_\infty)\ell_{\lambda_\infty}(h_\infty)} < +\infty$$

since the quantities ℓ and ι vary with continuity. □

Recall that the action of the mapping class group of S on

$$\mathrm{Teich}_\epsilon(S) := \{h \in \mathrm{Teich}(S) \mid \mathrm{injrad}(h) \geq \epsilon\}$$

is co-compact, by [Mum71]. As the volume $\mathrm{Vol}(\mathcal{C}(M_{h,h'})$ is invariant under the diagonal action of the mapping class group on $\mathrm{Teich}(S) \times \mathrm{Teich}(S)$, we deduce the following stronger version of Proposition 4.5.4.

Corollary 4.5.5. *Given any $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ such that*

$$C(\epsilon)\exp(d_{\mathrm{Th}}(h, h')) - C(\epsilon) \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'}))$$

for every $h \in \mathrm{Teich}_\epsilon(S)$ and every $h' \in \mathrm{Teich}(S)$, where $\mathrm{Teich}_\epsilon(S)$ is the ϵ-thick part of Teichmüller space of S.

We will now discuss the optimality of the multiplicative constant in the upper bound of Theorem 4.5.2. More precisely, we will exhibit a sequence of examples, in a surface S_g for any genus g, so that the volume grows actually like $|\chi(S_g)|\exp(\min\{d_{\mathrm{Th}}(h, h'), d_{\mathrm{Th}}(h', h)\})$.

Proposition 4.5.6. *There exist universal constants $C, g_0 > 0$ and there exist sequences of hyperbolic metrics h_g, h'_g in $\mathrm{Teich}(S_g)$, where S_g is the closed orientable surface of genus g, such that:*

$$\mathrm{Vol}(\mathcal{C}(M_{h_g,h'_g})) \geq C|\chi(S_g)|\exp(d_{\mathrm{Th}}(h_g, h'_g))$$

for every $g \geq g_0$.

Proof. Fix a pant decomposition P_g of S_g, which is composed of $3g-3$ disjoint simple closed curves $\alpha_1, \ldots, \alpha_{3g-3}$. Consider a hyperbolic metric h_g for which all the the simple closed curves $\alpha_1, \ldots, \alpha_{3g-3}$ have the same length, say u (independently of g). Let us consider the hyperbolic metric $h' = E_l^\lambda(h)$, where λ is the multicurve $\alpha_1, \ldots, \alpha_{3g-3}$, where all the curves are endowed with the same weight $w > 0$. Now, given any other simple closed curve α, we have (as in the proof of Proposition 4.5.4):

$$\frac{\ell_\alpha(h')}{\ell_\alpha(h)} \leq 1 + \frac{\iota(\lambda, \alpha)}{\ell_\alpha(h)} = 1 + w\frac{\iota(P, \alpha)}{\ell_\alpha(h)} .$$

Now, observe that every time the curve α crosses a curve α_i of P, α needs to exit the pair of pants adjacent to α_i through some boundary component of the same pair of pants. Hence the length of α is at least the intersection number $\iota(P,\alpha)$ times the distance between two boundary components. Since we chose h_g so that all pairs of pants in the decomposition have the same length u for all boundary components, the distance between two boundary components can be computed, as in Proposition 4.5.3, as:

$$r(u) = 2\text{arcsinh}\left(\frac{\cosh(\frac{u}{4})}{\sinh(\frac{u}{2})}\right) = 2\text{arcsinh}\left(\frac{1}{2\sinh(\frac{u}{4})}\right) .$$

Thus we obtain

$$\ell_\alpha(h) \geq \iota(P,\alpha)\cdot r(u) .$$

On the other hand, observe that $\ell_\lambda(h) = w(3g-3)u$. Hence we get:

$$\frac{\ell_\alpha(h')}{\ell_\alpha(h)} \leq 1 + w\frac{\iota(P,\alpha)}{\iota(P,\alpha)r(u)} = 1 + \frac{w}{r(u)} = 1 + \frac{\ell_\lambda(h)}{(3g-3)ur(u)} .$$

Since this inequality holds for every simple closed curve α, recalling Equation (4.13), we obtain:

$$\exp(d_{\text{Th}}(h_g,h'_g)) \leq 1 + \frac{2}{(3ur(u))}\frac{1}{|\chi(S_g)|}\ell_\lambda(h) \leq 1 + \frac{8}{(3ur(u))}\frac{1}{|\chi(S_g)|}\text{Vol}(\mathcal{C}(M_{h_g,h'_g})) ,$$

where in the last step we have used Theorem 4.2.7. In particular this shows that

$$\frac{\text{Vol}(\mathcal{C}(M_{h_g,h'_g}))}{\exp(d_{\text{Th}}(h_g,h'_g)) - 1} \geq C_0|\chi(S_g)| ,$$

for some constant $C_0 > 0$. Since $\ell_\lambda(h)$ (and thus also the volume) is going to infinity, it follows that

$$\text{Vol}(\mathcal{C}(M_{h_g,h'_g})) \geq C|\chi(S_g)|\exp(d_{\text{Th}}(h_g,h'_g))$$

for every constant $C < C_0$, if $g \geq g_0$. This concludes the claim. $\square$

Remark 4.5.7. *The proof of Proposition 4.5.6 actually produces sequences h_g, h'_g such that*

$$\text{Vol}(\mathcal{C}(M_{h_g,h'_g})) \geq C|\chi(S_g)|\exp\left(\max\{d_{\text{Th}}(h_g,h'_g), d_{\text{Th}}(h'_g,h_g)\}\right) .$$

In fact, h_g was chosen so that all pairs of pants in the pant decomposition P have a certain shape, and h'_g is obtained by earthquake along P. Hence for the metric h'_g, the pairs of pants also have this shape as well (in other words, h_g and h'_g only differ by twist coordinates in the Fenchel-Nielsen coordinates provided by P). Hence, switching left earthquakes with right earthquakes, the proof holds analogously for the other Thurston's distance.

4.6 Weil-Petersson distance

In this section we study the relation between the volume of a GHMC AdS_3 manifold and the Weil-Petersson distance between its left and right metric.

4.6.1 Weil-Petersson metric on Teichmüller space

The Weil-Petersson metric is a Riemannian metric on Teichmüller space, which connects the hyperbolic and the complex geometry of surfaces.

Given a Riemann surface (S, X), let us denote by K the canonical line bundle of S, that is the holomorphic cotangent bundle. It is known that the vector space $QD(X) = H^0(S, K^2)$ of holomorphic quadratic differentials on (S, X) has complex dimension $3g - 3$ and can be identified with the cotangent space $T^*_{[X]}\mathrm{Teich}(S)$. We recall briefly this identification for the convenience of the reader. A Beltrami differential μ is a smooth section of the vector bundle $\overline{K} \otimes K^{-1}$. In local coordinates, we can write $\mu = \mu(z)\frac{d\bar{z}}{dz}$. Beltrami differentials can be interpreted as $(0, 1)$-forms with value in the tangent bundle of S and correspond to infinitesimal deformations of the complex structure X. If we denote with $BD(X)$ the vector space of Beltrami differentials and with $BD_{tr}(X)$ the subspace corresponding to trivial deformations of the complex structure X, we have an identification:

$$T_{[X]}\mathrm{Teich}(S) \cong BD(X)/BD_{tr}(X) \ .$$

The duality pairing between a Beltrami differential μ and a holomorphic quadratic differential Φ

$$\langle \mu, \Phi \rangle = \int_S \mu(z)\phi(z) dz \wedge d\bar{z} \ ,$$

where in local coordinates $\Phi = \phi(z)dz^2$, induces the aforementioned isomorphism $QD(X) \cong T^*_{[X]}\mathrm{Teich}(S)$.

Let h be the unique hyperbolic metric on S compatible with the complex structure X. If we write in local coordinates $h = \sigma_0^2(z)|dz|^2$, the Weil-Petersson metric on $\mathrm{Teich}(S)$ arises from the real part of the Hermitian product on $QD(X)$, namely:

$$\langle \Phi, \Psi \rangle_{\mathrm{WP}} = \int_S \frac{\phi(z)\overline{\psi}(z)}{\sigma_0^2(z)} dz \wedge d\bar{z}$$

via the above duality pairing.

The Weil-Petersson metric is geodesically convex ([Wol87]), it has negative sectional curvature ([Wol86], [Tro86]) and the mapping class group acts by isometries ([MW02]). However, the Weil-Petersson metric is not complete ([Wol75]) and its completion gives rise to the augmented Teichmüller space $\overline{\mathrm{Teich}(S)}$, obtained by adding noded Riemann surfaces ([Mas76]). The Weil-Petersson distance from a point $X \in \mathrm{Teich}(S)$ to a noded Riemann surface Z with nodes along a collection of curves $\alpha_1, \cdots, \alpha_k$ is estimated by (see [Wol08, Section 4] and [CP12, Theorem 2.1])

$$d_{\mathrm{WP}}(X, Z) \leq \sqrt{2\pi\ell} \ , \tag{4.15}$$

where

$$\ell = \ell_{\alpha_1}(h) + \cdots + \ell_{\alpha_k}(h)$$

is the sum of the lengths of the curves α_j computed with respect to the unique hyperbolic metric h compatible with the complex structure on X.

4.6.2 A negative result

The failure of completeness of the Weil-Petersson metric at limits of pinching sequences in Teichmüller space implies that it is not possible to find an upper-bound for the volume of a GHMC AdS_3 manifold $M_{h,h'}$ in terms of the Weil-Petersson distance $d_{\mathrm{WP}}(h, h')$, as the following proposition shows.

Proposition 4.6.1. *It is not possible to find a continuous, increasing and unbounded function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\mathrm{Vol}(M_{h,h'}) \leq f(d_{\mathrm{WP}}(h, h')) \ .$$

Proof. It is sufficient to exibit a sequence of maximal globally hyperbolic AdS^3 manifolds $M_n = M_{h_n, h'_n}$ such that

$$\lim_{n \to +\infty} \mathrm{Vol}(M_n) = +\infty \qquad \text{but} \qquad d_{\mathrm{WP}}(h_n, h'_n) \leq C \quad \forall n \in \mathbb{N}$$

for some constant $C > 0$.
An example can be constructed as follows. Fix a hyperbolic metric $h \in \mathrm{Teich}(S)$ and a pants decomposition $P = \{\alpha_1, \cdots \alpha_{3g-3}\}$ of S. Consider a sequence of hyperbolic metrics h'_n obtained by letting the lengths of the curves α_j go to 0 for every $j = 1, \dots 3g - 3$. By construction, the sequence h'_n leaves every compact subset in $\mathrm{Teich}(S)$ and it is converging to the noded Riemann surface Z in the augmented Teichmüller space $\overline{\mathrm{Teich}(S)}$ where all the curves of the pants decomposition P are pinched. Therefore, by Equation (4.15),

$$d_{\mathrm{WP}}(h, h'_n) \leq d_{\mathrm{WP}}(h, Z) + d_{\mathrm{WP}}(h'_n, Z) \leq C \ ,$$

where $C = 2\sqrt{2\pi\ell_P(h)}$. On the other hand, the volume of the AdS^3 manifolds M_n is diverging because

$$\mathrm{Vol}(M_n) \geq \mathrm{Vol}(\Omega_n) = \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}\int_S \mathrm{trace}(b_n) d\mathrm{A}_h = \frac{\pi^2}{2}|\chi(S)| + \frac{1}{4}F(h, h'_n)$$

and the functional $F(h, \cdot) : \mathrm{Teich}(S) \to \mathbb{R}^+$ is proper ([BMS15, Proposition 1.2]). Notice that we can actually make the constant C arbitrarily small by choosing the metric h appropriately. $\square$

4.6.3 A lower bound on the volume

We can bound the volume of a GHMC AdS_3 manifold in terms of the Weil-Petersson distance between its left and right metric from below.

Theorem 4.6.2. *Let $M_{h,h'}$ be a GHMC AdS_3 manifold. Then there exist some positive constants $a, b, c > 0$ such that*

$$\exp\left(\frac{a}{|\chi(S)|} d_{\mathrm{WP}}(h, h') - b|\chi(S)|\right) - c \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \ .$$

The proof relies on a precise estimate of the norm of the Weil-Petersson gradient of the length function, whose proof is postponed to the next section.

Theorem 4.6.3. *There exists a universal constant $a > 0$ such that for every $\lambda \in \mathcal{ML}(S)$ and for every $h \in \mathrm{Teich}(S)$, we have*

$$\|\mathrm{grad}\ell_\lambda(h)\|_{\mathrm{WP}} \geq \frac{a}{|\chi(S)|} \ell_\lambda(h) \ . \tag{4.16}$$

We will also need the following result by Bers ([Ber74], see also [Bus10, Theorem 5.13,5.14]):

Theorem 4.6.4. *Let S be a closed surface of genus $g \geq 2$. For every hyperbolic metric $h \in \mathrm{Teich}(S)$ there is a pants decomposition P such that $\ell_\alpha(h) < L_g$ for every $\alpha \in P$, where $L_g = 6\sqrt{3\pi}(g-1) = 3\sqrt{3\pi}|\chi(S)|$.*

We will refer to the constant L_g as Bers' constant. Given $h \in \mathrm{Teich}(S)$ we can find a pants decomposition $P = \{\alpha_1, \dots \alpha_{3g-3}\}$, such that $\ell_{\alpha_j}(h) < L_g$ for every $j = 1, \dots 3g-3$. If we perturb the metric h using an earthquake, we can estimate how the lengths of the curves α_j change only in terms of the Bers' constant and of the length of the lamination.

Lemma 4.6.5. *Let $h, h' \in \mathrm{Teich}(S)$. Let λ be the measured geodesic lamination such that $h' = E_l^\lambda(h)$. Fix a pants decomposition $P = \{\alpha_1, \dots \alpha_{3g-3}\}$ such that $\ell_{\alpha_j}(h) < L$ for every $j = 1, \dots 3g-3$. Then there exists a constant $d(L) > 0$ depending only on L such that*

$$\ell_{\alpha_j}(h') \leq L + \frac{\ell_\lambda(h)}{d(L)} \ ,$$

for every $j = 1, \dots 3g-3$.

