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Geometric aspects of Higgs bundles

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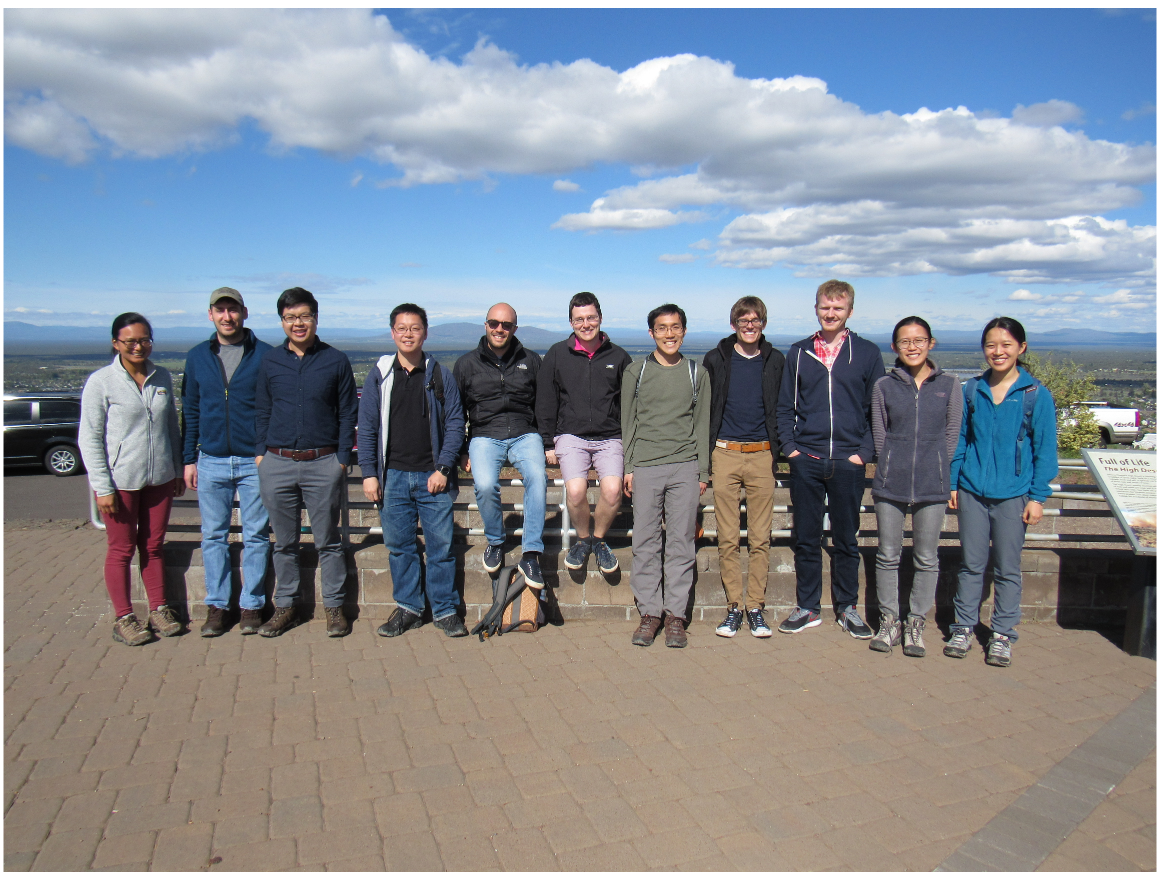
The following notes were written by the participants of the RTG Log Cabin Workshop *Geometric Aspects of Higgs Bundles*, which took place in Sunriver, Oregon in May 2019.

The goal of the workshop was to bring together early career mathematicians from different backgrounds, but with common research interests in representations and geometric structures. The basics behind Higgs bundles and opers were covered in early talks, along with various geometric structures such as convex real projective and $\mathbb{CP}^1$ structures. Later talks focused on recent developments concerning Hitchin, Anosov, and Θ-positive representations.

It is our hope that these notes will serve as a useful resource to those interested in learning about the basics of Higgs bundles, geometric structures and the higher Teichmüller theory. We kindly thank and acknowledge all the participants for attending the workshop, giving excellent talks and providing detailed notes of their lectures. We wish to thank the NSF for giving us the opportunity to organize this workshop.

The organizers,
X. Dai, C. Ouyang and A. Tamburelli



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Chapter 1

Lie groups and symmetric spaces

Michelle Chu (UCSB)

These notes are based on a lecture given at the workshop Geometric aspects of Higgs bundles in Sunriver, OR in May 2019. Unfortunately, we skip proofs and references. However, this material can be found in many books and notes.

In these notes, manifolds and vector spaces are always finite dimensional. Throughout the notes, F will be either $\mathbb{R}$ of $\mathbb{C}$.

1.1 Lie groups and Lie algebras

Definition 1.1.1. A Lie group G is a group which is also a smooth manifold and such that group multiplication and group inversion are smooth. A complex Lie group is a Lie group which is a complex manifold and multiplication and inversion are holomorphic.

Definition 1.1.2. A Lie algebra $\mathfrak{g}$ over a field K is a vector space over K equipped with a bilinear map called the Lie bracket

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
$$(X, Y) \mapsto [X, Y].$$

satisfying

- $[X, X] = 0$ for all $X \in \mathfrak{g}$
- (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $Z, Y, Z \in \mathfrak{g}$.

The conditions in the definition imply that the Lie bracket is skew-symmetric, i.e. $[X, Y] = -[Y, X]$.

Lie groups and Lie algebras are related via left-invariant vector fields. Let ℓ_g denote left multiplication by g, i.e. $\ell_g : h \mapsto gh$.

Definition 1.1.3. A vector field X on a Lie group G is left invariant if $d(\ell_g)_h(X(h)) = X(gh)$ for all $g, h \in G$.

The Lie bracket $[X, Y]$ of two vector fields X, Y on G is the vector field $[X, Y]$ defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \text{ for all } f \in C^\infty(G).$$

The Lie bracket of two left-invariant vector fields is again left-invariant. Therefore, we can associate to a Lie group G the Lie algebra $\mathfrak{g}$ of left-invariant vector fields.

Proposition 1.1.4. The linear map $L : \mathfrak{g} \to T_e G$ defined by $L(X) = X(e)$ is an isomorphism.

1.1.1 Classical Lie groups

Exercise. Let V be an n-dimensional vector space over $F = \mathbb{R}$ or $\mathbb{C}$. The set $\mathrm{GL}(V)$ of all invertible F-linear maps is a Lie group. Note that it is a manifold since it is an open subset of $M_n(F)$. Multiplication and inversion are polynomial and rational in terms of the matrix entries, hence smooth. The Lie algebra of $\mathrm{GL}(V)$ is $\mathfrak{gl}_n(F) = M_n(F)$. The Lie bracket is given by the commutator $[X, Y] = XY - YX$.

Exercise. The Special linear group and its Lie algebra

$$\mathrm{SL}_n(F) = \{A \in \mathrm{GL}_n(F) | \det(A) = 1\}$$

$$\mathfrak{sl}_n(F) = \{A \in \mathfrak{gl}_n(F) | \mathrm{tr}(A) = 0\}.$$

Exercise. Let V be a vector space over F with non-degenerate bilinear form with associated matrix Q (for some choice of basis).

$$\mathrm{O}(Q) = \{A \in \mathrm{GL}_n(F) | A^T Q A = Q\}$$

$$\mathfrak{o}(Q) = \{A \in \mathfrak{gl}_n(F) | A^T Q = -QA\}$$

These are complex Lie groups and complex Lie algebras when $K = \mathbb{C}$.

Here are some examples arising in this way:

- The orthogonal group $\mathrm{O}(n)$ and its Lie algebra $\mathfrak{o}(n) = \mathfrak{so}(n)$ given by $Q = I_n$.
- The indefinite orthogonal group $\mathrm{O}(p, q)$ and its Lie algebra $\mathfrak{o}(p, q) = \mathfrak{so}(p, q)$ given by $Q = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.
- The symplectic group $\mathrm{Sp}(2n, F)$ and ts Lie algebra $\mathfrak{sp}(2n, F)$ given by $Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Exercise. For $A \in M_n(\mathbb{C})$ let A^* denote the complex conjugate transpose. The unitary group is $\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) | A^* A = I_n\}$. Its Lie algebra is $\mathfrak{u}(n) = \{A \in \mathfrak{gl}_n(\mathbb{C}) | A^* = -A\}$.

The special unitary group is $\mathrm{SU}(n) = \{A \in \mathrm{SL}_n(\mathbb{C}) | A^* A = I_n\}$. Its Lie algebra is $\mathfrak{su}(n) = \{A \in \mathfrak{gl}_n(\mathbb{C}) | A^* = -A \text{ and } \mathrm{tr}(A) = 0\}$.

The unitary groups are real Lie groups.

1.1.2 Homomorphisms and subgroups

Definition 1.1.5. A Lie group homomorphism $\phi : H \to G$ is a group homomorphism between two Lie groups which is also smooth. A Lie algebra homomorphism $\varphi : \mathfrak{h} \to \mathfrak{g}$ is a homomorphism between Lie algebras which is

- linear: $\varphi(ax + by) = a\varphi(x) + b\varphi(y)$, and
- compatible with the Lie bracket: $\varphi([x, y]_{\mathfrak{h}}) = [\varphi(x), \varphi(y)]_{\mathfrak{h}}$.

Proposition 1.1.6. If $\phi : H \to G$ is a Lie group homomorphism, then $d\phi_e : T_e H \to T_e(G)$ is a Lie algebra homomorphism.

A linear representation of a Lie group is a Lie group homomorphism $\phi : G \to \mathrm{GL}(V)$. Similarly a linear representation of a Lie algebra is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}_n(F)$.

Theorem 1.1.1 (Ado's theorem)**.** Every Lie algebra has a faithful linear representation.

Definition 1.1.7. A Lie subgroup H of G is an abstract subgroup which is also a Lie group and such that the inclusion is a smooth immersion. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace which is closed under Lie bracket, i.e. $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$.

Proposition 1.1.8. If H is a Lie subgroup of G, then $\mathfrak{h} \simeq T_eH$ is a Lie subalgebra of $\mathfrak{g} \simeq T_eG$. Alternatively, if $\mathfrak{h}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}$ for a Lie group G, then there exist a unique connected Lie subgroup H of G.

The above Proposition shows that there is a bijection between Lie subalgebras of $\mathfrak{g}$ and connected Lie subgroups of G.

Theorem 1.1.2. If H is any closed subgroup of a Lie group G, then H is a Lie subgroup.

We saw before that to any Lie group we can associate a Lie algebra. The reverse is also true.

Theorem 1.1.3 (Lie's third theorem)**.** For every Lie algebra $\mathfrak{g}$, there is a Lie group G such that $\mathfrak{g}$ is Lie algebra isomorphic to its associated Lie algebra.

We might ask whether every Lie group has a linear representation, and it turns out the answer is no. The canonical examples is $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, the universal cover of $\mathrm{SL}_2(\mathbb{R})$.

1.1.3 The exponential map and the adjoint representation

For each $X \in T_eG$ there exists a one-parameter subgroup γ_X such that $d\gamma_X(t) = tX$, so $\gamma_X'(0) = X$.

Definition 1.1.9. The exponential map is defined as

$$\mathfrak{g} \to G$$
$$\exp(X) = \gamma_X(1).$$

For each $X \in \mathfrak{g}$, $\gamma(t) = \exp(tX)$ is a one-parameter subgroup with $\gamma'(0) = X$. The exponential map is smooth and $d(\exp)_0 = Id$.

Definition 1.1.10. For $g \in G$ let $C_g : h \mapsto ghg^{-1}$ be the conjugation map. The adjoint representation of G is the Lie group homomorphism defined as

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$
$$g \mapsto d(C_g)_e.$$

The adjoint representation of $\mathfrak{g}$ is the Lie algebra homomorphism

$$\mathrm{ad} : g \to \mathfrak{gl}(\mathfrak{g})$$
$$X \mapsto \mathrm{ad}_X = [X, \cdot].$$

Proposition 1.1.11. The adjoint representation satisfies

- $C_g(\gamma_X(t)) = \gamma_{\mathrm{Ad}(g)(X)}(t)$
- $d\mathrm{Ad} = \mathrm{ad}$
- $\exp(\mathrm{Ad}(g)(X)) = g \exp(X) g^{-1}$
- if G is connected, the kernel of Ad is the center of G.

Going back to the general linear group, for $A \in \mathrm{GL}_n$, we have $\exp(A) = \sum_0^\infty \frac{A^k}{k!}$. For any $g \in \mathrm{GL}_n$ and $X \in \mathfrak{gl}_n$, $\mathrm{Ad}(g)(X) = gXg^{-1}$.

Proposition 1.1.12. If G is any closed subgroup in GL_n, then G is a Lie group and its Lie algebra is given by

$$\mathfrak{g} = \{A \in M_n(F) | \exp(tA) \in G \text{ for all } t \in \mathbb{R}\}.$$

1.1.4 Structure

Definition 1.1.13. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

- An ideal $\mathfrak{i}$ of $\mathfrak{g}$ is a vector subspace such that $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$.
- The derived series of $\mathfrak{g}$ is defined inductively as $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}_{k-1}]$ (these are ideals in $\mathfrak{g}$).
- $\mathfrak{g}$ is solvable if there is some k for which $\mathfrak{g}_k = 0$
- The radical of $\mathfrak{g}$, denoted $\mathrm{rad}(\mathfrak{g})$, is the unique maximal solvable ideal.
- $\mathfrak{g}$ is simple if the only ideals are $\{0\}$ and $\mathfrak{g}$ itself.
- $\mathfrak{g}$ is semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$, equivalently if $\mathfrak{g}$ has no solvable ideals, equivalently if $\mathfrak{g}$ has no abelian ideals.

Proposition 1.1.14. For any Lie algebra $\mathfrak{g}$, the quotient Lie algebra $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple.

Definition 1.1.15. The Killing form of $\mathfrak{g}$ is the symmetric bilinear form

$$\begin{aligned} B : \mathfrak{g} \times \mathfrak{g} &\to F \\ (X, Y) &\mapsto \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \end{aligned}$$

Theorem 1.1.4 (Cartan's Second Criterion)**.** $\mathfrak{g}$ is solvable if and only if $B = 0$ on $[\mathfrak{g}, \mathfrak{g}]$. $\mathfrak{g}$ is semisimple if and only if B is non-degenerate.

Exercise. The Lie algebras $\mathfrak{sl}_n(F), \mathfrak{o}(p, q), \mathfrak{sp}(2n, F), \mathfrak{u}(n), \mathfrak{su}(n)$ are semisimple because their Killing forms are non-degenerate.

Theorem 1.1.5. If $\mathfrak{g}$ is semisimple, then

- $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ where each $\mathfrak{g}_i$ is a simple ideal.
- if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then it decomposes uniquely as $\oplus_{j \in I} g_j$ for $I \subset \{1, \ldots, k\}$.
- $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Definition 1.1.16. A Lie algebra is compact if it is the Lie algebra of a compact Lie group, equivalently if its Killing form is negative definite.

Exercise. The Lie algebras $\mathfrak{o}(n)$, $\mathfrak{u}(n)$, and $\mathfrak{su}(n)$ are compact.

1.1.5 Cartan subalgebra and roots

Definition 1.1.17. A Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra such that if $X \in \mathfrak{h}$ then ad_X is diagonalizable.

Cartan subalgebras of semisimple Lie algebras are unique up to automorphisms, so all Cartan subalgebras have the same dimension.

Definition 1.1.18. The rank of a Lie algebra $\mathfrak{g}$ is the dimension of a Cartan subalgebra. The rank of a Lie group is the rank of its Lie algebra.

For the remainder of this section we focus on the case of complex semisimple Lie algebras.

Exercise. For $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{gl}_n(\mathbb{C})$, the subset of diagonal matrices is a Cartan subalgebra.

In general, non-trivial Cartan subalgebras might not exist.

Theorem 1.1.6. Every complex semisimple Lie algebra $\mathfrak{g}$ has a Cartan subalgebra $\mathfrak{h}$. Moreover, $\mathfrak{h}$ is unique up to inner automorphism and $\mathfrak{h}$ is equal to its normalizer and centralizer in $\mathfrak{g}$.

Definition 1.1.19. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. For a linear map $\alpha \in \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$, the root space of α is

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, α is called a root of $\mathfrak{g}$ (with respect to $\mathfrak{h}$). Define $\Delta = \{\alpha \in \mathfrak{h}^* | \mathfrak{g}_\alpha \neq 0\}$, the set of non-zero roots.

Theorem 1.1.7. Let $\mathfrak{g}$ be complex semisimple, with Cartan subalgebra $\mathfrak{h}$ and Δ the non-zero roots.

- The root space decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$.
- If $\alpha, \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
- $\{\alpha | \alpha \in \Delta\}$ spans $\mathfrak{h}^*$.
- If $\alpha, \beta \in \Delta$ and $\alpha + \beta \neq 0$ then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.
- If $\alpha \in \Delta$ then $-\alpha \in \Delta$.
- For $\alpha \in \Delta$, there is some $H_\alpha \in \mathfrak{h}$ such that $B(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$.

Exercise. A Cartan subalgebra for $\mathfrak{sl})n(\mathbb{C})$ is given by

$$\mathfrak{h} = \{H = \mathrm{diag}(h_1, \ldots, h_n) | h_i \in \mathbb{C} \text{ and } \sum h_i = 0\}.$$

Let $\epsilon_i \in \mathfrak{h}^*$ be the linear function on $\mathfrak{h}$ which takes the (i, i) entry Let $E_{i,j}$ denote the matrix with 1 in the (i, j) entry and 0 elsewhere. If $H \in \mathfrak{h}$ it is diagonal and then $[H, E_{i,j}] = (\epsilon_i - \epsilon_j)(H)E_{i,j}$. So $(\epsilon_i - \epsilon_j)$ is a root of $\mathfrak{g}$ with root space $\mathfrak{g}_{(\epsilon_i-\epsilon_j)} = \mathbb{C}E_{i,j}$ and $\Delta = \{(\epsilon_i - \epsilon_j) | i \neq j\}$.

The Killing form for $H = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{h}$ is given by

$$B(H, H) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(H) = 2n \sum h_i^2$$

and $H_{\epsilon_i-\epsilon_j} = n(E_{i,i} - E_{j,j})$.

1.1.6 Cartan decomposition

Definition 1.1.20. If $\mathfrak{g}$ is a real Lie algebra, the complexification of $\mathfrak{g}$ is the complex Lie algebra $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes \mathbb{C}$ with

$$[U + iV, X + iY]_{\mathfrak{g}_\mathbb{C}} = [U, X]_\mathfrak{g} - [V, Y]_\mathfrak{g} + i\left([U, Y]_\mathfrak{g} + [V, X]_\mathfrak{g}\right).$$

Exercise. The complexification of $\mathfrak{gl}_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{C})$.

Definition 1.1.21. If $\mathfrak{g}$ is a complex Lie algebra, a real form of $\mathfrak{g}$ is a real Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}_\mathbb{C}$ is isomorphic to $\mathfrak{g}$

Theorem 1.1.8. Every complex semisimple Lie algebra has a compact real form.

In general, a complex Lie algebra may not have a real form. Let $\mathfrak{g}$ be the complex Lie algebra spanned by X, Y, Z with $[X, Y] = 0$, $[X, Z] = X$, and $[Y, Z] = aY$. Then $\mathfrak{g}$ has a real form if and only if $a \in \mathbb{R}$ or $|a| = 1$ (reference: Lie groups, representation theory, and symmetric spaces by Wolfgan Ziller.)

Definition 1.1.22. Let $\mathfrak{g}$ be a real semisimple Lie algebra with killing form $B(\cdot,\cdot)$. An involution θ on $\mathfrak{g}$ is a Lie algebra automorphism with $\theta^2 = Id$. Such an involution is a Cartan involution if $B_\theta(X,Y) = -B(X,\theta(Y))$ is positive definite.

Proposition 1.1.23. Any real semisimple Lie algebra has a Cartan involution, unique up to inner automorphism.

Exercise. For $\mathfrak{sl}_n\mathbb{R}$, $\theta(X) = -X^T$ defines the Cartan involution.

Complex conjugation on a complex Lie algebra is an involution. If this complex Lie algebra has a compact real form $\mathfrak{g}$, then complex conjugation on $\mathfrak{g}_\mathbb{C}$ is the Cartan involution.

Let θ be an involution of real Lie algebra $\mathfrak{g}$. As a linear map, θ has two eigenvalues $+1$ and -1. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the corresponding eigenspaces.

Definition 1.1.24. A Cartan decomposition of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The pair $(\mathfrak{k},\mathfrak{p})$ is called a Cartan pair.

If $(\mathfrak{k},\mathfrak{p})$ is a Cartan pair, then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$.

Proposition 1.1.25. Let $(\mathfrak{k},\mathfrak{p})$ be a Cartan pair

- The Killing form is negative definite on $\mathfrak{k}$.
- $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, and $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$
- $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$.

Exercise. For $\mathfrak{sl}_n\mathbb{R}$, $\theta(X) = -X^T$ defines the Cartan involution. The corresponding Cartan decomposition is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{so}_n\mathbb{R}$ and $\mathfrak{p}$ are the traceless symmetric matrices.

1.2 Symmetric Spaces

Definition 1.2.1. A Riemannian manifold (M,g) is a real, smooth manifold M equipped with an inner product g_p on the tangent space T_pM at each point p that varies smoothly from point to point.

Definition 1.2.2. Let (M,g) be a Riemannian manifold

- M is symmetric if for all $p \in M$ there is some isometry $s_p : M \to M$ with $s_p^2 = Id$ and p is an isolated fixed point. This isometry s_p is called an involution at p.
- M is locally symmetric if for all $p \in M$ there is some radius $r > 0$ and some isometry $s_p : B_r(p) \to B_r(p)$ with $s_p(p) = p$ and $d(s_p)_p = -Id$.

Proposition 1.2.3. Let (M,g) be a symmetric space.

- The involution s_p reverses geodesics through p, i.e. γ is a geodesic with $\gamma(0) = p$ then $s_p(\gamma(t) = \gamma(-t)$.
- M is complete.
- M is homogeneous.
- The isometry group $\mathrm{Isom}(M)$ is a Lie group and the stabilizer $\mathrm{Stab}_G(p)$ of a point p is compact.
- The identity component of the isometry group $\mathrm{Isom}_0(M)$ also acts transitively on M.

We can therefore write $M = G/K$ where $G = \mathrm{Isom}_0(M)$ and $K = \mathrm{Stab}_G(p)$.

1.2.1 Cartan involutions

Proposition 1.2.4. Let M be a symmetric space. The involution s_p gives rise to an involutive automorphism

$$\sigma = \sigma_p : G \to G$$
$$g \mapsto s_p g s_p.$$

Furthermore, if G^σ denotes the fixed points of σ, $G_0^\sigma \subset K \subset G^\sigma$.

Proposition 1.2.5. Let $M = G/K$ with $G_0^\sigma \subset K \subset G^\sigma$ for an involutive automorphism σ of G. If $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces of $d\sigma$ then $(\mathfrak{k}, \mathfrak{p})$ is a Cartan pair for $\mathfrak{g}$.

We saw before how a symmetric space M can be written as $M = G/K$ where $G = \mathrm{Isom}_0(M)$ and $K = \mathrm{Stab}_G(p)$. We now construct a symmetric space from a Lie group.

Proposition 1.2.6. Let G be a connected Lie group and $\sigma : G \to$ and involutive automorphism such that G_0^σ is compact. Then for any compact subgroup K with $G_0^\sigma \subset K \subset G^\sigma$, G/K is a symmetric space with any G-invariant metric.

Proposition 1.2.7. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and K the connected subgroup of G with Lie algebra $\mathfrak{k}$.

- There exist and involutive automorphism $\sigma : G \to G$ such that $K = G_0^\sigma$.
- If K is compact, then every G-invariant metric on G/K is symmetric.

We have seen now that a symmetric space gives rise to a Cartan decomposition and also a Cartan decomposition defines a symmetric space.

1.2.2 Examples of symmetric spaces

Exercise. Simply connected manifolds of constant curvature are symmetric spaces. For example, hyperbolic space $\mathbb{H}^n$ is a symmetric space. We can see this using the Lorentzian model

$$\{v \in \mathbb{R}^{n+1} | (v, v) = -1 \text{ and } x_{n+1} > 0\}$$

with inner product $(x, y) = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$. The isometry group is the Lie group $\mathrm{O}^+(n, 1)$ with identity component $\mathrm{SO}^+(n, 1)$. The stabilizer at a point is the subgroup $\mathrm{SO}(n)$. An involution at a point p is given by $s_p(v) = -v - 2(v, p)p$. We can identify $\mathbb{H}^n = \mathrm{SO}^+(n, 1)/\mathrm{SO}(n)$. The involutive autmorphism is conjugation by $I_{n,1}$.

Exercise. The set M of inner products on $\mathbb{R}^n$ is a non-compact symmetric space. If $\langle \cdot, \cdot \rangle_0$ is the standard inner product on $\mathbb{R}^n$ then any other inner product can be written as $\langle u, v \rangle = \langle Lu, v \rangle_0$ for L some self adjoint linear map. Therefore $M = \{A \in \mathrm{GL}_n(\mathbb{R}) | A = A^T, A > 0\}$ is the space of positive definite symmetric matrices. The involution at the identity is $s_{Id}(A) = A^{-1}$.

The Lie group $\mathrm{GL}_n(\mathbb{R})$ acts transitively by isometries on M via $g \cdot A = gAg^T$. The stabilizer of Id is $\mathrm{O}(n)$ and $M = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$. The Cartan involution is given by $\theta(X) = -X^T$. The involutive automorphism is given by $\sigma(A) = (A^T)^{-1}$ and has fixed point set $\mathrm{O}(n)$. We get the Cartan decomposition for $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k}$ the skew symmetric matrices and $\mathfrak{p}$ the symmetric matrices.

The subset $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ is a totally geodesic submanifold.

1.2.3 Symmetric pairs and geodesic submanifolds

Definition 1.2.8. A symmetric pair (G, K, σ) is Lie group G, a compact subgroup K, and an involution σ with $G_0^\sigma \subset K \subset G^\sigma$ where G acts on G/K with discrete kernel.

The condition of G acting on G/K with discrete kernel is equivalent to saying that in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{g}$ and $\mathfrak{p}$ do not have any ideal in common.

Exercise. If G/K is a symmetric space, (G, K) may not always be a symmetric pair. One non-example is the pair $(\mathrm{SU}(n), \mathrm{SU}(n-1)$ with symmetric space $\mathbb{S}^{2n-1}$ because there are no automorphisms σ with $\mathrm{SU}(n)^\sigma = \mathrm{SU}(n-1)$. However, for $\mathbb{S}^n$, $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ is a symmetric pair.

If on $M = G/K$ with $K = \mathrm{Stab}_G(p_0)$ we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathrm{Ad}_K(\mathfrak{p}) \subset \mathfrak{p}$. We can then identify $\mathfrak{p}$ with $T_{p_0}M$ via $X \to X^*(p_0)$ where $X^*(p) = \frac{d}{dt}_{|t=0}(\exp(tX) \cdot p)$.

Proposition 1.2.9. Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. If $X \in \mathfrak{p}$ then $\gamma(t) = \exp(tX) \cdot p_0$ is the geodesic in M with $\gamma(0) = p_0$ and $\gamma'(0) = X \in \mathfrak{p} \simeq T_{p_0}M$ (from §1.1.3).

Proposition 1.2.10.
- If M is a symmetric space and $N \subset M$ is a submanifold such that for all $p \in M$, $s_p(N) = N$, then N is totally geodesic and symmetric.
- Let $\sigma : G \to G$ be an involutive automorphism and G/K the corresponding symmetric space. If $L \subset G$ with $\sigma(L) \subset L$, then $L/(L \cap K)$ is a symmetric space such that $L/(L \cap K) \subset G/K$ is totally geodesic.

Proposition 1.2.11. Let $\mathbf{G}/K$ be a symmetric space corresponding to the Cartan involution σ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. If $\mathfrak{a} \subset \mathfrak{p}$ is a linear subspace with $[[\mathfrak{a}, \mathfrak{a}], \mathfrak{a}] \subset \mathfrak{a}$ then $\exp(\mathfrak{a})$ is a totally geodesic submanifold.

Definition 1.2.12. A flat in M is a totally geodesic Euclidean submanifold in M. It is called maximal is it is not properly contained in another flat.

The next proposition relates the rank of a symmetric space with rank of a Lie algebra.

Proposition 1.2.13. Let M be a symmetric space with $G = \mathrm{Isom}_0(M)$.

- Let $p \in M$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition corresponding to p. There is a bijection between maximal abelian subalgebras of $\mathfrak{p}$ and maximal flats of M containing p induced by the map $\mathfrak{a} \mapsto \exp(\mathfrak{a}) \cdot p$.
- Every geodesic is contained in some maximal flat.
- For any two maximal flats F_1, F_2 in M, there is some $g \in G$ such that $g \cdot F_1 = F_2$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra for $G = \mathrm{Isom}_0(M)$. For all $X \in \mathfrak{p}$, $\mathrm{ad}(X)$ is self-adjoint with respect to the B_θ from the Cartan involution.

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{p}$, i.e. a maximal abelian subalgebra and define for any $\alpha \in \mathfrak{a}^*$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \mathrm{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

$\Delta = \{\alpha \in \mathfrak{a}^* | \mathfrak{g} \neq 0\}$.

Definition 1.2.14. The restricted root space decomposition is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Unlike the root space decomposition for complex semisimple Lie algebras, $\mathfrak{g}_0$ might not be $\mathfrak{a}$ and $\mathfrak{g}_\alpha$ might not always be 1-dimensional.

1.2.4 Classification

Definition 1.2.15. Let (G, K) be a symmetric pair with B the Killing form of $\mathfrak{g}$. The pair is of compact type if $B|_{\mathfrak{p}} < 0$, of Euclidean type if $B|_{\mathfrak{p}} = 0$, or of non-compact typeif $B|_{\mathfrak{p}} > 0$.

A symmetric space is irreducible if it is not a product of two or more symmetric spaces.

Proposition 1.2.16. Let (G, K) be a symmetric pair and $M = G/K$.

- If M is irreducible, (G, K) is either of compact type, Euclidean type, or non-compac type.
- If M is simply connected, then M is isometric to a Riemannian product $M = M_0 \times M_1 \times M_2$ with M_0 of Euclidean type, M_1 of compact type, and M_2 of non-compact type.
- If (G, K) is of compact type then G is compact and semisimple and M is compact with non-negative curvature.
- If (G, K) is of non-compact type then G is non-compact and semisimple and M is non-compact with non-positive curvature.
- (G, K) is of Euclidean type if and only if $[\mathfrak{p}, \mathfrak{p}] = 0$. If M is simply connected then it must be isometric to $\mathbb{R}^n$ with the Euclidean metric.

Proposition 1.2.17. If (G, K) is a symmetric pair of non-compact type then the inner product $B^*(X, Y) = -B(\sigma(X), Y)$ of $\mathfrak{g}$ satisfies

- B^* is positive definite.
- If $X \in \mathfrak{k}$ then $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is skew symmetric.
- If $X \in \mathfrak{p}$ then $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is symmetric.

Proposition 1.2.18. If (G, K) is a symmetric pair of non-compact type with Cartan involution σ.

- G is non-compact and semisimple and G^σ and K are connected.
- K is a maximal compact subgroup of G.
- The center of G is in K.
- G is diffeomorphic to $K \times \mathbb{R}^n$ and G/K is simpl-connected and diffeomorphic to $\mathbb{R}^n$.

For a symmetric pair of non-compact type, the Cartan involution and corresponding Cartan decomposition are unique up to inner automorphism.

Proposition 1.2.19. Let (G, K) be a symmetric pair of non-compact type where G acts on G/K with trivial kernel. Then there exists an isometric embedding of G/K into $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ with totally geodesic image, given by $gK \mapsto \mathrm{Ad}(g) \cdot \mathrm{SO}(n)$.

Chapter 2

Introduction to non-abelian Hodge theory

Lorenzo Ruffoni (FSU)

It is a theory that gives a correspondence between

(i) (representations $\rho : \pi_1 S \to G$) where S a closed surface, $g \geq 2$, G reductive complex Lie group (e.g. $\mathrm{GL}_n\mathbb{C}$), and

(ii) (Higgs bundles on Riemann surfaces) i.e. X a Riemannan surface structure on S + a bunch of holomorphic objects on it.

Note that the object (i) is topological, and easy to define but "hard to access" ("few tools available"); object (ii) is holomorphic, and harder to define, but "easier to access".

2.1 (Abelian) Hodge theory

(Often $G = \mathrm{GL}(V)$; (non)-abelian refers to G being (non)-abelian. $\mathrm{GL}(V)$ is abelian iff V 1-dimensional, i.e. $G = \mathbb{C}^* = \mathrm{GL}_1\mathbb{C}$.) Classically, we have the moduli spaces

$\mathcal{M}_{Betti}(S, G) = \mathrm{Hom}(\pi_1 S, \mathbb{C}^*) = \mathrm{Hom}(H_1(S), \mathbb{C}^*) = H^1(S, \mathbb{C}^*)$.

$\mathcal{M}_{dR}(S, G) = H^1_{dR}(S, \mathbb{C})$ i.e. closed 1-forms mod exact 1-forms.

$\mathcal{M}_{Dol}(X) = H^{1,0}(X) \oplus H^{0,1}(X)$ (after fixing X some complex structure on S, where $H^{p,q}(X) = \frac{\bar\partial\text{-closed } (p,q)\text{-forms}}{\bar\partial\text{-exact } (p,q)\text{-forms}}$).

These moduli spaces are diffeomorphic (the Betti modul space is the baby version of (i); the Dolbeaut moduli space is the baby version of (ii).

2.1.1 de Rham complex

Consider the bundle $\bigwedge^k T^*M \to M$ whose smooth sections are k-forms. We write $\mathcal{A}^k(M)$ for the space of smooth k-forms on M. If we can equip M with a complex structure $J : TM \to TM$ (i.e. a $\mathbb{R}$-linear endomorphism such that $J^2 = \mathrm{id}$). Then $TM \otimes \mathbb{C}$ splits into eigenspaces for J: $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ (eigenvalues $\pm i$ resp.) Linear algebra then gives us $\bigwedge^k (T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=k} \bigwedge^{p,q}(T^*M \otimes \mathbb{C})$. Let $\mathcal{A}^{p,q}(M)$ denote the space of smooth (p,q)-forms on M, i.e. sections of $\bigwedge^{p,q}(T^k M \otimes \mathbb{C})$. By definition, a (p,q)-form locally looks like a sum of $\alpha_{i_1 \cdots i_p j_1 \cdots j_q}(z)\, dz_{i_1} \cdots dz_{i_p} d\bar{z_{j_1}} \cdots d\bar{z_{j_q}}$. Moreover, we have splittings

$$\mathcal{A}^1 = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1} \ ;$$

$$\mathcal{A}^2 = \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1} \oplus \mathcal{A}^{0,2} \ .$$

If J is integrable [1] (i.e. if J comes from a complex atlas on M) we can also define holomorphic sections: $T^*M \otimes \mathbb{C} \to M$ is a holomorphic bundle (the projection is holomorphic.)

We have an exterior differential $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$. Using J, the differential splits (on a surface):

$$d = \partial \oplus \bar{\partial}$$

with

$$\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q} \quad \text{and} \quad \bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1} \ .$$

Note that $\bar{\partial} = 0$ on surfaces. We get the Dolbeault complex

$$\cdots \to \mathcal{A}^{p,q}(S) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,q+1}(S) \to \ldots \ .$$

which gives the Dolbeault cohomology groups

$$H^{p,q}(X) = \frac{\ker \bar{\partial}}{\mathrm{im} \bar{\partial}}$$

2.1.2 Abel-Jacobi theorem

Recall that $H^1(S, \mathbb{C}^*)$ is the space of representations $\pi_1 S \to \mathbb{C}^*$. We want to relate this object with $T^* \operatorname{Jac}(X)$, where X is a fixed complex structure on S and $\operatorname{Jac}(X)$ is the Jacobian of X i.e. the space of degree-0 line bundles on X.

