Applications of Affine Differential Geometry to RP(2) Surfaces

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John Churchill Loftin

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In this thesis, we use a fundamental result of Cheng and Yau in affine differential geometry to study $\mathbb{RP}^n$ manifolds. In the 1970s, Cheng and Yau completely described hyperbolic affine spheres in $\mathbb{R}^{n+1}$. All such hypersurfaces are asymptotic to the boundary of a sharp convex cone, and to each such cone there exists (up to scaling) only one affine sphere asymptotic to the boundary. The affine sphere is invariant by any unimodular linear automorphism of the cone, and thus the structure descends to the quotient. By projectivizing, we find a canonical Riemannian metric and two canonical projectively flat connections on any properly convex $\mathbb{RP}^n$ manifold. One connection represents the given $\mathbb{RP}^n$ structure and the other the $\mathbb{RP}^n$ structure of the projective dual manifold.

The main application of this approach comes when $n = 2$. In this case, we use a description due to C.-P. Wang of hyperbolic affine spheres which cover compact surfaces to prove that a compact oriented convex $\mathbb{RP}^2$ surface of genus $g > 1$ is equivalent to a conformal structure on the surface and a holomorphic section of K^3. This recovers a theorem of Goldman on the deformation space of such surfaces, and we study the induced structure on the deformation space.

Contents

To my first mathematics teacher,
my father, John G. Loftin.

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1. Introduction

Affine differential geometry and the study of $\mathbb{RP}^n$ structures on manifolds are two fields in geometry each with a long and rich history. In short, affine geometry is the study of those properties of hypersurfaces in $\mathbb{R}^{n+1}$ which are invariant under the unimodular affine group generated by $\mathbf{SL}(n+1,\mathbb{R})$ and translations. The field was very active in the early part of this century with Blaschke's monograph on the subject [3], and subsequently, mathematicians such as Calabi, Cheng, and Yau have made important contributions.

An $\mathbb{RP}^n$ structure on an n-manifold, on the other hand, is a system of coordinate charts in $\mathbb{RP}^n$ glued together by transition maps in $\mathbf{PGL}(n+1,\mathbb{R})$. Ehresmann studied these in the 1930s. The study of $\mathbb{RP}^2$ structures has been particularly strong. Kuiper, Benzécri, Kobayashi, and Thurston have all done important work in the field. Recently, the field has been quite active, led by Goldman and Choi.

The connection between the two fields is this: For a large and important class of manifolds M with $\mathbb{RP}^n$ structure, the convex ones, $M = \Omega/\Gamma$, with Ω a convex domain in some $\mathbb{R}^n \subset \mathbb{RP}^n$ and $\Gamma \subset \mathbf{PGL}(n+1,\mathbb{R})$ an appropriate subgroup. Cheng and Yau prove that any bounded convex Ω uniquely determines a special hypersurface called a hyperbolic affine sphere which is asymptotic to the cone over Ω in $\mathbb{R}^{n+1}$. (For example, if Ω is a disk, then the affine sphere is just a hyperboloid and we recover the familiar hyperbolic structure on the disk.) The $\mathbf{SL}(n+1,\mathbb{R})$ invariance the affine geometry provides upstairs means that, under the natural map $\mathbb{R}^{n+1}\setminus 0 \to \mathbb{RP}^n$, we get a lot of canonical structure on Ω that is invariant under $\mathbf{PGL}(n+1,\mathbb{R})$. Therefore, all of it descends to our convex $\mathbb{RP}^n$ manifold M. (We should remark that in [14] Darvishzadeh-Goldman use other invariant structures on the cone and study a different hypersurface to investigate $\mathbb{RP}^2$ surfaces.)

Remark. The PDE required to produce a hyperbolic affine sphere is a Monge-Ampère equation similar to the one used in complex geometry to construct a Kähler-Einstein metric of negative scalar curvature. The affine sphere formulation plays a similar role in the study of manifolds with convex $\mathbb{RP}^n$ structures as Kähler-Einstein metrics do in the study of Kähler manifolds with $c_1 < 0$.