Proof. It is well known ([Ker83]) that the first variation of the length of a simple closed curve γ along an earthquake path is given by the integral over γ of the cosines of the angles formed by γ with the lamination λ. As a consequence,

$$\left| \frac{d}{dt}\bigg|_{t=t_0} \ell_\gamma(E_l^{t\lambda}(h)) \right| \leq \iota(\gamma, \lambda) \ ,$$

for every t_0. Hence
$$|\ell_\gamma(h') - \ell_\gamma(h)| \leq \iota(\lambda, \gamma) \ .$$

Therefore for every $j = 1, \dots 3g-3$ we can give an upper-bound for the lengths of the curves α_j:
$$\ell_{\alpha_j}(h') \leq \ell_{\alpha_j}(h) + \iota(\alpha_j, \lambda) \leq L + \iota(\alpha_j, \lambda) \ . \tag{4.17}$$

We only need to estimate the intersection between the curves α_j and the lamination λ in terms of the length of the lamination. We claim that
$$\ell_\lambda(h) \geq d \cdot \iota(\lambda, \alpha_j)$$
for some constant $d = d(L)$. To prove the claim, suppose first that $\lambda = (c, w)$ consists of a weighted simple closed geodesic. By the Collar Lemma, since $\ell_h(\alpha_j) \leq L$, there exist disjoint tubular neighborhoods $T_{\alpha_j, d(L)}$ of the geodesics α_j of width
$$d(L) = \operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{L}{2}\right)}\right) \ . \tag{4.18}$$

The intersection of c with $T_{\alpha_j, d(L)}$ is the disjoint union of $\#(c \cap \alpha_j)$ geodesic arcs of length at least $d(L)$. We deduce that for every $j = 1, \cdots 3g-3$ we have
$$\ell_\lambda(h) = w\ell_c(h) \geq wd(L) \sum_{j=1}^{3g-3} \#(c \cap \alpha_j) = d(L)\iota(\lambda, \alpha_j) \ . \tag{4.19}$$

The general case of the claim follows by a standard approximation argument using the well-known fact that weighted simple closed curves are dense in the space of measured geodesic laminations. The proof then follows by combining Equation (4.17) and Equation (4.19). □

Given a pants decomposition $P = \{\alpha_1, \dots, \alpha_{3g-3}\}$ and a real number $L > 0$, we define
$$V_L(P) = \{h \in \text{Teich}(S) \mid \ell_{\alpha_j}(h) \leq L \text{ for every } j = 1, \dots, 3g-3\}$$

Proposition 4.6.6 (Proposition 2.2 [Bro03]). *For every pant decomposition, the set $V_L(P)$ has bounded diameter for the Weil-Petersson metric. More precisely, for every pant decomposition P of S,*
$$\text{diam}_{\text{WP}}(V_L(P)) \leq 2\sqrt{2\pi L} \ .$$

We can estimate the Weil-Petersson distance between points lying in different level sets $V_m(P)$.

Proposition 4.6.7. *Let $h_0 \in V_m(P)$, for some $m > L$. Then*

$$d_{\mathrm{WP}}(h_0, V_L(P)) \leq \frac{|\chi(S)|}{a} \log\left(\frac{m(3g-3)}{L}\right) ,$$

where a is the constant provided by Theorem 4.6.3.

Proof. Let us denote with $\ell_P : \mathrm{Teich}(S) \to \mathbb{R}^+$ the function

$$\ell_P(h) = \sum_{i=1}^{3g-3} \ell_{\alpha_i}(h)$$

which computes the total length of the curves α_i in the pants decomposition P. In the above notation, P is considered as a measured lamination, composed of the multicurve $\alpha_1, \dots, \alpha_{3g-3}$, each with unit weight. By Theorem 4.6.3 we have

$$\|\mathrm{grad}\ell_P\|_{\mathrm{WP}} \geq \frac{a}{|\chi(S)|}\ell_P,$$

thus

$$\|\mathrm{grad}(\log \ell_P)\|_{\mathrm{WP}} \geq \frac{a}{|\chi(S)|} .$$

Let X be the vector field on $\mathrm{Teich}(S)$ defined by

$$X = -\frac{\mathrm{grad}(\log \ell_P)}{\|\mathrm{grad}(\log \ell_P)\|_{\mathrm{WP}}}$$

and let γ be an integral curve of X such that $\gamma(0) = h_0$. By the previous estimates, the function $\phi(t) = (\log \ell_P)(\gamma(t))$ satisfies the differential equation

$$\phi'(t) = \langle \mathrm{grad}(\log \ell_P), \gamma'(t)\rangle_{\mathrm{WP}} = -\|\mathrm{grad}(\log \ell_P)\|_{\mathrm{WP}} \leq -\frac{a}{|\chi(S)|} .$$

We deduce that

$$\phi(t) \leq \phi(0) - \frac{at}{|\chi(S)|} \leq \log(m(3g-3)) - \frac{at}{|\chi(S)|} ,$$

and that the curve $\gamma(t)$ intersects the set $V_L(P)$ after a time

$$t_0 \leq \frac{|\chi(S)|}{a} \log\left(\frac{m(3g-3)}{L}\right) ,$$

which implies the claim. □

We have now all the ingredients to prove Theorem 4.6.2:

Proof of Theorem 4.6.2. Let h be a hyperbolic metric on S and $h' = E_l^\lambda(h)$. Fix a pants decomposition P such that $h \in V_{L_g}(P)$, where L_g is as in the statement of Theorem 4.6.4. By Proposition 4.6.6 and Proposition 4.6.7 we have

$$\begin{aligned} d_{\mathrm{WP}}(h,h') &\leq d_{\mathrm{WP}}(h', V_{L_g}(P)) + \mathrm{diam}_{\mathrm{WP}} V_{L_g}(P) \\ &\leq \frac{|\chi(S)|}{a}\log\left(\frac{m(3g-3)}{L_g}\right) + 2\sqrt{2\pi L_g} \\ &\leq \frac{|\chi(S)|}{a}\log\left(\frac{m}{2\sqrt{3}\pi}\right) + 2\sqrt{2\pi L_g}\ , \end{aligned}$$

for some $m \in \mathbb{R}$ such that $h' \in V_m(P)$. We can choose m such that

$$m \leq L_g + \ell_\lambda(h)\ .$$

Hence

$$2\sqrt{3}\pi d(L_g)\exp\left(\frac{a}{|\chi(S)|}(d_{\mathrm{WP}}(h,h') - 2\sqrt{2\pi L_g})\right) - d(L_g)L_g \leq \ell_\lambda(h)\ .$$

Now from Equation (4.18),

$$\exp(-\delta - 2\sqrt{3}\pi(g-1)) \leq d(L_g) \leq \exp(-2\sqrt{3}\pi(g-1))$$

for some constant δ, and thus (using again the definition of L_g)

$$2\sqrt{3}\pi d(L_g)\exp\left(\frac{a}{|\chi(S)|}d_{\mathrm{WP}}(h,h') - 2\sqrt{2\pi L_g})\right) \geq \exp\left(\frac{a}{|\chi(S)|}d_{\mathrm{WP}}(h,h') - b|\chi(S)|\right)\ ,$$

for some constant $b > 0$. In conclusion, since $d(L_g)L_g \to 0$ as $g \to \infty$, there is a constant $c > 0$ such that

$$\exp\left(\frac{a}{|\chi(S)|}d_{\mathrm{WP}}(h,h') - b|\chi(S)|\right) - c \leq \ell_\lambda(h)\ .$$

The main statement of Theorem 4.6.2 then follows by applying Theorem 4.2.7, up to changing the constants b and c. □

4.7 Gradient of length function

This section is devoted to the proof of Theorem 4.6.3, which we recall here:

Theorem 4.7.1. *There exists a universal constant $a > 0$ such that for every $\lambda \in \mathcal{ML}(S)$ and for every $h \in \mathrm{Teich}(S)$, the following estimate holds:*

$$\|\mathrm{grad}\ell_\lambda(h)\|_{\mathrm{WP}} \geq \frac{a}{|\chi(S)|}\ell_\lambda(h)\ . \tag{4.20}$$

First, it suffices to prove the inequality (4.16) when λ is a simple closed curve (with weight 1). In fact, the inequality (4.16) is homogeneous with respect to multiplication of λ by some positive scalar. Hence if (4.16) holds for a simple closed curve $(c, 1)$, then it holds for every (c, w), where $w > 0$ is any weight. In this case, the inequality then holds also for every measured geodesic lamination, since weighted simple closed curves are dense in $\mathcal{ML}(S)$, and both sides of the inequality vary with continuity.

Moreover, we notice that Theorem 4.6.3 clearly holds if we restrict to the thick part of Teichmüller space. Namely, the function

$$g : \mathrm{Teich}(S) \times (\mathcal{ML}(S) \setminus \{0\}) \to \mathbb{R}^+$$
$$(h, \lambda) \mapsto \frac{\|\mathrm{grad}\ell_\lambda(h)\|^2_{\mathrm{WP}}}{\ell^2_\lambda(h)}$$

is invariant under the action of the Mapping Class Group and under rescaling of the measure of λ, hence if restricted to

$$\mathrm{Teich}_{\epsilon_0}(S) \times (\mathcal{ML}(S) \setminus \{0\}) = \{h \in \mathrm{Teich}(S) \mid \mathrm{injrad}(h) \geq \epsilon_0\} \times (\mathcal{ML}(S) \setminus \{0\})$$

it admits a minimum, since $\mathrm{Teich}_{\epsilon_0}(S)$ projects to a compact set in the moduli space

$$\mathcal{M}(S) = \mathrm{Teich}(S)/MCG(S)$$

and the quotient $(\mathcal{ML}(S) \setminus \{0\})/\mathbb{R}^+$ is compact.

This observation motivates the fact that main difficulty will thus arise when dealing with hyperbolic metrics with small injectivity radius. Let us recall that it is possible to choose a (small) constant ϵ_0, such that on any hyperbolic surface (S, h) of genus g, there are at most $3g - 3$ simple closed geodesics of length at most ϵ_0. We will fix such ϵ_0 later on. Notice that any $\epsilon_0 \leq 2\mathrm{arcsinh}(1)$ works. By the Collar Lemma, for every simple closed geodesic α of length ϵ, the tube

$$T_{\alpha,d} = \{x \in (S, h) \,|\, d_h(x, \alpha) \leq d\} \ , \tag{4.21}$$

is an embedded cylinder for any $d \leq d(\epsilon)$, where

$$d(\epsilon) := \mathrm{arcsinh}\left(\frac{1}{\sinh(\frac{\epsilon}{2})}\right) \ . \tag{4.22}$$

Moreover, if $\alpha_1, \ldots, \alpha_{3g-3}$ are pairwise disjoint, then $T_{\alpha_1,d(\epsilon_1)}, \ldots, T_{\alpha_{3g-3},d(\epsilon_{3g-3})}$ are pairwise disjoint. Hence we obtain a thin-thick decomposition of any hyperbolic surface (S, h), that is, we have

$$S = S_h^{\mathrm{thin}} \cup S_h^{\mathrm{thick}}$$

where

$$S_h^{\mathrm{thin}} = \bigcup_i T_{\alpha_i,d(\epsilon_i)} \ , \tag{4.23}$$

where the union is over all simple closed geodesics α_i of length $\epsilon_i \leq \epsilon_0$, and

$$S_h^{\mathrm{thick}} = S \setminus S_h^{\mathrm{thin}} \ . \tag{4.24}$$

It then turns out that the injectivity radius at every point $x \in S_h^{\mathrm{thick}}$ is at least $\epsilon_0/2$.

4.7.1 Riera's formula

We are going to prove the inequality of Equation (4.16) for a simple closed curve c on (S, h). If we denote by γ the h-geodesic representative of c, we will prove the inequality first in the case

$$\mathrm{length}_h(\gamma \cap S_h^{\mathrm{thin}}) \leq \mathrm{length}_h(\gamma \cap S_h^{\mathrm{thick}}) \ ,$$

and then in the opposite case, provided ϵ_0 is small enough. In both cases, a key tool will be the following theorem. This was proved by Riera in [Rie05] in a more general setting; the statement below is specialized to the case of closed surfaces.

Theorem 4.7.2. *Given a closed hyperbolic surface* (S, h)*, let us fix a metric universal cover* $\pi : \mathbb{H}^2 \to (S, h)$*, which thus identifies* $\pi_1(S)$ *to a Fuchsian subgroup of* $\mathrm{Isom}(\mathbb{H}^2)$*. Given a simple closed curve* c *in* S*, let* $C \in \pi_1(S)$ *be an element freely homotopic to* c*. Then*

$$\|\mathrm{grad}\ell_c(h)\|^2_{\mathrm{WP}} = \frac{2}{\pi}\ell_c(h) + \frac{2}{\pi} \sum_{\substack{D \in \langle C\rangle\backslash\pi_1(S)/\langle C\rangle \\ D \neq [\mathrm{id}]}} \left(u(D) \log\left(\frac{u(D)+1}{u(D)-1}\right) - 2\right) \ , \quad (4.25)$$

where for $D \in \langle C\rangle\backslash\pi_1(S)/\langle C\rangle$ *(not in the double coset of the identity) the function* u *is defined as*

$$u(D) = \cosh(d(\mathrm{Axis}(C), \mathrm{Axis}(DCD^{-1}))\ . \quad (4.26)$$

First of all, observe that the function u in Equation (4.26) is well-defined, since if $D' = ADB$ for $A, B \in \langle C\rangle$, then

$$\mathrm{Axis}(D'CD'^{-1}) = \mathrm{Axis}(ADCD^{-1}A^{-1})\ . \quad (4.27)$$

Thus

$$d(\mathrm{Axis}(C), \mathrm{Axis}(DCD^{-1}) = d(\mathrm{Axis}(C), \mathrm{Axis}(D'CD'^{-1})\ ,$$

since A stabilizes the axis of C.

Another equivalent way to express the summation in Equation (4.25) is the following. Let γ be the h-geodesic representative of c in S. Let $\mathcal{G}(\mathbb{H}^2)$ be the set of (unoriented) geodesics of $\mathbb{H}^2$ and let

$$\mathcal{A} = \{(\tilde\gamma_1, \tilde\gamma_2) \in \mathcal{G}(\mathbb{H}^2) \times \mathcal{G}(\mathbb{H}^2) : \pi(\tilde\gamma_1) = \pi(\tilde\gamma_2) = \gamma\}/\pi_1(S)\ , \quad (4.28)$$

where $\pi_1(S)$ acts diagonally on pairs $(\tilde\gamma_1, \tilde\gamma_2)$.

The set $\mathcal{A}$ is in bijection with $\langle C\rangle\backslash\pi_1(S)/\langle C\rangle$, by means of the function:

$$[D] \mapsto (\mathrm{Axis}(C), \mathrm{Axis}(DCD^{-1}))\ ,$$

which is well-defined since, if $D' = ADB$ for $A, B \in \langle C \rangle$, then from Equation (4.27),

$$(\mathrm{Axis}(C), \mathrm{Axis}(D'CD'^{-1})) = A \cdot (\mathrm{Axis}(C), \mathrm{Axis}(DCD^{-1})) \ .$$

The map is easily seen to be surjective since for every pair of geodesics $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ projecting to γ, up to composing with an element in $\pi_1(S)$ one can find a representative with $\tilde{\gamma}_1 = \mathrm{Axis}(C)$. Finally, it is injective since, supposing

$$(\mathrm{Axis}(C), \mathrm{Axis}(DCD^{-1})) = A \cdot (\mathrm{Axis}(C), \mathrm{Axis}(D'CD'^{-1})) \ ,$$

this implies that A stabilizes Axis(C) (namely, $A \in \langle C \rangle$) and that

$$\mathrm{Axis}(DCD^{-1}) = \mathrm{Axis}(AD'CD'^{-1}A^{-1}) \ ,$$

that is, $D^{-1}AD' \in \mathrm{Stab}(\mathrm{Axis}(C)) = \langle C \rangle$ and therefore $D = AD'B^{-1}$ for some $B \in \langle C \rangle$.