Definition 2.1.1. X a Riemann surface: $L \to X$ a line bundle if it is holomorphic vector ball of rank 1. L is degree 0 if and only if $L \cong X \times \mathbb{C}$ smoothly (i.e. bundle is topologically / smoothly [but not necc. holomorphically] trivial.)

Theorem 2.1.1 (Abel-Jacobi)**.** $\operatorname{Jac}(X)$ is diffeomorphic to a complex torus, $\cong \mathbb{C}^g/\mathbb{Z}^{2g}$

This is the main idea of the correspondence between $T^* \operatorname{Jac}(X)$ and $H^1(S, \mathbb{C}^*)$.

$$H^1(S, \mathbb{C}^*) = \frac{H^1(S, \mathbb{C})}{H^1(S, \mathbb{Z})}$$

by universal coefficient theorem applied to the exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \underset{\exp}{\to} \mathbb{C}^* \to 0$. Then

$$\frac{H^1_{dR}(S, \mathbb{C})}{H^1(S, \mathbb{Z})} = \frac{H^{1,0}(X) \oplus H^{0,1}(X)}{H^1(S, \mathbb{Z})} \ .$$

The last step is a black box from harmonic theory (Hodge theory): every 1-form (resp. (p,q)-form) has a unique harmonic representative. We can rewrite the last term as $H^{1,0}(X) \oplus H^{0,1}(X) = H^0(X, \Omega^1) \oplus H^1(X, \Omega^0)$ where Ω^k the sheaf of holomorphic k-forms on X. The first summand is the space of holomorphic 1-forms; the second is $H^0(X, \Omega^1)^*$ by Serre duality. Hence,

$$H^1(S, \mathbb{C}^*) = \cdots = H^0(X, \Omega^1) \oplus H^0(X, \Omega^1)^* / H_1(S, \mathbb{Z}) = T^* \operatorname{Jac}(X)$$

by Abel–Jacobi theory.

In summary:

$$\operatorname{Hom}(\pi_1 S, \mathbb{C}^*) \leftrightarrow H^1_{dR}(S, \mathbb{C}^*) \leftrightarrow T^* Jac(X)$$

[1] This is free on orientable surfaces

Nonabelian version. $\mathrm{Hom}(\pi_1 S, G) \leftrightarrow$ flat bundles $(E, \nabla) \leftrightarrow$ Higgs bundles $(E, \mathcal{F}, \varphi)$
(There is the big hammer of harmonic theory hiding in the second $\leftrightarrow$.)

2.2 Bundles

Definition 2.2.1. Say $G \curvearrowright F$. A fiber bundle with fiber F and structure group G is $E \xrightarrow{\pi} B$ where E and B are spaces, π is a continuous surjection, and $\exists \mathcal{U} = \{U_\alpha\}$ an open cover of B s.t. $\exists \varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism, s.t. for all α, β with $U_\alpha \cap U_\beta \neq \varnothing$ we have $\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ given by $(u, f) \mapsto (u, g_{\alpha\beta}(u)(f))$ with $g_{\alpha\beta}(u) \in G$
$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ defines a 1-cocycle $\{g_{\alpha\beta}\} \in H^1(B, G)$ in the sense of Čech cohomology.

Remark 2.2.2. If G is abelian, $H^1(B, G)$ is a group. If G is not, $H^1(B, G)$ makes sense as a set but is not a group.

If G is a complex Lie group, then E is smooth / holomorphic / flat if $\{g_{\alpha\beta}\}$ is smooth / holomorphic / locally constant (resp.)

Definition 2.2.3. A flat bundle over S is a smooth bundle with locally constant cocycles $g_{\alpha\beta}$.

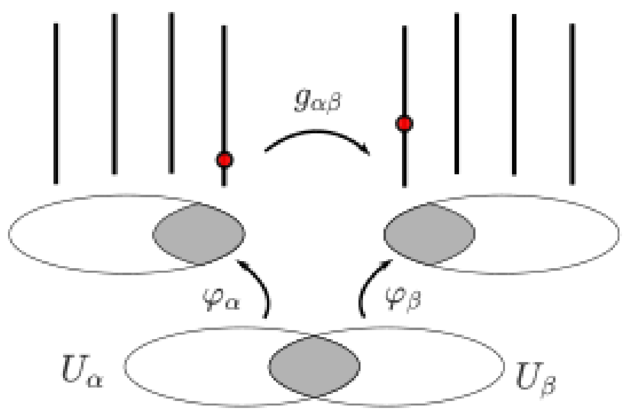


Figure 2.1: Flat bundles, a schematic

Example 2.2.4. Cylinder vs. Möbius band, viewed as flat $\mathbb{R}$-bundles over S^1:

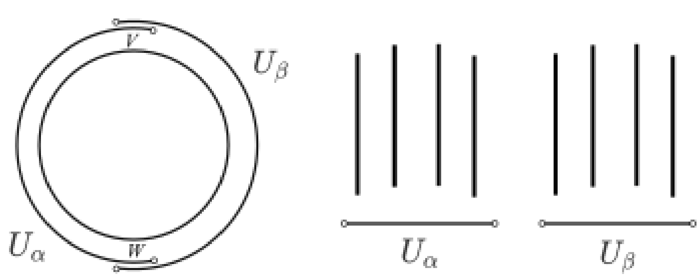


Figure 2.2: For the cylinder, $g_{\alpha\beta} \equiv 1$; for the Möbius band, $g_{\alpha\beta} = 1$ on V and -1 on W, where $U_\alpha \cap U_\beta =: V \coprod W$ (as labelled above.)

In this example $G = \mathbb{Z}_2 \curvearrowright \mathbb{R}$ ($1 \mapsto \mathrm{id}$, $-1 \mapsto (t \mapsto -t)$)

Let $E \to S$ be a flat bundle. Recall that $(U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap \beta) \times F$ given by $(u, f) \mapsto (u, g_{\alpha\beta}(f))$ where $g_{\alpha\beta} \in G$. Define on U_α a ("horizontal") foliation $\mathcal{F}_\alpha = \coprod_{f \in F} \mathcal{F}_{\alpha,f}$, where $\mathcal{F}_{\alpha,f} = \{(u, f) \mid u \in U_\alpha\}$.

Fact 2.2.5. $g_{\alpha\beta}$ constant $\implies$ $\mathcal{F}_\alpha$ glue to a global foliation $\mathcal{F}$ on E transverse to fibers.

In particular, get ("horizontal") complements to vertical[2] subspaces of TE. This induces a natural lift of paths from S to E with paths contained in $\mathcal{F}$ and a on of parallel transport. In particular a map from (loops in S) (in fact, $\pi_1 S$) to $\mathrm{Aut}(F) = G$ (the holonomy of the flat bundle.)

This gives us a map {flat bundles with structure grp G} $\hookrightarrow \chi(\pi_1(S), G)$.

To construct the inverse: let $\rho : \pi_1 S \to G$; choose any F s.t. $G \curvearrowright F$. Then consider the bundle E_ρ defined by

$$E_\rho = \tilde{S} \times F / \sim$$

where $(s, f) \sim (\gamma s, \rho(\gamma) f)$.

From now on: E smooth complex vector bundle; $G = \mathrm{GL}_n\mathbb{C}$; $F = \mathbb{C}^n$. On vector bundles, flat structures can be easily recorded by linear / differential tools.

Definition 2.2.6. A connection on E is a map $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$\nabla(fs) = df \cdot s + f \nabla s$$

where $s \in \mathcal{A}^0(E)$ and $f \in C^\infty(S)$.

∇ extends to all $\mathcal{A}^k(E)$ imposing Leibiniz rule.

Definition 2.2.7. The curvature of ∇ is $F_\nabla = \nabla^2 : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$. ∇ is flat if $F_\nabla = 0$.

Remark 2.2.8. A connection is a map $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$. The difference of 2 connections ∇_1, ∇_2 is a C^∞-linear map $\mathcal{A}^0(E) \to \mathcal{A}^1(E)$;

$$\nabla_1 - \nabla_2 \in \mathcal{A}^1(\mathrm{End}E).$$

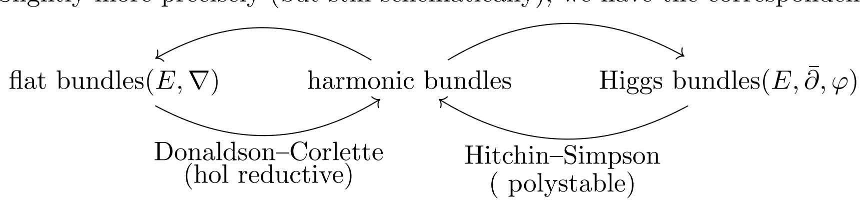
2.2.1 Higgs bundles

We describe the correspondence {flat rank-n degree-0 complex vector bundles on S} $\leftrightarrow$ {Higgs bundles on X} (given a Riemann surface structure X on S).

Definition 2.2.9. A Higgs bundle on X is a triple $(E, \bar{\partial}, \varphi)$ where $(E, \bar{\partial}) \to X$ is a holomorphic vector bundle on X (rank-n, degree-0), and $\varphi \in \Omega^1(X, \mathrm{End}E)$ a holomorphic 1-form valued in $\mathrm{End}E$.

This correspondence goes through harmonic bundles; the $\to$ direction was done by Corlette–Donaldson in '86; the $\leftarrow$ direction follows from results of Hitchin in '87 and Simpson in '92)

Slightly more precisely (but still schematically), we have the correspondences



Theorem 2.2.10. Given E a smooth complex vector bundle on X, $\bar{\partial}$ a holomorphic structure on E, h a Hermitian metric on E, there exists a unique connection ∇ (the Chern connection) on E such that

[2] i.e. $\ker d\pi$

(1) $\nabla h = 0$ i.e. ∇ is unitary / compatible w.r.t. h;

(2) $\nabla^{0,1} = \bar{\partial}$

However, $F_\nabla \neq 0$ in general. We twist this connection in order to get a flat one.

Any other connection D on E can be split into $A + \phi$ (using metric h: $Ah = 0$.) Using complex structure X, split $A = A^{1,0} + A^{0,1} + \phi^{1,0} + \phi^{0,1}$.

Definition 2.2.11. A harmonic bundle on X is $(E, \nabla, \bar{\partial}, \varphi, h)$ where

- E is a smooth complex vector bundle,
- ∇ is a flat connection,
- $\bar{\partial}$ is a holomorphic structure,
- h is a Hermitian metric (harmonic), and
- φ is a holomorphic 1-form with values in End$E$

such that $\nabla = A^{1,0} + A^{0,1} + \phi^{1,0} + \phi^{0,1}$, $\varphi = \phi^{1,0}$, $\bar{\partial} = A^{0,1}$, $A^{0,1}\phi^{1,0} = 0$

Definition 2.2.12. A metric h is harmonic if and only if the $(\pi_1(S), \rho)$-equivariant map $h : \tilde{X} \to \mathrm{GL}_n\mathbb{C}/\mathrm{SU}(n)$ is harmonic.

If (E, ∇) has reductive holonomy, i.e. writing $\Gamma := \overline{\mathrm{hol}_\nabla(\pi_1 S)}^Z$, if $\mathrm{Ad}\,\Gamma \curvearrowright \mathfrak{g}$ is completely reducible, then (by Donaldson-Corlette) such harmonic map exists, and hence a harmonic bundle.

For the reverse, given $(E, \bar{\partial}, \varphi)$ a Higgs bundle, can we find h harmonic (i.e. ∇ flat)? Yes, (by Hitchin-Simpson) if and only if $(E, \bar{\partial}, \varphi)$ is polystable:

Definition 2.2.13. Let E be a holomorphic vector bundle of degree 0. A Higgs bundle $(E, \bar{\partial}, \varphi)$ is stable if for any holomorphic subbundle $L \subset E$, $\varphi(L) \subset L \implies \deg L < 0$. It is polystable if it is a direct sum of stable bundles.

Remark 2.2.14. Stable bundles correspond to irreducible representations; polystable bundles correspond to completely reducible representations.

Example 2.2.15. If $\varphi = 0$, any stable (i.e. any subbundle $L \subset E$ has $\deg L < 0$) holomorphic vector bundle E gives a stable Higgs bundle $(E, 0)$. This recovers the following theorem

Theorem 2.2.16 (Narasimhan-Seshadri)**.** $\{$(stable) rank-n degree-0 holomorphic vector bundles on $X\} \leftrightarrow \{$unitary representations $\pi_1 X \to \mathrm{SU}(n)\}$

Chapter 3

Complex Projective Structures

Jane Wang (MIT)

3.1 Introduction

In these notes, we introduce complex projective structures and describe various approaches to the subject. These notes largely follow Dumas's Complex Projective Structures [Dum09].

We will start by defining projective structures in section 3.2 and giving some examples. We will let $\mathcal{P}(S)$ denote the set of projective structures on a surface, and $P(X)$ be the set of projective structures with a particular underlying Riemann surface structure. Then, we will use the theory of Schwarzian derivatives in section 3.3 to better understand the structure of $P(X)$. One of the main results that we will discuss is that there is an isomorphism,

$$P(X) \cong Q(X),$$

from the set of projective structures on a Riemann surface X to the set of holomorphic quadratic differentials on the same surface. Moving on, in section 3.4, we will introduce the grafting construction that creates projective structures from a complex structure and a measured lamination

$$\text{Gr: } \mathcal{ML} \times \mathcal{T}(S) \xrightarrow{\sim} \mathcal{P}(S),$$

and will see that this map is a homeomorphism. Furthermore, fixing any $\lambda \in \mathcal{ML}$ above defines a homeomorphism. We will also see that there is a natural conformal metric on a projective structure coming from grafting and will give some properties of this metric. In section 3.5, we will examine the holonomy map $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbb{C})$, and how the holonomy map interacts with the Schwarzian parametrization and with the grafting construction. We will see that the projective structures with Fuchsian holonomy can be parametrized exactly by pairs (λ, X) where $\lambda \in \mathcal{ML}_{\mathbb{Z}}$ the integral measured laminations, and $X \cong \mathbb{H}^2/\Gamma$ where Γ is a Fuchsian group, and every $Z \in \mathcal{P}(S)$ can be uniquely represented by grafting

$$Z = \mathrm{Gr}_{2\pi\lambda}(X).$$

Similarly, we will find that those projective structures with quasi-Fuchsian holonomy also break up nicely into

$$\mathcal{P}_{\mathcal{QF}}(S) = \bigcup_{\lambda \in \mathcal{ML}_{\mathbb{Z}}(S)} \mathcal{P}_\lambda(S),$$

where hol: $\mathcal{P}_\lambda(S) \xrightarrow{\sim} \mathcal{QF}(S)$, the space of quasi-Fuchsian holonomies on S. In section 3.6, we briefly review the different perspectives that we have covered, and finish with some results about the compactification of the space of projective structures. We will see that the Schwarzian and grafting perspectives are related via their compatifications of $P(X)$.

3.2 The basics

3.2.1 Definition of a projective structure

We start with S an oriented surface. Then, a complex projective structure Z on S is a maximal atlas of charts on S mapping into $\mathbb{CP}^1$ such that the transition maps are Möbius transformations, maps of the form

$$f(z) = \frac{az+b}{cz+d}.$$

We will often refer to complex projective structures as projective structures, for short. We say that two projective structures Z_1 and Z_2 on S are isomorphic if there is an orientation preserving diffeomorphism between Z_1 and Z_2 that pulls pack that projective charts of one surface to the other, and marked isomorphic if the diffeomorphism can be made homotopic to the identity.

Example 3.2.1.

1. **Genus 0.** Up to isotopy, the sphere has a unique projective structure given by $S^2 \cong \mathbb{CP}^1$. Here, we take a maximal atlas of open charts on S^2 and the transition functions are just the identity function.

2. **Genus 1.** Projective structures on a torus are all affine structures. For example, think of the structure on the torus coming from $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ or $\mathbb{C}^*/(z \sim 2z)$. In the first example, the transition functions are translations. In the second, the transition functions are of the form $f(z) = 2^k z$ for $z \in \mathbb{Z}$.

3. **Fuchsian groups and hyperbolic manifolds.** A Fuchsian group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Then, there is a natural projective structure on the hyperbolic manifold $\mathbb{H}/\Gamma$.

4. **Kleinian groups and the ideal boundary of hyperbolic 3-manifolds.** A Kleinian group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Let $\Omega(\Gamma)$ be the domain of discontinuity of Γ, the complement of the limit set of in $\mathbb{CP}^1$ of Γp, for any $p \in \mathbb{CP}^1$. Then, $\Omega(\Gamma)/\Gamma$ is the ideal boundary of a hyperbolic 3-manifold and has a natural projective structure.

5. **Quasi-Fuchsian groups.** A special type of Kleinian group is a quasi-fuchsian group $Q(X,Y)$, a group whose limit set in $\mathbb{CP}^1$ is a Jordan curve. This Jordan curve separates the sphere into two regions, whose quotients by $Q(X,Y)$ are the Riemann surfaces X and $\overline{Y}$.

In general, we will only be working with genus $g \geq 2$ surfaces in these notes.

Remark 3.2.2. Projective structures can also be thought of in the context of (G,X) structures, manifolds M described by an atlas of charts into X with transition functions in the group G. Then, projective structures are $(\mathrm{PSL}_2(\mathbb{C}), \mathbb{CP}^1)$ structures.

Remark 3.2.3. A projective structure is always subordinate to a complex structure, so there is a forgetful map

$$\pi : \mathcal{P}(S) \to \mathcal{T}(S)$$

from the space of projective structures to the Teichmüller space of complex structures. This map is surjective because every point in Teichmüller space arises as $\mathbb{H}/\Gamma$ for some Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$.

3.2.2 Holonomy and developing map

A projective structure Z on a surface S always lifts to a projective structure $\tilde{Z}$ on the univeral cover $\tilde{S}$. For any Z on S, we can find a developing map $f : \tilde{S} \to \mathbb{CP}^1$ such that the restriction of f to sufficiently small open sets on S are projective charts. Developing maps are unique up to postcomposition by a Möbius transformations, and can be thought of intuititively as a way to "unroll" the geometric structure of Z onto $\mathbb{CP}^1$.

Let $f : \tilde{S} \to \mathbb{CP}^1$ be a developing map for the projective structure Z on S. Then, for any $\gamma \in \pi_1(S)$, $f \circ \gamma$ is another developing map, and

$$f \circ \gamma = A_\gamma \circ f.$$

The map $\gamma \mapsto A_\gamma$ is a homomorphism $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ called the holonomy representation of Z. It is well defined up to the action of conjugation by $\mathrm{PSL}_2(\mathbb{C})$.

Every projective structure Z can be represented as a development-holonomy pair (f, ρ) where f is the developing map and ρ is the holonomy representation. This pair is uniquely determined up to the action $(f, \rho) \mapsto (A \circ f, A\rho A^{-1})$ by elements $A \in \mathrm{PSL}_2(\mathbb{C})$. We can give pairs (f, ρ) the compact-open topology, and $\mathcal{P}(S)$ then inherits the quotient topology.

3.3 The complex analytic approach

We saw in the previous section that there is surjective map $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ from the space of projective structures to the space of complex structures on a surface. The complex analytic approach that we pursue in this section will show us that we can identify the fibers of this map with the space of quadratic differentials on a Riemann surface.

Quadratic differentials. A quadratic differential is a section of the square of the cotangent bundle of a Riemann surface. Often, we will be concerned only with those quadratic differentials that are holomorphic. The vector space of holomorphic quadratic differentials on X will be denoted as $\mathcal{Q}(X)$, and the space of all quadratic differentials on a surface S is a complex vector bundle $\mathcal{Q}(S) \to \mathcal{T}(S)$. The bundle $\mathcal{Q}(S)$ is identified with the holomorphic cotangent bundle of Teichmüller space and is homeomorphic to $\mathbb{R}^{12g-12}$. Teichmüller space is homeomorphic to $\mathbb{R}^{6g-6}$.

The Schwarzian derivative. Let $f : \Omega \to \mathbb{CP}^1$ be a locally injective holomorphic map from a connected open subset of $\mathbb{C}$ into $\mathbb{CP}^1$. Then, the Schwarzian derivative of f is the holomorphic quadratic differential

$$S(f) := \left[\left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right] dz^2.$$

The Schwarzian derivative satisfies two nice properties:

1. Cocycle property. For locally injective holomorphic maps f and g such that $f \circ g$ is defined, we have that
$$S(f \circ g) = g^* S(f) + S(g).$$

2. Möbius invariance. f is a restriction of a Möbius function if and only if $S(f) \equiv 0$.

We notice that maps f are almost determined by their Schwarzian derivatives. That is, two functions with the same Schwarzian derivative are related to one another by a Möbius transformation. The intuition behind the Schwarzian derivative is that it somehow measures the failure of a holomorphic map to be the restriction of a Möbius transformation.

Identifying projective structures and quadratic differentials. Let $X \in \mathcal{T}(S)$, and let $P(X) = \pi^{-1}(X) \subset \mathcal{P}(S)$. Suppose that we want to parametrize the fiber $P(X)$. We can do this with the Schwarzian derivative as follows. Given $Z \in P(X)$, we first identify $\tilde{X} \cong \mathbb{H}$, where $\pi_1(X)$ acts on $\mathbb{H}$ as a Fuchsian group. Then, the developing map $f : \tilde{X} \cong \mathbb{H} \to \mathbb{CP}^1$ of Z is a meromorphic function on $\mathbb{H}$. The Schwarzian derivative $\tilde{\varphi} = S(f)$ is then a holomorphic quadratic differential on $\tilde{X}$ that is invariant under the action of $\pi_1(X)$. Hence, $\tilde{\varphi}$ descends to a holomorphic quadratic differential on X, and is called the Schwarzian of the projective structure. Thus, the Schwarzian defines a map

$$P(X) \to Q(X).$$

This map is bijective since given a quadratic differential $\tilde{\varphi}(z)\,dz^2$ on $\mathbb{H}$ that is a lift of a quadratic differen tial on X, there exists a locally injective function f for which $S(f) = \varphi(z)\,dz^2$. Such a solution is found by taking the quotient $f(z) = u_1(z)/u_2(z)$ of two basis solutions to the Schwarzian equation

$$u''(z) + \frac{1}{2}\varphi(z)u(z) = 0.$$

This discussion shows that fibers of the map $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ are naturally parametrized by a complex vector space.

Affine naturality. There are actually many ways in which $P(X) \cong Q(X)$. Our identification above depended on our choice of coordinate for $\tilde{X}$ coming from the standard Fuchsian structure on X. We could have started with a different coordinate and obtained a different Schwarzian derivative. If $Z_1, Z_2 \in P(X)$, then we can define $\varphi(Z_2 - Z_1)$ in local charts by $z_1^* S(z_2 \circ z_1^{-1})$ as the Schwarzian of Z_2 relative to Z_1. That is, we are computing the Schwarzian derivative for Z_2 in the coordinates of Z_1. Then, $P(X)$ has a natural structure of an affine space modeled on $Q(X)$. Choosing a basepoint $Z_0 \in P(X)$ gives an isomorphism $P(X) \xrightarrow{\sim} Q(X)$ by $Z \mapsto \varphi(Z - Z_0)$.

Schwarzian parametrization. Now, if we choose any section $\sigma : \mathcal{T}(S) \to \mathcal{P}(S)$, we can form a bijective Schwarzian parametrization between $\mathcal{P}(S)$ and $\mathcal{Q}(S)$,

$$Z \mapsto (\pi(Z), \varphi(Z - \sigma(\pi(Z)))).$$

For any continuous section σ, the Schwarzian parametrization is a homeomorphism. The Schwarzian parametrization also transports the complex manifold and holomorphic vector bundle structure of $\mathcal{Q}(S)$ to $\mathcal{P}(S)$. Two σ_i induce the same complex structure if and only if $\varphi(\sigma_1 - \sigma_2)$ is a holomorphic section of $\mathcal{Q}(S)$.

3.4 The geometric approach

In this section, we will examine how we can create projective structures by gluing together simple pieces. This will give us a way to visualize the geometry of projective structures.

Conformal grafting. Before discussing grafting related to projective structures, let us first define conformal grafting. We first recall that a conformal metric is an equivalence class of metrics, where two metrics f and g are equivalent if and only if $f = \rho g$ for some positive function $\rho > 0$. We will now describe a grafting procedure along curves in $\mathcal{S}$, the set of the simple closed curves on a surface,

$$\mathrm{gr} : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \to \mathcal{T}(S).$$

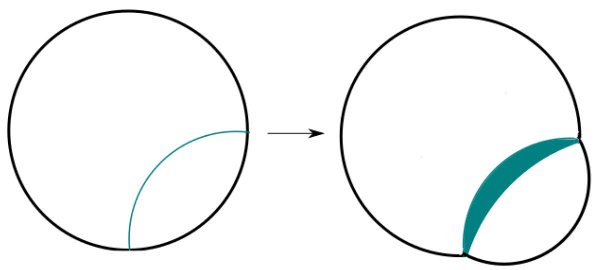
Suppose that we start with a curve $\gamma \in \mathcal{S}$, $t \in \mathbb{R}^+$, and $X \in \mathcal{T}(S)$. Then, to perform the grafting map, we equip X with its hyperbolic metric and find the geodesic representative of γ. Then, we cut along γ and glue in a Euclidean cylinder of height t. The hyperbolic metric on $X - \gamma$ and the Euclidean metric on the cylinder then give us a well-defined conformal structure on $\mathrm{gr}_{t\gamma}X$, and therefore also give us a complex structure.

Projective grafting. We can use a variant of this construction to create projective structures starting from complex structures and simple closed curves. This will be a map

$$\mathrm{Gr} : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \to \mathcal{P}(S).$$

To do this, let γ be a simple closed curve on a Riemann surface X. Up to conjugation, the holonomy of γ is $z \mapsto e^\ell z$, where ℓ is the hyperbolic length of the curve γ. Then, for any $0 < t < 2\pi$, let $\tilde{A}_t$ be a sector of angle t in the complex plane with vertex at 0. The quotient annulus $A_t := \tilde{A}_t/(z \sim e^\ell z)$ is then an annulus with a projective structure and holonomy $z \mapsto e^\ell z$ through the radius curve. We can then create a projective structure by inserting A_t into the standard Fuchsian structure on X.

In the universal cover, this looks like inserting a copy of $\tilde{A}_t$ for every lift of γ, and transforming the complementary regions by Möbius transformations so that the pieces fit together.



In the above figure, we show a schematic of this process where we graft in one copy of $\tilde{A}_t$ along one lift of γ in $\tilde{X}$. This process looks like inserting a lune at every lift of γ.

Projective grafting along simple closed curves is a lift of conformal grafting through the forgetful map $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$. That is,

$$\pi \circ \mathrm{Gr} = \mathrm{gr}.$$

Grafting along laminations. We recall that there is a space $\mathcal{ML}(S)$ of measured laminations, a completion of the space of the weighted simple closed curves. The topology of this space is given by realizing points $\lambda \in \mathcal{ML}(S)$ as points in $\mathbb{R}^{\mathcal{S}}$ by taking their intersection numbers with simple closed curves. In this way, $\mathcal{ML}(S)$ is also a piecewise linear manifold homeomorphic to $\mathbb{R}^{6g-g}$. There is a continuous extension

$$\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S).$$

Similarly, there is a continuous extension of the conformal grafting map. Grafting along a measured lamination is like taking the leaves of the measured lamination and thickening each one.

3.4.1 Thurston's Theorem

It turns out that every projective structure can be obtained from projective grafting in a unique way. This is the content of the following theorem.

Theorem 3.4.1 (Thurston)**.** The projective grafting map $\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$ is a homeomorphism.

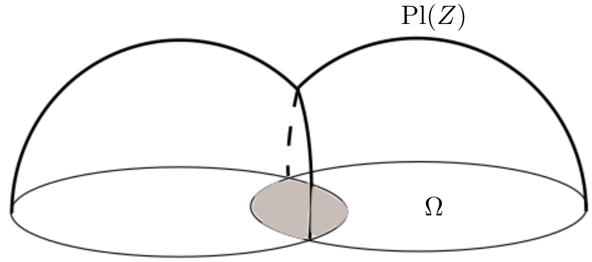
Proof sketch. Let us first suppose that $Z \in \mathcal{P}(S)$ is a projective structure such that the developing map $f : \tilde{S} \to \mathbb{CP}^1$ is an embedding. Let Ω be the image of the developing map. We will now describe a process to invert the projective grafting map and obtain a measured lamination and complex structure from Z.

1. Thinking of $\mathbb{CP}^1$ as the ideal boundary of $\mathbb{H}^3$, let $\mathrm{Pl}(Z)$ denote the boundary of the convex hull of $\mathbb{CP}^1 - \Omega$. $\mathrm{Pl}(Z)$ is then a convex pleated plane in $\mathbb{H}^3$, an copy of $\mathbb{H}^2$ mapped isometrically into $\mathbb{H}^3$, except that it is bent along some geodesics. With the path metric, $\mathrm{Pl}(Z)$ is actually isometric to $\mathbb{H}^2$. We notice that $\pi_1(X)$ acts on $\mathrm{PL}(Z)$.

2. We can then define the nearest-point projection map $\kappa : \Omega \to \mathrm{Pl}(Z)$ that sends $z \in \Omega$ to the first point in $\mathrm{Pl}(Z)$ that is touched by an expanding family of horoballs in $\mathbb{H}^3$ based at z.

3. The canonical stratification of Ω then decomposes Ω into 1-dimensional strata that are circular arcs that map homeomorphically by κ onto bending lines of $\mathrm{Pl}(Z)$ and 2-dimensional strata that map are bounded by circular arcs and that map homeomorphically via κ to the totally geodesic pieces of $\mathrm{Pl}(Z)$.

4. In the case that the projective structure came from grafting along a measured lamination λ that is a single simple closed curve, the 1-dimensional strata will sweep out lunes and the 2-dimensional strata will correspond to the complementary regions of $\tilde{\lambda}$ in $\tilde{X}$, and we have recovered the grafting structure.

   In the schematic picture above, Ω is two pieces of $\mathbb{H}$ glued along the sides of the shaded lune. $\mathrm{PL}(Z)$ then looks like a copy of $\mathbb{H}$ creased along one geodesic. The nearest point projection will project the left piece of Ω to the left piece of Pl(Z), the right piece to the right piece, and the lune to the bend line.

5. This process also works if λ is a collection of disjoint simple closed curves, and a limiting argument shows that we can also do this for arbitrary measured laminations.

In general, if the developing map is not an embedding, we can repeat a variation of this process where we do everything locally. We work with patches of Ω at a time and develop to get a locally convex pleated plane in place of $\mathrm{Pl}(Z)$ that might not be embedded. Our nearest-point projection process can also be adapted to work locally. □

3.4.2 The Thurston metric

There is a natural metric on a projective surface called the Thurston metric or the projective metric. We can see this metric either geometrically through grafting, or intrinsically as a variant of the Kobayashi metric. Recall that the Kobayashi metric on a complex manifold is defined in a way such that the length of any vector v in the tangent space is the infimum over all lengths given to it via holomorphically immersed disks equipped with the hyperbolic metric.

With this, the Thurston metric can be defined as either of the following:

1. In a projective structure $\mathrm{Gr}_{t\gamma}X$ resulting from grafting along a simple closed curve, the Thurston metric is the metric that combines the hyperbolic metric on X with the Euclidean metric on the grafted cylinder. This can be extended to projective structures grafted along measured laminations by taking appropriate limits.

2. The Thurston metric is also the same as the projective Kobayashi metric, where the length of a vector in the tangent space is the infimum of its length given by projectively immersed disks with the hyperbolic metric.

We might wonder why these two definitions give the same metric. The intuition goes something like this: for a projective structure resulting from grafting along a simple closed curve γ, the developed image of X looks like $\mathbb{H}$ with lunes inserted along every lift of γ. At points in $\mathbb{H}\backslash\gamma$, the maximal immersed disk matches with the patch of $\mathbb{H}\backslash\gamma$ that the point is in, and therefore the projective Kobayashi metric is just the hyperbolic metric. Each lune is $\mathbb{R}\times[0,1]$ where each $\mathbb{R}$ is a hyperbolic geodesic. There's a one-parameter family of maximally immersed disks such that each of these $\mathbb{R}$s is a geodesic in one of these disks.

Properties of the Thurston metric.

- The Thurston metric is a conformal metric on the Riemann surface, and locally looks like $\rho(z)|dz|$ where $\rho(z)$ is a positive density function.
- There is a continuous function that takes projective structures $Z\in\mathcal{P}(Z)$ to the space of density functions with the topology of local uniform convergence.
- The area form of a conformal metric is $\rho(z)^2|dz|^2$. The Thurston metric gives the surface an area of $4\pi(g-1)+\ell(\lambda,X)$, where $\ell(\lambda,X)$ is the length of the measured lamination with respect to the hyperbolic metric on X. This is because the union of the 2-dimensional strata has area $4\pi(g-1)$, the area of a genus g hyperbolic surface. The grafted region if λ is a simple closed curve then has area $t\ell(\gamma,X)$, and we can extend this to $\mathcal{ML}(S)$ by continuity.

3.4.3 Some theorems about conformal grafting

We finish with some properties of the conformal grafting map and consequences for projective grafting.

Theorem 3.4.2 (Scannell, Wolf)**.** For each $\lambda\in\mathcal{ML}(S)$, the λ-grafting map $\mathrm{gr}_\lambda:\mathcal{T}(S)\to\mathcal{T}(S)$ is a diffeomorphism.

A nice consequence of this theorem is that for any $\lambda\in\mathcal{ML}(S)$, the set of projective structures with grafting lamination λ projects homeomorphically to $\mathcal{T}(S)$ by the forgetful map. Alternatively, we can say that for each $\lambda\in\mathcal{ML}(S)$, there is a smooth section

$$\sigma_\lambda:\mathcal{T}(S)\to\mathcal{P}(S),$$

given by

$$\sigma_\lambda(X)=\mathrm{Gr}_\lambda(\mathrm{gr}_\lambda^{-1}(X)).$$

If we instead fix a complex structure X and vary the measured lamination, the following holds.

Theorem 3.4.3 (Dumas, Wolf)**.** For each $X\in\mathcal{T}(S)$, the X-grafting map $\mathrm{gr}_{\cdot}X:\mathcal{ML}(S)\to\mathcal{T}(S)$ is a homeomorphism.

3.5 Holonomy

We will now examine the holonomy representations of projective structures and how they relate to grafting and the Schwarzian coordinate systems for $\mathcal{P}(S)$.

The character variety. Let $\mathcal{R}(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))$. This is an affine C-algebraic variety as a subset of $(\mathrm{PSL}_2(\mathbb{C}))^N$. The group $\mathrm{PSL}_2(\mathbb{C})$ acts algebraically on $\mathcal{R}(S)$ by conjugation, and we can form the quotient character variety

$$\mathcal{X}(S) := \mathcal{R}(S)//\mathrm{PSL}_2(\mathbb{C}),$$

where the quotient is defined in the sense of geometric invariant theory. This is, the quotient is defined as the algebraic variety Y such that the ring of regular functions on Y is isomorphic to the ring of $\mathrm{PSL}_2(\mathbb{C})$-invariant regular functions on $\mathcal{R}(S)$.

One can check that the points of $\mathcal{X}(S)$ are in bijection with the set of characters, $\mathbb{C}$-valued functions on $\pi_1(S)$ of the form $\gamma \mapsto \mathrm{tr}^2(\rho(\gamma))$. We note that $\mathcal{X}(S)$ is not the same as the quotient $\mathcal{R}(S)/\mathrm{PSL}_2(\mathbb{C})$. Two conjugacy classes in $\mathcal{R}(S)$ may give rise to the same point in $\mathcal{X}(S)$. However, if we restrict our attention to non-elementary representations $\rho \in \mathcal{R}(S)$, those that do not fix a point or ideal point, and do not preserve an oriented geodesic, then things are a bit nicer. In this cases, there is a one-to-one correspondence between conjugacy classes of non-elementary representations and their characters.

3.5.1 The holonomy map

We recall that for any projective structure, the holonomy representation ρ is determined up to conjugacy, and is therefore an element of $\mathcal{R}(S)$. The holonomy representation is always non-elementary. For hyperbolic structures on complex manifolds, the holonomy representation always determines the geometric structure. For projective structures, this only holds locally.