Theorem 1. *The data of a compact Riemann surface of genus $g > 1$, together with a holomorphic section of K^3, the tricanonical bundle, is equivalent to the data of an oriented, compact surface with convex $\mathbb{RP}^2$ structure.*

Hence, the deformation space of convex $\mathbb{RP}^2$ structures on the surface (which we refer to as *Goldman space*) has the structure of a holomorphic $5g-5$ dimensional vector bundle over Teichmüller space (which of course has complex dimension $3g-3$). We thereby recover

Corollary 1.0.1 (Goldman [19]). *The deformation space of convex $\mathbb{RP}^2$ structures on an oriented compact surface of genus $g > 1$ is topologically a real $16g-16$ dimensional ball.*

C.P. Wang shows that an $\mathbf{SL}(3,\mathbb{R})$ invariant called the Pick form on a 2-dimensional hyperbolic affine sphere is equivalent to a holomorphic section of K^3, with the conformal structure given by a natural invariant metric. Using this fact, together with a classification theorem of Wang on hyperbolic affine spheres which cover a surface of genus $g > 1$ in an appropriate way, we get Theorem 1.

In particular, when the Pick form vanishes, the $\mathbf{RP}^2$ structure is the one descending from the hyperbolic disk Δ. (The Klein model of the hyperbolic metric on Δ shows that the hyperbolic isometries of Δ are given exactly by the action of those elements of $\mathbf{PGL}(3,\mathbb{R})$ which preserve the set $\Delta \subset \mathbb{RP}^2$.) Therefore, the uniformization theorem shows that the structure preserved is exactly the conformal structure of the Riemann surface Σ. From this, we see that Teichmüller space naturally sits in Goldman space as the zero section of the vector bundle with fiber $H^0(\Sigma, K^3)$.

Also, using result on the conormal map of affine spheres, we find that by replacing a section U of K^3 by $-U$, we recover the projective dual surface. This recovers the fact that Teichmüller space is exactly the fixed locus of the action of projective duality on Goldman space.

Another approach is given by Hitchin in [22]. By means of stable Higgs bundles on a Riemann surface Σ, he studies the connected components of the space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ for a simple Lie group G split over $\mathbb{R}$. An $\mathbb{RP}^2$ structure on an oriented surface induces a holonomy representation into $G = \mathbf{PSL}(3,\mathbb{R})$. Using this fact, Goldman and Choi [12] prove that $\mathcal{G}(S)$ coincides with one component of this representation space, which Hitchin had shown is parametrized by the space of sections $H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^3)$. This gives another complex structure on $\mathcal{G}(S)$, but again it is not clear how it is related to the one we find in Theorem 1. In particular, Hitchin's construction depends on an a priori choice of conformal structure on Σ, while in the construction used in Theorem 1, the $\mathbb{RP}^2$ structure determines a metric and therefore a conformal structure on Σ.

2. $\mathbb{RP}^n$ STRUCTURES

$\mathbb{RP}^n$ is defined as the space of all lines passing through 0 in $\mathbb{R}^{n+1}$. There is a natural map from $\mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ given by $p \mapsto \ell$, where ℓ is the unique line through p. We also use the notation $[p]$ to denote this line. The linear automorphisms of $\mathbb{RP}^n$ are given by the group $\mathbf{PGL}(n+1,\mathbb{R})$, which is equivalence classes of matrices $A \in \mathbf{GL}(n+1,\mathbb{R})$ with $A \sim \lambda A$ for real constants λ.

We say that an n-dimensional manifold M has an $\mathbb{RP}^n$ *structure* if it admits coordinate charts represented by open sets in $\mathbb{RP}^n$ and the transition maps between these coordinate charts are given by maps in $\mathbf{PGL}(n+1,\mathbb{R})$. We also say M is an $\mathbb{RP}^n$ *manifold.* A path in M is called a *geodesic* with respect to the $\mathbb{RP}^n$ structure if it is a straight line in each of the coordinate charts.

The $\mathbb{RP}^n$ structure on M can clearly be lifted to an $\mathbb{RP}^n$ structure on its universal cover $\tilde{M}$. Then we can define the *developing map* as a local diffeomorphism from $\tilde{M} \to \mathbb{RP}^n$ in the following manner. Any coordinate map for a neighborhood $\mathcal{U}_0$ of $x \in \tilde{M}$ serves to define the developing map $\mathcal{U}_0 \to \mathbb{RP}^n$. For any adjacent coordinate chart $\mathcal{U}$, the transition map ensures that there is a unique way to define a map from $\mathcal{U}$ to $\mathbb{RP}^n$ which agrees on the overlap of $\mathcal{U}$ and $\mathcal{U}_0$. Repeating this process, we define the developing map from $\tilde{M}$ to $\mathbb{RP}^n$. Deck transformations of $\tilde{M}$ are taken to linear automorphisms of $\mathbb{RP}^n$ by the developing map, and so define a holonomy map $\pi_1(M) \to \mathbf{PGL}(n+1,\mathbb{R})$. The developing map is unique up to the action of $\mathbf{PGL}(n+1,\mathbb{R})$.