Let us now observe that there is a well defined function

$$\mathfrak{u} : \mathcal{A} \to [1, +\infty)$$

such that $\mathfrak{u}[\tilde{\gamma}_1, \tilde{\gamma}_2] = \cosh d(\tilde{\gamma}_1, \tilde{\gamma}_2)$. Moreover the bijection between $\langle C \rangle \backslash \pi_1(S) / \langle C \rangle$ and $\mathcal{A}$ transforms u in $\mathfrak{u}$. In conclusion, the summation of Equation (4.25) is equal to:

$$\|\mathrm{grad}\ell_c(h)\|^2_{\mathrm{WP}} = \frac{2}{\pi}\ell_c(h) + \frac{2}{\pi}\sum_{\Gamma \in \mathcal{A} \backslash \Delta}\left(\mathfrak{u}(\Gamma)\log\left(\frac{\mathfrak{u}(\Gamma)+1}{\mathfrak{u}(\Gamma)-1}\right) - 2\right) \ . \tag{4.29}$$

where $\Delta \in \mathcal{A}$ denotes the class of $(\tilde{\gamma}_1, \tilde{\gamma}_1)$.

4.7.2 Estimates in the thick part of the hyperbolic surface

Let us begin with the case in which $\mathrm{length}_h(\gamma \cap S_h^{\mathrm{thin}}) \leq \mathrm{length}_h(\gamma \cap S_h^{\mathrm{thick}})$. In this case, the proof will use the following preliminary lemma:

Lemma 4.7.3. *There exist $\epsilon_0 > 0$ small enough and $n_0 > 0$ large enough such that, for every choice of:*

- *A hyperbolic metric h on a closed orientable surface S;*
- *A number $\delta > 0$;*
- *An embedded h-geodesic arc α of length at most ϵ_0, such that the δ-neighborhood of α is embedded;*
- *A simple closed curve c, whose h-geodesic representative γ intersects α at least n_0 times;*

one has:

$$||\mathrm{grad}\ell_c(h)||^2_{\mathrm{WP}} \geq C\delta(\#(\alpha \cap \gamma))^2 \ ,$$

for some constant $C = C(n_0)$ (independent of the genus of S).

Proof. Recall that γ denotes the h-geodesic representative of c, let $\pi : \mathbb{H}^2 \to (S, h)$ be a fixed metric universal cover, and let us fix a lift $\tilde{\alpha}$ of the geodesic arc α, so that $\pi|_{\tilde{\alpha}}$ is a homeomorphism onto α. We suppose that $\#(\alpha \cap \gamma) > n_0 > 0$, and we will determine n_0 later on. Let us denote

$$\mathcal{A}_{\tilde{\alpha}} = \{[\tilde{\gamma}_1, \tilde{\gamma}_2] \in \mathcal{A} : \tilde{\gamma}_1 \cap \tilde{\alpha} \neq \emptyset, \tilde{\gamma}_2 \cap \tilde{\alpha} \neq \emptyset\} \ . \tag{4.30}$$

Denote moreover $E = \gamma \cap \alpha$ and define a function

$$\varphi : E \times E \to \mathcal{A}_{\tilde{\alpha}}$$

such that $\varphi(p, q) = [\tilde{\gamma}_p, \tilde{\gamma}_q]$, where $\tilde{\gamma}_p$ is the unique geodesic of $\mathbb{H}^2$ such that $\pi(\tilde{\gamma}_p) = c$ and $\pi(\tilde{\gamma}_p \cap \tilde{\alpha}) = p$. Clearly φ is surjective and maps the diagonal in $E \times E$ to $\mathcal{A}_{\tilde{\alpha}} \cap \Delta$. We claim that, for $[\tilde{\gamma}, \tilde{\gamma}'] \in \mathcal{A}$:

$$\sinh d[\tilde{\gamma}, \tilde{\gamma}'] \leq 2(\sinh \epsilon_0)\exp\left(-\frac{\delta}{2} \cdot (\#(\varphi^{-1}[\tilde{\gamma}, \tilde{\gamma}']) - 1)\right) . \tag{4.31}$$

To prove the claim, suppose the cardinality of $\varphi^{-1}[\tilde{\gamma}, \tilde{\gamma}']$ is $n + 1$. Therefore there are $n + 1$ pairs (p_i, q_i) such that $[\tilde{\gamma}_{p_i}, \tilde{\gamma}_{q_i}]$ are all equivalent to $(\tilde{\gamma}, \tilde{\gamma}')$ in $\mathcal{A}$. This means that there exists $g_i \in \pi_1(S)$ such that $g_i(\tilde{\gamma}_{p_i}, \tilde{\gamma}_{q_i}) = (\tilde{\gamma}, \tilde{\gamma}')$. It follows that, for every $i \neq j$, the arcs $g_i(\tilde{\alpha})$ and $g_j(\tilde{\alpha})$ are distinct. Indeed, if $g_i(\tilde{\alpha}) = g_j(\tilde{\alpha})$, then $g_i \circ g_j^{-1}$ would send $\tilde{\alpha}$ to itself and move at least one point of $\tilde{\alpha}$, which is impossible since α is embedded. Now, the arcs $g_i(\tilde{\alpha})$ intersect $\tilde{\gamma}$ in the $n + 1$ different points $g_i(p_i)$, which are at distance at least δ from one another since the δ-neighborhood of α is embedded. Let r and r' be the feet of the common perpendicular of $\tilde{\gamma}$ and $\tilde{\gamma}'$. Then, at least one of the points $g_i(p_i)$, is at distance at least $n\delta/2$ from r. Denote by $p_0 = g_{i_0}(p_{i_0})$ be such point, and let q_0 be the projection of p_0 to $\tilde{\gamma}'$. The quadrilateral with vertices in p_0, r, r', q_0 is a Lambert quadrilateral, that is, it has right angles at r, r', q_0. See Figure 4.3. Hence the following formula holds:

$$\sinh d(p_0, \tilde{\gamma}') = \cosh d(p_0, r) \sinh d(\tilde{\gamma}, \tilde{\gamma}') \ .$$

This concludes the claim, since $d(p_0, \tilde{\gamma}') \leq \mathrm{length}(\alpha) \leq \epsilon$ and $d(p_0, r) \geq n\delta/2$, and thus

$$\sinh d(\tilde{\gamma}, \tilde{\gamma}') \leq \frac{\sinh \epsilon_0}{\cosh(n\delta/2)} \ ,$$

from which the claim follows.

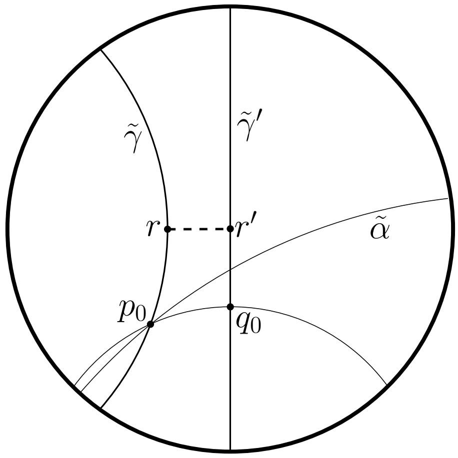

Figure 4.3: The Lambert quadrilateral in the proof of Lemma 4.7.3.

Now we can conclude the proof. By Theorem 4.7.2, we have:

$$\begin{aligned}
\|\mathrm{grad}\ell_c\|^2_{\mathrm{WP}} &\geq \frac{2}{\pi} \sum_{[\tilde{\gamma}_1,\tilde{\gamma}_2]\in\mathcal{A}_{\tilde{\alpha}}\setminus\Delta} \left(\cosh d[\tilde{\gamma}_1,\tilde{\gamma}_2] \log\left(\frac{\cosh d[\tilde{\gamma}_1,\tilde{\gamma}_2]+1}{\cosh d[\tilde{\gamma}_1,\tilde{\gamma}_2]-1}\right) - 2\right) \\
&\geq \frac{2}{\pi} \sum_{[\tilde{\gamma}_1,\tilde{\gamma}_2]\in\mathcal{A}_{\tilde{\alpha}}\setminus\Delta} \left(\log\left(\frac{\cosh d[\tilde{\gamma}_1,\tilde{\gamma}_2]+1}{\sinh d[\tilde{\gamma}_1,\tilde{\gamma}_2]}\right)^2 - 2\right) \\
&\geq \frac{2}{\pi} \sum_{[\tilde{\gamma}_1,\tilde{\gamma}_2]\in\mathcal{A}_{\tilde{\alpha}}\setminus\Delta} (2\log 2 - 2\log\sinh d[\tilde{\gamma}_1,\tilde{\gamma}_2] - 2) \\
&\geq \frac{4}{\pi} \sum_{[\tilde{\gamma}_1,\tilde{\gamma}_2]\in\mathcal{A}_{\tilde{\alpha}}\setminus\Delta} \left(-\log\sinh\epsilon_0 + \frac{\delta}{2}(\#(\varphi^{-1}[\tilde{\gamma},\tilde{\gamma}']) - 1) + \log 2 - 1\right)
\end{aligned}$$

where in the last line we have used Equation (4.31). Therefore, if we suppose that ϵ_0 is small enough so that $C_1(\epsilon_0) := -\log(\sinh(\epsilon_0)) - 1 + \log 2 > 0$, we get

$$\begin{aligned}
\|\mathrm{grad}\ell_c\|^2_{\mathrm{WP}} &\geq \frac{2\delta}{\pi} \sum_{[\tilde{\gamma}_1,\tilde{\gamma}_2]\in\mathcal{A}_{\tilde{\alpha}}\setminus\Delta} \big(\#(\varphi^{-1}[\tilde{\gamma},\tilde{\gamma}'])\big) + C_1(\epsilon_0)(\#(\alpha\cap\gamma) - 1) \\
&\geq \frac{2\delta}{\pi} \cdot \#(E \times E \setminus \varphi^{-1}(\Delta)) + C_1(\epsilon_0)(\#(\alpha\cap\gamma) - 1) \\
&= \frac{2\delta}{\pi}(\#(\alpha\cap\gamma))(\#(\alpha\cap\gamma) - 1) + C_1(\epsilon_0)(\#(\alpha\cap\gamma) - 1) \\
&= \frac{2\delta}{\pi}(\#(\alpha\cap\gamma))^2 + \left(C_1(\epsilon_0) - \frac{2\delta}{\pi}\right)\#(\alpha\cap\gamma) - C_1(\epsilon_0) \\
&\geq \frac{2\delta}{\pi}(\#(\alpha\cap\gamma) - 1)^2 \ ,
\end{aligned}$$

provided $\#(\alpha\cap\gamma) > 1$. The last quantity is certainly larger than $C\cdot\delta\cdot(\#(\alpha\cap\gamma))^2$, if $\#(\alpha\cap\gamma) > n_0$, for some suitable choices of n_0 and $C = C(n_0)$. □

We will now replace the constant ϵ_0 which gives the thin-thick decomposition (see Equations (4.23) and (4.24)) by a smaller constant (if necessary), so that ϵ_0 is smaller than the constant given by Lemma 4.7.3.

Proposition 4.7.4. *Let* $\epsilon_0 < \operatorname{arcsinh}(\exp(\log(2) - 1))$ *a constant inducing a thin-thick decomposition of* S*. There exists a constant* $a = a(\epsilon_0)$ *(independent of the genus of* S*) such that for every hyperbolic metric* h *on* S *and every simple closed curve* c*, if the* h*-geodesic representative* γ *of* c *satisfies:*

$$\text{length}_h(\gamma \cap S_h^{\text{thin}}) \leq \text{length}_h(\gamma \cap S_h^{\text{thick}}) ,$$

then

$$\|\text{grad}\ell_c(h)\|_{\text{WP}} \geq \frac{a}{|\chi(S)|}\ell_c(h) .$$

Proof. By an adaptation of the argument of [BS12, Lemma A.1], there exists a constant β_0 (independent on h) such that the subset

$$\widehat{\gamma} := \left\{ x \in \gamma \mid \#(\alpha_x^r(\epsilon_0/4) \cap \gamma) \leq \frac{\beta_0}{|\chi(S)|}\ell_c(h) \right\} \subseteq \gamma$$

has h-lenght at most $\ell_c(h)/2$, where $\alpha_x^r(\epsilon_0/4)$ is the h-geodesic arc orthogonal to γ starting at x_0, on the right with respect to a chosen orientation of γ, of length $\epsilon_0/4$. The constant β_0 only depends on the initial choice of ϵ_0.
Now, in our hypothesis, since $\text{length}_h(\gamma \cap S_h^{\text{thin}}) + \text{length}_h(\gamma \cap S_h^{\text{thick}}) = \ell_c(h)$, the length of $\gamma \cap S_h^{\text{thick}}$ is at least $\ell_c(h)/2$. Therefore there exists some point $x \in (\gamma \setminus \widehat{\gamma}) \cap S_h^{\text{thick}}$. Since x is in the ϵ_0-thick part of (S, h), the arc $\alpha_x^r(\epsilon_0/4)$ is embedded. Moreover, the $(\epsilon_0/4)$-neighborhood of $\alpha_x^r(\epsilon_0/4)$ is embedded, for otherwise there would be a closed loop starting from x of length less than ϵ_0, which contradicts x being in the ϵ_0-thick part. Recall that, since by construction $x \in (\gamma \setminus \widehat{\gamma}) \cap S_h^{\text{thick}}$, $\#(\alpha_x^r(\epsilon_0/4) \cap \gamma) \geq (\beta_0 \ell_c(h))/|\chi(S)|$. Hence, from Lemma 4.7.3, there exist constants $n_0 > 0$ and $K = K(\epsilon_0, n_0) > 0$ such that

$$||\text{grad}\ell_c(h)||_{\text{WP}} \geq K\#(\alpha_x^r(\epsilon_0/4) \cap \gamma) \geq \frac{K\beta_0}{|\chi(S)|}\ell_c(h) , \tag{4.32}$$

whenever $\#(\alpha_x^r(\epsilon_0/4) \cap \gamma) \geq n_0$, which occurs under the hypothesis that

$$\ell_c(h) \geq \frac{n_0}{\beta_0}|\chi(S)| . \tag{4.33}$$

On the other hand, we have the inequality $\|\text{grad}\ell_c(h)\|_{\text{WP}} \geq \sqrt{(2/\pi)\ell_c(h)}$ from Equation (4.25). Observe that if

$$\ell_c(h) \leq \frac{n_0}{\beta_0}|\chi(S)| , \tag{4.34}$$

then

$$\sqrt{(2/\pi)\ell_c(h)} \geq \sqrt{\frac{2\beta_0}{\pi n_0}}\frac{1}{|\chi(S)|^{1/2}}\ell_c(h) .$$

Hence, putting together the cases (4.33) and (4.34), there exists a constant $a > 0$ such that

$$\|\text{grad}\ell_c(h)\|_{\text{WP}} \geq \frac{a}{|\chi(S)|}\ell_c(h)$$

thus concluding the proof. □

4.7.3 Estimates in the thin part of the hyperbolic surface

We are left to consider the case when

$$\text{length}_h(\gamma \cap S_h^{\text{thin}}) \geq \text{length}_h(\gamma \cap S_h^{\text{thick}}) \ ,$$

where γ is the h-geodesic representative of c. For this purpose, suppose γ enters into a tube $T_{\alpha,d(\epsilon)}$, where α is a simple closed geodesic of length $\ell_h(\alpha) = \epsilon \leq \epsilon_0$.