Theorem 3.5.1 (Hejhal, Earle, Hubbard)**.** The holonomy map hol: $\mathcal{P}(S) \to \mathcal{X}(S)$ is a local biholomorphism.

Some properties of the holonomy map.

- The holonomy map is not injective. One way to see this is by grafting in annular regions along a simple closed curve with angle a multiple of 2π. This does not change the holonomy representation.
- The holonomy map is not a covering of its image, because the path-lifting property fails (Hejhal).
- An element of $\mathcal{X}(S)$ arises from a holonomy representation if and only if it is non-elementary and liftable to $\mathrm{SL}_2(\mathbb{C})$ (Gallo, Kapovich, Marden).

Connection to the Schwarzian parametrization. Let $\mathcal{X}_0'(S)$ be the subset of the character variety consisting of liftable non-elementary representations. If $\rho \in \mathcal{X}_0'(S)$, then we can find a projective structure $Z \in \mathcal{P}(S)$ with holonomy representation ρ. By our discussion of the Schwarzian parametrization, there is a whole family of projective structures $Z + \varphi$ for $\varphi \in Q(X)$, where X is the complex structure of Z. Then,

$$\mathrm{hol}(\varphi) = \mathrm{hol}(Z + \varphi)$$

gives a holomorphic embedding of $\mathbb{C}^{3g-3}$ into $\mathcal{X}(S)$, a family of projective deformations of ρ.

3.5.2 Bending Fuchsian groups

We will now move into describing how grafting changes the holonomy representation of a projective structure. To begin, we will describe algebraically how to bend a Fuchsian group. Suppose that $\gamma \in \pi_1(S)$ is a simple closed curve on the surface that separates S into S_1 and S_2. Then,

$$\pi_1(S) = \pi_1(S_1) *_{\langle\gamma\rangle} \pi_1(S_2),$$

the Z-amalgamated free product of the fundamental groups of the two sides. Let $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ be a Fuchsian representation and A be an elliptic element having the same axis as $\rho(\gamma)$. Then, the homomorphism

$$\rho'(x) = \begin{cases} \rho(x), & x \in \pi_1(S_1) \\ A\rho(x)A^{-1}, & x \in \pi_1(S_2) \end{cases}$$

is a bending deformation of ρ. The geometry of this new action is that instead of preserving a hyperbolic plan $\mathbb{H}^2 \subset \mathbb{H}^3$, it preserves a locally convex pleated plane. This gives a map

$$\beta : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \to \mathcal{X}(S),$$

that extends continuously to

$$\beta : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{X}(S).$$

We note that there is 2π periodicity of the map when dealing with simple closed curves, but this periodicity does not extend to measured laminations.

Bending cocycles. Another definition of the bending deformation that emphasizes the geometry involves the bending cocycle. Suppose that we are bending along a simple closed curve γ by some fixed angle t. Then, we let $\tilde{\gamma}$ be the lfit of γ to $\mathbb{H}^2 \cong \tilde{S}$. For any two points $x, y \in (\mathbb{H}^2 - \tilde{\gamma})$, let $g_1, \ldots, g_n$ be the sequence of lifts of γ that cross the geodesic connecting x and y. Then, we define the bending cocycle $B(x, y) \in \mathrm{PSL}_2(\mathbb{C})$ as

$$B(x, y) = E(g_1, t)E(g_2, t) \cdots E(g_n, t),$$

where $E(g_i, t)$ is the elliptic Möbius transformation through axis g_i with angle of rotation t. One can check that B satisfies the cocycle relation and that $B(\delta x, \delta y) = \rho_0(\delta)B(x, y)\rho_0(\delta)^{-1}$, for all $\delta \in \pi_1(S)$. Then, if ρ_0 is the holonomy representation of a Fuchsian group, ρ is the holonomy representation of the same group bent by t a long γ, and O is any point in $(\mathbb{H}^2 - \tilde{\gamma})$, we have that

$$\rho(\gamma) = B(O, \gamma O)\rho_0(\gamma).$$

That is, the bending cocycle records the difference between the Fuchsian holonomy and the bent Fuchsian holonomy.

Bending and grafting. There is a fundamental relationship between grafting, bending, and the holonomy map:

$$\mathrm{hol}(\mathrm{Gr}_\lambda Y) = \beta_\lambda(Y).$$

That is, the holonomy of the of the projective structure resulting from grafting a complex structure Y along a measured lamination λ is exactly the Fuchsian representation of Y bent along the measured lamination λ.

Fuchsian holonomy. Let $\mathcal{P}_{\mathcal{F}}(S) = \mathrm{hol}^{-1}(\mathcal{F}(S))$ be the set of all projective structures with Fuchsian holonomy. We recall that we can construct such projective structures by starting with the projective structure on $\mathbb{H}^2/\Gamma$ where Γ is a Fuchsian group, and then grafting in annular regions with $2\pi n$ angle along simple closed curves. It turns out that all projective structures with Fuchsian holonomy arise this way. Let $\mathcal{ML}_{\mathbb{Z}}$ denote the subset of $\mathcal{ML}(S)$ consisting of collections of disjoint simple closed curves with positive integral weights. We note that $\mathcal{ML}_{\mathbb{Z}}$ is countable. Then, the following holds.

Theorem 3.5.2 (Goldman)**.** Let $Z \in \mathcal{P}_{\mathcal{F}}(S)$ and $Y = \mathbb{H}^2/\mathrm{hol}(Z)$ be the hyperbolic surface associated to the Fuchsian holonomy representation. Then, $Z = \mathrm{Gr}_{2\pi\lambda}Y$ for some $\lambda \in \mathcal{ML}_{\mathbb{Z}}$.

We recall that we used that we had sections $\sigma_\lambda : \mathcal{T}(S) \to \mathcal{P}(S)$ taking a complex structure X to a the unique projective structure Z grafted along λ with underlying complex structure X. Then, one can also show the following:

$$\mathcal{P}_{\mathcal{F}}(S) = \bigcup_{\lambda \in \mathcal{ML}_{\mathbb{Z}}} \sigma_{2\pi\lambda}(\mathcal{T}(S)).$$

We notice that these two results imply that $\mathcal{P}_{\mathcal{F}}(S)$ splits up countably many copies of $\mathcal{T}(S)$ indexed by $\mathcal{ML}_{\mathbb{Z}}$ in two ways:

1. Those elements of $\mathcal{P}_{\mathcal{F}}(S)$ with underlying complex structure $X \in \mathcal{T}(S)$ can be naturally identified with $\mathcal{ML}_{\mathbb{Z}}$.
2. Those elements of $\mathcal{P}_{\mathcal{F}}(S)$ with underlying holonomy representation that of $X \in \mathcal{T}(S)$ can be naturally identified with $\mathcal{ML}_{\mathbb{Z}}$.

Quasi-Fuchsian holonomy. We can ask if we can parametrize those projective structures with quasi-Fuchsian holonomy in a similar way. We recall that a quasi-Fuchsian group $Q(X, Y)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ whose limit set in $\mathbb{CP}^1$ is a Jordan curve. This curve then separates the sphere into two regions Ω_+ and Ω_- whose quotients by $Q(X, Y)$ are X and $\overline{Y}$. We note here that $\mathcal{P}_{\mathcal{QF}}(S)$ is an open subset of $\mathcal{P}(S)$.

Theorem 3.5.3 (Goldman)**.** Let $Z \in \mathcal{P}_{\mathcal{QF}}(S)$ have quasi-Fuchsian holonomy and developing map $f : \tilde{Z} \to \mathbb{CP}^1$. Let $\Lambda \subset \mathbb{CP}^1$ be the Jordan curve that is the limit set of the holonomy group, and $\Omega_\pm$ be the complementary regions. Then,

(1) The quotient of the developed preimage of the limit set $\Lambda(Z) := f^{-1}(\Lambda)/\pi_1(S)$ consists of finitely many disjoint simple closed curves.

(2) The quotient of the developed preimage of Ω_-, denoted $Z_- := f^{-1}(\Omega_-/\pi_1(S))$ is a finite collection of disjoint annuli with homotopically nontrivial core curves, bounded by pairs of curves in $\Lambda(Z)$.

We notice that in the case where $Z = X/\Omega(X, Y)$ is a standardy quasi-Fuchsian structure, the developed $\tilde{Z}$ maps onto Ω_+ and $f^{-1}(\Lambda)$ and Z_- are empty.

In general, Z_- is a collection of n disjoint annuli with core curves $\gamma_1, \dots \gamma_n$, where the γ_i are nontrivial and can be repeated. Then, we can define the wrapping invariant of a projective structure Z with quasi-Fuchsian holonomy as

$$\mathrm{wr}(Z) := \sum_i \gamma_i \in \mathcal{ML}_{\mathbb{Z}}(S).$$

For Fuchsian projective structures, we have that

$$Z = \mathrm{Gr}_{2\pi \mathrm{wr}(Z)} Y.$$

We can see this by noticing that if λ is a simple closed curve with multiplicity n, then $\tilde{Z}$ looks like $\mathbb{H}^2$ with a lune of angle $2\pi n$ glued in at every lift of the underlying curve γ of λ. A lune of angle $2\pi n$ will map to an n-times cover of $\mathbb{CP}^1$, and Ω_- will be n sub-lunes. Quotienting out by the Fuchsian group, this will correspond to n disjoint annuli with core curve γ.

Let us move back to quasi-Fuchsian holonomy. Because the limit set Λ of $Q(X,Y)$ varies continuously, the wrapping invariant is also locally constant on $\mathcal{P}_{\mathcal{QF}}(S)$. Then, $\mathcal{P}_{\mathcal{QF}}(S)$ breaks up into countably many components

$$\mathcal{P}_{\mathcal{QF}}(S) = \bigcup_{\lambda \in \mathcal{ML}_{\mathbb{Z}}(S)} \mathcal{P}_\lambda(S),$$

where $\mathcal{P}_\lambda(S) = \mathrm{wr}^{-1}(\lambda)$. The main result here is that the holonomy map

$$\mathrm{hol} : \mathcal{P}_\lambda(S) \xrightarrow{\sim} \mathcal{QF}(S)$$

is a diffeomorphism when restricted to any component. The inverse of this map can either be constructed by generalizing the projective grafting construction or by deforming Fuchsian representations into quasi-Fuchsian representations.

3.6 Recap of perspectives, and some connections

In this section, we will review the various perspectives on projective structures that we have taken, and give some results that tie some of these perspectives together.

Using the Schwarzian derivative, we found that choosing any section $\sigma : \mathcal{T}(S) \to \mathcal{P}(S)$, we can form a bijective Schwarzian parametrization between $\mathcal{P}(S)$ and $\mathcal{Q}(S)$, given by

$$Z \mapsto (\pi(Z), \varphi(Z - \sigma(\pi(Z))).$$

Using projective grafting, we found a homeomorphism

$$\text{Gr: } \mathcal{ML} \times \mathcal{T}(S) \xrightarrow{\sim} \mathcal{P}(S).$$

We could ask if these two parametrizations of $\mathcal{P}(S)$ are related in any way. The answer is yes, and one of these relations involves the compactifications of the spaces involved.

3.6.1 Compactifications

We have the compactification

$$\overline{\mathcal{ML}(S)} = \mathcal{ML}(S) \cup \mathbb{P}\mathcal{ML}(S),$$

by saying that that $\lambda_i \in \mathcal{ML}(S)$ converges to $[\lambda] \in \mathbb{P}\mathcal{ML}(S)$ if there exists a sequence of positive c_i for which $c_i\lambda_i \to \lambda$ and $c_i \to 0$. We also have the Thurston compactification

$$\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S),$$

where a sequence $X_n \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathbb{P}\mathcal{ML}(S)$ if for every pair of simple closed curves $\alpha, beta$,

$$\frac{\ell(\alpha, X_i)}{\ell(\beta, X_i)} \to \frac{i(\alpha, \lambda)}{i(\beta, \lambda)}.$$

On Teichmüller space, ℓ denotes the hyperbolic length function, and on the space of measured laminations, $i(\alpha, \lambda)$ is the mass of α with respect to the transverse measure on λ.

We also have that

$$\overline{Q(X)} - Q(X) \cup \mathbb{P}^+ Q(X),$$

where $\mathbb{P}^+Q(X)$ is the space of positive rays.

Quadratic differentials and measured laminations. There is a natural map

$$\Lambda : Q(X) \to \mathcal{ML}(S)$$

that takes a quadratic differential to its horizontal foliation, which is then straightened to a measured lamination. In the local coordinate, $\varphi = dz^2$, the foliation is induced by the horizontal lines in $\mathbb{C}$ with transverse measure $|dy|$. We also have a map to the vertical foliation of a quadratic differential, $\Lambda(-\varphi)$. The following theorem will be relevant to us.

Theorem 3.6.1 (Hubbard, Masur)**.** For each $X \in \mathcal{T}(S)$, the map $\Lambda : Q(X) \to \mathcal{ML}(S)$ is a homeomorphism.

One consequence of the Hubbard-Masur theorem is that we can define antipodal maps on $\mathcal{ML}(S)$ for every $X \in \mathcal{T}(S)$. Every $\lambda \in \mathcal{ML}(S)$ is the horizontal foliation of a unique quadratic differential $q \in Q(X)$. The antipodal map i_X then takes λ to the vertical foliation of q.

Limits of fibers. Here are some results on the limits of $\mathcal{P}(S)$.

- Using the projective grafting homeomorphism, Gr: $\mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$, the grafting compactification of $\mathcal{P}(S)$ is just $\overline{\mathcal{ML}(S)} \times \overline{\mathcal{T}(S)}$.
- By a result of Dumas, the boundary of $P(X)$ in the grafting compactification of $\mathcal{P}(S)$ is the graph of the antipodal involution $i_X : \mathbb{P}\mathcal{ML}(S) \to \mathbb{P}\mathcal{ML}(S)$. That is, if $\mathrm{Gr}_{\lambda_n} Y_n \in P(X)$ for all n, then
$$\lim_{n\to\infty} \lambda_n = [\lambda] \text{ and and only if } \lim_{n\to\infty} Y_n = [i_X(\lambda)].$$
- Another result of Dumas states that the grafting and Schwarzian compactifications of $P(X)$ are naturally homeomorphic, and the boundary map $\mathbb{P}^+Q(X) \to \mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(S)$ is given by
$$[\varphi] \mapsto ([\Lambda(-\varphi)], [\Lambda(\varphi)]).$$
- This compactification result is about comparing two maps from $\mathcal{ML}(S) \to Q(X)$, the one coming from the Hubbard-Masur theorem, and the one coming from the Schwarzian parametrization $\lambda \mapsto \varphi(\sigma_\lambda(X) - \sigma_0(X))$, where $\sigma_\lambda(X)$ is the a projective structure with grafting lamination λ and complex structure X. These two maps $\mathcal{ML}(S) \to Q(X)$ are not the same, but in a way, they are asymptotically proportional.

We finish with a result comparing the three maps

$$\begin{aligned} \pi &= \text{projection map from } \mathcal{P}(S) \text{ to } \mathcal{T}(S) \\ p_{\mathcal{ML}} &= \text{projection to first factor of } \mathrm{Gr}^{-1} : \mathcal{P}(S) \to \mathcal{ML}(S) \times \mathcal{T}(S) \\ p_{\mathcal{T}} &= \text{projection to second factor of } \mathrm{Gr}^{-1} : \mathcal{P}(S) \to \mathcal{ML}(S) \times \mathcal{T}(S) \end{aligned}$$

Theorem 3.6.2 (Dumas, Wolf)**.** The maps $\pi, p_{\mathcal{ML}}$, and $p_{\mathcal{T}}$ have transverse fibers, and the product of any two of these maps gives a homeomorphism from $\mathcal{P}(S)$ to a product of two spaces of real dimension $6g - 6$.

Chapter 4

Introduction to Opers

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This note is from talk of the same title given in log cabin workshop on Geometric Aspects of Higgs Bundles organized by Xian Dai, Charles Ouyang and Andrea Tamburelli in May 2019. The material is mostly based on introductory parts of the notion of SL_n-opers and G-opers from [Aco16, Dal08, San18, Wen16].

Opers plays an important role in the famous and mysterious geometric Langlands conjecture, which for Σ algebraic curve and G a reductive group states that there is a correspondence between derived category of quasicoherent sheaves of $\mathcal{O}_X$-modules over moduli stack of ${}^LG$-local systems over Σ on one hand and $\mathcal{D}$-modules over moduli stack of G-bundles over Σ on the other hand:

$$\mathcal{O}\mathrm{Mod}\left(\mathrm{Loc}_{{}^LG}\left(\Sigma\right)\right) \xrightarrow{\sim} \mathcal{D}\mathrm{Mod}\left(\mathrm{Bun}_G\left(\Sigma\right)\right) \tag{4.0.1}$$

where the important objects on the right side called Hecke eigensheaves corresponds to ${}^LG$-opers on the left side. In this talk however we shall focus more on the classical origin from study of ordinary differential equations on compact Riemann surfaces, which is where oper got its name. We will introduce SL_n opers and more generally G-opers for G a connected complex simple Lie group of adjoint type and consider some interesting bundle-theoretic properties of these objects. Throughout the talk X will be a closed Riemann surface of genus $g \geq 2$. Let $\Gamma = \pi_1(X)$, and K_X the canonical line bundle or equivalently the holomorphic cotangent bundle of X.

The name 'oper' was coined by Beilinson and Drinfeld [BD05] from differential **oper**ators between line bundles [1]. Indeed perhaps the simplest examples, the SL_2-opers, comes from study of Sturm-Liouville operators and is closely related to $\mathbb{C}P^1$-structure on closed Riemann surfaces.

4.1 SL2 oper, Schwarzian derivative and complex projective structure

We start with an elementary exercise due to Lagrange [2] [Lag79]: given second order ODE

$$y'' + Qy = 0 \tag{4.1.1}$$

on a domain $\Omega \subset \mathbb{C}$, let $y_1(z), y_2(z)$ be linearly independent solutions with y_2 having no zeros in Ω, write their ratio as

$$f(z) = \frac{y_1(z)}{y_2(z)} \tag{4.1.2}$$

[1] see page 2 [BD05]

[2] For more history, see [OT09]

Question 4.1.1. Is it possible to recover coefficient Q in terms of f?

Without loss of generality assume Wronskian $W = y_1 y_2' - y_1' y_2 \equiv 1$ [3], then by simple calculation we see

$$f' = -\frac{1}{y_2^2} \tag{4.1.3}$$

and since we have recovered one of the solutions, we simply take $Q = -y_2''/y_2$, giving

$$Q = \frac{1}{2}\left[\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right] \tag{4.1.4}$$

Definition 4.1.1. Let f be a univalent [4] holomorphic function in $\Omega \subset \mathbb{C}$, the Schwarzian derivative is defined by

$$S(f) = \{f, z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \tag{4.1.5}$$

where the primes means ∂_z. Note that Schwarzian derivative does depend on choice of coordinate.

It is an easy albeit tedious calculation to check that S satisfies

- Cocycle property:

$$S(f \circ g) = (S(f) \circ g)(g')^2 + S(g) \tag{4.1.6}$$

  which is just like chain rule except with an extra term $S(g)$

- Möbius invariance: $S(f) = 0$ iff f is restriction of a Möbius transformation

It follows from these that $S(f) = S(g)$ iff $f = \varphi \circ g$ for some φ restriction of Möbius transformation.

Definition 4.1.2. A collection of holomorphic functions [5] $\{F(z)\}$ over a choice of coordinate chart on X is called a projective connection [6] if they obey transformation law

$$F(w)(w')^2 = F(z) - \{w, z\} \tag{4.1.7}$$

Note that the second term in transformation law $\{w, z\}$ is independent of choice of the collection $\{F\}$, therefore difference between any two such collections is a collection of holomorphic functions transforming like coefficient of dz^2 of a quadratic differential. Therefore we see that the set of projective connections on X form an affine space modelled on $H^0(X, K_X^2)$

Recall now the notion of $\mathbb{C}P^1$-structure on X: a coordinate chart of X that takes value in $\mathbb{C}P^1$ and transitions by Möbius transformations, equivalently it is given by a developing pair (f, ρ) where

$$\rho : \Gamma \to PSL(2, \mathbb{C}) = \text{Möb} \tag{4.1.8}$$

is a representation of fundamental group of X and

$$f : \tilde{X} \to \mathbb{C}P^1 \tag{4.1.9}$$

is a locally injective, holomorphic and ρ-equivariant map called developing map where $\tilde{X}$ is a universal cover of X. Basically f globalizes coordinate charts and ρ globalizes coordinate transition.

[3] since there is no y' term, Wronskian is a constant
[4] means f' has no zero
[5] Suppose $(U, z : U \to \mathbb{C})$ is one coordinate neighborhood, $F(z)$ is actually function on $z(U)$ not U.
[6] 'connection' here is in the style of Cartan connection

In style of the first view-point, given projective connection $\{F(z)\}$ we may solve

$$y'' + \frac{1}{2}Fy = 0 \tag{4.1.10}$$

locally on coordinate neighborhood and let $\xi := y_1/y_2$ be ratio between choice of linearly independent solutions y_1, y_2. This function ξ viewed as coordinate on $\mathbb{C}P^1$, gives coordinate charts valued in $\mathbb{C}P^1$ whose transition is given by Möbius transformation.

In the style of second view-point, we need only find out how to construct developing map f. Fix a uniformization $\tilde{X} \xrightarrow{\sim} \mathbb{H}$ to globalize coordinate chart on X. This gives via pull back from the projective connection $\{F\}$, a single function $\tilde{F} : \mathbb{H} \to \mathbb{C}$. [7] Now by solving

$$y'' + \frac{1}{2}Fy = 0 \tag{4.1.11}$$

on $\mathbb{H}$ and choosing two linearly independent y_1, y_2, we obtain the developing map

$$\begin{aligned} f : \mathbb{H} &\to \mathbb{C}P^1 \\ z &\mapsto [y_1(z) : y_2(z)] \end{aligned}$$

Therefore we have shown that there is a one-to-one correspondence between

$$\left\{\begin{matrix}\text{projective connections} \\ \text{on } X\end{matrix}\right\} \overset{\text{1-1}}{\longleftrightarrow} \left\{\begin{matrix}\mathbb{C}P^1 \text{ structures} \\ \text{on } X\end{matrix}\right\} \tag{4.1.12}$$

We can also recast the differential operator $Dy = y'' + \frac{1}{2}fy$, independent of choice of u, as a global object on X (instead of on $\mathbb{H}$). Start with the ansatz that our solutions y transforms according to

$$y(z) = y(w)\left(\frac{dw}{dz}\right)^{-1/2} \tag{4.1.13}$$

then the nasty terms that inevitably shows up in corresponding formula for y'' is precisely absorbed by transformation rule of $F(z)$, giving

$$y''(z) + \frac{1}{2}F(z)y(z) = \left(y''(w) + \frac{1}{2}F(w)y(w)\right)\left(\frac{dw}{dz}\right)^{3/2} \tag{4.1.14}$$

therefore we have a well-defined differential operator [8]

$$K^{-1/2} \xrightarrow{D} K^{3/2} \tag{4.1.15}$$

where we fixed a choice $K^{1/2}$ of square root [9] of canonical line bundle K. Note that the ambiguity in choice of square root of K lies in the ambiguity in choosing a square root $\sqrt{dw/dz}$.

The kernel of this differential operator map, viewed as map between sheaves of sections of line bundles, is a local system $\mathbb{V}$, which is a $\underline{\mathbb{C}}$-module. That is we have short exact sequence of sheaves over X

$$0 \to \mathbb{V} \to K^{-1/2} \xrightarrow{D} K^{3/2} \to 0 \tag{4.1.16}$$

[7] Here's why we get a single function $\tilde{F}$, start with any holomorphic coordinate chart $\{(U_\alpha, z_\alpha)\}_{\alpha\in\mathcal{A}}$ on X and the collection $\{F\}$ is not a collection of function on U_α but actually on $z_\alpha(U_\alpha)$, and given any other coordinate $U_\alpha \to \mathbb{C}$ we may use the transfrmation $(*)$ to get another function. Now let $p : \mathbb{H} \to X$ be the projection of universal cover $p^{-1}U_\alpha$ form an open cover of $\mathbb{H}$ itself, and for each $p^{-1}U_\alpha$, pull back via p the corresponding function in the collection with coordinate map $U_\alpha \to p^{-1}U_\alpha \subset \mathbb{H} \subset \mathbb{C}$ (e.g. viewing $\mathbb{H}$ as Poincarè d isk). Now when we go from one $p^{-1}U_\alpha$ to the next one on $\mathbb{H}$, the transition of coordinate downstairs on X is actually by identity function.

[8] takes a smooth (local) section of line bundle $K^{-1/2}$ to a smooth (local) section of line bundle $K^{3/2}$. Note this is not holomorphic homomorphism between line bundles.

[9] multiplication of line bundles is given by tensor product of line bundles. A choice of square root of cotangent bundle is also called a spin structure. Recall the space of degree zero line bundles as a group is a torus $\mathrm{Pic}^0 X \cong (S^1)^{2g}$, given any line bundle L of even degree $2d$, let $L_0 \in \mathrm{Pic}^d X$ such that $L_0^{\otimes 2} \cong L$, then for any 2-torsion point $N \in \mathrm{Pic}^0 X$, i.e. $N^{\otimes 2} \cong \mathcal{O}$, we have $(L_0 \otimes N)^{\otimes 2} \cong L$, and there are precisely 2^{2g} 2-torsion points.

4.2 Crash course on (holomorphic) bundles and connection

From now on we enter the world of bundles and connections, and the language of sheaves of modules will become inevitable, so here is a crash course in some of the basic concepts.

Definition 4.2.1.

- A complex rank n local system is a locally constant sheaf taking value in $\mathbb{C}^n$. Equivalently it is a sheaf of $\underline{\mathbb{C}}$-modules where $\underline{\mathbb{C}}$ denote the constant sheaf valued in $\mathbb{C}$.
- A holomorphic rank n vector bundle is a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ where $\mathcal{O}_X$ is sheaf of holomorphic functions on X. Locally free means we may fix some local trivializations

$$\mathcal{E}_U \xrightarrow{\sim} \mathcal{O}_U^{\oplus n} \tag{4.2.1}$$

  and local sections will then be identified with n-tuple of holomorphic functions, and we may recover the bundle from transition matrices valued in $GL(n, \mathcal{O})$, i.e. with entries holomorphic functions.

- A connection on smooth bundle E [10] is a $\mathbb{C}$-linear sheaf homomorphism satisfying Leibniz rule

$$D : \Omega^0(E) \to \Omega^1(E), D(fs) = df \otimes s + f\nabla s \tag{4.2.2}$$

  where $\Omega^k(E)$ denote (local) k-forms taking value in E. Curvature operator is defined to be connection operator applied twice:

$$F(D) = D \circ D \tag{4.2.3}$$

  and one easily checks that the differentiation in Leibniz rule cancels out, and $\mathbb{C}$-linearity becomes linearity with smooth coefficient, thus $F(D)$ is tensorial, $F(D) \in \Omega^2(X, \mathrm{End}(E))$

- A $\bar{\partial}$-operator [11] is an operator

$$\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E) \tag{4.2.4}$$

  satisfying Leibniz rule with df replaced by $\bar{\partial} f$.

- A holomorphic connection is a sheaf map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes K \tag{4.2.5}$$

  satisfying Leibniz rule with restriction that f be holomorphic

Remark 4.2.2. An equivalent definition for holomorphic vector bundle is a smooth vector bundle plus an integrable $\bar{\partial}$-operator, i.e. $\bar{\partial}_E \circ \bar{\partial}_E = 0$. On Riemann surfaces there are no nonzero (0,2)-forms $\Omega^{0,2}(X) = 0$, integrability is automatic.

Similarly since $\Omega^{2,0}(X) = 0$ as well, a holomorphic connection on Riemann surface is also automatically flat, $\nabla \circ \nabla = 0$. Moreover given holomorphic vector bundle with holomorphic connection $(\mathcal{E}, \nabla)$ we automatically have a flat connection by combining $\bar{\partial}$-operator

$$D = \bar{\partial}_E + \nabla \tag{4.2.6}$$

Equivalently holomorphic connection is flat because in local holomorphic frame connection matrices have holomorphic entries, and curvature is expressed as $\bar{\partial}$ of this matrix.

[10] simply replace the sheaf $\mathcal{O}$ in above definition with sheaf of smooth functions!
[11] sometimes called pseudo-connection

Theorem 4.2.3 ([Del06] page 12)**.** There is an equivalence between category of local systems and category of holomorphic bundles with holomorphic connections

The notions of stability are important in study of moduli problem, the easiest kind of which involves slope:

Definition 4.2.4. Let $\mathcal{E}$ be a holomorphic vector bundle of rank n. Degree of E is defined as first Chern class of its determinant line bundle, i.e. top exterior power:

$$\deg(\mathcal{E}) = c_1(\Lambda^n \mathcal{E}) \tag{4.2.7}$$

and slope is defined as the ratio

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\text{rank}\mathcal{E}} \tag{4.2.8}$$

which is a topological invariant since both degree and rank are topological invariant. Now $\mathcal{E}$ is called (slope) stable if for any nonzero proper subbundle $\mathcal{F} \subset \mathcal{E}$ we have

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \tag{4.2.9}$$

and if $<$ is replaced by $\leq$ we call $\mathcal{E}$ (slope) semistable. A bundle that is not semistable is called unstable.

Lemma 4.2.5. Let $\mathcal{E}_1, \mathcal{E}_2$ be semistable bundles and suppose $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$ then there is no non-zero holomorphic homomorphism $f : \mathcal{E}_1 \to \mathcal{E}_2$

Proof. See Prop 4.3 of [NS65] page 547 for details. Roughly speaking holomorphic vector bundles are algebraic on closed Riemann surfaces over $\mathbb{C}$, we may replace the sheaf $\mathcal{O}$ of holomorphic functions with that of regular functions. If $f \neq 0$, we consider a kind of Smith normal form for f since $\mathcal{O}$ is now a sheaf of PIDs,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V}_1 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\ & & & & & & \big\downarrow g & & \\ 0 & \longleftarrow & \mathcal{W}_2 & \longleftarrow & \mathcal{E}_2 & \longleftarrow & \mathcal{W}_1 & \longleftarrow & 0 \end{array}$$

where $\mathcal{V}_1$ is (saturation of) kernel of f and g is of full rank (generically an isomorphism) between bundle $\mathcal{V}_2$ and $\mathcal{W}_1$ of same rank, therefore induces holomorphic homomorphism of line bundles $\det \mathcal{V}_2 \to \det \mathcal{W}_1$, thus [12] $\deg\mathcal{V}_2 \geq \deg\mathcal{W}_1$ and we have by semistability $\mu(\mathcal{E}_1) \geq \mu(\mathcal{V}_2) \geq \mu(\mathcal{W}_1) \geq \mu(\mathcal{E}_2)$ which contradicts assumption. ☐

Definition/Proposition For any holomorphic vector bundle $\mathcal{E}$ there is a unique filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_\ell = \mathcal{E} \tag{4.2.10}$$

Let $\mathcal{Q}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ and $\mu_i = \mu(\mathcal{Q}_i)$, then this filtraion is required to satisfy

- Each $\mathcal{Q}_i$ is semistable
- $\mu_i > \mu_{i+1}$ for all i

[12] Think of degree as counting number of zeros minus poles of meromorphic sections. Now holomorphic homomorphism can have only zeros, so degree can only go up along it.

This filtration is called Harder-Narasimhan filtration of bundle $\mathcal{E}$, and the vector

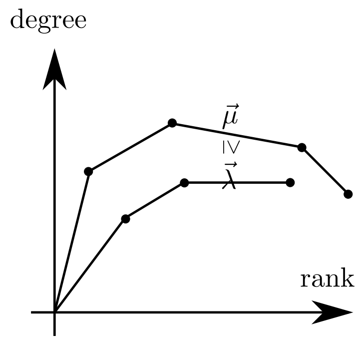
$$\vec{\mu} = (\mu_1, \dots, \mu_1, \dots, \mu_\ell, \dots, \mu_\ell) \tag{4.2.11}$$

where μ_i is repeated rank$(\mathbb{Q}_i)$ times, is called HN type of the bundle. Roughly speaking it is constructed inductively by taking 'maximally destabilizing subbundle' which is a subbundle with maximal possible slope and rank and hence automatically semistable. Note HN filtration for $\mathcal{E}$ is semistable is just $0 \subset \mathcal{E}$.

There is a partial order on the HN types, given by

$$\vec{\lambda} \leq \vec{\mu} \quad \text{iff} \quad \sum_{j\leq k} \lambda_j \leq \sum_{j\leq k} \mu_j \ \ \forall k \tag{4.2.12}$$

Equivalently if we represent HN type as broken lines connecting (deg, rank) on the plane, each $\vec{\mu}$ will be part of a convex polygon and the partial order $\vec{\lambda} \leq \vec{\mu}$ is equivalent to one broken line staying above the other:



We will see later that SL_n-opers are characterized precisely by being those unstable bundles that has maximal HN type.

Proposition 4.2.6. Let $\mathcal{V}$ be unstable bundle with an irreducible holomorphic connection ∇, then HN type $\vec{\mu}$ is bounded from above,

$$\mu_i - \mu_{i+1} \leq 2g - 2 \tag{4.2.13}$$

Proof. Let $0 = \mathcal{V}_0 \subset \dots \subset \mathcal{V}_\ell = \mathcal{V}$ be HN filtration, then $\mathcal{Q}_i = \mathcal{V}_i/\mathcal{V}_{i=1}$ are semistable. We'll find a nonzero map between quotients which are semistable and invoke the above lemma.

By irreducibility $\mathcal{V}_i \to \mathcal{V}/\mathcal{V}_i \otimes \mathcal{K}$ is nonzero for all i. Let $j \leq i$ be the smallest such that $\mathcal{V}_j \to \mathcal{V}/\mathcal{V}_i \otimes \mathcal{K}$ is nonzero, then $\mathcal{Q}_j \to \mathcal{V}/\mathcal{V}_i \otimes \mathcal{K}$ is nonzero. Now let $k \geq i$ be largest such that $\mathcal{Q}_j \to \mathcal{V}/\mathcal{V}_k \otimes \mathcal{K}$ nonzero. Then we see that

$$\mathcal{Q}_j \to \mathcal{Q}_{k+1} \otimes \mathcal{K} \tag{4.2.14}$$

is nonzero. By lemma above and definition of HN filtration we have

$$\mu_i \leq \mu_j \leq \mu_{k+1} + \frac{2g-2}{\text{rank}\mathcal{Q}_{k+1}} \leq \mu_{k+1} + (2g-2) \leq \mu_{i+1} + (2g-2) \tag{4.2.15}$$

□

Each holomorphic vector bundle is expressible as direct sum of indecomposable bundles, the following is a characterization of all those bundles admitting holomorphic conenction:

Theorem 4.2.7 (Weil [Wei38], Atiyah [Ati57])**.** Holomorphic vector bundle $\mathcal{V}$ admits holomorphic connection ∇ iff each indecomposable factor has deg$\mathcal{V} = 0$

It is also useful to recall the notion of extension of holomorphic bundles. Suppose we have short exact sequence of holomorphic bundles

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0 \tag{4.2.16}$$

we say that $\mathcal{E}$ is an extension of $\mathcal{Q}$ by $\mathcal{S}$. Here $\mathcal{S}$ is a holomorphic subbundle and $\mathcal{Q}$ a holomorphic quotient bundle. Any complex vector bundle admits a hermitian metric, and by taking orthogonal complement, $\mathbb{Q}$ may be realized as a smooth subbundle, and $\mathcal{E} = \mathcal{S} \oplus \mathcal{Q}$ as smooth bundles. This allows us to write the $\bar{\partial}$-operator of $\mathcal{E}$ in block form

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_S & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix} \tag{4.2.17}$$

where β is called second fundamental form and the class $[\beta] \in H^1(X, \mathcal{S} \otimes \mathcal{Q}^*)$ may equivalently be identified as image of $1_{\mathcal{Q}}$ under connecting map of the long exact sequence of sheaf cohomology

$$H^0(X, \mathrm{Hom}(\mathcal{Q}, \mathcal{Q})) \to H^1(X, \mathrm{Hom}(\mathcal{Q}, \mathcal{S})) \tag{4.2.18}$$

The projectivised space $\mathbb{P}H^1(X, \mathrm{Hom}(\mathcal{Q}, \mathcal{S}))$ parametrizes all inequivalent [13] non-split extensions.