2.1. Convex $\mathbb{RP}^n$ structures.

An $\mathbb{RP}^n$ manifold is *convex* if its developing map is a diffeomorphism onto a domain Ω convex in some affine $\mathbb{R}^n \subset \mathbb{RP}^n$. In this case, we can realize $M = \Omega/\Gamma$, where Γ is a subgroup of $\mathbf{PGL}(n+1,\mathbb{R})$ which acts discretely and properly discontinously on Ω. M is *properly convex* if Ω is bounded in some such $\mathbb{R}^n$. Below we find a canonical projectively flat connection on a properly convex $\mathbb{RP}^n$ manifold.

2.2. The tautological bundle.

We define $\mathbb{RP}^n$ as the space of all lines ℓ passing through 0 in $\mathbb{R}^{n+1}$. Then the subset of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ consisting of all (p,ℓ) with $p \in \ell$ is the total space for the *tautological line bundle* τ of $\mathbb{RP}^n$. Given an $\mathbb{RP}^n$ manifold M, $\mathrm{dev}^{-1}\tau$ defines the tautological bundle on $\tilde{M}$. We say M *admits a tautological bundle* if this structure descends to M, i.e. if there is a line bundle on M which pulls back to $\tilde{M}$ under the action of π_1 to $\mathrm{dev}^{-1}\tau$. For simplicity, we denote this line bundle as τ also.

The only $\mathbb{RP}^n$ manifolds which we consider in this paper are those with convex structure. In this case, we have

Proposition 2.2.1. *A manifold M with convex $\mathbb{RP}^n$ structure admits an oriented tautological bundle.*

Proof. Since M is convex, we have $M = \Omega/\Gamma$, where $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$ and $\Gamma \subset \mathbf{PGL}(n+1,\mathbb{R})$ is a representation of $\pi_1 M$ which acts discretely and properly discontinously. Introduce inhomogeneous coordinates on $\mathbb{R}^n \supset \Omega$, and consider the cone over Ω in $\mathbb{R}^{n+1}$ defined by

$$C(\Omega) = \{(tx, t) : x \in \Omega \subset \mathbb{R}^n,\, t > 0\}.$$

It is enough to prove that we can lift the action of Γ to a linear action Γ' on $C(\Omega) \subset \mathbb{R}^{n+1}$:

We assume we can lift Γ to Γ'. Then let $C^{\pm}(\Omega) = C(\Omega) \sqcup \{0\} \sqcup -C(\Omega)$ be the union of all lines in Ω. Then Γ' acts on $C^{\pm}(\Omega)$. The total space of the tautological bundle is all pairs

$$\{(p, \ell) \in C^{\pm}(\Omega) \times \Omega : p \in \ell\}$$

modulo the action of $\Gamma' \times \Gamma$. Since Γ' preserves the set $C(\Omega)$, the line bundle is oriented. Sections which take values in $C(\Omega)/\Gamma'$ can be thought of as positive sections.

Now we prove we can lift the action of Γ: Consider the natural map $\pi : \mathbf{SL}(n+1,\mathbb{R}) \to \mathbf{PGL}(n+1,\mathbb{R})$ induced by the projection on $\mathbf{GL}(n+1,\mathbb{R})$. Then there are two cases: If n is even, then π is an isomorphism; if n is odd, then $\mathbf{PGL}(n+1,\mathbb{R})$ has two components and π is a two-to-one map onto the identity component of $\mathbf{PGL}(n+1,\mathbb{R})$. However, we can define

$$\mathbf{SL}^{\pm}(n+1,\mathbb{R}) = \{A \in \mathbf{GL}(n+1,\mathbb{R}) : |\det A| = 1\}.$$

$$\tilde{\pi} \colon \mathbf{SL}^{\pm}(n+1,\mathbb{R}) \to \mathbf{PGL}(n+1,\mathbb{R})$$

is always a two-to-one map with kernel $\{\pm I\}$.

Now consider $\gamma \in \Gamma$ which acts on Ω. Then there are two lifts of γ in $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$. One of them will preserve $C(\Omega)$ and the other will interchange $C(\Omega)$ and $-C(\Omega)$. The former gives a canonical choice of lift which acts on $C(\Omega)$. □

2.3. **Projectively flat connections.** There is an equivalent way of defining $\mathbb{RP}^n$ manifolds in terms of affine and projective connections. A projectively flat connection on TM defines an $\mathbb{RP}^n$ structure on M. The geodesics for the $\mathbb{RP}^n$ structure are the geodesics for ∇. A key element in this correspondence follows from Cartan's theory of normal projective connections. We outline a simplified version which works for connections which are projectively flat. For the general case, see Kobayashi [23] or Hermann [21].