Let us fix a metric universal cover $\pi : \mathbb{H}^2 \to (S, h)$ and a lift $\tilde{\alpha}$ of α, that is, an entire geodesic in $\mathbb{H}^2$. Let $A \in \pi_1(S)$ be a primitive element which corresponds to a hyperbolic isometry with axis $\tilde{\alpha}$. We will denote (in analogy with the notation of (4.30), but with the difference that here $\tilde{\alpha}$ covers α):

$$\mathcal{A}_{\tilde{\alpha}} = \{[\tilde{\gamma}_1, \tilde{\gamma}_2] : \tilde{\gamma}_1 \cap \tilde{\alpha} \neq \emptyset, \tilde{\gamma}_2 \cap \tilde{\alpha} \neq \emptyset\} \ ,$$

which is a subset of the set of equivalence classes defined in Equation (4.28).

Lemma 4.7.5. *Let $[\tilde{\gamma}_1, \tilde{\gamma}_2], [\tilde{\gamma}_1', \tilde{\gamma}_2'] \in \mathcal{A}_{\tilde{\alpha}}$. If $[\tilde{\gamma}_1, \tilde{\gamma}_2] = [\tilde{\gamma}_1', \tilde{\gamma}_2']$ and $d(\tilde{\gamma}_1 \cap \tilde{\alpha}, \tilde{\gamma}_2 \cap \tilde{\alpha}) \geq \text{length}_h(\alpha)$, then there exists $k \in \mathbb{Z}$ such that $\tilde{\gamma}_1' = A^k(\tilde{\gamma}_1)$ and $\tilde{\gamma}_2' = A^k(\tilde{\gamma}_2)$.*

Proof. Suppose that the equivalence classes of $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ and $(\tilde{\gamma}_1', \tilde{\gamma}_2')$ coincide, and there does not exist any $k \in \mathbb{Z}$ such that $\tilde{\gamma}_1' = A^k(\tilde{\gamma}_1)$ and $\tilde{\gamma}_2' = A^k(\tilde{\gamma}_2)$. We will then prove that $d(\tilde{\gamma}_1 \cap \tilde{\alpha}, \tilde{\gamma}_2 \cap \tilde{\alpha}) < \text{length}_h(\alpha)$. We first consider the case in which $d(\tilde{\gamma}_1 \cap \tilde{\alpha}, \tilde{\gamma}_2 \cap \tilde{\alpha}) = \text{length}_h(\alpha)$, which occurs if $\tilde{\gamma}_2 = A(\tilde{\gamma}_1)$ (or $\tilde{\gamma}_2 = A^{-1}(\tilde{\gamma}_1)$, which will be completely analogous). This means that there exists $D \in \pi_1(S)$ such that $D(\tilde{\gamma}_i') = \tilde{\gamma}_i$ for $i = 1, 2$, but D is not in the stabilizer of $\tilde{\alpha}$. Hence $D(\tilde{\alpha})$ is a geodesic of $\mathbb{H}^2$, different from $\tilde{\alpha}$, which intersects both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.
We can also assume that D is such that $0 < d(\tilde{\gamma}_1 \cap \tilde{\alpha}, \tilde{\gamma}_1 \cap D(\tilde{\alpha})) < d(\tilde{\gamma}_2 \cap \tilde{\alpha}, \tilde{\gamma}_2 \cap D(\tilde{\alpha}))$. By this assumption, and the action by isometry of $\langle A \rangle$, it follows that $A \circ D(\tilde{\alpha})$ intersects $\tilde{\gamma}_2$ in a point which is closer to $\tilde{\alpha}$ than $D(\alpha) \cap \tilde{\gamma}_2$. On the other hand, $A \circ D(\tilde{\alpha})$ either intersects $\tilde{\gamma}_1$ in a point which is further from $\tilde{\alpha}$ than $D(\tilde{\alpha}) \cap \tilde{\gamma}_1$ (by the choice of D), or is disjoint from $\tilde{\gamma}_1$. In both cases, it follows that $A \circ D(\tilde{\alpha})$ must intersect $D(\tilde{\alpha})$, which gives a contradiction since α is a simple closed geodesic. See Figure 4.4.
In the case $d(\tilde{\gamma}_1 \cap \tilde{\alpha}, \tilde{\gamma}_2 \cap \tilde{\alpha}) > \text{length}_h(\alpha)$, we get a contradiction *a fortiori*, since every translate $D(\tilde{\alpha})$ which intersects $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, must also intersect $A(\tilde{\gamma}_1)$ (or $A^{-1}(\tilde{\gamma}_1)$). This gives a contradiction as in the previous paragraph. □

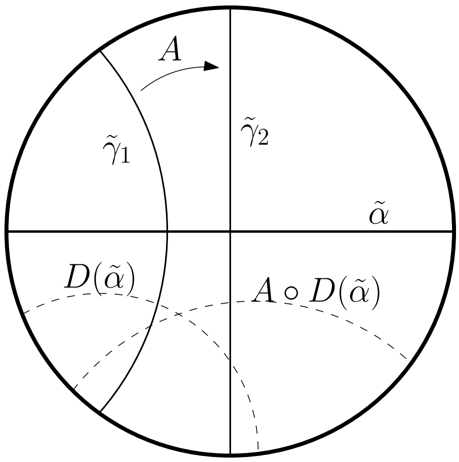

Figure 4.4: The contradiction in the proof of Lemma 4.7.5.

Let us fix a connected fundamental domain $\tilde{\alpha}_0$ for the action of $\langle A \rangle$ on $\tilde{\alpha}$, and let us denote $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ the lifts of γ which intersect $\tilde{\alpha}_0$, ordered according to an orientation of $\tilde{\alpha}_0$, where $n = \iota(\alpha, c)$. It follows from Lemma 4.7.5 and Equation (4.29) that

$$\|\mathrm{grad}\ell_c(h)\|^2_{\mathrm{WP}} \geq \frac{2}{\pi}\ell_c(h) + \frac{2}{\pi} \sum_{1 \leq i \leq j \leq n} \sum_{k=1}^{+\infty} \left(\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) \log \left(\frac{\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) + 1}{\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) - 1} \right) - 2 \right) . \tag{4.35}$$

The next step thus consists of providing a uniform estimate on the multiple summation in the above inequality (4.35).

Lemma 4.7.6. *Let α be a simple closed geodesic on (S, h) of length $\epsilon \leq \epsilon_0$ and let $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ be lifts of γ which intersect the fundamental domain $\tilde{\alpha}_0$ in $\tilde{\alpha}$. Then there exists a universal constant $K > 0$ such that*

$$\sum_{k=1}^{+\infty} \left(\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) \log \left(\frac{\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) + 1}{\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) - 1} \right) - 2 \right) \geq K \max \left\{ \frac{1}{\epsilon}, \frac{1}{\epsilon} |\log(\sin\theta)|^2 \right\} , \tag{4.36}$$

where θ is the angle formed by $\tilde{\gamma}_i$ and $\tilde{\alpha}$, and

$$\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) = \cosh d(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) .$$

Proof. By a simple application of hyperbolic trigonometry, we have (see Figure 4.5):

$$\begin{aligned} \sinh d(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) &\leq \sinh d(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j) \cap \tilde{\alpha}) \\ &= (\sin\theta) \sinh d(\tilde{\gamma}_i \cap \tilde{\alpha}, A^k(\tilde{\gamma}_j) \cap \tilde{\alpha}) \\ &\leq (\sin\theta) \sinh((k+1)\epsilon) . \end{aligned}$$

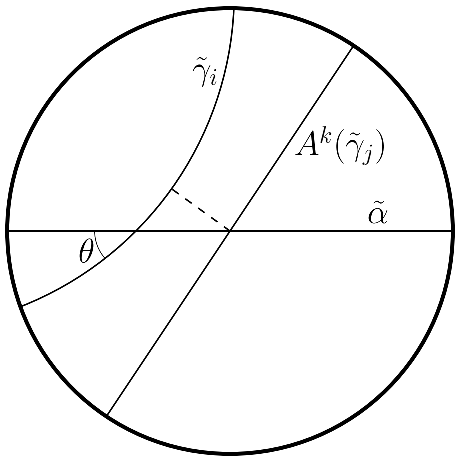

Figure 4.5: The inequality $\sinh d(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j)) \leq (\sin\theta)\sinh((k+1)\epsilon)$.

Let us denote

$$F(x) := \cosh(x)\log\left(\frac{\cosh(x)+1}{\cosh(x)-1}\right) - 2 ,$$

which is a positive, monotone decreasing function $F : (0, +\infty) \to (0, +\infty)$. Hence we have

$$\sum_{k=1}^{+\infty}\left(\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j))\log\left(\frac{\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j))+1}{\mathfrak{u}(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j))-1}\right) - 2\right) = \sum_{k=1}^{+\infty} F(d(\tilde{\gamma}_i, A^k(\tilde{\gamma}_j))) \geq \sum_{k=2}^{+\infty} F(\phi_\theta(k\epsilon)) ,$$

where

$$\phi_\theta(y) := \operatorname{arcsinh}(\sin\theta\sinh(y)) .$$

To show that the sum in Equation (4.36) is larger than K_1/ϵ, we observe that $\phi_\theta(y) \leq y$ and write:

$$\sum_{k=2}^{+\infty} F(\phi_\theta(k\epsilon)) \geq \int_2^{+\infty} F(\phi_\theta(x\epsilon))dx = \frac{1}{\epsilon}\int_{2\epsilon}^{+\infty} F(\phi_\theta(y))dy \geq \frac{1}{\epsilon}\int_{2\epsilon_0}^{+\infty} F(y)dy .$$

This concludes the claim, by declaring

$$K_1 = \int_{2\epsilon_0}^{+\infty} F(y)dy > 0 .$$

In light of the inequality we have just proved, to conclude the proof it suffices to show that there exists $\theta_0 > 0$ such that the sum in Equation (4.36) is larger than $(K_2/\epsilon)|\log(\sin\theta)|$, for all $\theta \leq \theta_0$ for some constant $K_2 > 0$.
For this purpose, let us start again from

$$\sum_{k=2}^{+\infty} F(\phi_\theta(k\epsilon)) \geq \int_2^{+\infty} F(\phi_\theta(x\epsilon))dx \geq \frac{1}{\epsilon}\int_{2\epsilon_0}^{+\infty} F(\phi_\theta(y))dy ,$$

and observe that, by a direct analysis, there exists a constant $C > 0$ such that

$$F(x) \geq C|\log(\sinh x)|$$

for $x \in (0, \operatorname{arcsinh}(1))$. Since $\phi_\theta(y) \in (0, \operatorname{arcsinh}(1))$ for $y \in (0, \operatorname{arcsinh}(1/\sin\theta))$, we can continue the inequality by:

$$\begin{aligned}
\sum_{k=2}^{+\infty} F(\phi_\theta(k\epsilon)) &\geq \frac{C}{\epsilon} \int_{2\epsilon_0}^{\operatorname{arcsinh}(\frac{1}{\sin\theta})} |\log(\sin\theta \sinh y)| dy \\
&\geq \frac{C}{\epsilon} \left(\int_{2\epsilon_0}^{\operatorname{arcsinh}(\frac{1}{\sin\theta})} |\log(\sin\theta)| dy - \int_{2\epsilon_0}^{\operatorname{arcsinh}(\frac{1}{\sin\theta})} \log(\sinh y) dy \right) \\
&\geq \frac{C}{\epsilon} \left(\int_{2\epsilon_0}^{|\log(\sin\theta)|} |\log(\sin\theta)| dy - \int_{1}^{\operatorname{arcsinh}(\frac{1}{\sin\theta})} y dy - C' \right) ,
\end{aligned}$$

where we have used that $\log(x) \leq \operatorname{arcsinh}(x)$, that $\log(\sinh y) \leq y$, and we put

$$C' := \int_{2\epsilon_0}^{1} |\log(\sinh y)| dy$$

Now, if we fix some small $\delta > 0$, we have

$$\int_{2\epsilon_0}^{|\log(\sin\theta)|} |\log(\sin\theta)| dy = (|\log(\sin\theta)| - 2\epsilon_0)|\log(\sin\theta)| \geq (1-\delta)|\log(\sin\theta)|^2$$

if θ is smaller than some $\theta_0 = \theta_0(\epsilon_0)$. On the other hand, since

$$\lim_{x \to +\infty} \frac{\log(x)}{\operatorname{arcsinh}(x)} = 1 ,$$

one has $|\log(\sin\theta)| \geq (1-\delta)\operatorname{arcsinh}(1/\sin\theta)$, for $\theta \leq \theta_0$ (up to replacing again θ_0) and therefore

$$\int_{1}^{\operatorname{arcsinh}(\frac{1}{\sin\theta})} y dy \leq \int_{0}^{\frac{|\log(\sin\theta)|}{1-\delta}} y dy = \frac{1}{2(1-\delta)^2} |\log(\sin\theta)|^2 .$$

In conclusion, we have

$$\sum_{k=2}^{+\infty} F(\phi_\theta(k\epsilon)) \geq \frac{C}{\epsilon} \left(\left((1-\delta) - \frac{1}{2(1-\delta)^2} \right) |\log(\sin\theta)|^2 - C' \right) \geq \frac{K_2}{\epsilon} |\log(\sin\theta)|^2 ,$$

for some constant K_2, provided $\theta \leq \theta_0$ and $\epsilon \leq \epsilon_0$. This concludes the proof. □

We are now ready to conclude the proof of the estimate of the Weil-Petersson gradient of the length function, in the case in which most of the length of the geodesic γ lies in the thin part of (S, h):