4.3 SLn-oper, differential equation and holonomy

Definition 4.3.1. An SL_n-oper is a triple $(\mathcal{V}, \nabla, 0 = \mathcal{V}_0 \subset \ldots \subset \mathcal{V}_n = \mathcal{V})$ where $\mathcal{V}$ is a holomorphic bundle of rank n with trivial determinant line bundle $\det \mathcal{V} = \mathcal{O}$, ∇ a holomorphic connection inducing trivial connection on $\mathcal{O}$, and a full filtration [14] by holomorphic subbundles satisfying

- Griffith transversality: $\nabla : \mathcal{V}_i \to \mathcal{V}_{i+1} \otimes \mathcal{K}$
- Non-degerenacy: induced linear map on quotient bundles $\bar{\nabla} : \mathcal{V}_i/\mathcal{V}_{i+1} \xrightarrow{\sim} \mathcal{V}_{i+1}/\mathcal{V}_i \otimes \mathcal{K}$ is isomorphism of line bundles

Let $\mathscr{G}^{\mathbb{C}}$ denote the group of automorphisms of $\mathcal{V}$ inducing identity map on $\det \mathcal{V}$, called complex gauge group. $\mathscr{G}^{\mathbb{C}}$ acts on the space of SL_n-opers in obvious manner [15] and let

$$\mathrm{Op}_n = \{SL_n\text{-opers}\} / \mathscr{G}^{\mathbb{C}} \tag{4.3.1}$$

Remark 4.3.2. The filtration above is called an *oper structure* on $(\mathcal{V}, \nabla)$. Not every holomorphic bundle admits holomorphic connection, and not all those that admits holomorphic connection admits oper structure.

Example 4.3.3. For $n = 2$. Recall we had short exact sequence

$$0 \to \mathbb{V} \xrightarrow{\varphi} K^{-1/2} \xrightarrow{D} K^{3/2} \to 0 \tag{4.3.2}$$

with $\mathbb{V}$ a rank n local system living inside [16] holomorphic line bundle $K^{-1/2}$. We may extend the scalar, consider $\tilde{\varphi} = \varphi \otimes_{\underline{\mathbb{C}}} \mathcal{O}$ to get a surjective sheaf map $\mathcal{V} := \mathbb{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O} \twoheadrightarrow K^{-1/2}$. Now $\mathcal{V}$ comes equipped with a canonical holomorphic connection:

$$\nabla(fs) := df \otimes s \in \mathcal{K} \otimes \mathcal{V} \tag{4.3.3}$$

and we see that kernel of this surjection is isomorphic to $K^{1/2}$, therefore we have short exact sequence

$$0 \to K^{1/2} \to \mathcal{V} \to K^{-1/2} \to 0 \tag{4.3.4}$$

[13] It is possible for inequivalent extension to give isomorphic bundle in the middle

[14] full means rank goes up by one each step, in particular successive quotients are all line bundles

[15] pulling back filtration and connection etc

[16] note the apparent discrepancy in rank: in the local system, we are only allowed to combine sections with constant coefficient whereas in line bundle we may combine them with holomorphic coefficients!

Claim. $(\mathcal{V}, \nabla, 0 \subset K^{1/2} \subset \mathcal{V})$ is an SL_2-oper

Proof. : (i) is vacuous. Choose local independent solutions y_1, y_2 with $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 1$, these gives sections s_1, s_2 of $\mathcal{V}$ and we may use these to trivialize $\mathcal{V}$ locally and $K^{1/2} = \ker \tilde{\varphi}$ is also trivialized by $y_2 s_1 - y_1 s_2$. The induced map

$$\bar{\nabla} : K^{1/2} = \mathcal{V}_1/\mathcal{V}_0 \to \mathcal{V}_2/\mathcal{V}_1 \otimes \mathcal{K} = K^{-1/2} \otimes K = K^{1/2} \tag{4.3.5}$$

is given by

$$f \cdot (y_2 s_1 - y_1 s_2) \mapsto f \cdot (y_2' y_1 - y_1' y_2) = f \tag{4.3.6}$$

which is indeed an isomorphism. □

More generally we may consider higher order ODE

$$Dy = y^{(n)} + Q_2 y^{(n-2)} + \ldots + Q_n y = 0 \tag{4.3.7}$$

by imposing similar but much more complicated transformation laws on $Q_{j\geq 2}$, the local system of solutions (the kernel of D as map between two line bundles) is realized in line bundle $K^{(1-n)/2}$.

Theorem 4.3.4 (Hejhal 1975 [Hej75], Di Franesco, Itzykson, Zuber 1991 [DFIZ91])**.** $D : K^{(1-n)/2} \to K^{(1+n)/2}$ is well-defined differential operator, then $\frac{12Q_2}{n(n^2-1)}$ is a projective connection. Moreover, there exists linear combinations w_k of $Q_n, Q_{n-1}, \ldots, Q_2$ and their derivatives with coefficients polynomials in Q_2 and its derivatives that transforms as k-differentials for all $k \geq 3$. The first two are given by

$$\begin{aligned} w_3 &= Q_3 - \frac{n-2}{2} Q_2' \\ w_4 &= Q_4 - \frac{n-3}{2} Q_3' + \frac{(n-2)(n-3)}{10} Q_2'' - \frac{(n-2)(n-3)(5n+7)}{10n(n^2-1)} Q_2^2 \end{aligned}$$

Space of such operators D is affine space modeled on the Hitchin base

$$B = \bigoplus_{j=2}^{n} H^0(X, K_X^j) \tag{4.3.8}$$

We again get a short sequence

$$0 \to \mathbb{V} \xrightarrow{\varphi} K^{\frac{1-n}{2}} \xrightarrow{D} K^{\frac{1+n}{2}} \to 0 \tag{4.3.9}$$

Let (L, ∇) be a flat line bundle (a line bundle equipped with a holomorphic connection), we may instead consider

$$0 \to \mathbb{V} \xrightarrow{\varphi} L \otimes K^{\frac{1-n}{2}} \xrightarrow{D} L \otimes K^{\frac{1+n}{2}} \to 0 \tag{4.3.10}$$

The following result answer the question: which local systems can be realized as monodromy of differential equations, which we may view as a restricted version of Riemann-Hilbert correspondence

Theorem 4.3.5. A representation $\rho : \Gamma \to SL(n, \mathbb{C})$ can be realized in L (by nth order ODE as above) iff (1) ρ is irreducible, (2) $H^0(X, \mathcal{V}_\rho^* \otimes L) \neq 0$ and (3) $L^n = K^{-n(n-1)/2}$

Proof. ⇒: (1) follows from a theorem of Hejhal, monodromy of differential operator must be irreducible. We'll be able to prove later. (2) also clear since $\mathcal{V}_\rho \xrightarrow{\varphi \otimes_{\mathbb{C}} \mathcal{O}} L$ is a nonzero section. Given linearly independent solutions $y_1, \ldots, y_n$ the Wronskian

$$\begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{4.3.11}$$

provides a nowhere vanishing section of $L^n \otimes K^{n(n-1)/2}$ thus globally trivializes it [17].
$\Leftarrow$: by (2) we may take $\varphi : \mathcal{V}_\rho \to L$ and restrict to local system $\mathcal{V}_\rho \xrightarrow{\varphi} L$. By (1) irreducibility, i.e. there is no proper sub $\underline{\mathbb{C}}$-modules in $\mathcal{V}_\rho$, $\ker \varphi = 0$ so φ is injective. Now (3) says L differs from $K^{-(n-1)/2}$ by an n-torsion line bundle [18] we have

$$L = L_0 \otimes K^{-(n-1)/2} \tag{4.3.12}$$

L_0 has flat connection since its degree is zero. Local sections of L is of the form $y = 1 \otimes w$ with 1 representing parallel section of L_0 wrt flat connection and we may define derivative to be

$$y^{(j)} := 1 \otimes w^{(j)} \tag{4.3.13}$$

Take local frame (v_i) of $\mathbb{V}_\rho$ and we get by setting

$$Dy = \begin{vmatrix} \varphi(v_1) & \dots & \varphi(v_n) & y \\ \vdots & & \vdots & \vdots \\ \varphi(v_1)^{(n)} & \dots & \varphi(v_n)^{(n)} & y^{(n)} \end{vmatrix} \tag{4.3.14}$$

that $Dy \in LK^n$ and $\ker D = \mathbb{V}_\rho$. $\square$

Now for rank n local system $\mathbb{V}$ (realized by nth order ODE) in $K^{(1-n)/2}$ we get a holomorphic bundle $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}$ with canonical holomorphic connection and a natural SL_n-oper structure by defining

$$\mathcal{V}_i = \left\{ \sum_i f_i \otimes v_i \middle| \sum_i f_i^{(j)} \varphi(v_i) = 0 \;\; \forall j \leq k-1 \right\} \tag{4.3.15}$$

with (v_i) a local frame of $\mathbb{V}$. We get exact sequences

$$0 \longrightarrow \mathcal{V}_{n-k-1} \longrightarrow \mathcal{V}_{n-1} \longrightarrow \mathcal{K}^{\frac{1-n}{2}+k} \longrightarrow 0$$

$$\textstyle\sum_i f_i \otimes v_i \longmapsto \sum_i f_i^{(k)} \varphi(v_i)(dz)^k$$

For $k = 0$ we recover the sequence

$$0 \to \mathbb{V}_{n-1} \to \mathbb{V} \xrightarrow{\varphi} \mathcal{K}^{\frac{1-n}{2}} \to 0 \tag{4.3.16}$$

Let's check oper conditions: (i) given $\sum_i f_i \otimes v_i \in \mathcal{V}_{n-k}$ we have

$$\nabla \left(\sum_i f_i \otimes v_i \right) = \sum_i (df_i) \otimes v_i = \sum_i (f_i' \otimes v_i)\, dz \tag{4.3.17}$$

which is indeed in $\mathcal{V}_{n-k+1} \otimes \mathcal{K}$, this proves Griffith transversality. (ii) For non-degeneracy: on the one hand

$$\mathcal{V}_{n-k-1} \xrightarrow{\;\nabla\;} \mathcal{V}_{n-k} \otimes \mathcal{K} \longrightarrow \mathcal{V}_{n-k}/\mathcal{V}_{n-k-1} \otimes \mathcal{K} = \mathcal{K}^{\frac{1-n}{2}+k+1}$$

$$\textstyle\sum_i f_i \otimes v_i \longmapsto \sum_i f_i^{(k+1)} \varphi(v_i)(dz)^{k+1}$$

[17] of course $y_1, \dots, y_n$ are only defined locally but we may arrange it such that they transition via constant SL_n matrices so Wronskian will remain constant

[18] i.e. a line bundle whose nth tensor product by itself is trivial line bundle

so $\bar{\nabla} : \mathcal{V}_{n-k-1}/\mathcal{V}_{n-k-2} \to \mathcal{V}_{n-k}/\mathcal{V}_{n-k-1} \otimes \mathcal{K}$ is indeed isomorphism.
Next we will prove Hejhal's theorem

Theorem 4.3.6. Holonomy representation of SL_n-oper is irreducible

First a lemma about the last quotient line bundle in the filtration and determinant line bundles of members of the filtration:

Lemma 4.3.7. $L^n \cong K^{-n(n-1)/2}$ and $\det V_j = L^j \otimes K^{nj-j(j+1)/2}$ with $L = \mathcal{V}/\mathcal{V}_n$

Proof. This is a simple computation using isomorphism provided in non-degeneracy condition:

$$\begin{aligned}\det \mathcal{V}_j &= (\det \mathcal{V}_{j-1} \otimes (\mathcal{V}_j/\mathcal{V}_{j-1})) = \ldots = \bigotimes_{\ell=1}^{j} \mathcal{V}_\ell/\mathcal{V}_{\ell-1} \\ &= \bigotimes_{\ell=1}^{j} LK^{n-\ell} = L^j K^{nj-\sum_1^n \ell} = L^j K^{nj-j(j+1)/2}\end{aligned}$$

where we have $\mathcal{V}_\ell/\mathcal{V}_{\ell-1} \cong \mathcal{V}_{\ell+1}/\mathcal{V}_\ell \otimes K \cong \ldots \cong \mathcal{V}/\mathcal{V}_{n-1} \otimes K^{n-\ell} = LK^{n-\ell}$. Take $j = n$ and we have $\mathcal{O} = \det \mathcal{V}_n = L^n K^{n^2-n(n+1)/2} = L^n K^{n(n-1)/2}$ □

of Hejhal's theorem. Let $0 \neq \mathcal{W} \subset \mathcal{V}$ be ∇-invariant holomorphic subbundle then $(\mathcal{W}, \nabla|_{\mathcal{W}})$ is a holomorphic bundle with holomorphic connection. Consider $\mathcal{W} \cap \mathcal{V} =: \mathcal{W}_i$.

$$\begin{array}{ccc} \mathcal{W}_i/\mathcal{W}_{i-1} & \longrightarrow & \mathcal{W}_{i+1}/\mathcal{W}_i \otimes \mathcal{K} \\ \big\downarrow & & \big\downarrow \\ \mathcal{V}_i/\mathcal{V}_{i-1} & \xrightarrow{\sim} & \mathcal{V}_{i+1}/\mathcal{V}_i \otimes \mathcal{K} \end{array}$$

From commutative diagram above we see that $\mathcal{W}_i/\mathcal{W}_{i-1} \hookrightarrow \mathcal{W}_{i+1}/\mathcal{W}_i \otimes \mathcal{K}$ is injective as sheaf map. Let $r_i = \text{rank}(\mathcal{W}_i/\mathcal{W}_{i-1})$ and let ℓ be the smallest such that $r_\ell = 1$, so $0 = r_1 = \ldots = r_{\ell-1}, r_\ell = \ldots = r_n = 1$. We have sequence of injective sheaf maps between line bundles

$$\mathcal{W}_i/\mathcal{W}_{i+1} \to \ldots \to \mathcal{W}_n/\mathcal{W}_{n-1} \otimes \mathcal{K}^{n-i} \to \mathcal{V}/\mathcal{V}_{n-1} \otimes \mathcal{K}^{n-i} = LK^{n-i} \tag{4.3.18}$$

thus

$$\deg(\mathcal{W}_i/\mathcal{W}_{i-1}) \leq \deg{(\mathcal{V}/\mathcal{V}_{n-1})} \otimes \mathcal{K}^{n-i} = \deg L + \deg K^{n-i} = (g-1)(n-2i+1) \tag{4.3.19}$$

and

$$deg\mathcal{W} = \sum_{i=\ell}^{m} \deg \mathcal{W}_i/\mathcal{W}_{i-1} = -(\ell-1)(n-\ell+1)(g-1) \tag{4.3.20}$$

this must vanish since $\mathcal{W}$ carries holomorphic connection, but it occurs only if $\ell = 1$ thus $\mathcal{W} = \mathcal{V}$ □

Next we show a uniqueness theorem for SL_n-opers

Theorem 4.3.8. An SL_n-oper structre on (V, ∇) is uniquely determined by $L = \mathcal{V}/\mathcal{V}_{n-1}$. In particular isomorphism class of $\mathcal{V}$ is constant on any connected component of Op_n.

Strategy: basically we show that each $\mathcal{V}_j$ is successively the unique non-split extension from the one below. The following diagram will be referred to often in the proof:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & LK^{n-j} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{V}_{j-1} & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{R}_{j-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{V}_{j} & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{R}_{j} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & LK^{n-j} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Lemma 4.3.9.

$$H^1(X, LK^{n-j} \otimes R_i^*) = \begin{cases} 0 & i \geq j-1 \\ H^1(X, K) & i = j \end{cases} \qquad (\star)$$

Proof. Fix j and use induction on i backwards. Note by Serre duality we have $H^1(X,K) \cong H^0(X,\mathcal{O})^*$ which is isomorphic to $\mathbb{C}$, since the only global holomorphic functions on closed Riemann surfaces are constants [19]. For first step in induction let $i = n-1$, we have $\mathcal{R}_{n-1} = L$ and

$$H^1(X, LK^{n-j}R_{n-1}^*) = H^1(X, K^{n-j}) = \begin{cases} 0 & n-j-1 > 0 \\ H^1(X,K) & n = j+1 \end{cases} \qquad (4.3.21)$$

Consider second column in the big diagram above with j replaced by i, dualized and tensored by LK^{n-j} and its associated long exact sheaf cohomology sequence, we have

$$\to H^1(X, LK^{n-k} \otimes \mathcal{R}_i^*) \to H^1(X, LK^{n-j}\mathcal{R}_{i-1}^*) \xrightarrow{\sim} H^1(X, K^{i-j}) \to 0 \qquad (4.3.22)$$

where first group is zero by induction. ☐

Lemma 4.3.10. $H^1(\mathcal{V}_j \otimes \mathcal{R}_i^*) = (\star)$ as in previous Lemma

Proof. Fix i, proceed by induction on j. Start with $j = 1$, $\mathcal{V}_1 = LK^{n-1}$ by previous Lemma. For induction step, consider long exact sequence associated with first row of big diagram after -$\otimes\mathcal{R}_i^*$, we get

$$\ldots \to H^1(\mathcal{V}_{j-1} \otimes \mathcal{R}_i^*) \to H^1(\mathcal{V}_j \otimes \mathcal{R}_i^*) \xrightarrow{\sim} H^1(LK^{n-j} \otimes \mathcal{R}_i^*) \to 0 \qquad (4.3.23)$$

where the first group is zero by induction ☐

Lemma 4.3.11. $H^1(\mathcal{V}_{j-1} \otimes (LK^{n-j})^*) = H^1(X,K) \cong \mathbb{C}$

Proof. Dualize second column of the big diagram and -$\otimes\mathcal{V}_{j-1}$ then take long exact sheaf cohomology sequence gives

$$\to H^1(\mathcal{V}_{j-1} \otimes \mathcal{R}_i^*) \to H^1(\mathcal{V}_{j-1} \otimes \mathcal{R}_{j-1}^*) \xrightarrow{\sim} H^1(\mathcal{V}_{j-1} \otimes (LK^{n-j})^*) \to 0 \qquad (4.3.24)$$

where the first group is zero by previous lemma. ☐

[19] and for higher powers of K by same argument we see that H^1 will be zero since the corresponding space will be dual to space of global sections for some line bundle with negative degree

Now the group mentioned in last lemma is precisely the Ext group for the first column of the big diagram, therefore we see that either the successive extension for the filtration in oper structure is trivial i.e. split or it is the unique non-split extension. I'll skip the argument that the extension class is indeed not split which is not hard to show, see [Wen16] for the rest of the diagram-chasing.

Proof. Now we finish proof of the theorem. Each $\mathcal{V}_j$ is shown to be the unique non-split extension of LK^{n-j} by $\mathcal{V}_{j-1}$, therefore $\mathcal{V}$ is uniquely determined by L which by $L^n \cong K^{-n(n-1)/2}$ is uniquely determined by an n-torsion line bundle, the n^{2g} choices of which are discrete, so must remain locally constant on Op_n. □

Corollary 4.3.12. *L is uniquely determined by holonomy ρ, so we get embedding,*

$$\mathrm{Op}_n \hookrightarrow \mathcal{M}_B^{(n)}$$
$$(\mathcal{V}, \nabla) \mapsto \rho$$

Proof. Suppose $(\mathcal{V}, \nabla_1)$, $(\mathcal{V}, \nabla_2)$ have same monodromy representation ρ, but correspond to different line bundles $L \ncong M$. Consider short exact sequence $0 \to L^*M \to \mathcal{V}^* \otimes M \to V_{n-1}^* \otimes M \to 0$ and corresponding long exact sequence of sheaf cohomology,

$$0 \to H^0(L^*M) \to H^0(\mathcal{V}^* \otimes M) \to H^0(\mathcal{V}_{n-1}^* \otimes M) \to \dots \tag{4.3.25}$$

where first group is zero [20], and by assumption second group is non-trivial, so there must also be, by exactness of above sequence a nonzero holomorphic homomorphism $\mathcal{V}_{n-1} \to M$. Now consider short exact sequence $0 \to L^*K^{j-n} \to \mathcal{V}_j^* \to \mathcal{V}_{j-1}^* \to 0$, after $\otimes M$ for $j < n$ we get

$$0 \to H^0(L^*K^{j-n}M) \to H^0(\mathcal{V}_j^* \otimes M) \to H^0(\mathcal{V}_{j-1}^* \otimes M) \to \dots \tag{4.3.26}$$

where first group is zero by degree reason, and nonvanishing of second group will always imply nonvanishing of the third, by induction we see that eventually we'll have $H^0(\mathcal{V}_1^*M) \neq 0$ but now $\mathcal{V}_1$ is a line bundle and $\deg(\mathcal{V}_1^*M) = -2(n-1)(g-1) < 0$ which is a contradiction. □

We have seen by now that to each nth order differential operator one can associate an SL_n-oper,

$$\left\{ \begin{matrix} LK^{\frac{n-1}{2}} \xrightarrow{D} LK^{\frac{1+n}{2}} \\ \text{n-th order ordinary differential operator} \end{matrix} \right\} \leftrightarrow \mathrm{Op}_n = \left\{ SL_n\text{-opers} \right\} / \mathscr{G}^{\mathbb{C}} \tag{4.3.27}$$

and given an SL_n-oper the three conditions for local system $\mathbb{V}_\rho$ to be realizable in line bundle L are satisfied. Now it follows from Hejhal's theorem cited earlier that

Theorem 4.3.13 (Beilinson-Drinfeld [BD05])**.** *The embedding $\mathrm{Op}_n \hookrightarrow \mathcal{M}_B^{(n)}$ gives an isomorphism between connected components of Op_n and (affine) Hitchin base $\bigoplus_{j=2}^n H^0(X, K_X^j)$*

Remark 4.3.14. We have seen that for $n \geq 2$, SL_n-opers must be unstable, e.g. line subbundle $\mathcal{V}_1$ has degree $(n-1)(g-1) > 0$ whereas $\deg \mathcal{V} = 0$, so the subbundle has a bigger slope.

Recall we have

Theorem 4.3.15 (Narasimhan-Seshadri 1965 [NS65])**.** *A holomorphic vector bundle $\mathcal{V}$ of degree zero is stable iff it arise from irreducible unitary representation of fundamental group.*

it is now an easy consequence that any SL_n-oper for $n \geq 2$, the bundle $\mathcal{V}$ cannot admit flat unitary connection

[20] since these are global holomorphic sections of a degree zero non-trivial line bundle, any non-zero such sections must not have any zeros (or else its degree will be positive) but that gives a global trivialization!

Corollary 4.3.16 (Teleman 1960 [Tel60])**.** Monodromy of a differential equation

$$y^{(n)} + Q_2 y^{(n-2)} + \ldots + Q_n y = 0 \tag{4.3.28}$$

cannot be unitary

Recall HN type of holomorphic bundle with holomorphic connection is bounded from above by the condition $\mu_i - \mu_{i+1} \leq 2g-2$, now by what we've proved above, the SL_n-oper filtration saturates this bound:

$$\deg \mathcal{V}_j/\mathcal{V}_{j-1} = (n+1-2j)(g-1) \tag{4.3.29}$$

In fact the if we write the slope of successive quotients repeated as many times as rank, as when we defined HN type for HN filtratoin, we will get the tuple

$$\vec{\mu} = ((n-1)(g-1), \ldots, -(n-1)(g-1)) \tag{4.3.30}$$

jumping down by $2g-2$ from each component to the next. In fact one can show:

Proposition 4.3.17. $(\mathcal{V}, \nabla)$ is an SL_n-oper, then the oper filtration is precisely the Harder-Narasimhan filtration.

Proof. Suffices to show $\mathcal{V}_{j+1}/\mathcal{V}_j \subset \mathcal{V}/\mathcal{V}_j$ is maximal destabilizer subbundle (which is how HN filtration was constructed). Let $\mu_{\max}(\mathcal{V}/\mathcal{V}_j)$ denote the maximal slope of any subsheaf [21]. Proceed by induction on j from $n-1$. $j = n-1$ case is clear since $\mathcal{V}/\mathcal{V}_{n-1}$ is a line bundle. Now assume

$$\mu_{\max}(\mathcal{V}/\mathcal{V}_{j+1}) = (n-1-2(j+1))(g-1)(= \mu(\mathcal{V}_{j+2}/\mathcal{V}_{j+1})) \tag{4.3.31}$$

Let $\mathcal{F} \subset \mathcal{V}/\mathcal{V}_j$ be maximally destabilizer subsheaf which is necessarily semistable. Since $\mathcal{V}_{j+1}/\mathcal{V}_j$ is also a subsheaf we have

$$\mu(\mathcal{F}) \geq \mu(\mathcal{V}_{j+1}/\mathcal{V}_j) > \mu(\mathcal{V}_{j+2}/\mathcal{V}_{j+1}) = \mu_{\max}(\mathcal{V}/\mathcal{V}_{j+1}) \tag{4.3.32}$$

we see that $\mathcal{F} \to \mathcal{V}/\mathcal{V}_{j+1}$ must be zero by Lemma 4.2.5. □

It can also be shown that given holomorphic bundle with holomorphic connection $(\mathcal{V}, \nabla)$ that has maximal Harder-Narasimhan type, it admits an SL_n-oper structure and the filtration is precisely given by HN filtration.

Theorem 4.3.18. Among holomorphic vector bundles with holomorphic connection, SL_n-opers are characterized by having maximal HN type.

Fact Given any algebraic family of vector bundles $\{\mathcal{V}_t\}_{t\in T}$ parametrized by e.g. a variety T, HN type is upper-semicontinuous: i.e.

$$\{t \in T | \vec{\mu}\,(\mathcal{V}_t) \geq \vec{\mu}_0\} \tag{4.3.33}$$

is closed for any fixed HN type $\vec{\mu}_0$, or more intuitively HN type can only jump up not down. It follows that the set of maximal HN type is closed and further

Corollary 4.3.19.

$$\mathrm{Op}_n \hookrightarrow \mathcal{M}_B^{(n)} \tag{4.3.34}$$

is a proper embedding

[21] also of any subbundle, since any subsheaf can be saturated, i.e. taking the smallest subbundle containing it, which is of degree ≥ that of the subsheaf

4.4 G-opers

More generally we may define G-oper for G connected complex simple Lie group, I will briefly go over its definition and introduce some results about its properties reminiscent of what we proved above.

First we will recall some basic notions

Definition 4.4.1. A (holomorphic) principal G-bundle $P_G \to X$ is a G (holomorphic) fibre bundle over X equipped with right G action that covers 1_X (i.e. action respects fibres). A holomorphic connection on P_G is a holomorphic 1-form [22] taking value in Lie algebra of G

$$\omega : TP_G \to \mathfrak{g} \tag{4.4.1}$$

satisfying

- $R_g^*\omega = \mathrm{Ad}(g^{-1}) \cdot \omega$, i.e. equivariant wrt G-action
- $\omega(X^\sharp) = X$ for $X \in \mathfrak{g}$ and $X^\sharp$ vertical vector field corresponding to infinitesimal G-action.

Given $\rho : G \to GL(V)$ representation on a vector space V we may associate a vector bundle [23] to P_G via $P_G(V) = P_G \times_\rho V := PG \times V/\sim$ where

$$(p, v) \sim (p \cdot g, \rho(g)^{-1}v) \tag{4.4.2}$$

A V-valued holomorphic k-differential $\bar{\beta}$ is called G-equivariant if $R_g^*\bar{\beta} = \rho(g)^{-1} \cdot \bar{\beta}$ for all $g \in G$, and called horizontal if $\iota_{\bar{Y}} \circ \bar{\beta} = 0$ for all $\bar{Y}$ vertical vector field [24], i.e. if it vanishes whenever contracted with vertical vectors

Proposition 4.4.2. Any horizontal G-equivariant k-differential $\bar{\beta}$ comes from a unique $\beta \in \Omega^k(P_G(V))$ via pullback

Example The curvature tensor for a principal G-connection ω is defined by

$$F(\omega) = d\omega + \frac{1}{2}[\omega \wedge \omega] \tag{4.4.3}$$

which is G-equivariant and horizontal, therefore it comes from a 2-form in $\Omega^2(P_G \times_{\mathrm{Ad}} \mathfrak{g})$.

Definition 4.4.3. Let $H < G$ be a subgroup, a reduction of structure group of P_G to H is simply a sub-fibre bundle with fibres H which is itself a principal H-bundle via restriction of original right action, i.e. $P_H \subset P_G$.

Given G-connection ω on P_G we may use any reduction to H to get

$$TP_H \to TP_G \xrightarrow{\omega} \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \tag{4.4.4}$$

which is G-equivariant and horizontal, thus by above proposition comes from $\psi \in \Omega^1(P_H \times_{\mathrm{Ad}} \mathfrak{g}/\mathfrak{h}$ called second fundamental form of ω.

Now G-connection ω is called irreducible if $\psi \equiv 0$ for any $H = P$ some proper parabolic subgroup [25].

[22] we may think of its kernel as giving a notion of horizontal directions at each point

[23] P_G itself may be glued from copies of $U \times G$ and the transitions are functions $U \cap V \to G$. One way to think of associated bundle is a bundle whose transition are these functions composed with ρ.

[24] a vector field is vertical if it is along fibre, i.e. killed by pushforward p_*

[25] a subgroup $P < G$ is parabolic if G/P is compact

Let $B < G$ be a Borel subgroup, and $\mathfrak{b} < \mathfrak{g}$ corresponding Lie algebras and $x \in \mathfrak{b}$ a regular semisimple element, eigenspace decomposition of $\mathfrak{g}$ corresponding to it

$$\mathfrak{g} = \bigoplus_{i=-K}^{K} \mathfrak{g}_i \tag{4.4.5}$$

is called height-grading, let $\mathfrak{g}^j := \bigoplus_{i=j}^{k} \mathfrak{g}_i$, we get filtration

$$\mathfrak{g}^K \subset \ldots \subset \mathfrak{g}^0 \subset \mathfrak{g}^{-1} \subset \ldots \subset \mathfrak{g}^{-K} = \mathfrak{g} \tag{4.4.6}$$

where summing up to $\mathfrak{g}^0$ gives back the Borel subalgebra $\mathfrak{b}$ and $\mathfrak{b}/\mathfrak{g}^1 = \mathfrak{t}$ is a Cartan subalgebra, with respect to which the positively indexed part correspond to positive root spaces. This also gives, after quotient out by $\mathfrak{b}$ the filtration

$$\mathfrak{g}^{-1}/\mathfrak{b} \subset \mathfrak{g}^{-2}/\mathfrak{b} \subset \ldots \subset \mathfrak{g}/\mathfrak{b} \tag{4.4.7}$$

which is B-invariant and can be shown to be independent of choice of x.

We have G-equivariant isomorphism identifying tangent space to flag variety G/B to associated bundle viewing G as principal-B-bundle over G/B:

$$T(G/B) \cong G \times_B \mathfrak{g}/\mathfrak{b} \tag{4.4.8}$$

Proposition 4.4.4. There exists a unique dense open B-orbit $O \subset \mathfrak{g}^{-1}/\mathfrak{b}$ with respect to adjoint B-action corresponding to projection onto each $\mathfrak{g}_{-\alpha}$ being nonzero, where α is any simple root.

Definition 4.4.5. A G-oper is a triple (P_G, P_B, ω), where P_B is a reduction of structure group of P_G to B, and ω is a holomorphic flat connection on P_G such that

- There exists unique $\tilde{\Psi}$ making the diagram commute

$$\begin{array}{ccc} & & P_B \times_{\mathrm{Ad}} \mathfrak{g}^{-1}/\mathfrak{b} \\ & \overset{\tilde{\Psi}}{\nearrow} & \downarrow \\ TX & \xrightarrow{\Psi} & P_B \times_{\mathrm{Ad}} \mathfrak{g}/\mathfrak{b} \end{array}$$

- for all $v \in TX$, $v \neq 0$ we have $\tilde{\Psi}(v) \in P_B \times_{\mathrm{Ad}} O$ where O is the dense open B-orbit in above proposition.

I will end by mentioning a theorem in Beilinson-Drinfeld original paper that started the modern study on opers which now looks quite familiar after our hard work on SL_n opers.

Theorem 4.4.6 (Beilinson Drinfeld 1991, §3.1.4 [BD91])**.** Given (P_G, P_B, ω) a G-oper

- Oper structure is unique, i.e. $P_B \subset P_G$ is the Harder-Narasimhan flag
- $\mathrm{Aut}(P_G, \omega) = 1$
- (P_G, ω) cannot be reduced to non-trivial parabolic subgroup, i.e. the connection ω is irreducible.

Chapter 5

Convex projective structures on surfaces

Martin Bobb (UT Austin)

Motivation: in higher dimension hyperbolic geometry is rigid, so we should be open to allow deformation into real projective structures to have any hope for a deformation theory.

5.1 (G, X)-structures

Definition 5.1.1. Let G be a Lie group acting strongly effectively and transitively [1] on a connected manifold X. A (G, X)-structure on M is an atlas $\{(U, \varphi_U)\}$ with images in X and transition functions in G.

This gives a developing map $\mathrm{dev} : \tilde{M} \to X$, and a holonomy representation $\mathrm{hol} : \pi_1(M) \to G$ (see Chapter 3 on complex projective structures).

Theorem 5.1.1 (Ehresmann-Thurston principle)**.** If M is a compact (G, X)-manifold and ρ_0 is its holonomy, then ρ sufficiently close to ρ_0 is also the holonomy of some (G, X)-structure on M.

Example 5.1.2. $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{H}^2)$-structures, are (Riemannian) hyperbolic metrics
$(\mathrm{PGL}(n + 1, \mathbb{R}), \mathbb{RP}^n)$-structures are real projective structures
$(\mathrm{Aff}(n), \mathbb{A}^n)$ are affine structures.

5.2 Convex projective structures

We care about $(\mathrm{SL}(3, \mathbb{R}), \mathbb{RP}^2)$-structures.

Definition 5.2.1. A subset $\Omega \subset \mathbb{RP}^n$ is a convex projective domain when it is open, does not contain a projective line and is convex.

Facts:

1. We can define a Hilbert metric on Ω. Take two points x, y. There is a unique line going through intersecting Ω at z and w. Define the distance between x and y as

$$d_\Omega(x, y) = \frac{1}{2} \log([z : x : y : w])$$

[1] $g|_U = h|_U$ on some open nbd U if and only if $g = h$

2. $\mathrm{Aut}(\Omega) = \mathrm{Stab}(\Omega)$ in $\mathrm{SL}^{\pm}(n+1, \mathbb{R})$ acts by isometries for this metric;
3. if Ω is an ellipsoid, then you get the Klein model for hyperbolic space.

Definition 5.2.2. Let $\Gamma < \mathrm{Aut}(\Omega)$ be discrete and cocompact, then we say Ω is divisible and Γ is dividing.

Remark 5.2.3. A random convex domain will have trivial automorphism group.

Definition 5.2.4. $\Omega^* = \{w \in (\mathbb{RP}^n)^* \mid w \cap \Omega = \varnothing\}$

The dual domain can be thought as hyperplanes disjoint from the domain.

Theorem 5.2.1. If Ω is convex, then Ω^* is convex. Moreover if $\Gamma \curvearrowright \Omega$ dividing, then $\Omega^* \curvearrowleft \Gamma$ dividing.

Definition 5.2.5. A convex projective manifold M is a $(\mathrm{SL}^{\pm}(n+1, \mathbb{R}), \mathbb{RP}^n)$-manifold such that the developing map is a diffeomorphism onto some convex projective domain.

Remark 5.2.6. So these are actually complete structures, i.e. quotients of the domain Ω. So they are determined by their holonomy. Convexity is nevertheless natural for real projective structures on a closed surface: Choi proved in his thesis that any projective structure on a surface decomposes into π-annuli (specific non-convex projective structures on annuli) and convex structures with boundary having diagonalizable holonomy with distinct eigenvalues.