Two connections ∇^1 and ∇^2 on TM are *projectively equivalent* if there is a one-form ρ such that

$$\nabla^1_X Y = \nabla^2_X Y + \rho(X)Y + \rho(Y)X,$$

where X and Y are tangent vector fields. This action of ρ is called a *projective transformation*. An equivalent condition is that the geodesics of ∇^1 and ∇^2 are the same as sets. A connection is *projectively flat* if it is locally projectively equivalent to a flat connection.

Proposition 2.3.1. *Let L be a trivial line bundle over M and ξ be a nonvanishing section of L. A torsion-free connection ∇ on TM is projectively flat if and only if there exists a connection $\tilde{\nabla}$ on the vector bundle $E = TM \oplus L$ such that for some $(0,2)$-tensor β*

$$\begin{cases} \tilde{\nabla}_X Y = \nabla_X Y + \beta(X,Y)\xi \\ \tilde{\nabla}_X \xi = X \end{cases} \tag{2.3.1}$$

and the curvature $\tilde{R}$ of $\tilde{\nabla}$ must satisfy

$$\tilde{R} = \lambda \otimes I, \tag{2.3.2}$$

where λ is a two-form on M and I is the identity in End$(TM \oplus L)$. The connection $\tilde{\nabla}$ satisfying these conditions is unique.

We call this $\tilde{\nabla}$ the *normal connection* associated to ∇.

Sketch of proof. This result is standard. The *curvature tensor* R is defined in the usual way:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

R is an End(TM)-valued two-form. The *Ricci tensor* is defined by

$$\mathrm{Ric}(Y,Z) = \mathrm{tr}\{X \mapsto R(X,Y)Z\}.$$

Note that unlike in the case of Riemannian curvature, Ric is not always symmetric in Y and Z. Define $P(X,Y)$ by

$$P(X,Y) = \tfrac{1}{n^2-1}[n\mathrm{Ric}(X,Y) + \mathrm{Ric}(Y,X)].$$

Then the *Weyl tensor*, which is invariant under projective transformations, is defined by

$$W(X,Y)Z = R(X,Y)Z - P(X,Y)Z + P(Y,X)Z - P(Y,Z)X + P(X,Z)Y.$$

The condition on projective flatness, i.e. the existence of ρ giving a projective transformation to a flat connection, is equivalent to $W = 0$ and

$$(\nabla_X P)(Y,Z) = (\nabla_Y P)(X,Z). \tag{2.3.3}$$

We note that if $n > 2$, then $W = 0$ implies (2.3.3).

On the other hand, compute the $\tilde{R}$ from (2.3.1)

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \beta(Y,Z)X - \beta(X,Z)Y + \gamma(X,Y,Z)\xi,\\ \tilde{R}(X,Y)\xi &= [\beta(X,Y) - \beta(Y,X)]\xi,\end{aligned}$$

where γ is a certain skew-symmetrization of $\nabla\beta$. Using these equations for $\tilde{R}$, we see the curvature constraint (2.3.2) is equivalent to (1) $W = 0$, (2) $\beta = -P$, and (3) $\gamma(X,Y,Z) = -(\nabla_X P)(Y,Z) + (\nabla_Y P)(X,Z) = 0$. Therefore, by (2.3.3) the projective flatness of ∇ is equivalent to the existence of the normal connection, which satisfies the curvature condition (2.3.2). ☐

This proposition allows us to construct an $\mathbb{RP}^n$ structure on a manifold equipped with a projectively flat connection ∇. Consider a base point x in M, and the universal cover $\tilde{M}$. Any two paths from x to y in $\tilde{M}$ induce by $\tilde{\nabla}$-parallel transport linear maps between E_x and E_y. Then (2.3.2) and the Ambrose-Singer holonomy theorem [24] show that these maps are equivalent up to homothety; i.e. they define a projective isomorphism from $P(E_x)$ to $P(E_y)$. Then ξ, as a section of E, defines the developing map from $\tilde{M}$ to $\mathbb{RP}^n$ by $[\xi] \in P(E_y)$ for all $y \in \tilde{M}$. See e.g. Goldman [20] for details.

The geodesics with respect to the $\mathbb{RP}^n$ structure are the same (as sets) as the geodesics of ∇, and in fact an $\mathbb{RP}^n$ structure on a manifold M is equivalent to a projective equivalence class of projectively flat connections on M.