Proposition 4.7.7. *There exists a constant a, depending only on the choice of a sufficiently small ϵ_0 inducing a thin-thick decomposition of S, such that for every hyperbolic metric h on S and every simple closed curve c, if the h-geodesic representative γ satisfies:*

$$\mathrm{length}_h(\gamma \cap S_h^{\mathrm{thick}}) \leq \mathrm{length}_h(\gamma \cap S_h^{\mathrm{thin}}) \ ,$$

then

$$\|\mathrm{grad}\ell_c(h)\|_{\mathrm{WP}} \geq \frac{a}{|\chi(S)|}\ell_c(h) \ .$$

Proof. Choosing ϵ_0 small enough, we have assured that there are at most $3g-3$ simple closed geodesics $\alpha_1, \ldots, \alpha_{3g-3}$ on (S,h) of length at most ϵ_0. Hence, the thin part of (S,h) is composed by at most $3g-3$ tubes $T_{\alpha_i, d(\epsilon_i)}$, where ϵ_i is the length of α_i and the tubes were defined in Equation (4.21). Let $\alpha = \alpha_{i_0}$ be one of such simple closed geodesics, of length ϵ, such that

$$\mathrm{length}_h(\gamma \cap T_{\alpha,d(\epsilon)}) \geq \frac{1}{3g-3}\mathrm{length}_h(\gamma \cap S_h^{\mathrm{thin}}) \geq \frac{1}{6g-6}\ell_c(h) \ .$$

We will denote $T = T_{\alpha,d(\epsilon)}$ for convenience. Observe that, for every connected component η of $\gamma \cap T$, such that the angle formed by η and α is θ, we have

$$\sinh\left(\frac{\mathrm{length}_h(\eta)}{2}\right) = \frac{\sinh d(\epsilon)}{\sin\theta} = \frac{1}{\sin\theta\sinh(\frac{\epsilon}{2})} \ , \tag{4.37}$$

by using the definition of $d(\epsilon)$ from (4.22).
Let us choose the connected component η whose length is minimal — which corresponds to choosing the connected component whose angle θ of intersection with α is maximal. Then it is easy to see that all the other connected components have length less than $\mathrm{length}_h(\eta) + \epsilon$, since they lift to geodesic segments in $\mathbb{H}^2$ connecting two points in the two boundary components of $\widetilde{T}$. See Figure 4.6.
Hence we have

$$\mathrm{length}_h(\gamma \cap T_{\alpha,d(\epsilon)}) \leq \iota(\alpha, c)(\mathrm{length}_h(\eta) + \epsilon) \ . \tag{4.38}$$

On the other hand, from Equation (4.37), we have

$$\frac{\mathrm{length}_h(\eta)}{2} = \mathrm{arcsinh}\left(\frac{1}{\sin\theta\sinh(\frac{\epsilon}{2})}\right)$$

and therefore

$$\mathrm{length}_h(\eta) + \epsilon \leq C\left|\log\left(\sin\theta\sinh\left(\frac{\epsilon}{2}\right)\right)\right| + \epsilon_0 \leq C'(|\log\epsilon| + |\log(\sin\theta)|) \tag{4.39}$$

for some suitable constants C, C', if ϵ is at most some small constant ϵ_0.

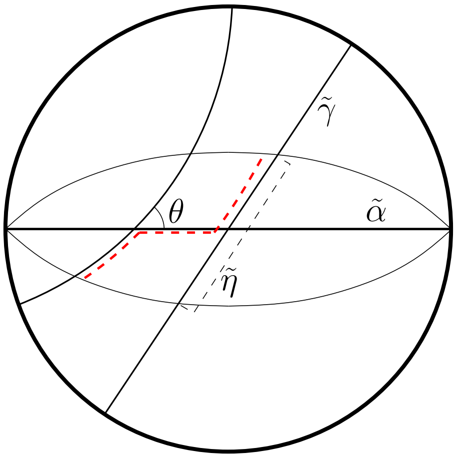

Figure 4.6: In the universal cover, the tube T is lifted to the set of points at bounded distance from $\tilde{\alpha}$. Using the fact that all components of $\gamma \cap T$ are disjoint, one sees that the length of every component of $\gamma \cap T$ is at most $\mathrm{length}_h(\eta) + \epsilon$, where η is the shortest component.

Now, using Equation (4.35) and Lemma 4.7.6, we obtain

$$\begin{aligned}\|\mathrm{grad}\ell_c(h)\|^2_{\mathrm{WP}} &\geq \frac{K}{\epsilon}\max\left\{1, |\log(\sin\theta)|^2\right\}\iota(\alpha,c)^2 \\ &\geq \frac{K}{2\epsilon}(1+|\log(\sin\theta)|^2)\iota(\alpha,c)^2 \\ &\geq K'\iota(\alpha,c)^2(|\log\epsilon|^2+|\log(\sin\theta)|^2) \\ &\geq \frac{K'}{2}\iota(\alpha,c)^2\left(|\log\epsilon|+|\log(\sin\theta)|\right)^2 \ ,\end{aligned}$$

Therefore, comparing with (4.38) and (4.39), we have obtained

$$\|\mathrm{grad}\ell_c(h)\|^2_{\mathrm{WP}} \geq K''(\mathrm{length}_h(\gamma \cap T_{\alpha,d(\epsilon)}))^2 \geq \left(\frac{K''}{6g-6}\right)^2 \ell_c(h)^2 \ ,$$

which concludes the proof. □

4.7.4 Conclusion of the proof and an application

The proof of Theorem 4.6.3 is now straightforward:

Proof of Theorem 4.6.3. By Propositions 4.7.4 and 4.7.7, we have (for a constant a which replaces the constants involved there)

$$\|\mathrm{grad}\ell_c(h)\|_{\mathrm{WP}} \geq \frac{a}{|\chi(S)|}\max\{\mathrm{length}_h(\gamma \cap S_h^{\mathrm{thick}}), \mathrm{length}_h(\gamma \cap S_h^{\mathrm{thin}})\}$$

and therefore

$$\|\mathrm{grad}\ell_c(h)\|_{\mathrm{WP}} \geq \frac{a}{2|\chi(S)|}\ell_c(h) \ ,$$

as claimed. □

We conclude by observing that, using Theorem 4.6.3, one can give another proof of Theorem 4.5.2. For this purpose, first observe that, by the density of simple closed curves in the space of measured geodesic laminations, Thurston's asymmetric distance $d_{\mathrm{Th}}(h,h') = \inf_f \log L(f)$ can also be computed by the following characterization of $L(f)$ (compare with Equation (4.13)):

$$L(f) = \sup_{\mu \in \mathcal{ML}(S)} \frac{\ell_\mu(h')}{\ell_\mu(h)} \ . \tag{4.40}$$

Now, given two metrics h and $h' = E_l^\mu(h)$, for some measured geodesic lamination λ, by convexity of the length function along earthquake paths, we have:

$$\ell_\mu(h') \leq \ell_\mu(h) + \left.\frac{d}{dt}\right|_{t=0} \ell_\mu(E_l^{t\lambda}(h)) = \ell_\mu(h) + \langle \mathrm{grad}\ell_\mu, \dot{E}_l^\lambda(h)\rangle_{\mathrm{WP}} \ , \tag{4.41}$$

where $\dot{E}_l^\lambda$ defines a vector field on Teich(S). Since it is known by a result of Wolpert ([Wol83]) that the symplectic gradient of the length function ℓ_λ is the infinitesimal earthquake along λ, that is:

$$\left.\frac{d}{dt}\right|_{t=0} \ell_\lambda(r(t)) = \omega_{\mathrm{WP}}(\dot{E}_l^\lambda(h), \dot{r}(t)) = \langle J\dot{E}_l^\lambda(h), \dot{r}(t)\rangle_{\mathrm{WP}} \ ,$$

where J is the almost-complex structure of Teich(S), from Equation (4.41) we get:

$$\ell_\mu(h') \geq \ell_\mu(h) + \langle \mathrm{grad}\ell_\mu(h), J\mathrm{grad}\ell_\lambda(h)\rangle_{\mathrm{WP}} \ .$$

In particular, if we choose μ as the measured geodesic lamination such that $\mathrm{grad}\ell_\mu(h) = J\mathrm{grad}\ell_\lambda(h)$,

$$\frac{\ell_\mu(h')}{\ell_\mu(h)} \geq 1 + \frac{||\mathrm{grad}\ell_\lambda(h)||_{\mathrm{WP}}||\mathrm{grad}\ell_\mu(h)||_{\mathrm{WP}}}{\ell_\mu(h)} \geq 1 + \frac{a^2}{|\chi(S)|^2}\ell_\lambda(h)$$

by Theorem 4.6.3. Using Theorem 4.2.7 and Equation (4.40), this concludes the alternative proof of the following:

Theorem 4.7.8. *Let $M_{h,h'}$ be a GHMC AdS_3 manifold. Then*

$$\mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{\pi^2}{2}|\chi(S)| + \frac{|\chi(S)|^2}{4a^2}\left(e^{d_{\mathrm{Th}}(h,h')} - 1\right) \ .$$

Chapter 5

Entropy degeneration of GHMC anti-de Sitter 3-manifolds

Using the parameterisation of the deformation space of GHMC anti-de Sitter structures on $S \times \mathbb{R}$ by the cotangent bundle of the Teichmüller space of S, we study how some geometric quantities, such as the Lorentzian Hausdorff dimension of the limit set, the width of the convex core and the Hölder exponent, degenerate along rays of quadratic differentials.

5.1 A parameterisation using maximal surfaces

In this chapter we use another parameterisation of the deformation space of GHMC anti-de Sitter structures on $S \times \mathbb{R}$, introduced by Krasnov and Schlenker ([KS07]). We recall here the main steps of their construction.

Let M be a GHMC anti-de Sitter 3-manifold. It is well-known ([BBZ07]) that M contains a unique embedded maximal surface Σ, i.e. with vanishing mean curvature. By the Fundamental Theorem of surfaces embedded in anti-de Sitter space, Σ is uniquely determined by its induced metric I and its shape operator $B : T\Sigma \to T\Sigma$, which are related to each other by the Gauss-Codazzi equations:

$$\begin{aligned} &d^{\nabla_I} B = 0 \\ &K_I = -1 - \det(B) \ , \end{aligned}$$

where we have denoted with K_I the curvature of the metric I. The first equation implies that the second fundamental form $II = I(B\cdot, \cdot)$ is the real part of a quadratic differential q, which is holomorphic for the complex structure compatible with the metric, in the following sense. For every couple of vector fields X and Y on Σ, we have

$$\mathcal{R}\mathrm{e}(q)(X,Y) = I(BX,Y) \ .$$

In a local conformal coordinate z, we can write $q = f(z)dz^2$, for some holomorphic function f, and $I = e^{2u}|dz|^2$. Thus, $\mathcal{R}\mathrm{e}(q)$ is the bilinear form that in the frame $\{\partial_x, \partial_y\}$ is represented by

$$\mathcal{R}\mathrm{e}(q) = \begin{pmatrix} \mathcal{R}\mathrm{e}(f) & -\mathcal{I}\mathrm{m}(f) \\ -\mathcal{I}\mathrm{m}(f) & -\mathcal{R}\mathrm{e}(f) \end{pmatrix} \ ,$$

and the shape operator B can be recovered as $B = I^{-1}\mathcal{R}\mathrm{e}(q)$. Therefore, we can define a map

$$\begin{aligned} \Psi : \mathcal{GH}(S) &\to T^*\mathrm{Teich}(S) \\ M &\mapsto (h,q) \end{aligned}$$

associating to a GHMC anti-de Sitter structure the unique hyperbolic metric in the conformal class of I and the quadratic differential q, constructed from the embedding data of the maximal surface Σ embedded in M.

In order to prove that Ψ is a homeomorphism, Krasnov and Schlenker found an explicit inverse. They showed that, given a hyperbolic metric h and a quadratic differential q that is holomorphic for the complex structure compatible with h, it is always possible to find a smooth map $u : S \to \mathbb{R}$ such that $I = e^{2u}h$ and $B = I^{-1}\mathcal{R}\mathrm{e}(q)$ are the induced metric and the shape operator of a maximal surface embedded in a GHMC anti-de Sitter manifold. This is accomplished by noticing that the Codazzi equation for B is trivially satisfied since q is holomorphic, and thus it is sufficient to find u so that the Gauss equation holds. Now,

$$\det(B) = \det(e^{-2u}h^{-1}\mathcal{R}\mathrm{e}(q)) = e^{-4u}\det(h^{-1}\mathcal{R}\mathrm{e}(q)) = -e^{-4u}\|q\|_h^2$$

and

$$K_I = e^{-2u}(K_h - \Delta_h u),$$

hence the Gauss equation translates into the quasi-linear PDE

$$\Delta_h u = e^{2u} - e^{-2u}\|q\|_h^2 + K_h \ . \tag{5.1}$$

Proposition 5.1.1 (Lemma 3.6 [KS07]). *There exists a unique smooth solution $u : S \to \mathbb{R}$ to Equation (5.1).*

In Section 5.3, we will give precise estimates for the solution u in terms of the quadratic differential q, and study its asymptotic along a ray $q = tq_0$ for a fixed non-trivial holomorphic quadratic differential q_0.

5.1.1 Relation with Mess' parameterisation

The theory of harmonic maps between hyperbolic surfaces provides a bridge between the two parameterisations of $\mathcal{GH}(S)$.

We recall that a diffeomorphism $m : (S, h_l) \to (S, h_r)$ is minimal Lagrangian if it is area-preserving and its graph is a minimal surface in $(S \times S, h \oplus h')$. These can also be characterised by the fact that can be factorised as $m = f' \circ f^{-1}$, where

$$f : (S, h) \to (S, h_l) \quad \text{and} \quad f' : (S, h) \to (S, h_r)$$

are harmonic with opposite Hopf differentials. We call h the center of the minimal Lagrangian map.

Proposition 5.1.2 ([BS10]). *Let h_r and h_l be hyperbolic metrics on S with holonomy ρ_r and ρ_l. The center of the minimal Lagrangian map $m : (S, h_l) \to (S, h_r)$ is the conformal class of the induced metric on the maximal surface Σ contained in the GHMC anti-de Sitter manifold M with holonomy $\rho = (\rho_l, \rho_r)$. Moreover, the second fundamental form of Σ is (up to a constant multiple) the real part of the Hopf differential of the harmonic map factorising m.*

5.2 Hölder exponent

In this section we introduce the Hölder exponent of a GHMC anti-de Sitter manifold and study its asymptotic behaviour along a ray of quadratic differentials.

Let M be a GHMC anti-de Sitter manifold. Its holonomy representation $\rho : \pi_1 \to \mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ gives rise, by projecting into each factor, to two discrete and faithful representations ρ_l and ρ_r. Let $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be the unique homeomorphism such that

$$\rho_r(\gamma) \circ \phi = \phi \circ \rho_l(\gamma) \qquad \text{for every } \gamma \in \pi_1(S) \ .$$

It is well-known ([Thu98]) that ϕ is quasi-symmetric, and, in particular, has Hölder regularity.