5.3 Bending and bulging

Bending is defined for any (G, X)-structure. Given $\rho_0 : \pi_1(M) \to G$ some holonomy, let $C \subseteq M$ be a separating submanifold. We get an amalgamated free product decomposition for $\pi_1(M)$. Consider the centralizer of the image of $\pi_1(C)$ in G. For any path in that, we can bend the representation and get a path of representations. Anytime you have non trivial centralizer you get at least locally holonomies of nearby geometric structures.

In the convex projective setting: suppose that $\rho_0 : \pi_1(S) \to \mathrm{SO}(2,1) < \mathrm{SL}(3, \mathbb{R})$ is the holonomy of a hyperbolic structure. The holonomy around a closed geodesic will be conjugate to a diagonal matrix.

$$\rho_0(\tilde{C}) \sim \begin{pmatrix} e^{\lambda} & & \\ & 1 & \\ & & e^{-\lambda} \end{pmatrix}.$$

The centralizer is

$$Z \begin{pmatrix} e^{\lambda} & & \\ & 1 & \\ & & e^{-\lambda} \end{pmatrix} = \left\langle \begin{pmatrix} e^{s} & & \\ & 1 & \\ & & e^{-s} \end{pmatrix}, \begin{pmatrix} e^{-t} & & \\ & e^{2t} & \\ & & e^{-t} \end{pmatrix} \right\rangle$$

and we can do bending. The geometric picture corresponding to it is that the convex domain gets bulged in the direction of some collection of points (coming from neutral fix points of the matrices in the centralizer). Bending using the second family of matrices, we get a so called bulge deformation. By Ehresmann-Thurston principle these are projective structures, but a priori it is not clear that these should be convex projective. They actually turn out to be convex projective for all times t.

5.4 Convex projective structures on surfaces

Let $\mathcal{B}(S)$ the space of convex projective structures on S. Let $\mathrm{Hit}_3(S)$ be the Hitchin component for $\mathrm{SL}(3,\mathbb{R})$ repesentations. The character variety has two more components: one containing the trivial representation and one containing Barbot representations (coming from upper block embedding of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(3,\mathbb{R})$), which are Anosov, at least for a bit. The Hitchin component is of course the one containing Teichmuller space, i.e. the irreducible embedding of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(3,\mathbb{R})$.

Theorem 5.4.1 (Goldman)**.** $\mathrm{hol} : \mathcal{B}(S) \to Hit_3(S)$ is a diffeomorphism.
In particular, $\dim(\mathcal{B}) = 16g - 16$.

Ehresmann-Thurston principle tells us that hol is a local diffeomorphism, but actually Goldman theorem implies it is a global diffeomorphism.

Remark 5.4.1. There is a retraction of $\mathcal{B}(S)$ onto $\mathcal{T}(S)$ which is mapping class group invariant; this is seen as a retraction of $Hit_3(S)$ onto the real locus. In particular the mapping class group acts properly on $\mathcal{B}(S)$. If you parametrize teichmüller space by quadratic differentials then fibers of this retraction are cubic differentials.

5.5 Flatness in higher dimension

Example 5.5.1. Let $\Omega = \Delta^n$ be the interior of the convex hull of a basis in $\mathbb{RP}^n$. So it is a triangle in dimension 2. Automorphisms of this domain are given by permutations of the points and all the diagonal matrices

$$\mathrm{Aut}(\Omega) = \mathbb{R}^n \rtimes \mathfrak{S}_n$$

This feels a lot like Euclidean isometries, and indeed this Ω is bilipschitz to Euclidean space. So there is flatness going on here. By Bieberbach the only compact convex projective manifolds coming from actions on this domain are virtually tori.

When $n > 2$, since the Hilbert metric only feels the plane on which these triangles live, these triangles are flat subspaces. Benoist (2006) proved that a divisible domain with properly embedded triangles in $\mathbb{RP}^3$ has a quotient which is an amalgam of cusped hyperbolic 3-manifolds glued along cusp cross sections (covered by these triangles).

5.6 Goldman parametrization of $\mathcal{B}(S)$

Let us start with a divisible convex domain $\Gamma \curvearrowright \Omega \subset \mathbb{RP}^2$.

Lemma 5.6.1. If there exists a segment in $\partial\Omega$, then Ω must be the triangle.

Proof. By contradiction let $\sigma = [a, b]$ be a segment in the boundary of Ω. Let $c \in \partial\Omega \setminus \sigma$. Consider the triangle $T = T(a, b, c)$ and a sequence of points y_i in it converging to some $y \in [a, b]$. Let K be a compact set such that $\Gamma K = \Omega$. Let $\gamma_i \in \Gamma$ st $\gamma_i y_i \in K$. Notice that in the Hilbert metric $d(y_i, [a, c]) \to \infty$ and $d(y_i, [b, c]) \to \infty$. Let $T_i = \gamma_i T$. Then $\gamma_i T$ subconverges to some T_∞, which is a triangle. Furthermore its edges at infinity are infinitely far from a point $y_i \in \Omega$, hence its boundary is contained in $\partial\Omega$, so it must be the whole thing. □

More generally (Danciger, Kassell.) if $\Omega \subset \mathbb{RP}^n$ is divisible and there is a segment in the boundary, then it contains a properly embedded triangle. This happens if and only if the domain is not δ-hyperbolic for the Hilbert metric. Notice that this does not say that the triangle contains the edge. Notice that as mentioned before the Hilbert metric on a triangle is bilipschitz (hence quasi-isometric) to the Euclidean plane. Indeed triangles are isometric to the hex-metric on $\mathbb{R}^2$.

Lemma 5.6.2. If Ω is divisible and not a triangle, then the boundary is C^1.

Proof. Suppose it is not at $z \in \partial\Omega$. Then there are at least two supporting lines, and all the positive linear combinations are supporting lines, so there is a segment in the boundary of the dual domain. By the previous lemma, Ω is a triangle. But then our domain is a triangle (in general $(\Omega^*)^* = \Omega$). □

Therefore we get that if $S = \Omega/\Gamma$ is a convex projective surface (closed and with negative Euler characteristic), then $\partial\Omega$ is C^1. This is an Anosov behavior.

Proposition 5.6.1 (Goldman)**.** $\forall \gamma \in \Gamma$ either it is the identity or it is conjugate to a diagonal matrix with distinct eigenvalues $(\mu, \frac{1}{\mu\lambda}, \lambda)$ with $0 < \mu < 1 < \lambda$.

Proof. We have to exclude the following cases:

1. there is a complex eigenvalue. This gives a fixed point, so the action is not poper. Discard.
2. there is a real jordan block of rank 2. This gives two fixed points; the lines between them supports the second one, hence we get segments in the boundary. Discard by previous lemmas.
3. a rank 3 jordan block. This gives a cusp, so a curve of length 0. Discard.

□

5.6.1 Goldman Parametrization

It is the analogue of Fenchel-Nielsen parametrization of Teichmuller space. Let $S = \Omega/\Gamma$ for a divisible convex set, be a closed surface with negative E uler characteristic. Choose $\{P_i\}$ a pants decomposition. We need to parametrize convex projective structures on a pair of pants P where holonomy of boundary curves is as in the above proposition. For each curve in the pants decomposition we get two parameters essentially coming from the eigenvalues of the holonomy, say we take

$$\frac{\lambda}{\mu} \text{ and } \lambda\mu ,$$

where the first measures the length of the curve in the Hilbert metric, and the second one measures how far you are from being hyperbolic (= 1 if and only the holonomy can be made into a hyperbolic isometry). But then there are two more internal parameters: given a pair of pants decompose into two ideal triangles. For triangles the situation is the following: on a triangle there is a unique projective structure; but the ones coming from a pants decomposition have more data, namely supporting hyperplanes. The deformation space of projective structures on an ideal triangle is $\mathbb{R}$. Hence the space of convex projective structure on the pair of pants P is 8-dimensional, actually equal to $\mathbb{R}^8$.

Now given a pair of pants we get 8 parameters, and there are $2g-2$ of them. But we over counted curves, so remove $3g-3$. Then keep track of gluing data (Fenchen-Nielsen and bulging parameters). So a total of $16(g-1)$ parameters.

Theorem 5.6.3 (Benoit,Koszul)**.** If $\mathcal{F}(M) \subset \chi(M, SL_n^{\pm}, \mathbb{R})$ are holonomies of convex projective structures, then it is open and closed.

Therefore the Hitchin component is entirely made of holonomies of convex projective structures.

Chapter 6

Analytic perspective on convex projective geometry

Xian Dai (Rice University)

Let S be a compact oriented surface of genus $g \geq 2$.

Theorem 6.0.1. There exists a correspondence between convex $\mathbb{RP}^2$-structures on S and pairs (J, U) of a complex structure and a holomoprhic cubic differential on it.

Idea of proof. For the forward direction, let $S = \Omega/\Gamma$ for some convex domain $\Omega \subset \mathbb{R}^2 \subset \mathbb{RP}^2$. Define a Monge-Ampere type PDE, and get a hyperbolic affine sphere H which is asymptotic to the cone over $\Omega \subseteq \mathbb{RP}^2$ in $\mathbb{R}^3$. Get an affine metric on H, hence one on S. We also get a connection, and from it we get the cubic differential. □

Outline:

1. Affine geometry, its invariants
2. Affine spheres, examples
3. Formulate hyperbolic affine spheres in $\mathbb{R}^3$, associated PDEs
4. John Loftin's Theorem

6.1 Affine geometry

Goal: study hypersurfaces in $\mathbb{R}^{n+1}$ and properties thereof which are invariant under the group of special affine transformations (i.e. SL(n+1,R) and translations, so volume is preserved).

Given H a smooth hypersurface, we want a vector field ζ along H which is transverse to H at all points and is invariant under special affine transformations in the sense that for any special affine φ we have

$$\varphi\zeta_H = \zeta_{\varphi(H)}$$

. Such vector is called affine normal.

We get equations of Gauss and Weingarten for affine normal ζ

$$D_X Y = \nabla_X Y + h(X,Y)\zeta, X, Y \in TH$$

$$D_X\zeta = -s(X) + \Gamma(X)\zeta$$

where D is the flat connection on $\mathbb{R}^{n+1}$. Requirement for the affine normal:

a) $\Gamma(x) = 0$ (equivalent to "equiaffine condition");

b) $|\det(X_1 X_2 \dots X_n J)| = 1$, $h(X_i, X_j) = \delta_{ij}$, $X_i \in T_y M$

The affine normal is well-defined up to sign; by convention, we pick ζ towards convex part of M.

Another geometric characterization due to Blaschke: take a point, take local parallel planes to the tangent plane, intersect with H and take center of mass of the resulting plane curve; this defines a curve starting at the point, the tangent of which at the point is the affine normal. (this is not quite the right thing, but kind of, up to constants and reasonable parametrizations of the curve).

Definition 6.1.1. ∇ is called the Blaschke connection; when H is strictly locally convex, $h(X, Y)$ is positive definite and is called the affine metric. S is the affine shape operator.

6.1.1 Affine spheres

Definition 6.1.2. If all of the affine normal lines meet at a point p, H is called an affine sphere (with center p.) When $p = \infty$ (i.e. affine normals all parallel) H is called an improper (or parabolic) affine sphere

Proposition 6.1.3. Affine sphere is improper $\iff S = \lambda I$, $\lambda = 0$.

Proof. If affine normals are parallel (in synthetic Euclidean sense), we can find λ nonvanishing function s.t. $\lambda\zeta$ is a parallel section (under Euclidean parallel transport):

$$0 = D_X(\lambda\zeta) = X(\lambda)\zeta + \lambda D_X\zeta = X(\lambda)\zeta - \lambda S(X)$$

for all X. This happens if and only is iff $S(X) = 0$ and λ is constant. □

Proposition 6.1.4. For a proper affine sphere, $S = \alpha I$ where $\alpha \neq 0$

Proof. $\Leftarrow$: If $S(X) = \lambda X$, consider the vector field $Y = f(x) + \frac{1}{\lambda}\zeta$. where $f : H \to \mathbb{R}^{n+1}$ is an isometric immersion. Then

$$D_X Y = D_X\left(f(x) + \frac{1}{\lambda}\zeta\right) = Xf + \frac{1}{\lambda}D_X\zeta = f_*X - \frac{1}{\lambda}f_*(SX) = 0$$

Hence $Y(x)$ is constant, which $\implies$ that affine normals meet at $Y(x)$.

$\implies$: if all affine normals meet, the $p = y(x) = f(x) + \lambda\zeta$ is the center and

$$0 = D_X p = D_X f(X) + D_X(\lambda\zeta) = F_*(X) + (X\lambda)\zeta + f_*(\lambda S(X))$$

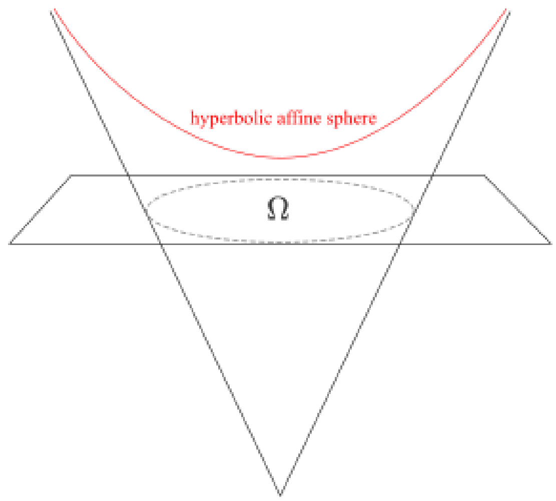
$f_*(X) = f_*(\lambda S(X)) \implies S(X) = \frac{1}{\lambda}X$. □

Proper affine spheres come in two types: if affine normals point away from the center, the affine sphere is called hyperbolic; else if they point towards the center, the affine sphere is elliptic.

Example 6.1.5. Improper: $f(x, y) = \left(x, y, \frac{1}{2}(x^2 + y^2)\right)$; $\zeta = (0, 0, 1)$.

Example 6.1.6. Elliptic: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (Ellipsoid; affine-equivalent to a sphere.)

Example 6.1.7. Hyperbolic: $x^2 + y^2 - z^2 = 1$



6.2 Hyperbolic affine spheres

Let us consider a hyperbolic affine sphere. Up to rescaling and translation we can assume the center is at 0 and $\lambda = -1$, so that $\zeta(x) = f(x)$, i.e. the affine normal is just the position vector.

Theorem 6.2.1 (Cheng-Yau)**.** Given a bounded convex domain $\Omega \subseteq \mathbb{R}^n$, there exists a unique properly embedded hyperbolic affine sphere[1] asymptotic to the cone over Ω. Indeed it will be given by

$$\mathcal{H} = \{-\frac{1}{u(x)}(1,x) \mid x \in \Omega\}$$

if we think of Ω as being embedded in the inhomogenous affine chart $x_0 = 1$, where u is a smooth solution, continuous up to the boundary, which is the unique convex soltuion of the Monge Ampere PDE

$$detD^2 u = \left(\frac{-1}{u}\right)^{n+2}, u = 0 \text{ on } \partial\Omega$$

where D^2 is the Hessian. Moreover any such hyp affine sphere is asymptotic to the boundary of the cone given by the convex hull of H itself.

The affine metric is given by

$$h = -\frac{1}{u(x)} \frac{\partial^2 u}{\partial x_i \partial x_j} dx^i dx^j$$

Also properness of the embedding is equivalent to completeness of h.

6.2.1 Pick form

Define a 3-tensor C, called Pick form on the tangent space TH to an affine sphere H as

$$C : TH^{\otimes 3} \to \mathbb{R}$$

$$C(X,Y,Z) = h(\nabla_X Y - \nabla^h_X Y, Z)$$

[1]From now on center at 0 and mean curvature $\lambda = -1$

where ∇ is Blaschke connection and ∇^h is the levi civita of h.

Let us see it is symmetric. Let $K_X = \nabla - \nabla^h$ be the endomorphism measuring the difference of the two connections.

$$\nabla_X h(Y,Z) = \nabla_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

and we have $\nabla^h_X h = 0$. By Codazzi we have

$$\nabla_X h(Y,Z) = \nabla_Y h(X,Z)$$

moreover by torsion freeness we have

$$K_X Y = K_Y X$$

hence we get that $h(Y, K_X Z)$ is symmetric in X, Y. This is actually just a scalar multiple of C. Moreover $tr(K_X) = 0$. (Apolarity condition). Follows from equiaffine condition (preservation of volume).

Remark 6.2.1. For the standard hyperboloid the pick form is identically zero. This is not obvious, follows for a long tricky computation.

6.3 Hyperbolic affine spheres in $\mathbb{R}^3$

Let's now consider case $n = 2$ (following C.P. Wang). What are the conditions for a 2-dimensional surface to be an affine hyperbolic sphere in terms of conformal geometry given by affine metric. Suppose we have a metric and we choose local conformal coordinates for it on the surface $x = x + iy$. The affine metric in this conformal class will be

$$h = e^{\psi}|dz|^2$$

for some function ψ. The parametrization for our surface can be written as

$$f : \mathbb{D} \to \mathbb{R}^3$$

The equations reduce now to

$$D_X Y = \nabla_X Y + h(X,Y)f$$

$$D_X f = X$$

because $s(X) = -X$ and we want also the condition that $det(X_1, X_2, f)^2 = 1$, with X_1, X_2 an orthnormal basis for h. Use $X_1 = e^{-1/2\psi} f_x$ and $X_2 = e^{-1/2\psi} f_y$. Now complexify everything by tensoring with $\mathbb{C}$ and consider the frame $\{f_z, f_{\bar{z}}, f\}$. Then we get

$$h(f_z, f_z) = 0$$

same for $\bar{z}$ and

$$h(f_z, f_{\bar{z}}) = 1/2e^{\psi}$$

$$D_{f_z} f_z = f_{zz}$$

and similarly

$$D_{f_{\bar{z}}} f_z = f_{\bar{z}z}$$

.

The structure equations become

$$\begin{aligned} f_{zz} &= \psi_z f_z + U e^{-\psi} f_{\bar{z}} \\ f_{\bar{z}\bar{z}} &= \bar{U} e^{-\psi} f_z + \psi_{\bar{z}} f_{\bar{z}} \\ f_{z\bar{z}} &= \frac{1}{2} e^{\psi} f \end{aligned}$$

where U is a cubic differential.

The Levi-Cività connection may be written in term of ψ.

From apolarity condition $\mathrm{tr} k_X = 0 \, \forall X \in T_x\mathcal{H}$: there are only two terms: $C_{iii}, C_{\overline{iii}} \neq 0$ (where i is for ∂_z, $\bar{i}$ for $\partial_{\bar{z}}$.) and $C_{111} = \frac{1}{2} e^{\psi} C^{\bar{1}}_{11} = U$, $\bar{\partial} U = 0$; $C_{\bar{1}\bar{1}\bar{1}} = \frac{1}{2} e^{\psi} C^{1}_{\bar{1}\bar{1}}$. Thus U is the complexified Pick differential.

Take the structure equations above + integrability condition (mixed partials commute, Frobenius theorem), to get

$$\psi_{z\bar{z}} + |U|^2 e^{-2\psi} - \frac{1}{2} e^{\psi} = 0 \qquad\qquad U_{\bar{z}} = 0$$

Remark 6.3.1. These are the Hitchin equations for $\mathrm{SL}(3, \mathbb{R})$ Higgs bundles.

Suppose we are given U which is holomorphic w.r.t. z, we can find ψ, h solving the PDE above. That is expressed in local coordinates, if we want a global PDE on a compact Riemann surface Σ of genus $g \geq 2$, we introduce $h_0 = e^{\phi}|dz|^2$ a hyperbolic metric of constant curvature -1. We then write the affine metric $h = e^{\phi + u}|dz|^2 = e^{\psi}|dz|^2$, $\phi + u = \psi$. Now u is a global function and we get ageometric PDE

$$\Delta_0 u + 4\|U\|_0^2 e^{-2u} - 2e^u + 2 = 0$$

.

Proposition 6.3.2. This has a unique C^∞ solution.

Proof. Existence: (from general PDE theory) suffices to find subsolution, supersolution. $u \equiv 0$ is a subsolution; for supersolution: define $G = \max g(x) \geq 0$, let m be positive root of $2x^3 - 2x^2 - G = 0$; $u \equiv \log m$ is a supersolution.

Smoothness: elliptic regularity.

Uniqueness: maximal principle. □

Hence, there esists a unique C^∞ affine metric with given U.

Now, if we know U and h, we can solve for f in the structure equations, and thus find the affine sphere in $\mathbb{R}^3$.

Proposition 6.3.3 (Wang)**.** A hyperbolic affine sphere $\mathcal{H}$ with mean curvature -1 and center 0 admitting $P \leq \mathrm{SL}_3\mathbb{R}$ acting properly discontinuously is entirely determined by a conformal structure on S and a holomorphic cubic differential U of K^3.

Theorem 6.3.4 (Loftin)**.** $\{$Convex $\mathbb{RP}^2$ structures on $S\} \leftrightarrow \{(J, U)\}$

Corollary 6.3.5 (Goldman)**.** The deformation space of convex $\mathbb{RP}^2$ structures is the bundle of holomorphic cubic differentials (a holomorphic vector bundle) over Teichmüller space (which is the Hitchin component.)

Remark 6.3.6. If we run the same process for genus 1: get

$$\Delta_0 u + 4\|U\|_0^2 e^{-2u} - 2e^u = 0$$

K^3 trivial; $\|U\|_0^2$ constant. If this constant is 0, there are no solutions to the PDE. If nonzero: u constant. Hence $h = e^u g_0$ is flat. But but all affine spheres in $\mathbb{R}^3$ with flat affine metrics are classified. In particular only one of them is hyperbolic, called the Titeica surface $xyz = C > 0$ $(x, y, z > 0.)$

$\implies$ any properly convex $\mathbb{RP}^2$ structure on a genus-1 surface has developing image Ω a triangle.

Chapter 7

Anosov representations

Max Riestenberg (UT Austin)

Significant parts of these notes were taken from a series of talks by Dick Canary at a "Graduate school on Geometry of Teichmuller spaces" held by the Simons Center for Geometry and Physics in April 2019. Any mistakes are my own. Many of the theorems discussed in these notes are much more general in the source material; here we almost exclusively focus on projective Anosov representations in $SL(d, \mathbb{R})$, and state the theorems in this context only.

7.1 Brief history

Anosov structures were introduced by Labourie in 2004 as dynamical structures on negatively curved manifolds [Lab06]. His primary application was to prove that Hitchen representations are discrete and faithful. In 2012, Guichard and Wienhard extended his definition to define Anosov representations for Gromov hyperbolic groups to semisimple Lie groups, and showed Anosov representations of surface groups model geometric structures [GW12]. In 2013, Kapovich, Leeb and Porti gave several new characterizations of Anosov representations as generalizations of convex cocompact representations in rank 1 and initiated a systematic study of geometric structures associated to Anosov representations [KLP18].

For any semisimple (even reductive) Lie group G, there are in general several (finitely many) possible notions of Anosov one can study. In this talk, we will mainly focus on the case of projectively Anosov representations of surface groups in $SL(d, \mathbb{R})$. In some sense this case is the most basic and crucial to the theory. We will also discuss the link between projective Anosov representations and convex cocompact actions on projective space due to Danciger, Gueritaud and Kassel [DGK18b], including the special case of pseudo-Riemannian hyperbolic geometry [DGK18a].

7.2 Motivation: Convex cocompact representations

Let Γ be a finitely generated free group or closed surface group of genus at least 2. Given a representation $\rho : \Gamma \to PSL(2, \mathbb{R})$, we consider the orbit map $\Gamma \to \mathbb{H}^2$ given by $\gamma \mapsto \gamma x_0$ for some basepoint x_0. We view Γ as a hyperbolic metric space by constructing its Cayley graph and using the word metric. We say that ρ is convex cocompact if this orbit map is a quasi-isometric embedding. (The terminology "convex cocompact" should call to mind a geometric structure; this will be addressed later.)

Example: If S is a closed surface of genus at least 2, and $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$ is discrete and faithful, then the orbit map is a quasi-isometry, and notice that $\mathbb{H}^2/\rho(\pi_1(S))$ is homeomorphic to

S. In particular, there is a dictionary between geometric structures on S and convex cocompact representations.

7.2.1 Stability

Lemma 7.2.1 (Stability of convex cocompact representations)**.** If $\rho_0 : \Gamma \to PSL(2, \mathbb{R})$ is convex cocompact, then there is a neighborhood of ρ_0 in $\mathrm{Hom}(\Gamma, PSL(2, \mathbb{R}))$ consisting entirely of convex cocompact representations.

Proof. We will use a local-to-global principle of hyperbolic metric spaces: Given (L, A), there exists s, L', A' such that if the restriction of the orbit map to an s-ball in the Cayley graph is an (L, A) quasi-isometric embedding, then the orbit map is (globally) a (L', A') quasi-isometric embedding.

Armed with that, assume ρ_0 is convex cocompact, so there exists (L_0, A_0) such that the orbit map is an (L_0, A_0)-quasi-isometric embedding. In particular, it is a (say) $(2L_0, A_0 + 1)$-quasi-isometric embedding when restricted to an $s = s(2L_0, A_0 + 1)$-ball. Now for ρ near enough ρ_0, ρ is also a $(2L_0, A_0 + 1)$ on an s-ball, hence by the local-to-global principle is convex cocompact. □

For a proof of the local-to-global principle, see my answer here.

7.2.2 Limit maps and dynamics

For a hyperbolic metric space X, define its Gromov boundary $\partial_\infty X$ to be the space of geodesic rays up to bounded equivalence. It has a topology with the informal description "geodesic rays that stay near for a long time are close together." A quasi-isometric embedding $f : X \to Y$ of hyperbolic metric spaces induces an embedding $\hat{f} : \partial_\infty X \to \partial_\infty Y$. If f is a quasi-isometry, $\hat{f}$ is a homeomorphism, so the boundary of a hyperbolic metric space is well-defined. We may immediately conclude the following:

Lemma 7.2.2 (Limit maps of convex cocompact representations)**.** If $\rho : \Gamma \to PSL(2, \mathbb{R})$ is convex cocompact, then there is a ρ-equivariant embedding $\xi_\rho : \partial_\infty \Gamma \to \partial \mathbb{H}^2$ called the limit map.

Moreover, ξ is dynamics-preserving. Dynamics-preserving means that ξ takes the attracting fixed point of γ to the attracting fixed point of $\rho(\gamma)$. We can see this easily in the case Γ is a closed hyperbolic surface group, because of equivariance.

7.2.3 Linear algebra

We now start looking towards higher rank Lie groups. Rather than slog through the general Lie theory, we will lean on our knowledge of linear algebra to study $G = SL(d, \mathbb{R})$. Any matrix $a \in SL(d, \mathbb{R})$ may be written in the form $a = kdk'$ where $k, k' \in SO(d)$ and d is a diagonal matrix with entries

$$d_{ii} = \sigma_i(a) \text{ and } \sigma_i(a) \geq \sigma_{i+1}(a) \text{ for all } i$$

Notice that $\sigma_1(a)$ is the operator norm of a and is the length of the major axis of $a(S^{d-1})$. In general if $i \geq 2$, $\sigma_i(a)$ is the length of the ith minor axis of $a(S^{d-1})$. Let $U(a) := \langle ke_1 \rangle$ be the major axis of $a(S^{d-1})$, and let $H(a) := \langle ke_1, ke_2, \dots ke_{d-1} \rangle$ be the "attracting hyperplane."

For actions on the hyperbolic plane as the upper half space with basepoint $x_0 = i$, we have

$$d(x_0, \rho(\gamma)x_0) = \log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))}.$$

(Points in $\mathbb{H}^2$ correspond to inner products on $\mathbb{R}^2$, which is why singular values show up here.) By equivariance, it's enough to ask that distances to the basepoint satisfy the quasi-isometry condition, so ρ is convex cocompact if and only if there exist $L \geq 1$ and $A \geq 0$ so that

$$\frac{1}{L}|\gamma| - A \leq \log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \leq L|\gamma| + A$$

where $|\gamma|$ is the word length of γ. In fact, an easy argument shows that

$$d(x_0, \gamma x_0) \leq \max_i d(x_0, \gamma_i x_0)|\gamma|,$$

where the maximum is over generators γ_i, holds for <u>any</u> action of a finitely generated group on a metric space, so we only ever need the lower bound.

Exercise (Cartan property)**.** If ρ is convex cocompact and $\gamma_n \to z \in \partial\Gamma$ then

$$U(\rho(\gamma_n)) \to \xi(z) \text{ and } \xi(\gamma_n^+) \to z$$

where γ^+ is the attracting fixed point of γ.

The minimal <u>translation length</u> on $\mathbb{H}^2$ of an element $g \in PSL(2, \mathbb{R})$ is given by

$$\alpha_1(g) := \log \frac{\lambda_1(g)}{\lambda_2(g)} = \inf_{x \in \mathbb{H}^2} d(x, gx)$$

where λ_i is the modulus of the ith eigenvalue of (a lift of) g. Write $\|\gamma\|$ for the translation length of γ on the Cayley graph

$$\|\gamma\| := \min_{x \in \Gamma} d(x, \gamma x).$$

The following somewhat surprising result tells us that the orbit map coarsely preserves distances to the basepoint if and only if the representation coarsely preserves translation lengths.

Exercise ("Proximality result")**.** An action $\rho : \Gamma \to PSL(2, \mathbb{R})$ is convex cocompact if and only if

$$(1/L)\|\gamma\| - A \leq \alpha_1(\rho(\gamma)) \leq L\|\gamma\| + A.$$

A far reaching generalization of this result is due Delzant-Guichard-Labourie-Mozes and [GGKW17a] prove an analogue of this result for Anosov representations.

7.2.4 Geometric structures

If S is a closed hyperbolic surface, the space of convex cocompact representations of $\pi_1(S)$ into $PSL(2, \mathbb{R})$ is both open (by what we proved) and closed (by the Margulis lemma). (The Margulis lemma implies that the set of discrete faithful representations of $\pi_1(S)$ is closed and discrete faithful representations of $\pi_1(S)$ are automatically convex cocompact.) The space of convex cocompact representations are two components of $\mathrm{Hom}(\pi_1(S), PSL(2, \mathbb{R}))$. The quotient of each of these is a copy of the Teichmüller space of S. The Teichmüller space of S is homeomorphic to a ball of (real) dimension $6g - 6$.

On the other hand, $\mathrm{Hom}(F_n, PSL(2, \mathbb{R})) = PSL(2, \mathbb{R})^n$ is connected and the set of convex cocompact representations is open but not closed.

Teichmüller theory enjoys interesting deformations (e.g. bending), limit maps, and associated geometric structures. For any word hyperbolic group Γ we can define $\rho : \Gamma \to PO(n, 1) \cong \mathrm{Isom}(\mathbb{H}^n)$ to be <u>convex cocompact</u> if some (equivalently, every) orbit map is a quasi-isometric embedding. The same theory goes through: the space of such representations is open, we have associated equivariant limit maps, and associated geometric structures. The name "convex cocompact" comes from the following fact:

Theorem 7.2.3. An action $\rho : \Gamma \to PO(n, 1)$ is convex cocompact if and only if there exists a convex subset C of $\mathbb{H}^n$ such that $\rho(\Gamma)$ acts properly discontinuously and cocompactly on C.

For example, quasifuchsian representations $\rho : \pi_1(S) \to \mathbb{H}^3$ are convex cocompact. In fact the theory of convex cocompact representations works for any rank 1 symmetric space of noncompact type: these are the real, complex, quaternionic and octonionic hyperbolic spaces.

7.2.5 Failures of naive generalizations

"Higher Teichmüller theory," perhaps better called higher rank geometry, is the study of representations of Gromov hyperbolic groups into higher rank Lie groups. Here we mean the real rank of the reductive Lie group G, which is the dimension of a maximal flat in its associated Riemannian symmetric space. Each such representations correspond to an action by isometries on the associated symmetric space. However, a naive notion of convex compactness for representations $\Gamma \to G$ fails to share the good properties we want.

If we use the quasi-isometric embedding definition, stability fails: the Morse lemma and local to global principle of hyperbolic metric spaces fails for the Euclidean plane, so it fails for any higher rank symmetric space. In fact, Guichard produced an example of a representation

$$\rho : F_2 \to SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \subset SL(4, \mathbb{R})$$

whose orbit map is a quasi-isometric embedding but ρ is a limit of representations which fail to be discrete and faithful. (Take two complete hyperbolic structures on a pair of pants, with one cuff being hyperbolic in one structure but a cusp in the other, and vice versa for another cuff.)

If we seek actions which act cocompactly on convex subsets of symmetric spaces, we find too few representations (with too limited deformation theory).

Theorem 7.2.4 (Kleiner-Leeb [KL06], Quint [Qui05])**.** If $\rho : \Gamma \to SL(d, \mathbb{R})$ acts properly and cocompactly on a convex subset C of the symmetric space, then ρ "essentially" comes from a representation of a rank one Lie group into $SL(d, \mathbb{R})$.

We will study "Anosov representations", which generalize convex cocompact representations in rank 1, are stable, have limit maps, and often have associated actions which are proper and cocompact on domains of discontinuity.

7.3 Anosov Representations

For a given semisimple Lie group G, there are several possible notions of Anosov that can be studied. We will focus on projective Anosov representations into $SL(d, \mathbb{R})$. We start with a simple characterization due to Guichard and Wienhard in terms of limit maps.

Each Grassmanian $Gr(k, d)$ of k-planes in $\mathbb{R}^d$ plays a role as a sort of boundary for the symmetric space associated to $SL(d, \mathbb{R})$.

Definition 7.3.1 (Limit map)**.** Let Γ be Gromov hyperbolic and $\rho : \Gamma \to SL(d, \mathbb{R})$ be a representation. A limit map ξ^k_ρ for ρ is a continuous ρ-equivariant map

$$\xi^k_\rho : \partial\Gamma \to Gr(k, d).$$

Two limit maps ξ^k and ξ^{d-k} are transverse if whenever $x \neq y$ in $\partial\Gamma$ then $\xi^k(x) \oplus \xi^{d-k}(y) = \mathbb{R}^d$. If $k \leq d - k$ then ξ^k and ξ^{d-k} are compatible if $\xi^k(x) \subset \xi^{d-k}(x)$ for all $x \in \partial\Gamma$.

Theorem 7.3.2 (Guichard-Wienhard [GW12])**.** Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be an irreducible representation of a hyperbolic group. Then ρ is projective Anosov if and only if it has compatible transverse limit maps ξ^1 and ξ^{d-1}.

We don't have a definition of Anosov yet, so we can't yet prove this theorem, but let's at least see what mileage we get out of having an irreducible representation and compatible transverse limit maps. Note that there are many flavors of Anosov besides projective Anosov, and they don't admit a characterization quite as simple as this one. Part of what this result tells us is that projective Anosov representations are easier to work with than other flavors.

Let $\rho : \Gamma \to SL(d, \mathbb{R})$ be an irreducible representation of an infinite hyperbolic group with a limit map $\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$. By equivariance, $\xi^1(\partial\Gamma)$ is preserved by $\rho(\Gamma)$, hence so is its span, which is then an invariant subspace and must be all of $\mathbb{R}^d$ by irreducibility. If γ^+ is the attracting fixed point of γ in $\partial\Gamma$, then $\xi^1(\gamma^+)$ is an attracting fixed point for the action of $\rho(\gamma)$ on $\mathbb{P}(\mathbb{R}^d)$, i.e. $\xi^1(\gamma^+)$ is an attracting eigenline for $\rho(\gamma)$. We say that ξ^1 is dynamics-preserving. It follows that $\rho(\gamma)$ is proximal, i.e. its eigenvalue of maximal modulus is real and has multiplicity one. Similarly, one may show that $\xi^{d-1}(\gamma^-)$ is the repelling hyperplane of $\rho(\gamma)$.

We've displayed nice dynamical properties of this representation. This is really the (original) defining feature of Anosov representations: they carry the good properties of the geodesic flow on a negatively curved space over to some associated bundle.

7.3.1 Labourie's original definition

Assume $\Gamma = \pi_1(M)$ for M a closed negatively curved manifold. Its unit tangent bundle UM has a geodesic flow which is Anosov. In detail:

We have the identification

$$U\widetilde{M} = \left(\partial\widetilde{M} \times \partial\widetilde{M} - \Delta\right) \times \mathbb{R} = (\partial\Gamma \times \partial\Gamma - \Delta) \times \mathbb{R}$$

and in these coordinates the geodesic flow is $\phi(x, y, s) = (x, y, s + t)$. The geodesic flow is clearly invariant under the Γ action $\gamma(x, y, s) = (\gamma x, \gamma y, s)$, as it must be since Γ acts by isometries; hence it descends to $UM \cong U\widetilde{M}/\Gamma$. We have the splitting into stable, invariant and unstable subbundles:

$$T(UM) = V_- \oplus V_0 \oplus V_+$$

where

$$\begin{aligned} &\text{if } v \text{ is a section of } V_- \text{ then } \|(\phi_t)_* v\| \le Ce^{-at}\|v\| \\ &\text{if } v \text{ is a section of } V_0 \text{ then } \|(\phi_t)_* v\| = \|v\| \\ &\text{if } v \text{ is a section of } V_+ \text{ then } \|(\phi_t)_* v\| \ge De^{bt}\|v\| \end{aligned}$$

for some constants $a, b, C, D > 0$. We say the flow is contracting on V_- and dilating on V_+. (In some places, the condition on V_+ is called "expanding," but we will reserve that terminology for a dynamical notion due to Sullivan.) (Draw a picture for $\widetilde{M} = \mathbb{H}^2$.)

These properties are independent of the norm $\|\cdot\|$ on UM; if they hold for one such norm, then they hold for any continuously varying norm. The idea of an Anosov representation $\rho : \Gamma \to G$ to a semisimple Lie group G is to mimic these dynamics on a bundle associated to ρ. In case Γ is a general Gromov hyperbolic group, Guichard and Wienhard extend this definition by using a flow space associated to Γ defined by Gromov in place of $U\widetilde{M}$ [GW12].

We will use the associated bundle $E_\rho = \widetilde{E}/\Gamma$ where $\widetilde{E}$ is the trivial bundle

$$\widetilde{E} = U\widetilde{M} \times \mathbb{R}^d$$

and Γ acts on $\widetilde{E}$ as the group of covering transformations in the first factor and via ρ in the second factor. The bundle E_ρ over UM has a "flat connection" which means that a smooth curve in UM

lifts to a curve which is locally constant in the (local) $\mathbb{R}^d$ factor. The geodesic flow ϕ_t on UM lifts to a flow ψ_t on E_ρ parallel to the flat connection. Explicitly, the flow $\widetilde{\psi}_t$ on $\widetilde{E}$ given by

$$\widetilde{\psi}_t((x,y,s),v) = ((x,y,s+t),v) = (\widetilde{\phi}_t(x,y,s),v)$$

descends to a flow ψ_t on E_ρ. If ρ has transverse limit maps ξ^1 and ξ^{d-1}, one gets an equivariant splitting

$$\widetilde{E} = \widetilde{\Xi} \oplus \widetilde{\Theta}$$

where $\widetilde{\Xi}$ is the line bundle over $U\widetilde{M}$ whose fiber over (x,y,s) is the line $\xi^1(x)$ and $\widetilde{\Theta}$ is the hyperplane bundle over $U\widetilde{M}$ whose fiber over (x,y,s) is the hyperplane $\xi^{d-1}(y)$. By equivariance of the limit maps, this descends to a splitting

$$E_\rho = \Xi \oplus \Theta.$$

Notice that the flow ψ_t preserves the splitting $E_\rho = \Xi \oplus \Theta$ by construction.

Definition 7.3.3 ((close cousin to) Labourie's definition of Anosov)**.** A representation $\rho : \Gamma \to PSL(d,\mathbb{R})$ with compatible transverse limit maps is projective Anosov if the flow is contracting on $\mathrm{Hom}(\Theta,\Xi) = \Xi \otimes \Theta^*$. Equivalently, for some (every) continuous norm $\|\cdot\|$ on E_ρ, there exists $C, a > 0$ such that if $m \in UM, v \in \Xi_m, w \in \Theta_m$ and $t > 0$ then

$$\frac{\|\psi_t(v)\|_{\phi_t(m)}}{\|\psi_t(w)\|_{\phi_t(m)}} \leq Ce^{-at}\frac{\|v\|_m}{\|w\|_m}.$$

The flow is also automatically contracting on Ξ, by a simple tensor analysis.

In this framework, stability of Anosov representations follows immediately from a standard result in dynamics that hyperbolic systems are stable [Lab06].

7.3.2 Examples

Any real hyperbolic convex cocompact representation is projective Anosov, e.g. Fuschian representations and quasifuchsian representations. Also any word hyperbolic group dividing a convex projective domain is projective Anosov by work of Benoist. Perhaps the most famous examples are Hitchen representations, defined as follows. For each d there is an irreducible representation $\tau : SL(2,\mathbb{R}) \to SL(d,\mathbb{R})$, unique up to conjugation. Now given any hyperbolic structure on a closed surface $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$ we can lift to a representation $\hat{\rho} : \pi_1(S) \to SL(2,\mathbb{R})$ and post-compose $\tau \circ \hat{\rho}$ to obtain a representation to $SL(d,\mathbb{R})$. These representations and their deformations are called Hitchen representations. (Hitchen proved these are a component of the character variety, and Labourie proved that all Hitchen representations are Anosov representations and therefore discrete and faithful.)

7.3.3 (Some) properties and characterizations

The following theorem records (some) basic properties of Anosov representations.

Theorem 7.3.4 (Labourie, Guichard-Wienhard, Gueritaud-Guichard-Kassel-Wienhard)**.** Let Γ be a Gromov hyperbolic group and let $\rho : \Gamma \to SL(d,\mathbb{R})$ be a projective Anosov representation. Then

- ρ is discrete with finite kernel; and
- Some (/every) orbit map from Γ to the associated symmetric space is a quasi-isometric embedding; and
- There is a neighborhood U of ρ in $\mathrm{Hom}(\Gamma, SL(d,\mathbb{R}))$ consisting of representations which are projective Anosov; and

- $Out(\Gamma)$ acts properly discontinuously on the space of (conjugacy classes) of projective Anosov representations of Γ into $SL(d,\mathbb{R})$; and
- There exists J and B such that

$$\alpha_1(\rho(\gamma)) = \log \frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))} \geq J\|\gamma\| - B$$

We also have a nice characterization of Anosov representations, in terms of linear algerba (in general, root theory). This result is parceled out in stages of increasingly strong hypotheses and conclusions.

Theorem 7.3.5 ([GGKW17a] Thm 1.1)**.** Let Γ be a word-hyperbolic group and $\rho : \Gamma \to SL(d,\mathbb{R})$ be any representation.

- If there exists a constant $C > 0$ such that

$$\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \geq 2\log|\gamma| - C$$

  then there exist continuous, ρ-equivariant limit maps $\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^{d-1} : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$.

- If moreover for any $\gamma \in \Gamma$

$$\log \frac{\sigma_1(\rho(\gamma^n))}{\sigma_2(\rho(\gamma^n))} - 2\log|n| \to +\infty$$

  as $|n| \to +\infty$ then ξ^1 and ξ^{d-1} are dynamics preserving.

- If moreover for any geodesic ray (γ_n) in the Cayley graph the sequence

$$\log \frac{\sigma_1(\rho(\gamma_n))}{\sigma_2(\rho(\gamma_n))}$$

  is CLI then ξ^1 and ξ^{d-1} are transverse and ρ is projective Anosov.

We say a sequence of real numbers is CLI (has coarsely linear increments) if it is a quasi-isometry $\mathbb{N} \to \mathbb{R}_+$.

The previous results are tied together into a few slightly different equivalance statements. Moreover, it is (surpisingly!) possible to use eigenvalues instead of singular values for forming equivalent characterizations.

Theorem 7.3.6 ([GGKW17a] Thm 1.3 and Thm 1.7)**.** Let Γ be Gromov hyperbolic. For any representation $\rho : \Gamma \to SL(d,\mathbb{R})$, the following conditions are equivalent:

- ρ is projective Anosov
- There exist continuous, ρ-equivariant, dynamics-preserving, and transverse maps $\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^{d-1} : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$, and we have

$$\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \to +\infty$$

  as $\gamma \to \infty$ in Γ.

- There exist continuous, ρ-equivariant, dynamics-preserving, and transverse maps $\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^{d-1} : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$, and there exists constants $c, C > 0$ such that

$$\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \geq c|\gamma| - C$$

  for all $\gamma \in \Gamma$.

- For any geodesic ray (γ_n) in the Cayley graph starting at the identity, the sequence

$$\log \frac{\sigma_1(\rho(\gamma_n))}{\sigma_2(\rho(\gamma_n))}$$

  is lower-CLI. The CLI constants are not assumed to be uniform (they can depend on the ray).

- There exist continuous, ρ-equivariant, dynamics-preserving, and transverse maps $\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^{d-1} : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$, and we have

$$\log \frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))} \to +\infty$$

  as $\|\gamma\| \to \infty$ in Γ.

- There exist continuous, ρ-equivariant, dynamics-preserving, and transverse maps $\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^{d-1} : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$, and there exists a constant $d > 0$ such that

$$\log \frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))} \geq d|\gamma|_\infty$$

  for all $\gamma \in \Gamma$. Here $|\gamma|_\infty := \lim_n |\gamma^n|/n$ is the stable length; it is comparable to the translation length. (In general, on the Lie group side, the relevant quantity comes from the Lyapunov projection.)

All of the above theorems hold for any flavor of Anosov representation in any reductive Lie group, suitably adjusted. For example, one considers the ith singular value gap, or in general any collection of simple restricted roots. Nonetheless, projective Anosov representations are in a sense the most basic and crucial to the theory.

Proposition 7.3.7 ([GW12] Prop 1.6)**.** A representation $\rho : \Gamma \to G$ is Anosov if and only if there is a real vector space V with a non-degenerate indefinite quadratic form F and a homomorphism $\phi : G \to O(V, F)$ such that $\phi \circ \rho$ is "projective Anosov in $O(V, F)$".

The notion "projective Anosov in $O(V, F)$" is equivalent to asking that $\rho(\Gamma)$ lands in $O(V, F)$ and after further postcomposing by the inclusion $O(V, F) \hookrightarrow SL(V)$ the result is projective Anosov. This notion is studied closely in [DGK18a].

7.3.4 Domains of discontinuity

Observe that if M has a (G, X) structure, M and X have the same dimension. So an "Anosov structure" on M, encoded by an Anosov representation $\rho : \pi_1(M) \to G$ is rarely a (G, X) structure on M, if X is the associated symmetric space; it is a looser "dynamical" structure. However, Anosov representations often do yield geometric structures on manifolds associated to M. For example, Hitchen representations of a surface group $\pi_1(S)$ in $PSL(4, \mathbb{R})$ correspond naturally to "properly convex foliated projective structures" on the unit tangent bundle of S [GW08].

A typical goal is to associate geometric structures to Anosov representations. Guichard and Wienhard proved this is possible when Γ is a free or surface group.

Choose a manifold Z that $G = SL(d,\mathbb{R})$ acts naturally on, e.g. $Z = \mathbb{P}(\mathbb{R}^d)$. To an Anosov representation $\rho : \Gamma \to G$, we want to associate a nonempty open domain of discontinuity $\Omega \subset Z$ which $\rho(\Gamma)$ preserves and such that the action is properly discontinuous and cocompact. Such a domain is called a proper cocompact domain of discontinuity.

Theorem 7.3.8 (Guichard-Wienhard [GW12])**.** Let Γ be the fundamental group of a connected surface of negative Euler characteristic, and let $\rho : \Gamma \to SL(d,\mathbb{R})$ be Anosov. Then there exists a nonempty proper cocompact domain of discontinuity $\Omega \subset G/AN \cong F^+$.

Here we have $G = KAN$ the Iwasawa decomposition of G, which for $G = SL(d,\mathbb{R})$ can be taken to be $K = SO(d)$, $A =$ positive diagonal matrices with entries in nondecreasing order, $N =$ upper triangular matrices with 1's on the diagonal. In this case the space G/AN is naturally isomorphic to the space F^+ of oriented complete flags on $\mathbb{R}^d$ (up to a simultaneous sign reversal on all subspaces if d is even).

Generally speaking, we look for proper cocompact domains of discontinuity in (partial) flag manifolds associated to Anosov representations. Kapovich, Leeb and Porti develop a systematic construction of such domains corresponding to "balanced ideals" in the Weyl group. Stecker proved that for fully Anosov representations, such as Hitchen representaions, every cocompact domain of discontinuity comes from a balanced ideal [Ste18]. The number of balanced ideals grows extremely quickly in d and the theory is expected to have rich combinatorics. This perspective is generalized by Stecker and Treib, allowing one to play the same game in oriented flag manifolds [ST18]. However not all existing domains are predicted by this theory; for example the domains we will study due to Danciger, Gueritaud and Kassel are not.

7.3.5 Proper actions on homogeneous spaces

[GGKW17a] also prove results about proper actions on homogenous spaces associated to Anosov representations. These results have a different flavor; rather then find some preferred subset, e.g. by throwing away the limit set, they show the action on the whole homogeneous space is proper. Their result is a corollary of a properness criterion due to Benoist and Kobayashi:

Theorem 7.3.9 ([Ben96], [Kob98])**.** Let G be a reductive Lie group and H, Γ two closed subgroups of G. Then Γ acts properly on G/H if and only if for any compact subset C of $\mathfrak{a}$ the intersection $(\mu(\Gamma) + C) \cap \mu(H) \subset \mathfrak{a}$ is compact.

Based on this properness criterion, a strengthening of the notion of properly discontinuity was introduced in [KK16]: a discrete subgroup $\Gamma \subset G$ is said to act sharply on G/H if the set $\mu(\Gamma)$ drifts away from $\mu(H)$ at infinity "with a nonzero angle," i.e. there are constants $c, C > 0$ such that for all $\gamma \in \Gamma$,

$$d_{\mathfrak{a}}(\mu(\gamma)), \mu(H)) \geq c\|\mu(\gamma)\| - C, .$$

One can roughly think of sharpness as a quantification of proper discontinuity; many but not all properly discontinuous actions are sharp. For an Anosov representaton we always have a set of roots $\alpha \in \theta$ such that $\alpha(\mu(\gamma))$ tend to infinity. Thus if a subgroup H has $\mu(H)$ in the kernel of every root in θ, we can witness the drift of $\mu(\Gamma)$ away from $\mu(H)$.

Corollary 7.3.10 ([GGKW17a] Cor 1.9)**.** Let Γ be a Gromov hyperbolic group, G a real reductive Lie group, and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of G. For any P_θ-Anosov representation $\rho : \Gamma \to G$, the group acts sharply on G/H for any closed subgroup H of G such that $\mu(H) \subset \cup_{\alpha\in\theta} \ker \alpha$.

We pick out just a sample of the applications given in [GGKW17a].

Corollary 7.3.11 ([GGKW17a] Cor 1.10)**.** Let $\rho : \pi_1(S) \to SL(d,\mathbb{R})$ be Hitchen.

- if $k < d-1$ and $H = SL(k,\mathbb{R})$ or
- if $|d-2k| > 1$ and $H = SO(d-k,k)$ or
- if $d = 2k$ and $H = SL(d,\mathbb{C}) \times U(1)$ then

$\rho(\Gamma)$ acts sharply on G/H.

Again, roughly speaking, the point is that for Hitchen representations, we have every singular value gap, so we can see $\rho(\Gamma)$ drift away from, say, the reducible copy of $SL(k,\mathbb{R})$ because it doesn't have enough singular value gaps.

7.4 Convex cocompactness in real projective geometry

Our goal in this section is to prove a theorem by Danciger, Geuritaud and Kassel that relates projective Anosov representations and a notion of convex cocompact actions on projective space [DGK18b]. Part of what makes this theorem special is that we can recover the Anosov property from a geometric structure; there is not a way to do that for Anosov representations in general. We will need some basic notions from projective geometry. We write $V = \mathbb{R}^d$ and we identify $\mathbb{P}(V^*)$ with the space of hyperplanes in $\mathbb{P}(V)$ by the map $[\alpha] \in \mathbb{P}(V^*) \mapsto \ker\alpha$.

We say a subset Ω of $\mathbb{P}(V)$ is properly convex if it its closure is contained in an affine chart and is convex in that chart. (Since affine lines in the chart are projective lines, this notion is independent of the chosen affine chart.) We say Ω is strictly convex if moreover its boundary contains no nontrivial projective line segment and we say Ω is C^1 if each point of its boundary has a unique supporting hyperplane (a hyperplane $H \in \mathbb{P}(V^*)$ is supporting at $z \in \partial\Omega$ if H is disjoint from the interior of Ω).

Any properly convex open subset Ω of $\mathbb{P}(V)$ can be endowed with the Hilbert metric

$$d_\Omega(x,y) := \frac{1}{2}\log[a,x,y,b]$$

where a,b are the intersections of a projective line from x to y with $\partial\Omega$, and $[a,x,y,b]$ is the cross ratio. The metric space (Ω, d_Ω) is proper and complete, and the group

$$\mathrm{Aut}(\Omega) = \{g \in PGL(V) \mid g\cdot\Omega = \Omega\}$$

acts by isometries. In particular, any discrete subgroup Γ preserving Ω acts on Ω properly discontinuously. The orbital limit set $\Lambda_\Omega^{orb}(\Gamma)$ is the set of accumulation points of some Γ-orbit in Ω; we have $\Lambda_\Omega^{orb}(\Gamma) = \overline{\Gamma z} \cap \partial\Omega$ for any $z \in \Omega$. By strict convexity, the orbital limit set doesn't depend on the choice of $z \in \Omega$.

Let Γ be an infinite discrete subgroup of $PGL(V)$, and let Ω be a Γ-invariant properly convex open subset of $\mathbb{P}(V)$ whose boundary is strictly convex and C^1. The action of Γ on Ω is convex cocompact if the convex hull in Ω of the orbital limit set $\Lambda_\Omega^{orb}(\Gamma)$ of Γ in Ω is nonempty and has compact quotient by Γ. The group Γ is strongly convex cocompact in $\mathbb{P}(V)$ if it admits a convex cocompact action on some nonempty properly convex open subset Ω of $\mathbb{P}(V)$ whose boundary is strictly convex and C^1. This notion is due to Crampon-Marquis, who studied more general geometrically finite actions in projective space.

We say a subgroup Γ of $PGL(V)$ is projective Anosov if the inclusion map is a projective Anosov representation.

Theorem 7.4.1 ([DGK18b]). Let Γ be an infinite discrete subgroup of $PGL(V)$. Then Γ is strongly convex cocompact if and only if Γ is projective Anosov and Γ preserves some nonempty properly convex open subset of $\mathbb{P}(V)$.

Examples: If the dimension of V is even then the image of a Hitchen representation in $PGL(V)$ is never convex cocompact, because the limit curve in projective space is not nulhomotopic.

Proof. We will assume Γ is irreducible; this is not essential but makes the argument a bit shorter because we can use the theorem of [GW12] mentioned above.

Let's assume that Γ is strongly convex cocompact and show that it is projective Anosov. By assumption there exists a properly convex open subset Ω of $\mathbb{P}(V)$ which is strictly convex and C^1, and there is a closed convex subset C of Ω such that Γ preserves C and Ω and acts cocompactly on C.

Here's the sketch: First we show that (C, d_Ω) is a Gromov hyperbolic metric space. Then Γ is hyperbolic by Milnor-Švarc. Also, we get a boundary map by identifying $\partial\Gamma$ with $\partial_i C$, and we get the dual boundary map by using the unique supporting hyperplane. Thus compatibility is immediate.

For transversality, suppose that $\xi(\eta) \in \xi^*(\eta')$. Our goal is to show that $\eta = \eta'$. The projective line segment $[\xi(\eta), \xi(\eta')]$ is contained in $\xi^*(\eta')$, hence in the ideal boundary $\partial_i C$. Again by assumption there are no nontrivial line segments in the boundary so we deduce that $\eta = \eta'$.

We assumed Γ is irreducible and we have compatible transverse limit maps, so by [GW12] we conclude Γ is projective Anosov. (Note that irreducibility is really not essential here. It is only slightly more work to demonstrate the singular value gap criterion in [GGKW17a] or [KLP14].)

For the converse, we now assume Γ is projective Anosov and preserves some nonempty properly convex open subset of $\mathbb{P}(V)$ and we will show that Γ is strongly convex cocompact.

We must have $\xi(\partial\Gamma) \subset \partial\Omega$ and we let C be the convex hull of $\xi(\partial\Gamma)$ in Ω.

To show that the action of Γ on C is cocompact, we use the fact (from [KLP18] or [GGKW17a]) that Anosov implies expanding, related to Sullivan's characterization of convex cocompactness in real hyperbolic space. That is, for any point $z \in \Lambda_\Gamma$, there exists an element $\gamma \in \Gamma$, a neighborhood U of z in $\mathbb{P}(V)$, and a constant $c > 1$ such that for any $x, y \in U$,

$$d(\gamma x, \gamma y) \geq c\, d(x, y).$$

Here, for a metric on projective space we can choose any inner product on V and use

$$d([v], [w]) := |\sin \angle(v, w)|.$$

A little exercise shows that if we change to any bilipschitz equivalent metric, the action is still expanding.

Suppose for the sake of contradiction that the action of Γ on C is not cocompact and let let (ϵ_n) be a sequence of real numbers converging to 0. For any m, the set $K_m := \{z \in C \mid d(z, \Lambda_\Gamma) \geq \epsilon_m\}$ is compact, hence there exists a Γ-orbit contained in $C \setminus K_m$. By proper discontinuity of the action on C, the supremum of $d(\cdot, \Lambda_\Gamma)$ on this orbit is achieved at some point $z_m \in C$ and by construction $0 < d(z_m, \Lambda_\Gamma) \leq \epsilon_m$. Then for all $\gamma \in \Gamma$,

$$d(\gamma z_m, \Lambda_\Gamma) \leq d(z_m, \Lambda_\Gamma).$$

Up to subsequence, we may assume that (z_m) converges to some $z \in \Lambda_\Gamma$. Since the action is expanding, we can choose γ, U, c as above. For any m we have $z'_m \in \Lambda_\Gamma$ such that $d(\gamma z_m, \Lambda_\Gamma) = d(\gamma z_m, \gamma z'_m)$. For large enough m we have $z_m, z'_m \in U$, so

$$d(\gamma z_m, \Lambda_\Gamma) = d(\gamma z_m, \gamma z'_m) \geq c d(z_m, z'_m) \geq c d(z_m, \Lambda_\Gamma) > 0$$

which contradicts $c > 1$.

Now in fact the domain Ω may not be strictly convex and C^1; however it is possible to build a "smoothification" since the action of Γ is properly discontinuous. This process is a bit technical and involves many choices and the smooth substitute Ω_{smooth} constructed for Ω isn't canonical in any way. A precise statement is

Lemma 7.4.2 ([DGK18b] Lemma 9.2)**.** Let Γ be an infinite discrete subgroup of $PGL(V)$ and Ω a nonempty Γ-invariant properly convex open subset of $\mathbb{P}(V)$. Suppose Γ acts convex cocompactly on Ω. Fix a uniform neighborhood C_u of C^{cor}, the convex hull of the (full) orbital limit set. Then the convex core C^{cor} admits a Γ-invariant, properly convex, closed neighborhood $C_{sm} \subset C_u$ which has C^1, strictly convex nonideal boundary.

Of course the interior of C_{sm} is nonempty, so we can use that as a suitable replacement for Ω. Note that C^{cor} must be contained in the interior of C_{sm}. □

[DGK18b] also deals with the case of convex cocompact actions for groups which are not hyperbolic, e.g. $\mathbb{Z}^2$ (see Theorem 1.17). Of course these groups cannot be Anosov, but this notion is stable and an interesting open question is to ask if there is a generalization of Anosov which accommodates these groups. DGK prove that such groups never have unipotent elements, so this notion is expected to be quite distant from the notion of "relatively Anosov" actions defined by Kapovich and Leeb [KL18].

7.5 Convex cocompactness in pseudo-Riemannian hyperbolic geometry

By a theorem of Guichard and Wienhard mentioned above, a representation is Anosov (any flavor) if and only if we can post-compose it to be projective Anosov in $O(p, q)$. Thus the case of projective Anosov representations in $PO(p, q)$ is especially interesting.

For any positive integers (p, q), let $\mathbb{R}^{p,q}$ be the vector space $\mathbb{R}^{p+q}$ endowed with a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ of signature (p, q). Let $PO(p, q)$ be the image of $O(p, q)$ in $PGL(p+q)$. As above, we say an infinite subgroup Γ of $PO(p, q)$ is projective Anosov if the inclusion $\Gamma \hookrightarrow PO(p, q) \hookrightarrow PGL(p + q)$ is projective Anosov. In particular Γ must be discrete.

Define
$$\mathbb{H}^{p,q-1} := \{[x] \in| \langle x, x \rangle < 0\}$$
and
$$\partial\mathbb{H}^{p,q} := \{[x] \in| \langle x, x \rangle = 0\}.$$

We say a subset Λ of the boundary is <u>negative</u> if for every $[x] \neq [y] \in \Lambda$, we have $\langle x, y \rangle < 0$. The following characterization of negative subsets shows its usefulness.

Lemma 7.5.1 ([DGK18a] Lemma 3.2)**.** Let Λ be a subset of $\partial\mathbb{H}^{p,q}$ with at least 3 points. Then the following are equivalent:

- Λ is negative;
- every triple of distinct points of Λ is negative;
- every triple of distinct points of Λ spans a triangle fully contained in $\mathbb{H}^{p,q}$ outside the vertices.

This lets us build a convex hull out of negative limit sets, even though $\mathbb{H}^{p,q-1}$ is not properly convex in general.

Theorem 7.5.2 ([DGK18a] Thm 1.11)**.** For any positive integers p, q and any irreducible discrete subgroup Γ of $PO(p, q)$, Γ is $\mathbb{H}^{p,q-1}$-convex cocompact if and only if Γ projective Anosov, and the proximal limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q-1}$ is negative.

It is worth observing that

Proposition 7.5.3. If a closed subset Λ of $\partial\mathbb{H}^{p,q-1}$ is connected and transverse then it is negative or positive.

We should mention here that the notion of transversality in pseudo-Riemannian hyperbolic geometry is slightly different from the notion in projective geometry. We say a subset $\Lambda \subset \partial\mathbb{H}^{p,q-1}$ is transverse if for every $x \neq y \in \Lambda$, we have $x \notin y^{\perp}$, and we say a limit map ξ^1 is transverse if it is transverse to $\xi^{d-1}(x) := \xi^1(x)^{\perp}$ in the usual sense. Roughly speaking, the proof works by defining a "sign" function on the space $\Lambda^{(3)}$ of unordered distinct triples in Λ. The function is continuous on a connected space (this is where the work is hidden), and by transversality never takes the value 0. Hence, the sign is either negative or positive.

Note that being positive in the boundary of $\mathbb{H}^{p,q-1}$ is equivalent to being negative in the boundary of $\mathbb{H}^{q,p-1}$. So if we have connected boundary but drop the negativity assumption, we still get the partial converse:

Theorem 7.5.4 ([DGK18a])**.** If Γ is a projective Anosov subgroup of $PO(p,q)$ and $\partial\Gamma$ is connected, then Γ is $\mathbb{H}^{p,q-1}$-convex cocompact or $\mathbb{H}^{q,p-1}$-convex cocompact.

They also construct lots of new examples of Anosov representations from right-angled Coxeter groups.

Chapter 8

An introduction to Θ-positivity

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This note was prepared to complement my May 2019 talk at the Workshop "Geometric aspects of Higgs bundles" organized by Xian Dai, Charles Ouyang, and Andrea Tamburelli. It is with great pleasure that I thank the organizers for inviting me and giving me the opportunity to speak. I would also like to thank all the participants of the workshop for contributing to such a great learning experience.

This note is primarily based on the paper "Positivity and higher Teichmüller theory" by Guichard and Wienhard [GW18].

8.1 Introduction

Let S be a connected, orientable, closed surface of genus $g > 1$ and let G be one of the Lie groups

$$\mathrm{PSL}(d,\mathbb{R}), \mathrm{PSp}(2n,\mathbb{R}), \mathrm{SO}(p,q).^1 \tag{8.1.1}$$

We are primarily interested in the space of G-conjugacy classes of group homomorphisms from the fundamental group of S into G. A well-known issue is that this space is not necessarily Hausdorff. One defines the representation variety $\mathcal{R}(S,G)$ as the Hausdorff space of G-conjugacy classes of reductive group homomorphisms from $\pi_1(S)$ to G.

Anosov representations [Lab06, GW] are preferred elements of $\mathcal{R}(S,G)$ sharing many algebraic, dynamical and geometric properties. A representation $\rho \in \mathcal{R}(S,G)$ is Anosov with respect to a set Θ of simple roots for G (see §8.2). In §§8.3-8.5 we discuss three of the prominent/motivating examples of Anosov representations: Fuchsian, Hitchin and maximal representations.

Notably, Anosov representations are "stable" in the sense that the space of Anosov representation is open in $\mathcal{R}(S,G)$. The spaces of Hitchin and maximal representations are also closed in $\mathcal{R}(S,G)$. However, for more general Lie groups, Anosov representations can fail to fill connected components of the representation variety. For example, Fuchsian representations can be deformed in the representation variety $\mathcal{R}(S,\mathrm{PSL}(2,\mathbb{C}))$ to representations that are not Anosov.

Guichard and Wienhard [GW18] introduced a unified framework to study Hitchin and maximal representations via the definition of Θ-positive structures for Lie groups G. Here, Θ is, again, a set of simple roots. A Θ-positive representation is then a representation "well-behaved" with respect to a Θ-positive structure. See §8.6.

The pairs (G,Θ) such that G admits a Θ-positive structure are classified by Theorem 8.6.2 [GW18, Theorem 4.3]. In particular, Hitchin and maximal representations are Θ-Anosov, they are

[1] We narrow our focus to these groups for the sake of brevity, referring to [GW18] for the general theory.

Θ-positive with respect to the same Θ, and they fill connected components of the corresponding representation varieties.

Conjecture 8.1.1. [GLW] The space of Θ-positive representations is closed in $\mathcal{R}(S, G)$.

Furthermore, the connected components of Hitchin and maximal representations are exotic in the respective representation varieties, in the sense that they are not distinguished by classical topological invariants [Hit92, GW].

Conjecture 8.1.2. [GW18] If G has a Θ-positive structure, the representation variety $\mathcal{R}(S, G)$ has exotic connected components.

In §8.8, we briefly discuss evidence in support of Conjecture 8.1.2 provided by [AABC+19].

8.2 Generalities on Lie theory

In this section, we collect the Lie theoretic tools we will need in the following sections. Standard references are [Ebe96, Hel78].[2]

8.2.1 Generalities on $\mathrm{PSL}(d, \mathbb{R})$

Let $G = \mathrm{PSL}(d, \mathbb{R})$, then $\mathfrak{g} = \mathrm{Lie}(G)$ is the space of traceless $d \times d$ matrices. We choose the Cartan subspace $\mathfrak{a}$ of traceless diagonal matrices. Let us identify $\mathfrak{a}$ with the set $\{(x_1, \dots, x_d) \colon x_1 + \dots + x_d = 0\}$ by recording the diagonal entries in decreasing order. We choose $\mathfrak{a}^+ = \{(x_1, \dots, x_d) \in \mathfrak{a} \colon x_1 \geq \dots \geq x_d\}$ to be fundamental Weyl chamber.

A positive (restricted) root $\alpha \in \Sigma^+ \subset \mathfrak{a}^*$ is a functional which is positive on $\mathfrak{a}^+$. Recall that Σ^+ is positively generated by elements in $\Delta = \{\alpha_1, \dots, \alpha_{d-1}\}$ with $\alpha_j(x_1, \dots, x_d) = x_j - x_{j+1}$. Given $\Theta \subset \Delta$, the standard Θ-parabolic subgroup of G is the Lie subgroup P_Θ whose Lie algebra is

$$\mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \mathrm{Span}(\Delta - \Theta) \cap \Sigma^+} \mathfrak{g}_{-\alpha}$$

where $\mathfrak{g}_\beta = \{Y \in \mathfrak{g} \colon [X, Y] = \beta(X)Y, \text{ for all } X \in \mathfrak{a}\}$ is the restricted root space of β. Note that for $G = \mathrm{PSL}(d, \mathbb{R})$, the standard Δ-parabolic subgroup is the subgroup of upper triangular matrices. The quotient G/P_Δ is identified with the space of (complete) flags in $\mathbb{R}^d$.

Finally, note that G is a real form of $\mathrm{PSL}(d, \mathbb{C})$ as $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $\mathrm{PSL}(d, \mathbb{C})$. Furthermore, G is a split real form because $\dim_{\mathbb{R}} \mathfrak{a}$ (the real rank of G) is equal to the dimension over $\mathbb{C}$ of a(ny) maximal abelian subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

8.2.2 Generalities on $\mathrm{PSp}(2n, \mathbb{R})$

Let $G = \mathrm{PSp}(2n, \mathbb{R})$. The symmetric space $\mathcal{X}_G$ of G admits a G-invariant complex structure, and it is biholomorphic to the Siegel upper half space

$$\mathcal{X}_G = \mathrm{Sym}(n, \mathbb{R}) + i\mathrm{Pos}(n, \mathbb{R}),$$

where $\mathrm{Sym}(n, \mathbb{R})$ is the vector space of symmetric $n \times n$ matrices and $\mathrm{Pos}(n, \mathbb{R})$ is the cone in $\mathrm{Sym}(n, \mathbb{R})$ of positive definite matrices. By definition, this means that $\mathrm{PSp}(2n, \mathbb{R})$ is a Hermitian Lie group of tube-type. The reader might be familiar with the case $n = 1$, when the Siegel upper

[2] Thanks to past and present editions of the Log Cabin workshops, many "specialized" surveys on the Lie theory background needed for our purposes are available at http://gear.math.illinois.edu/programs/workshops/logcabin-workshops.html.

half space reduces to the upper half space model of $\mathbb{H}^2$ of the hyperbolic plane. We choose a Cartan subspace and a positive Weyl chamber to be

$$\mathfrak{a} = \{(x_1, \dots, x_n, -x_n, \dots, -x_1) \in \mathbb{R}^{2n}\} \cong \mathbb{R}^n, \text{and}$$
$$\mathfrak{a}^+ = \{(x_1, \dots, x_n, -x_n, \dots, -x_1) \colon x_1 \geq \cdots \geq x_r \geq 0\}$$

We can define simple roots $\alpha_j(x_1, \dots, x_n, -x_n, \dots, -x_1) = x_j - x_{j+1}$, for $j = 1, \dots, n$. In the context of maximal representations, we will focus on the standard $\{\alpha_n\}$-parabolic subgroup P_{α_n}. The quotient G/P_{α_n} is the space of Lagrangians, i.e. maximal (with respect to inclusion) isotropic subspace in the given symplectic vector space $\mathbb{R}^{2n}$. Note that Lagrangians have dimension n.

Remark 8.2.1. The Lie group $\mathrm{PSp}(2n, \mathbb{R})$ is a split real form of $\mathrm{PSp}(2n, \mathbb{C})$. In particular, Theorem 8.6.2 states that it has two Θ-positive structures. For this note, we will narrow our focus to the $\{\alpha_n\}$-positive structure.

8.2.3 Generalities on $\mathrm{SO}(3,4)$

In this section, we follow the exposition in [GW18, §4.5]. We identify the group $SO(3,4)$ with the special orthogonal group associated to the quadratic form Q of signature $(3,4)$ given by

$$Q(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = 2(x_1x_7 - x_2x_6 + x_3x_5 - x_4^2).$$

With our usual identification between traceless diagonal matrices and vectors in $\mathbb{R}^7$, choices for a Cartan subpace and a positive Weyl chamber are

$$\mathfrak{a} = \{(x_1, x_2, x_3, 0, -x_3, -x_2, -x_1) \in \mathbb{R}^7\} \cong \mathbb{R}^3, \text{and}$$
$$\mathfrak{a}^+ = \{(x_1, x_2, x_3, 0, -x_3, -x_2, -x_1) \colon x_1 \geq x_2 \geq x_3 \geq 0\}.$$

The simple roots are $\alpha_j(x) = x_j - x_{j+1}$, $j = 1, 2, 3$.