Definition 5.2.1. *The Hölder exponent $\alpha(M)$ of M is the minimum between the Hölder exponents of ϕ and ϕ^{-1}.*

Remark 5.2.2. *This definition takes into account that ϕ and ϕ^{-1} have in general different Hölder exponents. On the other hand, the manifolds with holonomies (ρ_l, ρ_r) and (ρ_r, ρ_l) are isometric, because the map*

$$\begin{aligned} \mathbb{P}SL(2,\mathbb{R}) &\to \mathbb{P}SL(2,\mathbb{R}) \\ A &\mapsto A^{-1} \end{aligned}$$

induces an orientation-reversing isometry of AdS_3 which swaps the left and right holonomies in Mess' parameterisation. Hence, we expect a geometric interesting quantity to be invariant under this transformation.

An explicit formula for the Hölder exponent of ϕ is well-known:

Theorem 5.2.3 (Chapter 7 Proposition 14 [GH90], Theorem 6.5 [BS11])**.** *Let ρ_r and ρ_l be Fuchsian representations. The Hölder exponent of the unique homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that*

$$\rho_r(\gamma) \circ \phi = \phi \circ \rho_l(\gamma) \qquad \text{for every} \;\; \gamma \in \pi_1(S)$$

is

$$\alpha(\phi) = \inf_{\gamma \in \pi_1(S)} \frac{\ell_r(\gamma)}{\ell_l(\gamma)}$$

where $\ell_r(\gamma)$ and $\ell_l(\gamma)$ denote the lengths of the geodesic representatives of γ with respect to the hyperbolic metrics with holonomy ρ_r and ρ_l, respectively.

Therefore, the Hölder exponent of a GHMC anti-de Sitter manifold with holonomy $\rho = (\rho_l, \rho_r)$ is given by

$$\alpha(M) = \inf_{\gamma \in \pi_1(S)} \min \left\{ \frac{\ell_r(\gamma)}{\ell_l(\gamma)}, \frac{\ell_l(\gamma)}{\ell_r(\gamma)} \right\} . \tag{5.2}$$

Remark 5.2.4. *Since the formula for $\alpha(M)$ is homogeneous and weighted simple closed curves are dense in the space of measured foliations, the above formula is equivalent to*

$$\alpha(M) = \inf_{\mu \in \mathcal{ML}(S)} \min \left\{ \frac{\ell_r(\mu)}{\ell_l(\mu)}, \frac{\ell_l(\mu)}{\ell_r(\mu)} \right\} .$$

We easily deduce a rigity property of the Hölder exponent:

Proposition 5.2.5. *The Hölder exponent of a GHMC anti-de Sitter manifold is equal to* 1 *if and only if M is Fuchsian*

Proof. If M is Fuchsian $\ell_r(\gamma) = \ell_l(\gamma)$ for every $\gamma \in \pi_1(S)$, hence the Hölder exponent is equal to 1. On the other hand, if M is not Fuchsian, by a result of Thurston ([Thu98]), there exists a curve $\gamma \in \pi_1(S)$ such that $\ell_l(\gamma) > \ell_r(\gamma)$, hence $\alpha(M) < 1$. □

Before studying the asymptotics of the Hölder exponent along rays of quadratic differentials, we want to give a new interpretation of the Hölder exponent that is more related to anti-de Sitter geometry.
Let $\rho = (\rho_r, \rho_l)$ be the holonomy representation of a GHMC anti-de Sitter structure. Let us suppose first that $\rho_l \neq \rho_r$. Since ρ_l and ρ_r are the holonomies of hyperbolic structures on S, for every $\gamma \in \pi_1(S)$, the elements $\rho_l(\gamma)$ and $\rho_r(\gamma)$ are hyperbolic

isometries of the hyperbolic plane. Therefore, there exist $A, B \in \mathbb{P}SL(2,\mathbb{R})$ such that

$$A\rho_l(\gamma)A^{-1} = \begin{pmatrix} e^{\ell_l(\gamma)/2} & 0 \\ 0 & e^{-\ell_l(\gamma)/2} \end{pmatrix} \qquad B\rho_r(\gamma)B^{-1} = \begin{pmatrix} e^{\ell_r(\gamma)/2} & 0 \\ 0 & e^{-\ell_r(\gamma)/2} \end{pmatrix} .$$

We thus notice that the isometry of AdS_3 given by $\rho(\gamma) = (\rho_l(\gamma), \rho_r(\gamma))$ leaves two space-like geodesics invariant

$$\sigma^*(t) = A \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} B^{-1} \qquad \text{and} \qquad \sigma(t) = A \begin{pmatrix} 0 & e^t \\ e^{-t} & 0 \end{pmatrix} B^{-1} .$$

An easy computation shows that the isometry $\rho(\gamma)$ acts on σ^* by translation with translation length

$$\beta^*(\gamma) = \frac{|\ell_l(\gamma) - \ell_r(\gamma)|}{2}$$

and acts by translation on σ with translation length

$$\beta(\gamma) = \frac{\ell_l(\gamma) + \ell_r(\gamma)}{2} .$$

We claim that only the geodesic σ is contained in the convex hull of the limit set Λ_ρ. Recall that the limit set can be constructed as the graph of the homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that

$$\rho_r(\gamma) \circ \phi = \phi \circ \rho_l(\gamma) \qquad \text{for every} \quad \gamma \in \pi_1(S) .$$

In particular, ϕ sends the attactive (resp. repulsive) fixed point of $\rho_l(\gamma)$ into the attractive (resp. repulsive) fixed point of $\rho_r(\gamma)$. Therefore, we must have

$$\phi(A[1:0]) = B[1:0] \quad \text{and} \quad \phi(A[0:1]) = B[0:1] .$$

Now, the geodesic σ_2 has ending points

$$\sigma(-\infty) = (A[0:1], B[0:1]) \in \mathbb{RP}^1 \times \mathbb{RP}^1$$

and

$$\sigma(+\infty) = (A[1:0], B[1:0]) \in \mathbb{RP}^1 \times \mathbb{RP}^1 ,$$

whereas the geodesic σ_1 has ending points

$$\sigma^*(-\infty) = (A[0:1], B[1:0]) \in \mathbb{RP}^1 \times \mathbb{RP}^1$$

and

$$\sigma^*(+\infty) = (A[1:0], B[0:1]) \in \mathbb{RP}^1 \times \mathbb{RP}^1$$

hence only the ending points on σ lie on the limit curve Λ_ρ. As a consequnce, σ is contained in the convex hull of Λ_ρ and its projection is a closed space-like geodesic in the convex core of M. On the other hand, the geodesic σ^* does not even belong to the domain of dependence of Λ_ρ. In fact, it it easy to check that the dual space-like plane of any point of σ^* contains the geodesic σ, thus its boundary at infinity is not disjoint from the limit curve Λ_ρ.

In the special case, when $\rho_r = \rho_l$, the point $[Id] \in AdS_3$ is fixed and its dual space-like plane P_0 is left invariant. By definition of the dual plane (see Section 1.2),

$$P_0 = \{A \in \mathbb{P}SL(2,\mathbb{R}) \mid \mathrm{trace}(A) = 0\}$$

is the dual of $[Id] \in AdS_3$ and it is easy to check that it is a copy of the hyperbolic plane. With this identification, $\rho(\gamma)$ acts on P_0 as the hyperbolic isometry $\rho_r(\gamma) = \rho_l(\gamma)$ does on $\mathbb{H}^2$.

We thus obtain another way of computing the Hölder exponent of a GHMC anti-de Sitter manifold:

Proposition 5.2.6. *Let M be a GHMC anti-de Sitter manifold with holonomy ρ. Let $\beta(\gamma)$ and $\beta^*(\gamma)$ be the translation lengths of the isometries $\rho(\gamma)$ for every $\gamma \in \pi_1(S)$. Then*

$$\alpha(M) = \inf_{\gamma \in \pi_1(S)} \frac{\beta(\gamma) - \beta^*(\gamma)}{\beta(\gamma) + \beta^*(\gamma)} \ .$$

Proof. This is a direct consequence of the explicit formulas for $\beta(\gamma)$ and $\beta^*(\gamma)$ and Theorem 5.2.3. □

We can now describe the asymptotic behaviour of the Hölder exponent:

Theorem 5.2.7. *Let M_t be the family of GHMC anti-de Sitter manifolds parameterised by the ray $(h, tq_0) \in T^*\mathrm{Teich}(S)$ for a non-zero quadratic differential q_0. Then*

$$\lim_{t \to +\infty} \alpha(M_t) = 0 \ .$$

Proof. Let $\rho_t = (\rho_{l,t}, \rho_{r,t})$ be the holonomy representation of M_t. Let $h_{l,t}$ and $h_{r,t}$ be the hyperbolic metrics on S with holonomy $\rho_{l,t}$ and $\rho_{r,t}$, respectively. By Proposition 5.1.2, we can suppose that the identity maps

$$id : (S, h) \to (S, h_{l,t}) \qquad id : (S, h) \to (S, h_{r,t})$$

are harmonic with Hopf differentials tiq_0 and $-tiq_0$, respectively.
Associated to tiq_0 are two measured foliations λ_t^+ and λ_t^-: in a natural conformal coordinate $z = x + iy$ outside the zeros of q_0, we can express $tiq_0 = dz^2$. The foliations are then given by

$$\lambda_t^+ = (y = const, z^*|dy|) \quad \text{and} \quad \lambda_t^- = (x = const, z^*|dx|)$$

Notice, in particular, that the support of the foliation is fixed for every $t > 0$ and only the measure changes, being it multiplied by $t^{1/2}$. We can thus write

$$\lambda_t^+ = t^{1/2}\lambda_0^+ \quad \text{and} \quad \lambda_t^- = t^{1/2}\lambda_0^-$$

where $\lambda_0^\pm$ are the measured foliations associated to iq_0. Moreover, multiplying a quadratic differential by -1 interchanges the two foliations.
By Wolf's compactification of Teichmüller space (Section 4.2 [Wol89]), we know that

$$\lim_{t\to+\infty} \frac{\ell_{l,t}(\gamma)}{2t^{1/2}} = \iota(\lambda_0^+, \gamma)$$

for every $\gamma \in \pi_1(S)$. By density, the same holds for every measured foliation on S. Therefore, using Remark 5.2.4,

$$\begin{aligned} 0 \le \lim_{t\to+\infty} \alpha(M_t) &= \lim_{t\to+\infty} \inf_{\mu\in\mathcal{ML}(S)} \min\left\{\frac{\ell_{l,t}(\mu)}{\ell_{r,t}(\mu)}, \frac{\ell_{r,t}(\mu)}{\ell_{l,t}(\mu)}\right\} \\ &\le \lim_{t\to+\infty} \frac{\ell_{l,t}(\lambda_0^+)}{\ell_{r,t}(\lambda_0^+)} = \lim_{t\to+\infty} \frac{\ell_{l,t}(\lambda_0^+)}{2t^{1/2}} \frac{2t^{1/2}}{\ell_{r,t}(\lambda_0^+)} \\ &= \frac{\iota(\lambda_0^+, \lambda_0^+)}{\iota(\lambda_0^-, \lambda_0^+)} = 0 \end{aligned}$$

because every measured lamination has vanishing self-intersection and $\iota(\lambda_0^-, \lambda_0^+) \neq 0$ by construction. □

5.3 Entropy

In this section we study the asymptotic behaviour of the Lorentzian Hausdorff dimension of the limit curve Λ_ρ associated to a GHMC anti-de Sitter manifold.

5.3.1 Lorentzian Hausdorff dimension

Let M be a GHMC anti-de Sitter manifold with holonomy representation ρ. In Section 1.5, we saw that the limit set of the action of $\rho(\pi_1(S))$ is a simple closed curve Λ_ρ in the boundary at infinity of AdS_3. Moreover, Λ_ρ is the graph of a locally Lipschitz function, thus its Hausdorff dimension is always 1. Recently, Glorieux and Monclair defined a notion of Lorenztian Hausdorff dimension, that manages to describe how far the representiation ρ is from being Fuchsian. This resembles the usual definition of Hausdorff dimension, where instead of considering coverings consisting of Euclidean balls, they used Lorentzian ones ([GM16, Section 5.1]). They also gave an equivalent definition in terms of entropy of a quasi-distance in AdS_3.

Definition 5.3.1. *Let $\Lambda_\rho \subset \partial_\infty AdS_3$ be the limit set of the holonomy of a GHMC anti-de Sitter structure. The quasi-distance*

$$d_{AdS} : \mathcal{C}(\Lambda_\rho) \times \mathcal{C}(\Lambda_\rho) \to \mathbb{R}_{\geq 0}$$

is defined as follows. Let $x, y \in \mathcal{C}(\Lambda_\rho)$ and let $\gamma_{x,y}$ be the unique geodesic connecting x and y. We put

$$d_{AdS}(x,y) := \begin{cases} \text{length}(\gamma_{x,y}) & \text{if } \gamma_{x,y} \text{ is space-like} \\ 0 & \text{otherwise} \end{cases}$$

The function d_{AdS} is a quasi-distance in the following sense: it is symmetric, and there exists a constant k_ρ depending on the representation ρ such that

$$d_{AdS}(x,z) \leq d_{AdS}(x,y) + d_{AdS}(y,z) + k_\rho$$

for every $x, y, z \in \mathcal{C}(\Lambda_\rho)$ ([GM16, Theorem 3.4]).

Definition 5.3.2. *The entropy of the quasi-distance d_{AdS} is*

$$E(d_{AdS}) = \limsup_{R \to +\infty} \frac{1}{R} \log(\#\{\gamma \in \pi_1(S) \mid d_{AdS}(\rho(\gamma)x_0, x_0) \leq R\}) \ ,$$

where $x_0 \in \mathcal{C}(\Lambda_\rho)$ is a fixed base point.

The link between the entropy of the pseudo-distance d_{AdS} and the Lorentzian Hausdorff dimension is provided by the following result:

Theorem 5.3.3 (Thorem 1.1 [GM16])**.** *Let Λ_ρ be the limit set of the holonomy representation ρ of a GHMC anti-de Sitter structure. Then*

$$\mathcal{L}\mathcal{H}\text{dim}(\Lambda_\rho) = E(d_{AdS}) \ .$$

In particular, $E(d_{AdS})$ does not depend on the choice of the based point x_0.

5.3.2 Entropy of the maximal surface

Another natural quantity that can be associated to a GHMC anti-de Sitter structure is the volume entropy of the Riemannian metric induced on the unique maximal surface. We will use this in the next subsection to provide an upper-bound for the Lorentzian Hausdorff dimension of the limit set.