8.3 Fuchsian representations

For this section let $G = \mathrm{PSL}(2, \mathbb{R}) = \mathrm{PSp}(2, \mathbb{R})$.

8.3.1 Anosov property

A hyperbolic metric on S with holonomy ρ induces an isometry $\widetilde{S} \to \mathbb{H}^2$ which extends to a continuous $\pi_1(S)$-equivariant map between visual boundaries

$$\xi_\rho \colon \partial_\infty \widetilde{S} \to \partial_\infty \mathbb{H}^2 = \mathbb{RP}^1 = G/P_\Delta = \mathrm{Flag}(\mathbb{R}^2).$$

The limit map ξ_ρ is injective, continuous, and for every $\gamma \in \pi_1(S) - \mathrm{id}$, the image of the attracting fixed point $\xi_\rho(\gamma^+)$ is the attracting eigenline of $\rho(\gamma)$. By [GW12, Theorem 1.8], as G is a rank 1 Lie group, this suffices to show that Fuchsian representations are Anosov with respect to the parabolic subgroup $P_\Delta = P_{\alpha_1}$.

8.3.2 Maximality

Let us define the Euler number $e(\rho) \in \mathbb{Z}$ of a representation $\rho \in \mathcal{R}(S, G)$.

The representation ρ gives a $\pi_1(S)$-action on $\widetilde{S} \times \mathbb{H}^2$. Denote by E_ρ the quotient under this action, and note that E_ρ is a fiber bundle over S with fiber $\mathbb{H}^2$. Thus, there exists a section which we can write as $\widetilde{\sigma} \colon \widetilde{S} \to \mathbb{H}^2$. We use $\widetilde{\sigma}$ to define a $\pi_1(S)$-invariant closed two-form on $\widetilde{S}$ as follows.

Let g be the hyperbolic metric on $\mathbb{H}^2$ and let J be the natural complex structure. For $X, Y \in T_x\mathbb{H}^2$, set $\omega_1(X,Y) = g(JX,Y)$. Note that ω_1 is a two-form (in particular, it is closed). The pull-back $\widetilde{\sigma}^*\omega_1$ is a $\pi_1(S)$-invariant closed two-form on $\widetilde{S}$. The Euler number is $e(\rho) = \frac{1}{2\pi}\int_S \widetilde{\sigma}^*\omega_1$.

Theorem 8.3.1. The Euler number is independent on the choice of the section σ of E_ρ. Moreover, $e(\rho) \in \mathbb{Z}$, and

1. Milnor-Wood inequality [Mil]: for every $\rho \in \mathcal{R}(S,G)$, we have $|e(\rho)| \leq 2g-2$;
2. [Gol88] The representation ρ is Fuchsian if and only if $|e(\rho)| = 2g-2$.

In other words, a representation $\rho \in \mathcal{R}(S,G)$ is Fuchsian if and only if it has maximal Euler number.

8.3.3 Positivity on a projective line

The orientation on S induces a cyclic order on $\partial_\infty\widetilde{S}$. Let (x,y,z) be a triple of cyclically oriented points in $\partial_\infty\widetilde{S}$. We can choose a representative in the conjugacy class of the Fuchsian representation ρ with corresponding limit map ξ_ρ such that $\xi_\rho(x) = \begin{bmatrix}1\\0\end{bmatrix}$, and $\xi_\rho(z) = \begin{bmatrix}0\\1\end{bmatrix}$. Then, $\xi_\rho(y) = \begin{bmatrix}t\\1\end{bmatrix}$ with $t>0$. This fact can be rephrased as to say that $\xi_\rho(y) = u\cdot\xi_\rho(z)$ for u a matrix in the subsemigroup of $\mathrm{SL}(2,\mathbb{R})$

$$U^{>0} = \left\{\begin{bmatrix}1 & t\\0 & 1\end{bmatrix} : t>0\right\}$$

With some work, this observation leads to:

- a parametrization of the space of Fuchsian representations via positive crossratio coordinates, depending on a choice of a maximal geodesic lamination [Thu81, Bon96];
- a combinatorial proof, using the Perron-Frobenius theorem, of well-known properties of Fuchsian representations: e.g. for every non-trivial $\gamma \in \pi_1(S)$, $\rho(\gamma)$ has a simple positive real eigenvalue, and, therefore, it is diagonalizable.

8.4 Hitchin positivity

8.4.1 The Hitchin component

Let $G = \mathrm{PSL}(d,\mathbb{R})$. When $d \geq 3$, Hitchin [Hit92] proved that $\mathcal{R}(S,G)$ has either 3 or 6 connected components, depending on the parity of d. One of these components contains the image of the Fuchsian space of S under the irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL}(d,\mathbb{R})$. We denote this Hitchin component by $\mathcal{H}(d,S)$.

Theorem 8.4.1.

a) [Hit92] The Hitchin component is topologically trivial: $\mathcal{H}(d,S) \cong \mathbb{R}^{(d^2-1)(2g-2)}$.

b) [Lab06] Hitchin representations are Anosov with respect to the parabolic subgroup P_Δ.

c) [FG06] The limit map $\xi_\rho\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^d) = G/P_\Delta$ of a Hitchin representation is positive.

In this section we explain the statement of Theorem 8.4.1.c by defining positivity for triples of flags in G/P_Δ.

8.4.2 Flag positivity

We describe the positivity properties of the limit map ξ_ρ by studying the moduli space of generic triples of flags (F_1, F_2, F_3), i.e. flags such that

$$\dim(F_1^{(a)} + F_2^{(b)} + F_3^{(c)}) = \min\{d, a+b+c\}. \tag{8.4.1}$$

Condition 8.4.1 is our running assumption. We wish to define the subset $\mathrm{Flag}(\mathbb{R}^d)^3$ of positive triples of flags.

As a first step, set $d = 3$ and let F_1, F_2, F_3 be flags. Note that the flag F_i is a pair (p_i, ℓ_i) where $p_i \in \mathbb{RP}^2$ and ℓ_i is a projective line passing through p_i. Our genericity assumption corresponds to assuming $p_i \notin \ell_j \cup \ell_k$ for $\{i, j, k\} = \{1, 2, 3\}$.

Note that $p_1, p_2, p_3, \ell_1 \cap \ell_3$ form a projective frame in $\mathbb{RP}^2$. Moreover, the line ℓ_2, passing through p_2, cannot pass through the points p_1, p_3 because of condition 8.4.1. By keeping track of the intersection between the lines ℓ_1 and ℓ_3, one sees that the $\mathrm{PGL}(3, \mathbb{R})$-moduli space of triples of flags consists of two 1-dimensional connected components. The triple ratio parametrizes this space.

Choose representatives v_1, v_2, v_3 (resp. $\varphi_1, \varphi_2, \varphi_3$) for the projective classes p_1, p_2, p_3 (resp. ℓ_1, ℓ_2, ℓ_3). The triple ratio is the projective invariant

$$T(F_1, F_2, F_3) = \frac{\varphi_1(v_2)\varphi_2(v_3)\varphi_3(v_1)}{\varphi_1(v_3)\varphi_2(v_1)\varphi_3(v_2)} \in \mathbb{R} - \{0\}.$$

A triple of flags (F_1, F_2, F_3) is positive if $T(F_1, F_2, F_3) > 0$.

For $d > 3$, and $a, b, c \in \mathbb{Z}_{>0}$, $a+b+c = d$, one defines triple ratios $T_{abc}(F_1, F_2, F_3)$ by considering triple ratios of the quotients of F_1, F_2, F_3 in the three-dimensional spaces

$$\mathbb{R}^d / (F_1^{(a-1)} + F_2^{(b-1)} + F_3^{(c-1)}).$$

A triple of flags (F_1, F_2, F_3) in $\mathbb{R}^d$ is positive if all of its triple ratios are positive.

Theorem 8.4.1.c states that the limit map ξ_ρ of a Hitchin representation ρ sends cyclically oriented triples of points in $\partial_\infty \widetilde{S}$ to positive triples of flags. Fock and Goncharov [FG06] used this positivity property to bridge between Hitchin representations and Lusztig's theory of total positivity.

Finally, consider the subsemigroup $U^{>0}$ of $\mathrm{SL}(d, \mathbb{R})$ defined by

$$U^{>0} = \left\{ \begin{bmatrix} 1 & & \star \\ & \ddots & \\ 0 & & 1 \end{bmatrix} : \text{non-vanishing minors are positive} \right\}$$

where we think of a minor m as a function $\mathrm{SL}(d, \mathbb{R}) \to \mathbb{R}$ and a minor m is non-vanishing if it is not identically zero on the space of upper triangular matrices.

Given (F_1, F_2, F_3) a triple of flags, up to the action of $\mathrm{PGL}(d, \mathbb{R})$, we can assume

$$F_1^{(a)} = \mathrm{Span}(\vec{e}_1, \dots, \vec{e}_a), \text{and } F_2^{(b)} = \mathrm{Span}(\vec{e}_d, \dots, \vec{e}_{d-a+1})$$

The triple of flags (F_1, F_2, F_3) is positive if there exists a unipotent matrix $u \in U^{>0}$ such that $F_3 = u \cdot F_2$.

8.5 Maximal positivity

For this section we let $G = \mathrm{PSp}(2n, \mathbb{R})$.

8.5.1 Maximality of the Toledo invariant

We define the Toledo invariant by mimicking the discussion in §8.3.2. Consider $\rho \in \mathcal{R}(S, G)$ and define the bundle E_ρ given by the $\pi_1(S)$-action on the trivial bundle $\widetilde{S} \times \mathcal{X}_G$. Note that E_ρ is a flat bundle with contractible fiber $\mathcal{X}_G$. A smooth section of E_ρ corresponds to a $\pi_1(S)$-equivariant map $\widetilde{\sigma} \colon \widetilde{S} \to \mathcal{X}_G$.

As $\mathcal{X}_G$ is a Hermitian symmetric space, it has a G-invariant complex structure J. For X, Y in the tangent bundle $T\mathcal{X}_G$ and g the Riemannian metric on $\mathcal{X}_G$, define the closed (see [BILW05, Lemma 2.1]) 2-form $\omega_G(X, Y) = g(JX, Y)$.

Since $\widetilde{\sigma}$ is $\pi_1(S)$-equivariant, we can integrate $\widetilde{\sigma} * \omega_g$ on S. The resulting number $T(\rho)$ is the Toledo number of ρ.

Theorem 8.5.1.

1. The Toledo number $T(\rho)$ of $\rho \in \mathcal{R}(S, G)$ does not depend on the choice of σ. Moreover, $T(\rho)$ is an integer and it satisfies $|T(\rho)| \leq r(2g-2)$.

2. [BIW, BILW05] A representation ρ in $\mathcal{R}(S, G)$ is maximal if $|T(\rho)| = r(2g-2)$. Maximal representations are P_{α_n}-Anosov.

Recall from §8.2 that G/P_{α_n} is the space of maximal isotropic subspaces, called Lagrangians in the symplectic vector space $\big(\mathbb{R}^{2n}, \langle\ ,\ \rangle\big)$. Two Lagrangians L_1, L_2 are transverse if $L_1 \cap L_2 = \{0\}$. We now define a notion of positivity for a triple (L_1, L_2, L_3) of pairwise transverse Lagrangians. Consider the quadratic form Q_{L_1,L_2,L_3} on $L_1 \oplus L_2 \oplus L_3$ defined by

$$(x_1, x_2, x_3) \mapsto \sum_{i=1}^{3} \langle x_i, x_{i+1} \rangle$$

where $x_4 = x_1$, by convention. As we assume that the Lagrangians are pairwise transverse, the Maslov index of (L_1, L_2, L_3), denoted by $\tau(L_1, L_2, L_3)$ is the signature of Q_{L_1,L_2,L_3} and it is such that

$$\tau(L_1, L_2, L_3) \in \{-n, -n+2, \dots, n-2, n\}.$$

Definition 8.5.2. A triple of Lagrangian subspaces (L_1, L_2, L_3) is positive if $\tau(L_1, L_2, L_3) = n$.

Theorem 8.5.3 ([BIW])**.** A representation ρ is maximal if and only if the limit map ξ_ρ sends positive triples of points in $\partial_\infty \widetilde{S}$ to positive triples of Lagrangians.

Once again, one can describe positivity of a triple of Lagrangians in terms of a subgroup of $G = \mathrm{Sp}(2n, \mathbb{R})$.

Fix a symplectic basis $(e_1, \dots, e_r, f_1, \dots f_r)$ of $\mathbb{R}^{2n}$ and consider a triple of transverse Lagrangians (L_1, L_2, L_3). We can identify a Lagrangian L with a $2n \times n$ matrix with columns a basis of L. Up to the G-action, we can then assume that $L_1 = \begin{bmatrix} \mathrm{Id}_n \\ 0 \end{bmatrix}$ and $L_2 = \begin{bmatrix} 0 \\ \mathrm{Id}_n \end{bmatrix}$. Then, the triple (L_1, L_2, L_3) is positive if

$$L_3 = \begin{bmatrix} \mathrm{Id}_n & M \\ 0 & \mathrm{Id}_n \end{bmatrix} L_2$$

where M is a positive definite matrix in $\mathrm{Pos}(n, \mathbb{R})$. Finally, note that the matrices of the form $\begin{bmatrix} \mathrm{Id}_n & M \\ 0 & \mathrm{Id}_n \end{bmatrix}$ as above defines a subsemigroup $U^{\succ 0}$ of $\mathrm{Sp}(2n, \mathbb{R})$.

8.6 Θ-positivity

In sections §8.4,8.5, we described seemingly unrelated notions of positivity for generic triples in the flag spaces G/P_Θ for opportune choices of a subset Θ of simple roots. Guichard and Wienhard [GW18] provided a general framework to understand these notions via the introduction of Θ-positivity.

The Levi subgroup L_Θ of the parabolic subgroup P_Θ is the intersection between P_Θ and its opposite. Denote by L_Θ° the connected component of the identity of the Levi subgroup.

Set $\mathfrak{p}_\Theta = \mathrm{Lie}(P_\Theta)$ and $\mathfrak{l}_\Theta = \mathrm{Lie}(L_\Theta)$. Finally denote by $\mathfrak{u}_\Theta$ the direct sum of restricted root spaces such that $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{u}_\Theta$.

The adjoint action of L_Θ on $\mathfrak{u}_\Theta$ gives a weight space decomposition of $\mathfrak{u}_\Theta$ into vector subspaces $\mathfrak{u}_\beta$ for β in the dual of the center of $\mathfrak{l}_\Theta$. We say that β is indecomposable if $\beta \in \Theta$.

Definition 8.6.1. The Lie group G admits a Θ-positive structure if for all $\beta \in \Theta$ there exists an L_Θ°-invariant sharp convex cone in $\mathfrak{u}_\beta$.

Let us give more details in the concrete case of $\mathrm{PSL}(3,\mathbb{R})$. Observe that L_Δ is the set of diagonal matrices in $\mathrm{PSL}(3,\mathbb{R})$ and L_Δ° is isomorphic to $\mathbb{R}^2_{>0}$. Moreover,

$$\mathfrak{l}_\Delta = \mathfrak{a}, \text{and } \mathfrak{u}_\Delta = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}, \tag{8.6.1}$$

where

$$\mathfrak{g}_{\alpha_1} = \left\{ \lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},\ \mathfrak{g}_{\alpha_2} = \left\{ \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\},\ \mathfrak{g}_{\alpha_1+\alpha_2} = \left\{ \lambda \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

Note that as $\mathfrak{l}_\Delta = \mathfrak{a}$ is an abelian Lie algebra, the decomposition of $\mathfrak{u}_\Delta$ into weight spaces for the L_Δ adjoint action coincides with the decomposition in Equation 8.6.1. In this case, $\mathfrak{u}_\beta = \mathfrak{g}_\beta$ and β is indecomposable if $\beta \in \Delta = \{\alpha_1, \alpha_2\}$.

Theorem 8.6.2 (Theorem 4.3 [GW18])**.** A semisimple Lie group G admits a Θ-positive structure if and only if

1. G is a split real form (e.g. $\mathrm{PSL}(d,\mathbb{R})$) and $\Theta = \Delta$;
2. G is of Hermitian type of tube type (e.g. $\mathrm{PSp}(2n,\mathbb{R})$) and $\Theta = \{\alpha_r\}$;
3. G is locally isomorphic to $\mathrm{SO}(p,q)$, $p \neq q$ and $\Theta = \{\alpha_1, \dots, \alpha_{p-1}\}$.
4. Exceptional Lie group cases.

In [GW18, §4.3], Guichard and Wienhard use the Θ-positive structure of a group G to define a subsemigroup $U_\Theta^{>0}$ extending the subsemigroups defined in §§8.4-8.5.

Definition 8.6.3. Assume F_1, F_2, F_3 are pairwise transverse elements in G/P_Θ. The triple (F_1, F_2, F_3) is Θ-positive if there exists $u \in U_\Theta^{>0}$ such that $F_2 = u \cdot F_3$.

Definition 8.6.4. Let G be a semisimple Lie group with a Θ-positive structure. A representation $\rho\colon \pi_1(\Sigma_g) \to G$ is Θ-positive if there exists a ρ-equivariant positive map $\xi_\rho\colon \mathbb{RP}^1 \to G/P_\Theta$ sending positive triples in $\mathbb{RP}^1$ to Θ-positive triples in G/P_Θ.

Conjecture 8.6.5 ([GLW])**.** A Θ-positive representation is Θ-Anosov. The set of Θ-positive representations is open and closed in the representation variety.

Conjecture 8.6.6 ([GW18])**.** If G carries a Θ-positive structure, there are exotic components in the representation variety of G.

8.7 Positivity in $\mathrm{SO}(3,4)$

Let us briefly focus on the case $G = \mathrm{SO}(3,4) = \mathrm{SO}(Q)$. The case $G = \mathrm{SO}(3,q)$ is discussed in details in [GW18, §4.5], and we follow their exposition.

Theorem 8.6.2 leads us to consider $\Theta = \{\alpha_1, \alpha_2\}$. In this case, the quotient G/P_Θ is the space of isotropic flags $V_1 \subset V_2$ with $\dim V_i = i$. The Levi subgroup L_Θ is the subgroup of $\mathrm{SO}(Q)$ given by block-diagonal matrices of the form

$$\alpha_{\lambda,A} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & A & & \\ & & & 1/\lambda_2 & \\ & & & & 1/\lambda_1 \end{bmatrix}$$

where A is in $SO(q)$, for q the quadratic defined by $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

The weight space decomposition of $\mathfrak{u}_\Theta$ via the adjoint action of L_Θ gives indecomposable subspaces

$$\mathfrak{u}_{\alpha_1} = \{xE_{12} + xE_{67} \colon x \in \mathbb{R}\} \cong \mathbb{R}$$

$$\mathfrak{u}_{\alpha_2} = \left\{ X_v = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & v^\top & 0 & 0 \\ 0 & 0 & Jv & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right\} \cong \mathbb{R}^3$$

where we denote by E_{ij} the matrix with 1 in the ijth position and zeros elsewhere. One can check that the cone in $\mathfrak{u}_{\alpha_2}$ given by $\{(v_1, v_2, v_3) \colon v_1 > 0,\ v_2^2 < v_1 v_3\}$ verifies Definition 8.6.1.

8.8 Positivity and exotic components

The Non-Abelian Hodge correspondence between $\mathcal{R}(S, G)$ and (a subspace of) the space of Higgs bundles provides analytic tools for the study of the connected components of the representation variety. In the cases of G a split real form or a Hermitian Lie group of tube-type, the representation variety $\mathcal{R}(S, G)$ has "exotic" connected components that are not detected by classical bundle invariants. In the case of $G = \mathrm{PSL}(d, \mathbb{R}),\ \mathrm{PSp}(2n, \mathbb{R})$, the exotic components contain positive representations. This fact was extended in [AABC$^+$19] to the case $G = \mathrm{SO}(p, q)$. See also [Col17] for earlier results.

Theorem 8.8.1. Let $p < q$. Let $r \colon \pi_1 S \to \mathrm{SO}_0(p, p-1)$ be a Hitchin representation and let $\alpha \colon \pi_1 S \to O(q-p+1)$ be any representation. Then,

$$\rho = r \otimes \det \alpha \otimes \alpha \colon \pi_1 S \to \mathrm{SO}(p, q)$$

is a Θ-positive representation. For $q > p+1$, the representation variety $\mathcal{R}(S, \mathrm{SO}(p, q))$ has exotic components which contain the Θ-positive representation ρ.

Chapter 9

Anosov representations in affine geometry

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9.1 Affine geometry

Affine geometry is what remains of Euclidean geometry if we forget the metric notions of distance and angle. What is left: the notion of a straight line; parallelism; projective notions

We can give a more precise definition in terms of (G, X)-structures. Let X be a connected smooth manifold, and $G < \mathrm{Diffeo}(X)$ satisfy the following analyticity condition: if $g_1, g_2 \in G$ and there is an open subset $U \subset X$ such that g_1, g_2 are equal when restricted to U then $g_1 = g_2$

Definition 9.1.1. A (G, X)-structure on a topological manifold M is an atlas of charts on M with values in X and transition maps in G. Such an atlas may be called a (G, X)-atlas, and manifold equipped with such a structure a (G, X)-manifold.

This means that a (G, X)-structure gives M, locally, the geometry of X with it (sub)group of symmetries G.

Example 9.1.2. (i) A hyperbolic structure on a surface Σ is equivalent to a $(\mathrm{PSL}(2,\mathbb{R}), \mathbb{H}^2)$-structure on Σ.

(ii) A Euclidean structure on a n-manifold M^n is equivalent to a $(\mathrm{SO}(n) \ltimes \mathbb{R}^n, \mathbb{R}^n)$-structure on Σ.

Definition 9.1.3. An affine structure on a n-manifold M^n is a $(\mathrm{Aff}(\mathbb{R}^n), \mathbb{R}^n)$-structure on M^n, where $\mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}(\mathbb{R}^n) \ltimes \mathbb{R}^n)$ is the group of affine transformations

Equivalently, we can in terms of think of quotients of affine space:

Theorem 9.1.4 (Affine Killing-Hopf Theorem)**.** Let M be a geodesically complete flat affine manifold. Then M is affinely equivalent to $\mathbb{R}^n/\Gamma$, where $\Gamma \cong \pi_1 M$ is a discrete group of affine transformations of $\mathbb{R}^n$. In particular, Γ acts freely and properly discontinuously.

Example 9.1.5. Any Euclidean structure on M is an affine structure on M.

Example 9.1.6. Consider the quotient of the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by the map $x \mapsto 2x$ (or, equivalently, by the equivalence relation $\mathbf{x} \sim 2\mathbf{x}$.) This is called the Hopf torus. It is an example of an incomplete affine structure.

Why affine geometry? Historical interest

A natural interpolation between Euclidean and projective geometry (See also: equivalence of crystallographic groups, next)

Algebraic geometry

Theoretical physics

9.1.1 General motivating question

We can ask if there is any sort of classification, and/or nice structure theory, of affine manifolds.

Where would the representation theory come in?

The study of affine manifolds is already a mix of representation theory and geometry, so ... not entirely a surprise maybe? See also:

hyperbolic structures on closed surfaces $\longleftrightarrow$ Fuchsian representations

hyperbolic structures on 3-manifolds $\longleftrightarrow$ Kleinian groups

Let's start by looking at the special case of Euclidean ones, where there are some nice answers:

Definition 9.1.7. A (n-dimensional) crystallographic group is a discrete group of Euclidean isometries acting cocompactly on Euclidean (n-)space. Two crystallographic groups are equivalent if they are conjugate by affine transformations of $\mathbb{R}^n$

Equivalently, these are fundamental groups of compact Euclidean manifolds (or orbifolds, if we allow torsion.)

Example 9.1.8. For $n = 2$, the 17 wallpaper groups (Fedorov 1891; Pólya 1924)
For $n = 3$, 230 space groups (or 219, depending on if you allow equivalence via orientation-reversing transformations) (Fedorov; Schönflies 1891)

Theorem 9.1.9 (Bieberbach 1910)**.** Any n-dimensional crystallographic group contains a $\mathbb{Z}^n$ generated by translations as a finite-index subgroup.

Conversely, Zassenhaus (1948) showed that conversely any group that is the extension of $\mathbb{Z}^n$ by a finite group acting faithfully is crystallographic. Hence compact Euclidean manifolds are all, up to taking finite-index covers, tori.

If we move from the Euclidean to the affine world, things are rather more mysterious.

Definition 9.1.10. An affine crystallographic group is a discrete group of affine transformations acting cocompactly on affine (n-)space.

The first Bieberbach theorem as stated is no longer true in this setting:

Example 9.1.11. Some setup: if we write points $\mathbf{x} \in \mathbb{R}^n$ as $(n+1)$-vectors $(\mathbf{x}, 1)^T$, we may represent the action of $(\tau(\gamma), l(\gamma)) \in \mathbb{R}^n \rtimes \mathrm{GL}(\mathbb{R}^n)$ by the $(n+1)$-by-$(n+1)$ square matrix $\begin{pmatrix} \ell(\gamma) & \tau(\gamma) \\ \mathbf{0} & 1 \end{pmatrix}$, since we have

$$\begin{pmatrix} \ell(\gamma) & \tau(\gamma) \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} l(\gamma)\mathbf{x} + \tau(\gamma) \\ 1 \end{pmatrix}.$$

In this representation, the $\mathbb{R}^n$ appears as the subgroup $\left\{ \begin{pmatrix} I & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix} \right\}$. Now observe

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & a & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\} \subset \mathrm{Aff}(\mathbb{R}^3)$$

is an affine crystallographic group acting on $\mathbb{R}^3$: it is clearly discrete; cocompactness follows since any point in $\mathbb{R}^3$ is equivalent under the action of Γ to some point (x, y, z) with $x, y, z \in [-1, 1]$ (pick $a = -\lfloor x \rfloor$, $b = -\lfloor y \rfloor$, and $c = -\lfloor z - ay \rfloor$.) However,

- $\Gamma \cap \mathbb{R}^n$ spans only a 2-dimensional subspace (we must have $a = 0$), and
- the linear part $l(\Gamma) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \mid a \in \mathbb{Z} \right\}$ is not finite.

There is a conjecture about a potential replacement:

Conjecture 9.1.12 (Auslander 1964)**.** Affine crystallographic groups are virtually solvable.

Auslander's Conjecture has been proven in special cases: $n \leq 3$ (Fried–Goldman, 1983); when $\ell(\Gamma)$ preserves a quadratic form of signature $(n-1,1)$, i.e. $\ell(\Gamma) \subset \mathrm{Isom}(\mathbb{R}^{n-1,1})$ (Goldman–Kamishima, 1984; Carrière, 1989)[1] (Note that if $\ell(\Gamma)$ preserves a quadratic form of signature $(n,0)$, we are back in the Euclidean world.) Main idea behind all of these: "cohomological argument", using virtual cohomological dimension of Γ.

The state of the art on this Conjecture, using dynamical ideas:

Theorem 9.1.13 (Abels–Margulis–Soifer, 2012)**.** Auslander conjecture holds for $n \leq 6$.

Why the dimensional restrictions? Proof looks at semisimple part of Zariski closure of Γ and $\ell(\Gamma)$ and does casework based on that; list of possible semisimple parts gets longer with dimension ...

Milnor: Can we remove the cocompactness assumption, i.e. are all groups acting properly discontinuously on $\mathbb{R}^n$ by affine transformations virtually solvable?

Tits alternative: equivalent to non-abelian free group acting properly on $\mathbb{R}^n$.

Margulis constructed examples of such actions. These are complete affine manifolds with fundamental group isomorphic to non-abelian free groups (!)

Fried–Goldman (1983) show that the linear part of such an action on $\mathbb{R}^3$ must have linear part in (some conjugate of) $\mathrm{SO}(2,1)$ in $\mathrm{GL}(3,\mathbb{R})$. Drumm (1993) shows that any discrete subgroup of $\mathrm{SO}(2,1)$ can appear as the linear part.

Abels–Margulis–Soifer (2002) that there exist discrete subgroups of $\mathrm{SO}(n+1,n) \ltimes \mathbb{R}^{2n+1}$ acting properly on $\mathbb{R}^{2n+1}$ iff n is odd. Recent work of Smilga (2014–2016–) extends this to a characterisation for all $G \ltimes V$ (with suitable assumptions.)1

9.2 Building a Margulis spacetime

Remark 9.2.1. $\Gamma \curvearrowright \mathbb{R}^n$ a properly discontinuous affine action $\implies \forall \gamma$ of infinite-order, $\gamma \cdot v + \tau(\gamma) = v$ should have no solution in $v \in \mathbb{R}^n$, hence 1 is an eigenvalue of γ. $\implies$ 1 is an eigenvalue of all g in the Zariski closure (of the linear part of $\rho(\Gamma)$ in $\mathrm{GL}(V)$.) In Lie group language, 0 must be a weight of the linear part of ρ.

Idea of proof, as described by F. Guéritaud. "Refine the idea that 1 must be an eigenvalue."

$g \in \mathrm{SO}(2,1) \ltimes \mathbb{R}^{2,1}$ acts on $\mathbb{R}^{2,1}$ with expanding, contracting, and neutral eigendirections e^+, e^-, e^0 resp. with a well-defined sign for the translational part. The sign is the same for g^{-1}. Then show that translation lengths are coarsely additive under mutiplication in Γ ["at least for sufficiently transverse elements."] The signed translation length can be used to see that these representations are infinitesimal deformations of Fuchsian representations; the signed translation length is a derivative. □

[1] Goldman–Kamishima, 1984: Auslander conjecture holds for complete compact flat Lorentz manifolds. Carrière, 1989: Compact flat Lorentz manifolds are complete. Grunewald–Margulis, 1989: Classification is known.

Key tool to make this precise: the Margulis invariant. This is a function $\alpha_\rho : \Gamma \to \mathbb{R}$ associated to a representation $\rho : \Gamma \to \mathrm{SO}(2,1) \ltimes \mathbb{R}^3$, given by

$$\alpha_\rho(\gamma) := Q\left(\tau_\rho(\gamma), \nu(\xi(\gamma^-), \xi(\gamma^+))\right)$$

where τ_ρ denotes the translation part, ξ is the Anosov / Fuchsian limit map, Q is the Lorentzian bilinear form, and $\nu : \mathrm{SO}_0(1,2)/(P_0^+ \cap P_0^-) \to \mathbb{R}^3$ is a "neutral section" $[g] \mapsto [g \cdot (0_{n-1}, 1, 0_{n-1})^T]$.

Informally, this measures interaction between the translation part and the north-south dynamics of the linear part; more precisely, we may identify it as a *bona fide* (signed) translation length along a special axis, given by the eigenline for the eigenvalue 1, where we normalize the sign by choosing a positive eigenbasis (v_+, v_0, v_-) with $\langle v_+, v_- \rangle < 0$. We note that determining this signed translation length is equivalent to writing the translation part $\tau(\gamma)$ in this eigenbasis, and looking at the coefficient for v_0.

Key properties:

- $\alpha(\gamma^n) = n\alpha(\gamma)$ for all $n \geq 1$
- $\alpha(\gamma^{-1}) = \alpha(\gamma)$
- $\alpha(\gamma) = 0$ iff γ has a fixed point
- if γ is ϵ-hyperbolic (i.e. the attracting and repelling fixed points are at least ϵ apart), then $|\alpha(\gamma)| \sim \|\tau^0(\gamma)\|$ (where τ^0 is the translation along the central i.e. eigenvalue-1 direction) with constants depending only on ϵ
- ("almost-additivity") if γ and δ are ϵ-hyperbolic and ϵ-transverse, then

$$\alpha(\gamma\delta) - (\alpha(\gamma) + \alpha(\delta)) \leq F(d(x_0, \ell(\gamma)), d(x_0, \ell(\delta)), |\alpha(\delta)|, |\alpha(\gamma)|)$$

Lemma 9.2.2 (Opposite sign lemma)**.** Let γ, δ be ϵ-hyperbolic and ϵ-transverse with $\alpha(\gamma) < 0 < \alpha(\delta)$. Then $\langle \gamma, \delta \rangle$ does not act properly discontinuously on $\mathbb{R}^{2,1}$.

There is also a converse to this (see Theorem 9.4.13 below), and in this sense this obstruction characterizes proper discontinuity in this case.

Other counterexamples to Milnor

Danciger–Guéritaud–Kassel [DGK18c] constructed examples of proper affine actions of right-angled Coxeter groups. This is a large class of examples which includes, for instance, closed surface groups. Note these are still not counter-examples to the Auslander conjecture, e.g. the surface group actions they construct are in dimension $d \geq 6$ and not cocompact.

9.3 Anosov representations (and friends)

These have been the subjects of previous talks; roughly, they are classes of representations, of word-hyperbolic groups (often surface groups or free groups) into semisimple Lie groups (e.g. $\mathrm{PSL}(d, \mathbb{R})$, $\mathrm{SO}(n, n-1)$, or $\mathrm{PSO}(n, n)$) with "good" dynamical / geometric properties.

9.3.1 Parabolics and flags

A representation is Anosov with respect to some given / chosen parabolic subgroup $P < G$; G/P is naturally a flag manifold.

For example if we choose $P = P_1$ to be the stabilizer of a line in $G = \mathrm{PSL}(d, \mathbb{R})$ (up to conjugacy, this is the group of block upper-triangular matrices with a 1-by-1 block in the upper-left, and a large $(d-1)$-by-$(d-1)$ block in the lower-right), then G/P is (homeomorphic to) the projective space $\mathbb{P}(\mathbb{R}^d)$.

More generally, if $P = P_k$ is the stabilizer of a k-plane in $G = \mathrm{PSL}(d, \mathbb{R})$, then G/P is the Grassmannian of k-planes in $\mathbb{R}^d$; $P_i \cap P_j$ is the stabilizer of a flag consisting of an i-plane nested in a j-plane (assuming $i < j$), and $G/(P_i \cap P_j)$ is the space of such flags.

For example if $P = B$ is the stabilizer of a full flag in $G = \mathrm{PSL}(d, \mathbb{R})$ (up to conjugacy, this is the group of upper-triangular matrices; it is the Borel subgroup), then G/P is the space of full flags, which we will denote $\mathcal{F}_B$.

In $\mathrm{SO}(n, n-1)$, we define $\mathrm{Gr}_k(\mathbb{R}^{n,n-1})$ to be the Grassmannian of *isotropic* k-planes (if $k \leq n-1$; otherwise we define it as the Grassmannian of k-planes whose orthogonal complements are isotropic); $P''_k := P_k \cap \mathrm{SO}(n, n-1)$ is s.t. $\mathrm{SO}(n, n-1)/P''_k \cong \mathrm{Gr}_k(\mathbb{R}^{n,n-1}$.

The same holds in $\mathrm{PSO}(n, n)$ (and we write $P'_k := P_k \cap \mathrm{PSO}(n, n)$) except $\mathrm{PSO}(n, n) \cap \mathrm{Gr}_n(\mathbb{R}^{n,n})$ (the Grassmannian of isotropic n-planes) has 2 orbits $\mathrm{Gr}^{\pm}_n(\mathbb{R}^{n,n}) = \mathrm{PSO}(n, n)/P^{\pm}_n$, where P^+_n is the stabilizer of the span of $(e_1, \dots, e_n)$ and P^-_n is the stabilizer of the span of $(e_1, \dots, e_{n-1}, e_{n+1})$.

9.3.2 Anosov representations

Theorem 9.3.1 (Definition / Theorem; [GGKW17b])**.** $\rho : \Gamma \to G$ is P-Anosov if there exists a continuous, Γ-equivariant, dynamics-preserving, transverse limit map $\xi : \partial_\infty \Gamma \to G/P$, and the eigenvalue gaps $\frac{\lambda_1}{\lambda_2}(\rho(\gamma))$ grow uniformly exponentially in stable length $|\gamma|_\infty := \lim_{n\to\infty} \frac{1}{n}|\gamma^n|$.

In fact, it suffices to assume word-hyperbolicity of the abstract group and uniform exponential growth of the eigenvalue gaps (the existence of the limit maps then follows as a consequence:

Theorem 9.3.2 (Kassel–Potrie)**.** If that Γ is word-hyperbolic and $\rho : \Gamma \to G$ satisfies $\frac{\lambda_1}{\lambda_2}(\rho(\gamma)) \geq Ce^{\mu|\gamma|_\infty}$, then ρ is P_1-Anosov.