Let g be a Riemannian metric or a flat metric with conical singularity on the surface S. Let $\tilde{S}$ be the universal cover of S. The volume entropy of g can be defined as

$$E(g) = \limsup_{R \to +\infty} \frac{1}{R} \log(\#\{\gamma \in \pi_1(S) \mid d_g(\gamma \cdot x_0, x_0) \leq R\}) \in \mathbb{R}^+$$

where $x_0 \in \tilde{S}$ is an arbitrary base point.

We introduce the function $E : T^*\mathrm{Teich}(S) \to \mathbb{R}$ that associates to a point $(h, q) \in T^*\mathrm{Teich}(S)$ the volume entropy of the Riemannian metric $I = e^{2u}h$, where u is the solution to Equation (5.1). Namely, $E(h, q)$ is the volume entropy of the Riemannian metric induced on the unique maximal surface embedded in the GHMC anti-de Sitter manifold corresponding to (h, q). By identifying $T^*\mathrm{Teich}(S)$ with $\mathcal{GH}(S)$ (see Section 5.1), we will often denote this map as $E(\rho)$, where ρ is the holonomy representation of the corresponding GHMC anti-de Sitter structure.

Notice that, since in Equation (5.1) only the h-norm of the quadratic differential q appears, the function E is invariant under the natural S^1 action on $T^*\mathrm{Teich}(S)$ given by $(h, q) \mapsto (h, e^{i\theta}q)$. In particular, a complete understanding of this function is obtained by studing its behaviour along rays (h, tq_0) for a fixed unitary quadratic differential q_0 for $t \geq 0$.

5.3.3 Estimates for the induced metric on the maximal surface

In this section we study the asymptotic behaviour of the induced metric I_t on the maximal surface Σ_t along a ray tq_0 of quadratic differentials. We deduce also estimates for the principal curvatures of Σ_t.

Let us start finding a lower bound for I_t.

Proposition 5.3.4. *Let u_t be the solution to Equation (5.1) for $q = tq_0$. Then*

$$u_t > \frac{1}{2}\log(t\|q_0\|_h) \ .$$

In particular, $I_t > t|q_0|$.

Proof. The main idea of the proof lies on the fact that $\frac{1}{4}\log(\|tq_0\|_h^2)$ is a solution to Equation (5.1), outside the zeros of q_0. To be precise, let $s_t : S \to \mathbb{R}$ be the function such that

$$e^{2s_t}h = t|q_0|$$

at every point. Then, outside the zeros of q_0, we have

$$\begin{aligned}\Delta_h s_t &= \frac{1}{4}h^{-1}\bar{\partial}\partial\log(\|tq_0\|_h^2) = \frac{1}{4}h^{-1}\bar{\partial}\partial[\log(t^2 q_0\bar{q}_0) - \log(h^2)] \\ &= -\frac{1}{2}\Delta_h \log(h) = K_h\end{aligned}$$

and

$$e^{2s_t} - t^2e^{-2s_t}\|q_0\|_h^2 = t\|q_0\|_h - t\|q_0\|_h = 0 \ ,$$

hence s_t is a solution of Equation (5.1) outside the zeros of q_0. We observe, moreover, that at the zeros of q_0, s_t tends to $-\infty$. Therefore, by the comparison principle $u_t \geq s_t$.
Now, the strong maximum principle ([Jos07, Thereom 2.3.1]) implies that on any domain where s_t is continuous up to the boundary, we have either $u_t > s_t$ or $u_t \equiv s_t$. Thus if $u_t(p) = s_t(p)$ for some $p \in S$ (and clearly p cannot be a zero for q_0 in this case), then u_t and s_t must agree in the complement of the zeros of q_0, but this is not possible, since s_t diverges to $-\infty$ near the zeros, whereas u_t is smooth everywhere on S.
In particular, we deduce that $I_t = e^{2u_t}h > e^{2s_t}h = t|q_0|$. □

Corollary 5.3.5. *Let λ_t be the positive principal curvature of the maximal surface Σ_t, then $\lambda_t < 1$.*

Proof. Recall that the shape operator of Σ_t can be written as

$$B_t = I_t^{-1}II_t = e^{-2u_t}h^{-1}\Re\mathrm{e}(tq_0) \ .$$

Therefore, $\lambda_t^2 = -\det(B_t) = e^{-4u_t}t^2\|q_0\|_h^2 < 1$, by the previous proposition. □

In order to find an upper bound for I_t, we introduce a new metric on the surface S. Let U be a neighbourhood of the zeros of q_0. We consider a smooth metric g on S in the conformal class of h such that $g = |q_0|$ in the complement of U and $\|q_0\|_g^2 \leq 1$ everywhere on S. This is possible because $\|q_0\|_g^2 = 1$ on $S \setminus U$ and it vanishes at the zeros of q_0. Let w_t be the logarithm of the density of I_t with respect to g, i.e $w_t : S \to \mathbb{R}$ satisfies

$$e^{2w_t}g = I_t \ .$$

The function w_t is the solution of Equation (5.1), where the background metric on S is now g. We can give an upper-bound to the induced metric I_t by estimating the function w_t.

Proposition 5.3.6. *Let K be the minimum of the curvature of g and let S_t be the positive root of the polynomial $r_t(x) = x^2 + Kx - t^2$. Then $e^{2u_t} \leq S_t$.*

Proof. By compactness of S, the function w_t has maximum at some point $p \in S$. By the maximum principle, we have

$$\begin{aligned}
0 \geq \Delta_g w_t(p) &= e^{2w_t(p)} - t^2 e^{-2w_t(p)}\|q_0(p)\|_g^2 + K_g(p) \\
&= e^{-2w_t(p)}(e^{4w_t(p)} + e^{2w_t(p)}K_g(p) - t^2\|q_0(p)\|_g^2) \\
&\geq e^{-2w_t(p)}(e^{4w_t(p)} + Ke^{2w_t(p)} - t^2) = e^{-2w_t(p)}r_t(e^{2w_t(p)})
\end{aligned}$$

The biggest possible value in which this inequality is true is for $e^{2w_t(p)} = S_t$. Since p is a point of maximum of w_t we deduce that $e^{2w_t} \leq S_t$ everywhere on S. □

Corollary 5.3.7. *Along a ray tq_0, the induced metric I_t on the maximal surface satisfies*

$$I_t = t|q_0|(1 + o(1)) \qquad \text{for } t \to +\infty$$

outside the zeros of q_0.

Proof. Combining Proposition 5.3.4 and Proposition 5.3.6 we have

$$t|q_0| \leq I_t \leq S_t g \ .$$

Now, we notice that $\frac{S_t}{t}$ is the biggest positive root of the polynomial $\tilde{r}_t(x) = x^2 + \frac{K}{t}x - 1$, hence

$$\frac{S_t}{t} \to 1 \qquad \text{when } t \to +\infty \ .$$

Moreover, outside the zeros of q_0, by definition $g = |q_0|$, thus

$$|q_0| \leq \frac{I_t}{t} \leq \frac{S_t}{t}|q_0| \xrightarrow{t \to +\infty} |q_0|$$

and the proof is complete. □

We can actually be more precise about the way the induced metrics $\frac{I_t}{t}$ converge to the flat metric $|q_0|$.

Proposition 5.3.8. *Outside the zeros of q_0,*

$$\frac{I_t}{t} \to |q_0| \quad \textit{when} \quad t \to +\infty$$

monotonically from above.

Proof. Recall that we can write $I_t = e^{2u_t}h$, where u_t is the solution of Equation (5.1) for $q = tq_0$. By Proposition 5.3.4, we know that

$$u_t > \frac{1}{2}\log(t\|q_0\|_h) \ .$$

It is thus sufficient to show that $\varphi_t = u_t - \frac{1}{2}\log(t\|q_0\|_h) > 0$ is monotone decreasing in t. Outside the zeros of q_0, the function φ_t satisfies the differential equation

$$\begin{aligned}\Delta_h \varphi_t &= \Delta_h u_t - \frac{1}{2}\Delta_h \log(t\|q_0\|_h) = e^{2u_t} - t^2\|q_0\|_h^2 e^{-2u_t} \\ &= t\|q_0\|_h(e^{2u_t}t^{-1}\|q_0\|_h^{-1} - t\|q_0\|_h e^{-2u_t}) \\ &= t\|q_0\|_h(e^{2\varphi_t} - e^{-2\varphi_t}) = 2t\|q_0\|_h \sinh(2\varphi_t) \ .\end{aligned}$$

Taking the derivative at $t = t_0$, we obtain

$$\Delta_h \dot{\varphi}_{t_0} = 2\|q_0\|_h \sinh(2\varphi_{t_0}) + 4t_0\|q_0\|_h \cosh(2\varphi_{t_0})\dot{\varphi}_{t_0} \ . \tag{5.3}$$

We would like to apply the maximum principle to Equation (5.3), but up to now the function $\dot{\varphi}_t$ is defined only on the complement of the zeros of q_0, and may be unbounded. However, since $e^{2u_t}e^{-2\varphi_t} = t\|q_0\|_h$, taking the derivative in $t = t_0$ we deduce that

$$2\|q_0\|_h t_0(\dot{u}_{t_0} - \dot{\varphi}_{t_0}) = \|q_0\|_h \ ,$$

hence, outside the zeros of q_0, we have

$$\dot{\varphi}_{t_0} = \dot{u}_{t_0} - \frac{1}{2t} \ ,$$

which implies that $\dot{\varphi}_{t_0}$ extends to a smooth function at the zeros of q_0 because $\dot{u}_{t_0}$ does and, moreover, they share the same points of maximum and minimum.
In particular, we can show that $\dot{\varphi}_{t_0}$ does not assume maximum at a point p which is a zero of q_0. Otherwise, this would be also a point of maximum for $\dot{u}_{t_0}$ and we would have (cfr. Proposition 5.3.13)

$$\begin{aligned} 0 \geq \Delta_h \dot{u}_{t_0}(p) &= 2e^{2u_{t_0}(p)}\dot{u}_{t_0}(p) - 2t_0\|q_0(p)\|_h^2 e^{-2u_{t_0}(p)} + 2t_0^2 \dot{u}_{t_0(p)} e^{-2u_{t_0}(p)}\|q_0(p)\|_h^2 \\ &= 2e^{2u_{t_0}(p)}\dot{u}_{t_0}(p) \end{aligned}$$

which would imply that $\dot{u}_{t_0} \leq 0$. On the other hand, we will prove in Proposition 5.3.13 that $\dot{u}_{t_0} \geq 0$, everywhere on S, thus $\dot{u}_{t_0}$ would vanish identically. But then

$$0 = \Delta_h \dot{u}_{t_0} = -2t_0\|q_0\|_h^2 e^{-2u_{t_0}}$$

would give a contradiction.
Therefore, $\dot{\varphi}_{t_0}$ takes maximum outside the zeros of q_0, and we can apply the maximum principle to Equation (5.3). At a point p of maximum for $\dot{\varphi}_{t_0}$, we have

$$\begin{aligned} 0 \geq \Delta_h \dot{\varphi}_{t_0}(p) &= 2\|q_0(p)\|_h \sinh(2\varphi_{t_0}(p)) + 4t_0\|q_0(p)\|_h \cosh(2\varphi_{t_0}(p))\dot{\varphi}_{t_0}(p) \\ &> 4t_0\|q_0(p)\|_h \cosh(2\varphi_{t_0}(p))\dot{\varphi}_{t_0}(p) > 4t_0\|q_0(p)\|_h \dot{\varphi}_{t_0}(p) \ , \end{aligned}$$

which implies that $\dot{\varphi}_{t_0} < 0$ everywhere on S, and φ_t is monotone decreasing in t as desired. ☐

Corollary 5.3.9. *Let λ_t be the positive principal curvature of the maximal surface Σ_t. Then $\lambda_t \to 1$ monotonically outside the zeros of q_0, when t goes to $+\infty$.*

Proof. Recall that the shape operator of Σ_t can be written as

$$B_t = I_t^{-1} II_t = e^{-2u_t} h^{-1} \Re\mathrm{e}(tq_0) \ .$$

Therefore, $\lambda_t^2 = -\det(B_t) = e^{-4u_t}t^2\|q_0\|_h^2$ and this is monotonically increasing to 1 by the previous proposition. ☐

5.3.4 Asymptotics of the Lorentzian Hausdorff dimension

We now compare the Lorenztian Hausdorff dimension of the limit set of a GHMC anti-de Sitter manifold with the volume entropy of the unique maximal surface.

Lemma 5.3.10. *Let ρ be the holonomy representation of a GHMC anti-de Sitter manifold M with limit set Λ_ρ. Then*

$$\mathcal{L}\mathcal{H}\mathrm{dim}(\Lambda_\rho) \leq E(\rho) \ .$$

Proof. Let Σ be the unique maximal surface embedded in M. We identify the universal cover of M with the domain of dependence $\mathcal{D}(\Lambda_\rho)$ of the limit set. In this way, Σ is lifted to a minimal disc $\tilde{\Sigma}$ in AdS_3 with asymptotic boundary Λ_ρ, contained in the convex hull $\mathcal{C}(\Lambda_\rho)$. We fix a base point $x_0 \in \tilde{\Sigma}$. By definition,

$$E(\rho) = \limsup_{R\to+\infty} \frac{1}{R}\log(\#\{\gamma \in \pi_1(S) \mid d_I(\rho(\gamma)x_0, x_0) \leq R\}) \ ,$$

where I is the induced metric on $\tilde{\Sigma}$, and by Theorem 5.3.3

$$\mathcal{L}\mathcal{H}\mathrm{dim}(\Lambda_\rho) = \limsup_{R\to+\infty} \frac{1}{R}\log(\#\{\gamma \in \pi_1(S) \mid d_{AdS}(\rho(\gamma)x_0, x_0) \leq R\}).$$

Therefore, it is sufficient to show that for every couple of points $x, y \in \tilde{\Sigma}$, we have

$$d_I(x, y) \leq d_{AdS}(x, y) \ .$$

Since Σ is a Cauchy surface for M, the geodesic connecting x and y is space-like. We can thus find a Lorentzian plane $P \subset AdS_3$ containing x and y. In an affine chart, this is isometric to $(\mathbb{R} \times (-\pi/2, \pi/2), dt^2 - \cosh^2(t)ds^2)$, where t is the arc-length parameter of the space-like geodesic between x and y. By intersecting P with $\tilde{\Sigma}$ we obtain a curve $\gamma \subset \tilde{\Sigma}$ with length

$$\mathrm{length}(\gamma) = \int_0^{d_{AdS}(x,y)} \sqrt{1 - \cosh^2(t)s'(t)}dt \leq d_{AdS}(x, y) \ .$$

As a consequence, the distance between x and y in the induced metric of $\tilde{\Sigma}$ must be smaller than $d_{AdS}(x, y)$. □

Theorem 5.3.11. *Let M_t be the sequence of GHMC anti-de Sitter manifolds parameterised by the ray $(h, tq_0) \in T^*\mathrm{Teich}(S)$ for some non-zero holomorphic quadratic differential q_0. Let Λ_t be the limit sets of the corresponding holonomy representations. Then*

$$\lim_{t\to+\infty} \mathcal{L}\mathcal{H}\mathrm{dim}(\Lambda_t) = 0$$

Proof. By Lemma 5.3.10, it is sufficient to show that the entropy of the maximal surface tends to 0 when t goes to $+\infty$. Since the metrics $I_t = e^{2u_t}h$ are bounded from below by the flat metrics with conical singularities $g_t = t|q_0|$ (Proposition 5.3.4), we deduce that

$$E(\rho_t) \leq E(g_t) \ .$$

The proof is then completed by noticing that $E(t|q_0|) = t^{-1}E(|q_0|)$. □

In order to prove a rigidity result for the entropy of the maximal surface and the Lorentzian Hausdorff dimension, we study the derivative of the entropy of the maximal surface along a ray. To this aim, we need the following useful formula for the variation of the volume entropy along a path of smooth Riemannian metrics:

Theorem 5.3.12 ([KKW91]). *Let g_t be a smooth path of negatively curved Riemannian metrics on a closed manifold S. Then*

$$\frac{d}{dt}E(g_t)_{|t=t_0} = -\frac{E(g_{t_0})}{2}\int_{T^1S}\frac{d}{dt}g_t(v,v)_{|t=t_0}d\mu_{t_0}$$

for a suitable measure μ_{t_0} defined on the unit tangent bundle T^1S of S.