Very informally: the representation steers sufficiently clear of ~~flats~~ where complicated interactions between the flats happen, so that hyperbolic dynamics / behavior in the large is preserved.

Example 9.3.3.

- Fuchsian representations $\rho : \pi_1\Sigma_g \to \mathrm{PSL}_2\mathbb{R}$ (or $\rho : F_r \to \mathrm{PSL}_2\mathbb{R}$ with hyperbolic boundary holonomy (representing totally geodesic boundary) are P_1-Anosov (i.e, B-Anosov in this case.)
- (Benoist) strictly convex real projective holonomies $\rho : \pi_1 M \to \mathrm{PSL}_n\mathbb{R}$ are P_1-Anosov.

  (In dimension 2, this is again the same as B-Anosov; more generally, this is not true,)
- (Labourie) Hitchin representations $\rho : \pi_1\Sigma_g \to \mathrm{PSL}_n\mathbb{R}$ are B-Anosov.

Key properties of Anosov representations $\rho : \Gamma \to G$:

- $|\ker \rho| < \infty$;
- $\rho(\Gamma)$ discrete in G;
- orbit maps $\gamma \mapsto \rho(\gamma)x_0$ are quasi-isoemtric embeddings;
- Anosovness is an open condition.

9.3.3 Hitchin representations

Let G be $\mathrm{PSL}(2n,\mathbb{R})$, $\mathrm{PSO}(n,n)$, or $\mathrm{SO}(n,n-1)$. Let $\Gamma := \pi_1\Sigma_g$ be a surface group. $\chi(\Gamma,G) := \mathrm{Hom}(\Gamma,G)/G$ has two Teichmüller components of discrete faithful representations corresponding to hyperbolic structures on Σ_g. These come from a principal representation $\tau_G : \mathrm{PSL}(2,\mathbb{R}) \to G$.

(Informally: the principal representation is "as generic as possible" / "does not create additional symmetries".)

For example, for $G = \mathrm{PSL}(d,\mathbb{R})$, $\tau_G = \tau_d$ is defined via the induced action on the symmetric power $\mathrm{SL}(2,\mathbb{R}) \curvearrowright S^{d-1}\mathbb{R}^2 \cong \mathbb{R}^d$, which yields

$$\tau_d \begin{pmatrix} \lambda & \\ & 1/\lambda \end{pmatrix} = \begin{pmatrix} \lambda^{d-1} & & & \\ & \lambda^{d-3} & & \\ & & \ddots & \\ & & & \lambda^{1-d} \end{pmatrix}.$$

For $G = \mathrm{SO}(n,n-1)$, the same map works (i.e. we can check that it in fact lands in $\mathrm{SO}(n,n-1)$.)

For $G = \mathrm{PSO}(n,n)$, $\tau_G = \iota_{n,n} \circ \tau_d$, where $\iota_{n,n} : \mathrm{SO}(n,n-1) \hookrightarrow \mathrm{PSO}(n,n)$ is the natural inclusion induced by the splitting $\mathbb{R}^{n,n} = \mathbb{R}^{n,n-1} \oplus \mathbb{R}^{0,1}$. Note that

$$\tau_G \begin{pmatrix} \lambda & \\ & 1/\lambda \end{pmatrix} = \iota_{n,n} \circ \tau_d \begin{pmatrix} \lambda & \\ & 1/\lambda \end{pmatrix} = \begin{pmatrix} \lambda^{2(n-1} & & & & & & \\ & \lambda^{2(n-2)} & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & \lambda^{-2(n-1)} \end{pmatrix}$$

Definition 9.3.4. A <u>G-Hitchin reprsentation</u> is a representation $\rho : \Gamma \to G$ in the connected component of $\mathrm{Hom}(\Gamma,G)/G$ containing $\tau_G(\mathrm{PSL}(2,\mathbb{R}))$.

Theorem 9.3.5. *G-Hitchin representations are B-Anosov* (i.e. they are Anosov with respect to all the parabolics ...)

Moreover, we have the following stronger characterization, in terms of positivity:

Theorem 9.3.6 (Labourie, Guichard, Fock-Goncharov)**.** *$\rho : \Gamma \to G$ is G-Hitchin iff there exists a continuous ρ-equivariant limit map $\xi : \partial_\infty\Gamma \to G/B$ which sends (oriented) triples in $\partial_\infty\Gamma$ to positive triples in (the flag space) $\mathcal{F}_B$.*

This gives us, among other things, even more transversality [2] which will be useful later:

Proposition 9.3.7 ([DZ18],Corollary 3.7)**.** *Let $\varrho : \Gamma \to \mathrm{PSO}(n,n)$ be Hitchin; then the Anosov limit curve $\xi : \partial\Gamma \to \mathcal{F}_{B'}$ satisfies*

$$\xi^{(n-1)}(x) + \left(\xi^{(n-1)}(z) \cap \xi^{(n+2)}(y)\right) + \xi_\pm^{(n)}(y) = \mathbb{R}^{2n}$$

for all $(y,z,x) \in \partial\Gamma^{(3)}$.

[2] And transversality is key. Often, the more transversality the limit maps have, the "nicer" properties we can show for our representation.

9.4 Anosov representations in affine geometry

We recall that {affine structures on M} $\longleftrightarrow$ {proper actions of $\pi_1 M$ on $\mathbb{R}^n$ by affine transformations}. Below we (a) look briefly at use of Anosov reps to study the exotic affine structures which are Margulis spacetimes; (b) study more carefully the use of Anosov reps in [DZ18] to look at proper affine actions by closed surface groups (i.e. affine structures on manifolds homotopic to closed surfaces)

9.4.1 Affine Anosov representations and Margulis spacetimes

Anosov representations are representations of word-hyperbolic groups into semisimple Lie groups G. Extend this to $G \rtimes V$, where V is a vector space on which G acts by linear transformations. Why? Because these are / could be groups of affine transformations, and maybe we can then say things about affine geometry?

Definition 9.4.1. A faithful representation $\rho : \Gamma \to \mathrm{SO}_0(n-1,n) \ltimes \mathbb{R}^{2n-1}$ is <u>affine Anosov</u> if

1. there exist a continuous, injective, equivariant limit map $\xi : \partial_\infty \Gamma \to \mathcal{X}$ (the "generic" orbit of pairs of isotropic spaces);
2. (informally) there is contraction along the Gromov geodesic flow;
3. there is an equivariant map $s : \widetilde{U\Gamma} \to \mathbb{R}^{2n-1}$ which is Hölder-continuous and differentiable along flow lines, and the derivative of s along flow lines is not orthogonal, w.r.t. the Lorentz quadratic form, to $\nu(\xi^+, \xi^-)$ (informally, "the [variation in the] translation part is sufficiently transverse")

Theorem 9.4.2 ([GT17], Theorem 0.1 = 7.1 + 7.3)**.** A representation of a word-hyperbolic group Γ into $\mathrm{SO}^0(n, n-1) \ltimes \mathbb{R}^{n,n-1}$ is affine Anosov if and ony if its linear part is Anosov w.r.t the stabilizer of a maximal isotropic plane, and it acts properly on $\mathbb{R}^{n,n-1}$

Theorem 9.4.3 ([Gho18], Theorem 0.0.3)**.** If $\rho : \Gamma \to \mathrm{SO}_0(n,n)$ is Anosov w.r.t. the stabilizer of an oriented isotropic $(n-1)$-plane, then $\rho(\Gamma) \curvearrowright \mathrm{SO}_0(n,n)/\mathrm{SO}_0(n-1,n)$ is proper if and only if ρ is Anosov in $\mathrm{SL}(2n,\mathbb{R})$ wrt the stabilizer of an oriented n-plane.

In particular, if we let Γ be a nonabelian free group, then this says that affine Anosov representations give rise to Margulis spacetimes; conversely, Margulis spacetimes <u>with Anosov linear parts</u> give rise to affine Anosov representations.

We can interpret affine representations in $\mathrm{SO}_0(1,2) \ltimes \mathbb{R}^3$ as tangent directions in space of Anosov representations in $\mathrm{SO}_0(1,2)$, or as tangent directions of deformations in $\mathrm{SO}_0(2,2)$ of Anosov representations in $\mathrm{SO}_0(1,2)$. i.e. we can nterpret affine $\mathrm{SO}_0(n-1,n)$ representations as infinitesimal $\mathrm{SO}_0(n,n)$ representations.

Proposition 9.4.4 ([Gho18], Proposition 0.0.1, §2?)**.** Suppose Γ is word-hyperbolic and

- $\mathfrak{u} = \frac{d}{dt}\Big|_{t=0} \rho_t$ where $\rho_t : \Gamma \to \mathrm{SO}_0(n,n)$ is an analytic one-parameter family of representations;
- $v_0 \in \mathbb{R}^{2n}$ a fixed vector of $\mathrm{SO}_0(n-1,n)$; and
- $\rho_0(\Gamma)$ Anosov wrt stabilizer of a maximal isotropic plane in $\mathbb{R}^{n-1,n}$.

Then $\rho(\Gamma) := (\rho, \mathfrak{u} v_0)(\Gamma)$ is a subgroup of the right affine group s.t. for any $\gamma \in \Gamma \backslash \{\mathrm{id}\}$,the Margulis invariant $\alpha_\rho(\gamma)$ is proportional to $\frac{d}{dt}\Big|_{t=0} \log \frac{\lambda_n}{\lambda_{n+1}}(\rho_t(\gamma))$ (these are the middle eigenvalues.)

Also get affine actions "by deformation":

Theorem 9.4.5 ([Gho18], Theorem 0.0.4)**.** Suppose Γ is word-hyperbolic and

- $\mathfrak{u} = \frac{d}{dt}\big|_{t=0}\rho_t$ where $\rho_t : \Gamma \to \mathrm{SO}_0(n,n)$ is an analytic one-parameter family of representations;
- $\rho_0(\Gamma) \subset \mathrm{SO}_0(n-1,n) \subset \mathrm{SO}_0(n,n)$ such that affine action $(\rho_0, \mathfrak{u}v_0)(\gamma) \curvearrowright \mathbb{R}^{n-1,n}$ is proper.

Then there exists an $\epsilon > 0$ such that for all $t \in (0,\epsilon)$, the groups $\rho_t(\Gamma)$ act properly on

$$\mathrm{SO}_0(n,n)/\mathrm{SO}_0(n-1,n) \cong \mathbb{R}^{2n-1}.$$

Combining this with an earlier result of Abels–Margulis–Soifer, get

Corollary 9.4.6 ([Gho18], Corollary 0.0.5)**.** There exist proper affine actions of F_k, k even, on

$$\mathrm{SO}_0(n,n)/\mathrm{SO}_0(n-1,n)$$

which (when written as $2n$-by-$2n$ matrices) lie in $\mathrm{SO}_0(n,n)$.

9.4.2 Affine actions of surface groups: linear part not Hitchin

Theorem 9.4.7 (Danciger–Zhang, Theorem 1.1)**.** If $\pi_1 S \to \mathrm{GL}_d\mathbb{R} \ltimes \mathbb{R}^d$ is a proper affine action, then the linear part does not lie in a Hitchin component.

This restricts the situation described in the first Ghosh–Treib result when Γ is a surface group: there are no proper affine actions with Hitchin linear part (*but* it may still have Anosov linear part—e.g. Barbot representations are Anosov but not Hitchin), hence no corresponding affine Anosov representations (but those may still exist, just with non-Hitchin Anosov linear part.)

Compare this e.g. to the case of free groups acting properly by affine transformations on $\mathbb{R}^3$—those are B-Anosov, by Fried–Goldman. The Danciger–Zhang theorem tells us that for surface groups in higher rank, proper affine actions, if they exist, must necessarily look rather different.

Proof sketch for Danciger–Zhang : Four steps! Step 1 feels like trickery. Step 2 is neat, though it does not use Anosov representations. Step 3 requires substantial work with Margulis invariants / length functions. Step 4 is the main technical step which uses the theory of Anosov representations. Below we described the steps in slightly more detail.

9.4.3 Reduction to $\mathrm{SO}(n, n-1)$

By considering Zariski closures (or: mild trickery using Lie / algebraic groups), reduce to case of $\mathrm{SO}(n, n-1)$:

Theorem 9.4.8 (Guichard, in preparation)**.** If $\rho : \Gamma \to \mathrm{SL}_d(\mathbb{R})$ is Hitchin, then the Zariski closure $\overline{\rho(\Gamma)}^Z$ must contain the principal $\mathrm{SL}_2\mathbb{R}$, i.e. the image of the irreducible representation $\tau_d : \mathrm{SL}_2\mathbb{R} \to \mathrm{SL}_d\mathbb{R}$.

There is a shortish list of algebraic subgroups with this property:

1. $\mathrm{SL}_d\mathbb{R}$;
2. the principal $\mathrm{SL}_2\mathbb{R}$;
3. $\mathrm{SO}(n, n-1)$, if $d = 2n-1$ is odd;
4. $\mathrm{Sp}(2n, \mathbb{R})$, if $d = 2n$ is even
5. if $d = 7$, the 7-dimensional representation of G_2 in $\mathrm{SO}(4,3)$.

Suppose ρ is the linear part of a proper affine action and is the lift of a representation in the $\mathrm{PSL}_d\mathbb{R}$-Hitchin component. If ρ' is any lift of the same to $\mathrm{SL}_d\mathbb{R}$, then we can show that $\overline{\rho(\Gamma)}^Z \supset \overline{\rho'(\Gamma)}^Z$, so $\overline{\rho(\Gamma)}^Z$ contains the principal $\mathrm{SL}_2\mathbb{R}$. As observed above, any element of $\overline{\rho(\Gamma)}^Z$ has 1 as an eigenvalue. If d is even, the principal $\mathrm{SL}_2\mathbb{R}$ contains elements which do not satisfy this; hence in our case $= 2n-1$ must be odd.

Still using the observation that 1 is an eigenvalue, and via more mild trickery with algebraic groups, we obtain that the projection $\mathrm{GL}_{2n-1}\mathbb{R} \to \mathrm{SL}_{2n-1}\mathbb{R}$ is injective between the Zariski closures $\overline{\rho(\Gamma)}^Z \to \overline{\rho'(\Gamma)}^Z$, i.e. $\overline{\rho(\Gamma)}^Z = \overline{\rho'(\Gamma)}^Z$, so $\overline{\rho(\Gamma)}^Z$ must in fact be conjugate to (2), (3), or (5). In all of these cases $\overline{\rho(\Gamma)}^Z < \mathrm{SO}(n, n-1)$, so $\rho(\Gamma) < \mathrm{SO}(n, n-1)$.

Hence we have reduced our original statement to

Theorem 9.4.9 ([DZ18], Theorem 1.2)**.** If $\pi_1 S \to \mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{n,n-1}$ is an action by isometries of $\mathbb{E}^{n,n-1}$ with linear part a $\mathrm{SO}(n, n-1)$-Hitchin representation, then the action is not proper.

9.4.4 Geometric transition

By performing a geometric transition, we obtain a deformation path in $\mathrm{SO}(n, n)$: $\mathbb{E}^{n,n-1}$ may be seen as a geometric limit of (scaled copies of) $\mathbb{H}^{n,n-1}$ in the following sense. Both $\mathbb{H}^{n,n-1}$ and $\mathbb{E}^{n,n-1}$ naturally embed in real projective geometry. The projective model for $\mathbb{H}^{n,n-1}$ is given by

$$\mathbb{H}^{n,n-1} := \left\{[x] \in \mathbb{P}(\mathbb{R}^{2n}) : \langle x, x\rangle_{n,n} < 0\right\}$$

with metric coming from the restriction of $\langle \cdot, \cdot \rangle_{n,n}$ to the tangent spaces of the hyperboloid $\{x : \langle x, x\rangle_{n,n} = 1$ which double-covers $\mathbb{H}^{n,n-1}$. $\mathrm{PSO}(n, n) < \mathrm{PGL}(2n, \mathbb{R})$ preserves $\mathbb{H}^{n,n-1}$ (and its orientation) and is the (op) isometry group of this metric (a geodesically complete pseudo-Riemannian metric of signature $(n, n-1)$.) For the projective model of $\mathbb{E}^{n,n-1}$ we may take the parallel affine hyperplane

$$\mathbb{E}^{n,n-1} := \{[x_1 \dots L_{2n-1} : 1]\} \subset \mathbb{P}(\mathbb{R}^{2n})$$

with metric given by the restriction of $\langle \cdot, \cdot \rangle_{n,n}$ to the relevant vector subspace (this is a complete flat metric of signature $(n, n-1)$.) The subgroup of $\mathrm{PGL}(2n, \mathbb{R})$ which preserves this affine chart and its flat metric (and orientation) gives the isometry group

$$\mathrm{Isom}^+(\mathbb{E}^{n,n-1}) = \left\{ \begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix} \in \mathrm{PGL}(2n, \mathbb{R}) : A \in \mathrm{SO}(n, n-1) v \in \mathbb{R}^{2n-1} \right\}.$$

Consider a differentiable path $r \mapsto g_r := \begin{bmatrix} A_r & v_r \\ w_r^T & b_r \end{bmatrix}$ in $\mathrm{PSO}(n, n)$ based at $g_0 = \iota(h)$, where $h \in \mathrm{SO}(n, n-1)$. Here A_r is a $(2n-1)$-by-$(2n-1)$ matrix, $v_r, w_r \in \mathbb{R}^{2n-1}$, and $b_r \in \mathbb{R}$ (all well-defined up to simultaneously changing signs.)
$g_0 = \iota(h) \implies A_0 = h$, $v_0 = w_0 = 0$, $b_0 = 1$. Let $c_r \in \mathrm{PGL}(2n, \mathbb{R})$ be the projective transformation given by $\begin{bmatrix} \frac{1}{r}\,\mathrm{id}_{2n-1} & 0 \\ 0 & 1 \end{bmatrix}$.
Observe that $c_r \,\mathrm{PSO}(n, n) c_r^{-1} \to \mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1}$ (in the Chabauty topology on closed subgroups of $\mathrm{PSL}(2n, \mathbb{R})$), as $r \to 0$, since

$$\lim_{r\to 0} c_r g_r c_r^{-1} = \lim_{r\to 0} \begin{bmatrix} A_r & \frac{1}{r} v_r \\ r w_r^T & b_r \end{bmatrix} = \begin{bmatrix} h & u \\ 0 & 1 \end{bmatrix} \in \mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1}$$

where $u := \frac{d}{dr}\big|_{r=0} v_r$. Note moreover $[0 : \cdots : 0 : 1] =: x_0 \in \mathbb{H}^{n,n-1} \cap \mathbb{E}^{n,n-1}$.

In fact, the action $\mathrm{PSO}(n, n) \curvearrowright \mathbb{H}^{n,n-1}$ converges to the action $\mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1} \curvearrowright \mathbb{E}^{n,n-1}$ under conjugation by c_r, in the following sense: let $r \mapsto x_r$ be a differentiable path in $\mathbb{H}^{n,n-1}$ based at $x_0 = [0 : \cdots : 0 : 1]$ (note $\mathrm{Stab}(x_0) = \iota(\mathrm{SO}(n, n-1))$.) For all r small enough, $c_r x_r \in \mathbb{E}^{n,n-1}$, so $c_r x_r$ converges to some limit $x' \in \mathbb{E}^{n,n-1}$. (If we write $\mathbb{R}^{n,n-1} = T_{x_0}\mathbb{H}^{n,n-1}$, the tangent vector to x_r at $r = 0$ is precisely the displacement vector $x' - x_0$.) Moreover, $c_r g_r x_r = c_r g_r c_r^{-1}(c_r x_r) \to (h, u)x'$.

Hence $\mathbb{E}^{n,n-1}$ is a geometric limit of $\mathbb{H}^{n,n-1}$ as subgeometries of real projective geometry, in the sense of Cooper–Danciger–Wienhard.

Define $\varrho_r^{c_r} : \Gamma \to \mathrm{PSL}(2n, \mathbb{R})$ by $\varrho_r^{c_r}(\gamma) := c_r \varrho(\gamma) c_r^{-1}$. By the calculation above, $\lim_{r\to 0} \varrho_r^{c_r} = (\rho, u)$ is a representation into $\mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1}$ with linear part ρ and translation part the ρ-cocycle $u : \Gamma \to \mathbb{R}^{2n-1}$. Moreover, we may write any affine action with irreducible linear part as the limit of such a transition path:

Lemma 9.4.10 ([DZ18], Lemma 8.1)**.** If $(\rho, u) : \Gamma \to \mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1}$ is any surface group representation with irreducible linear part ρ, then there exists a path $\varrho_r : \Gamma \to \mathrm{PSO}(n, n)$ such that $\varrho_0 = \iota \circ \rho$ and $\lim_{r\to 0} \varrho_r^{c^r} = (\rho, u)$ as above.

Proof. ρ irreducible $\implies$ $\iota \circ \rho : \Gamma \to \mathrm{PSO}(n, n)$ has finite centralizer, $\implies$ $\iota \circ \rho$ is a smooth point of $\mathrm{Hom}(\Gamma, \mathrm{PSO}(n, n))$ by work of Goldman. Hence any tangent direction there is integrable; picking a suitable one, (see [DZ18] for details) we win. □

9.4.5 Interlude: isotropic Grassmannians and flag manifolds

We remark that $\mathrm{SO}(n, n-1)$-Hitchin representations are always $\mathrm{PSL}(2n-1)$-Hitchin (via the natural inclusion), and, after composing with the natural inclusion $\iota_{n,n} : \mathrm{SO}(n, n-1) \hookrightarrow \mathrm{PSO}(n, n)$ described above, are always $\mathrm{PSO}(n, n)$-Hitchin. However, $\mathrm{PSO}(n, n)$-Hitchin representations are not $\mathrm{PSL}(2n)$-Hitchin via the natural inclusion there.

We remark that $\mathrm{SO}(n, n-1)/B''$, where B'' is the Borel subgroup of $\mathrm{SO}(n, n-1)$, is the space of flags $F^{(1)} \subset \cdots \subset F^{(2n-1)}$ where $F^{(k)} \in \mathrm{Gr}_k(\mathbb{R}^{n,n-1})$ and $F^{(2n-1-k)} = (F^{(k)})^\perp$.

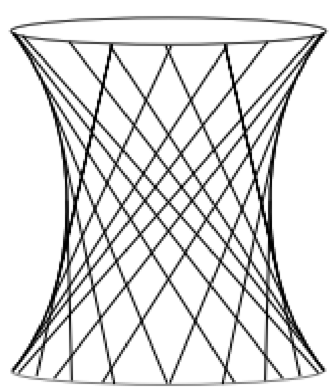
Also, $\mathrm{PSO}(n, n)/B'$, where B' is the Borel subgroup of $\mathrm{PSO}(n, n)$, is the space of flags

$$F^{(1)} \subset \cdots \subset F^{(n-1)} \subset F_+^{(n)}, F_-^{(n)} \subset F^{(n+1)} \subset \cdots \subset F^{(2n-1)}$$

where $F^{(k)} \in \mathrm{Gr}_k(\mathbb{R}^{n,n})$, $F_\pm^{(n)} \in \mathrm{Gr}_n^\pm(\mathbb{R}^{n,n})$, and $F^{(2n-k)} = (F^{(k)})^\perp$.

Any given $H_0 \in \mathrm{Gr}_{n-1}(\mathbb{R}^{n,n})$ is contained in a unique $H_\pm \in \mathrm{Gr}_n^\pm(\mathbb{R}^{n,n})$. This gives us natural projection maps $\bar{\omega}^p m : \mathrm{Gr}_{n-1}(\mathbb{R}^{n,n}) \to \mathrm{Gr}_n^\pm$ which are fiber bundles with $\mathbb{RP}^{n-1}$ fibers.

For example, (figure and description from [DZ18]) $\mathrm{Gr}_1(\mathbb{R}^{2,2}) \subset \mathrm{Gr}_1(\mathbb{R}^4) = \mathbb{RP}^3$ is the doubly-ruled hyperboloid. The lines of one of the rulings make up $\mathrm{Gr}_2^+(\mathbb{R}^{2,2})$ while the lines of the other make up $\mathrm{Gr}_2^-(\mathbb{R}^{2,2})$ The projection maps $\bar{\omega}_\pm$ map a point of $\mathrm{Gr}_1(\mathbb{R}^{2,2})$ to the line of the + / - (resp.) ruling containing it.



9.4.6 Unexpected Anosovness via Margulis invariant functional

In fact we get a deformation subpath of Anosovs in $\mathrm{PSL}_{2n}\mathbb{R}$:

Theorem 9.4.11 ([DZ18], Theorem 8.7)**.** Let $\rho : \Gamma \to \mathrm{SO}(n, n-1)$ be Anosov with respect to the stabilizer of an isotropic $(n-1)$-plane. Let $u : \Gamma \to \mathbb{R}^{2n-1}$ be a ρ-cocycle such that the affine action $(\rho, u) \curvearrowright \mathbb{R}^{n,n-1}$ is properly discontinuous. Let $\varrho_r : \Gamma \to \mathrm{PSO}(n, n)$ be any path so that $\varrho_0 = \iota_{n,n} \circ \rho$ and $\varrho_r^{c^r} \to (\rho, u)$. Then for all sufficiently small $r > 0$, $\iota_{2n} \circ \varrho_r : \Gamma \to \mathrm{PSL}(2n, \mathbb{R})$ is Anosov wrt the stabilizer P_n of a n-plane in $\mathbb{R}^{2n}$.

For this we use Margulis invariants (see above.) To obtain a converse to the opposite sign lemma (Lemma 9.2.2) we work with currents:

Let S be a hyperbolic surface, $\Gamma = \pi_1 S$, and let φ_t denote the geodesic flow on T^1S.

Definition 9.4.12. A(n oriented) geodesic current μ on S is a finite φ_t-invariant Borel emasure on the unit tangent bundle T^1S. We denote the space of geodesic currents on S by $\mathcal{C}(S)$

These are a nice space containing the space of (oriented) closed geodesics (a "completion" / "closure" of sorts.) Moreover, by Banach-Alaoğlu, the space of currents of total mass 1 is compact.

Theorem 9.4.13 (Goldman-Labourie-Margulis, Ghosh-Treib)**.** Suppose the linear part of an affine action $(\rho, u) \in \mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1}$ is Anosov wrt the stabilizer of an isotropic $(n-1)$-plane. Then

(1) There exists a unique continuous linear functional $\alpha_{(\rho,u)} : \mathcal{C}(S) \to \mathbb{R}$ such that $\alpha_{(\rho,u)}(\mu_c) = \alpha((\rho, u)(\gamma))$ for all $c = [\gamma] \in \mathcal{CG}(S)$.

(2) $(\rho, u)(\Gamma) \curvearrowright \mathbb{E}^{n,n-1}$ is properly discontinuous iff $\alpha_{(\rho,u)}(\mu) \neq 0$ for all $\mu \in \mathcal{C}(S)$.

$\alpha_{(\rho,u)}$ is variously called the Margulis invariant functional, the Labourie-Margulis invariant, or "Labourie's diffusion of the Margulis invariant".

Note (2) implies the Opposite Sign Lemma and its converse. Moreover, we can define an analogous length function $\mathcal{L}_\varrho : \mathcal{C}(S) \to \mathbb{R}$ which gives the translation length $\mathcal{L}$ on $\mathbb{H}^{n,n-1}$ (along a special "slow axis" which is the analogue of our neutral line) when restricted to $\mathcal{CG}(S)$.

By examining the two geometries and the relation between them, we obtain

Lemma 9.4.14 ([DZ18], Lemma 8.2)**.** Let $(\rho, u) : \Gamma \to \mathrm{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1}$ be any representation whose linear part $\rho : \Gamma \to \mathrm{SO}(n, n-1)$ is Anosov (wrt stabilizer of an isotropic $(n-1)$-plane.) Let $\varrho_r : \Gamma \to \mathrm{PSO}(n, n)$ be a differentiable path based at $\varrho_0 = \iota \circ \rho$ and satisfying $\lim_{r\to 0} \varrho_r^{c_r} = (\rho, r)$. Then the length functions $\alpha_{(\rho,u)}, \mathcal{L}_{\varrho_r} : \mathcal{C}(S) \to \mathbb{R}$ satisfy

$$\lim_{r\to 0} \frac{1}{r}\mathcal{L}_{\varrho_r}(\cdot) = \alpha_{(\rho,u)}(\cdot)$$

and the convergence is uniform on compact subsets of $\mathcal{C}(S)$.

Proof of Theorem 9.4.11. Theorem 9.4.13 gives us a well-defined, nowhere zero Margulis invariant functional $\alpha_{(\rho,u)}$. Since the space of current is connected, $\alpha_{(\rho,u)}$ always has the same sign, WLOG +. By compactness, there exists $\epsilon > 0$ such that $\alpha_{(\rho,u)}(\mathcal{C}_1(S)) > \epsilon > 0$. From Lemma 9.4.14, for all $r > 0$ sufficiently small, $\mathcal{L}_{\varrho_r}(\mathcal{C}_1(S)) > r\frac{\epsilon}{2}$. Below, assume r is sufficiently small for this to hold. Given a fixed hyperbolic metric on S, the stable length $|\gamma|_\infty$ is biLipschitz to the length $\ell([\gamma])$, with constant say M. Using that $\mathcal{L}_\varrho$ restricts to the translation length $\mathcal{L}$, we then have, for every $\gamma \in \Gamma\backslash\{\mathrm{id}\}$, an inequality on the Lyapunov projections

$$\lambda_n(\varrho_r(\gamma)) \geq Mr\frac{\epsilon}{2}|\gamma|_\infty > 0$$

Moreover we can build the Anosov limit map ξ by composing $\xi_+^{(n)}$ with the inclusion $\mathrm{Gr}_n^+(\mathbb{R}^{n,n}) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{2n})$ (since, writing $\gamma^+ = \lim_{m\to\infty} \gamma^m \in \partial\Gamma$, $\xi_+^{(n)}(\gamma^+)$ is the attracting fixed point for $\varrho(\gamma) \curvearrowright \mathrm{Gr}_n(\mathbb{R}^{2n})$.) Hence the [GGKW17b] characterization of Anosov reps in terms of limit maps + regular growth of Lyapunov projections (Definition / Theorem 9.3.1), we are done, $\square$

9.4.7 Ruling out Anosovness via lots of transversality

But nonesuch exist:

Theorem 9.4.15 ([DZ18], Theorem 1.3)**.** *If $\varrho : \pi_1 S \to \mathrm{PSO}(n,n)$ is a $\mathrm{PSO}(n,n)$-Hitchin representation, then $\iota_{2n} \circ \varrho : \pi_1 S \to \mathrm{PSL}(2n,\mathbb{R})$ is not P_n-Anosov.*

The proof here (or really, of the key Lemma below) uses Anosovness, Fock–Goncharov positivity: (although [DZ18] remark that you can replace this with Labourie's condition (H) here—basically, you need a little more transversality than the Anosov condition along gives you ...)

Proof of Theorem 1.3. Suppose $\iota_{2n} \circ \varrho$ is P_n-Anosov. ϱ, being Hitchin, is Anosov w.r.t. the Borel subgroup $B' < \mathrm{PSO}(n,n)$. This, combined with P_n-Anosov, implies that $\iota_{2n} \circ \varrho$ is B-Anosov in $\mathrm{PSL}(2n,\mathbb{R})$. Furthermore, the Anosov limit map are "preserved under the inclusion", with a consistent choice of sign (say, +) for the isotropic part.

If n is odd, transversality will fail for the isotropic part (to see: Remark 2.8), and we get a contradiction.

If n is even: use the following key

Lemma 9.4.16 ([DZ18], Lemma 5.1)**.** *Given a Hitchin representation $\rho : \Gamma \to \mathrm{PSO}(n,n)$, the subset $\xi^{(n-1)}(\partial\Gamma) \subset \mathrm{Gr}_{n-1}(\mathbb{R}^{n,n})$ is a differentiable submanifold that is everywhere tangent to the fibers of the natural projection $\bar{\omega}^+ : \mathrm{Gr}_{n-1}(\mathbb{R}^{n,n}) \to \mathrm{Gr}_n^+(\mathbb{R}^{n,n})$, and therefore contained in a single fiber.*

But now $\xi^{(n-1)}(\partial\Gamma)$ contained in a single fiber $\implies \xi^{(n)}(x_1) = \xi_+^{(n)}(x_1) = \xi_+^{(n)}(x_2) = \xi^{(n)}(x_2)$, which contradicts injectivity of $\xi^{(n)}$. Modulo the proof of the Lemma (below), we are done. ☐

Proof of Lemma 5.1. Consider the affine chart of $\mathrm{Gr}_{n-1}(\mathbb{R}^{2n})$ consisting of all $(n-1)$-planes $\pitchfork$ $\xi^{(n+1)}(y)$. We get local coordinates given by

$$\mathrm{Hom}\left(\xi^{(n-1)}(x), \xi^{(n+1)}(y)\right) = \bigoplus_{1\le i<n\le j\le 2n} \mathrm{Hom}(L_i, L_j)(x,y).$$

For each $y < z \le x < y$ in $\partial\Gamma$, $\xi^{(n-1)}(z)$ is in this affine chart; write in coordinates as $(u_{ij}(y,z,x))_{1\le i<n\le j\le 2n}$. Fock–Goncharov positivity (or Labourie's Property (H) with $k = n-2$?) yields the strengthened transversality statement in Proposition 9.3.7, and translating that statement into our coordinates we obtain that $u_{n-1,n}(y,z,x) \neq 0$ for all $y < z < x < y$ in $\partial\Gamma$ ([DZ18], Lemma 5.2.) Since the $u_{i,j}$ are ρ-equivariant, by compactness of T^1S, there exists $C > 0$ such that

$$\|u_{i,j}(y,z,x)\|_{(y,z,x)} \le C$$

for all $i \le i < n \le j \le 2n$, and in addition $\frac{1}{C} \le \|u_{n-1,n}(y,z,x)\|_{(y,z,x)}$ ([DZ18], Lemma 5.3.) Now write the tangent cone C_x at x to the curve $\xi^{(n-1)}(\partial\Gamma)$ in linear coordinates on our affine patch. This part is slightly mysterious to me. But once you've done it, we can use Anosov dynamics (and equivariance) to show that all of the coordinates of things in C_x are zero, except possibly for the $(n-1,n)$ coordinate ([DZ18], Lemma 5.4.). Hence C_x is contained in the line corresponding to $\mathrm{Hom}(L_{n-1}, L_n)(x,y)$ in our coordinates. A quick argument (the cocycle condition, + [DZ18], Lemma 5.2) shows that it is in fact the whole line ([DZ18], Lemma 5.5.). Hence $\xi^{(n-1)}(\partial\Gamma)$ is a differentiable submanifold of dimension one (although the parametrization by $\xi^{(n-1)}$ is not necessarily C^1!). Working in affine charts for $\mathrm{Gr}_{n-1}(\mathbb{R}^{2n})$, the tangent space to the fiber above $\xi^{(n-1)}(x)$ of the projection under consideration is $\mathrm{Hom}(\xi^{(n-1)}(x), L_n(x,y)) \supset \mathrm{Hom}(L_{n-1}, L_n)(x,y)$. Hence our limit curve is tangent to the fiber at an arbitrary point x, i.e. everywhere, as desired. ☐

Recap of argument!

(1) If we have a proper affine action with a Hitchin linear part, we may as well assume ($d = 2n - 1$ is odd and) the linear part is $\mathrm{SO}(n, n-1)$-Hitchin.

(2) Produce a deformation path of $\mathrm{PSO}(n, n)$-Hitchin representations

(3) Using the proper affine action, show that (possibly after passing to a shorter subpath) these are also P_n-Anosov.

(4) By messing around with limit maps, show that $\mathrm{PSO}(n, n)$-Hitchin representations can never be P_n-Anosov: contradiction.

9.4.8 What about the other components?

For $d \geq 3$, $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{PSL}_d\mathbb{R})$ has 3 or 6 components (depending on whether d is odd or even; in the even case pairs of them are identified by reversing orientation.) One (or two, resp.) of these are Hitchin; representations in the other are "very different and still quite mysterious" ...

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