Proposition 5.3.13. *The volume entropy of the maximal surface of a GHMC anti-de Sitter manifold is strictly decreasing along a ray tq_0 for $t \geq 0$.*

Proof. Along the ray tq_0, Equation (5.1) can be re-written as

$$\Delta_h u_t = e^{2u_t} - e^{-2u_t}t^2\|q_0\|_h^2 - 1 \ . \tag{5.4}$$

Taking the derivative at t_0 we obtain

$$\Delta_h \dot{u}_{t_0} = 2e^{2u_{t_0}}\dot{u}_{t_0} - 2t_0\|q_0\|_h^2 e^{-2u_{t_0}} + 2t_0^2\dot{u}_{t_0}e^{-2u_{t_0}}\|q_0\|_h^2 \ . \tag{5.5}$$

At a point p of minimum for $\dot{u}_{t_0}$ we have

$$0 \leq \Delta_h\dot{u}_{t_0}(p) = 2\dot{u}_{t_0}(p)(e^{2u_{t_0}(p)} + e^{-2u_{t_0}(p)}t_0^2\|q_0(p)\|_h^2) - 2t_0\|q_0(p)\|_h^2 e^{-2u_{t_0}(p)}$$

which implies, since $t_0 \geq 0$, that $\dot{u}_{t_0}(p) \geq 0$. Hence, $\dot{u}_{t_0} \geq 0$ everywhere on S.
Now, the induced metrics on the maximal surfaces are $I_t = e^{2u_t}h$, thus for every unit tangent vector $v \in T^1S$

$$\frac{d}{dt}I_t(v,v)_{|t=t_0} = 2\dot{u}_{t_0}e^{2u_{t_0}}h(v,v) \geq 0 \ .$$

Since the induced metrics I_t are negatively curved by the Gauss equation and Corollary 5.3.5, we can apply Theorem 5.3.12 and deduce that the volume entropy is decreasing.
To prove that it is strictly decreasing, we notice that

$$\frac{d}{dt}E(I_t)_{|t=t_0} = -\frac{E(I_{t_0})}{2}\int_{T^1S}\frac{d}{dt}I_t(v,v)_{|t=t_0}d\mu_{t_0} = 0$$

if and only if $\dot{u}_{t_0}$ vanishes identically on S. In this case, Equation (5.5) reduces to

$$0 = 2t_0 \|q_0\|_h^2 e^{-2u_{t_0}}$$

which implies that $t_0 = 0$, because q_0 is not identically zero. □

Corollary 5.3.14. *$E(h,q) \leq 1$ for every $(h,q) \in T^*\mathrm{Teich}(S)$ and $E(h,q) = 1$ if and only if $q = 0$.*

Proof. If $q = 0$, the function $u = 0$ is the unique solution to Equation (5.1). Hence, the induced metric on the maximal surface is hyperbolic, and it is well-known that the volume entropy of the hyperbolic metric is 1.
On the other hand, since the function $E(h, tq_0)$ is strictly decreasing for $t \geq 0$, for every non-zero quadratic differential q we have $E(h,q) < E(h,0) = 1$. □

The rigidy result for the Lorentzian Hausdorff dimension then follows:

Theorem 5.3.15. *Let M be a GHMC anti-de Sitter manifold and let Λ be its limit set. Then*

$$\mathcal{LH}\mathrm{dim}(\Lambda) = 1$$

if and only if M is Fuchsian.

Proof. If M is Fuchsian, the holonomy representation $\rho = (\rho_0, \rho_0)$ preserves the totally geodesic space-like plane P_0, that is isometric to the hyperbolic plane. Fix the base point x_0 on P_0. Since for every $\gamma \in \pi_1(S)$, the isometry $\rho(\gamma)$ acts on the plane P_0 like the hyperbolic isometry $\rho_0(\gamma)$ on $\mathbb{H}^2$ (see Section 5.2), the entropy of d_{AdS} coincides with the entropy of the hyperbolic metric associated to ρ_0, which is equal to 1.
Viceversa, suppose that $\mathcal{LH}\mathrm{dim}(\Lambda) = 1$, then by Lemma 5.3.10 the entropy of the maximal surface embedded in M is at least 1. By Corollary 5.3.14, we deduce that M is Fuchsian. □

5.4 Width of the convex core

Another geometric quantity associated to GHMC anti-de Sitter manifolds is the width of the convex core. This has already been extensively studied in [Sep17]. Combining the aformentioned work with our estimates in Section 5.3, we can describe its asymptotic behaviour.

We recall that the convex core of a GHMC anti-de Sitter manifold M is homeomorphic to $S \times I$, where I is an interval that can be reduced to a single point if M is Fuchsian. The width of the convex core expresses how far M is from being Fuchsian, as it measures the distance between the two boundary components of the convex core. More precisely, let Λ_ρ be the limit set of the holonomy representation ρ of M. The convex core can be realised as the quotient of the convex hull of Λ_ρ in AdS_3 by the action of $\rho(\pi_1(S))$.

Definition 5.4.1. *The width $w(M)$ of the convex core of M is the supremum of the length of a time-like geodesic contained in $\mathcal{C}(\Lambda_\rho)$.*

We can give an equivalent definition by introducting a time-like distance in AdS_3. Given two points $x, y \in AdS_3$, we denote with $\gamma_{x,y}$ the unique geodesic connecting the two points. We define

$$d_t : AdS_3 \times AdS_3 \to \mathbb{R}_{\geq 0}$$

as

$$d_t(x,y) = \begin{cases} \text{length}(\gamma_{x,y}) & \text{if } \gamma_{x,y} \text{ is time-like} \\ 0 & \text{otherwise} \end{cases}$$

where the length of a time-like curve $\gamma : [0,1] \to AdS_3$ is

$$\text{length}(\gamma) = \int_0^1 \sqrt{-\|\dot{\gamma}(t)\|} dt \ .$$

Therefore, Definition 5.4.1 is equivalent to

$$w(M) = \sup_{\substack{p \in \mathcal{C}(M)^+ \\ q \in \mathcal{C}(M)^-}} d_t(p,q)$$

where $\mathcal{C}(M)^\pm$ denotes the upper- and lower-boundary of the convex core. Notice, in particular, that $w(M) = 0$, if and only if M is Fuchsian.

Seppi found an estimate for the width of the convex core in terms of the principal curvatures of the maximal surface:

Theorem 5.4.2 (Theorem 1.B [Sep17])**.** *There exist universal constants $C > 0$ and $\delta \in (0,1)$ such that if Σ is a maximal surface in a GHMC anti-de Sitter manifold with principal curvatures λ satisfying $\delta \leq \|\lambda\|_\infty < 1$, then*

$$\tan(w(M)) \geq \left(\frac{1}{1 - \|\lambda\|_\infty} \right)^{\frac{1}{C}} \ .$$

We consider now a family of GHMC anti-de Sitter manifolds M_t parameterised by the ray $(h, tq_0) \in T^*\text{Teich}(S)$ for a non-zero holomorphic quadratic differential q_0.

Proposition 5.4.3. *The width of the convex core $w(M_t)$ converges to $\pi/2$ when t goes to $+\infty$.*

Proof. By Theorem 5.4.2, it is sufficient to show that the positive principal curvature λ_t of the maximal surface Σ_t embedded in M_t converges to 1. This is exactly the content of Corollary 5.3.9 □

Chapter 6

Perspectives and future work

Our results still leave a number of questions unanswered. We list them here as a conclusion of this thesis, hoping to work on them in the next future.

6.1 Prescription of metrics and measured laminations

In Chapter 2, we proved the following:

Theorem 2.3.3 *For every couple of smooth metrics* (g_+, g_-) *with curvature less than* -1 *on a closed, connected, oriented surface* S*, there exists a GHMC anti-de Sitter manifold* M*, which contains a convex compact subset* $K \cong S \times I$*, whose induced metrics on the boundaries are* g_+ *and* g_-*.*

It is natural to ask

Question 6.1.1. *Is* M *uniquely determined by* g_+ *and* g_-*?*

It would also be interesting to see if it is possible to remove the smoothness assumption in Theorem 2.3.3.

These questions are related to Mess' conjectures. Let M be a GHMC anti-de Sitter manifold. Recall that the convex core of M is homeomorphic to $S \times I$, where I is an interval that can be reduced to a point if M is Fuchsian. The boundary components are space-like surfaces endowed with hyperbolic metrics $m_\pm$ and pleated along (possibly empty) measured laminations $\lambda_\pm$. Mess asked the following:

Question 6.1.2. *Is* $\mathcal{GH}(S)$ *parameterised by the metrics* $(m_+, m_-) \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$*?*

Question 6.1.3. *Is* $\mathcal{GH}(S)$ *parameterised by the measured laminations* $(\lambda_+, \lambda_-) \in \mathcal{ML}(S) \times \mathcal{ML}(S)$*?*

For both questions, it is known that every couple of metrics ([Dia13]) and every couple of filling measured laminations ([BS12]) can be realised, but uniqueness is still open.

6.2 Convexity of volume and energy

In Chapter 4 we studied the volume of the convex core of a GHMC anti-de Sitter manifold as a function of the two parameters in Mess' parameterisation. Recall our main result:

Theorem 4.3.8 *Let $M_{h,h'}$ be a GHMC AdS_3 manifold. Then*

$$\frac{1}{4}E_{Sch}(h,h') - \pi|\chi(S)| \leq \mathrm{Vol}(\mathcal{C}(M_{h,h'})) \leq \frac{1}{4}E_{Sch}(h,h') + \frac{\pi^2}{2}|\chi(S)| \ .$$

The 1-Schatten energy functional between hyperbolic surfaces was introduced by Trapani and Valli [TV95] and was later studied by Bonsante, Mondello and Schlenker who proved the following:

Theorem 6.2.1 ([BMS15])**.** *The function $E_{Sch}(\cdot,h') : \mathrm{Teich}(S) \to \mathbb{R}^+$ is convex with respect to the Weil-Petersson metric on the Teichmüller space of S.*

Combining it with Theorem 4.3.8, it is then natural to ask

Question 6.2.1. *Is the volume of a GHMC anti-de Sitter manifold $M_{h,h'}$ convex with respect to the Weil-Petersson metric, as a function of $h \in \mathrm{Teich}(S)$?*

It would also be interesting to understand if other types of energies share the same convexity property. For instance, a more commonly used energy between hyperbolic surfaces is the classical L^2-energy

$$E_d(h,h') = \inf_f \int_S \|df\|^2 \ \mathrm{dA}_h$$

which is realised by the unique harmonic map between (S,h) and (S,h') isotopic to the identity. It is known that $E_d(h,\cdot) : \mathcal{T}(S) \to \mathbb{R}^+$ is convex with respect to the Weil-Petersson metric ([Tro96], [Yam99]). Unlike the holomorphic 1-energy, $E_d(\cdot,\cdot)$ is not symmetric, hence one could ask the following:

Question 6.2.2. *Is the function $E_d(\cdot,h') : \mathcal{T}(S) \to \mathbb{R}^+$ convex with respect to the Weil-Petersson metric on the Teichmüller space of S?*

In case of affermative answer, it would be worth studying possible generalisations on the realm of higher Teichmüller theory. For instance this result would lead to the existence of a unique minimal surface in $(\mathbb{H}^2)^k$.

6.3 Special foliation of Lorentzian 3-manifolds

Motivated by the necessity of modelling the presence and physical interactions of massive particles, different kinds of singularities in Lorentzian metrics have been introduced in the literature ([BBS11]). Recall that a GHMC manifold M is diffeomorphic to $S \times \mathbb{R}$. Fix a finite number of points $\{p_1, \dots, p_n\}$ on S. We say that M contains

- **static particles**, if the holonomy around a singular line is a rotation of angle smaller than π fixing the singular line pointwise;
- **interactive particles**: if the holonomy around a singular line is a rotation of angle bigger than π fixing the singular line pointwise;
- **spin particles**: if the holonomy around a singular line is an ellyptic transformation consisting of a rotation of angle smaller than π and a translation along the singular line.

From the above description, it is evident that the presence of mass perturbes the local geometry, but it is not clear if this has repercussions on the global geometry. As a consequence of my joint work (which has not been included in this thesis) with Qiyu Chen, together with some of her previous results, we obtained a description of the global geometry for static particles:

Theorem 6.3.1 ([CS16],[CS17],[CT17])**.** *Let M be GHMC manifold with cone singularities of angle less than π along time-like geodesics. Suppose that M is locally modelled on Minkowski, anti-de Sitter or de Sitter space. Then M admits a unique foliation by constant Gauss curvature (in the complement of its convex core) and constant mean curvature surfaces orthogonal to the singular lines.*

Nevertheless, the picture is far from being complete.

Question 6.3.1. *Is it possible to extend Theorem 6.3.1 to cone singularities of angles bigger than π along time-like geodesics?*

This seems to be a straightforward generalisation of Theorem 6.3.1, but new technical difficulties arise when trying to study this problem. For instance, if the conical singularities are bigger than π, the principal curvatures of a surface orthogonal to the singular lines diverge at the points of intersection with the singular locus.

More in general, we can ask the following:

Question 6.3.2. *Are GHMC manifolds with constant sectional curvature and spin particles foliated by constant mean curvature surfaces?*

Question 6.3.3. *Are GHMC manifolds with constant sectional curvature and spin particles foliated by constant Gauss curvature surfaces (outside their convex core)?*

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