2.4. Special results on $\mathbb{RP}^2$ structures.

In this section we record some results in the special case $n = 2$. See the survey article of Choi and Goldman for a good overview of this material [13]. Goldman also has notes [18] which are useful.

In the applications below, we will be mainly interested in the theory of compact oriented $\mathbb{RP}^2$ surfaces of genus $g \geq 2$. We begin with a fundamental result on such surfaces which are convex, which is due to Kuiper [25] and Benzécri [2]:

Proposition 2.4.1. *If S is a compact, oriented $\mathbb{RP}^2$ surface of genus $g \geq 2$, it must be properly convex. Furthermore, the boundary $\partial\Omega$ is always strictly convex and C^1, and must be either an ellipse or a Jordan curve which is nowhere C^2.*

Another fundamental result is the decomposition theorem of Goldman [19]. For any surface S as above, we can find a set of disjoint closed $\mathbb{RP}^2$-geodesics which cut the surface into many pairs of pants. Goldman explicitly determines real parameters needed to describe each pair of pants as an $\mathbb{RP}^2$ surface and the gluing data needed to put them back together to form S. This analog of Fenchel-Nielsen coordinates provides global coordinates on Goldman space and provides Goldman's original proof of Corollary 1.0.1.

We should also mention the work of Choi [9, 10, 11] that any compact $\mathbb{RP}^2$ surface with $\chi(S) < 0$ can be decomposed in a canonical way into convex pieces. Together with Goldman's coordinates, Choi's results extend the preceding proposition to every component of the space of $\mathbb{RP}^2$ structures on S, not just the one corresponding to convex surfaces.

2.5. **A symplectic form on Goldman space.** There is a natural symplectic form on Goldman space, which is described using the language of affine connections [20]. Given an $\mathbb{RP}^2$ structure on a compact surface S, we can always find a projectively flat, torsion-free connection on the tangent bundle TS whose geodesics are the $\mathbb{RP}^2$ geodesics. Any two such connections which are projectively equivalent give rise to the same $\mathbb{RP}^2$ surface. In order to study the deformation problem, we consider any two connections to be equivalent if they are related by an action of $\mathrm{Diff}_0(S)$, the identity component of the diffeomorphism group of S. The deformation space of $\mathbb{RP}^2$ structures on S is the space of all projectively flat, torsion-free connections on S, modulo projective equivalence and the action of $\mathrm{Diff}_0(S)$.

(Below we describe Goldman space slightly differently. For a given convex $\mathbb{RP}^2$ structure on S, we find a canonical metric on S, which gives it the structure of a Riemann surface. Also, the $\mathbb{RP}^2$ structure provides the extra data of a holomorphic section of K^3, and we are able to find within the projective equivalence class a canonical torsion-free projectively flat connection corresponding to the $\mathbb{RP}^2$ structure. Conversely, given a conformal structure and a section of K^3, we can construct the canonical projectively flat connection which determines the convex $\mathbb{RP}^2$ structure. Of course in this picture, the action of Diff_0 is taken care of by the Teichmüller theory of Riemann surfaces.)

Now we briefly describe Goldman's construction of the symplectic form on Goldman space. Let Conn be the affine space of all connections on TS. First consider the following natural symplectic structure on Conn:

$$\omega(\sigma_1, \sigma_2) = \int_S \tfrac{1}{3} \operatorname{tr} \sigma_1 \wedge \operatorname{tr} \sigma_2 - \operatorname{tr}(\sigma_1 \wedge \sigma_2). \tag{2.5.1}$$

(This differs in sign from the form Goldman defines.) Here the σ_i are in the tangent space to Conn, i.e. we view them as sections of $T^*(S) \otimes \mathrm{End}(TS)$. The traces are over the endomorphism part and the wedge products over the one-form part.

Now we perform two successive symplectic reductions. First the torsion tensor is used to define a map from Conn to $\Omega^1(S)^*$ which is an equivariant moment map for the action of $\Omega^1(S)$ on Conn by projective equivalance. Thus we have a symplectic structure on Conn_0, the space of torsion-free connections modulo projective equivalence.

There is a notion of projective curvature tensor which is used to define a map from Conn_0 to $\Omega^1(S)$. This in turn is an equivariant moment map for the natural action of Diff_0 on Conn_0. Therefore, the symplectic form above defines a symplectic structure on Goldman space. (This also works for other components of the space of $\mathbb{RP}^2$ structures on S.)

3. Affine differential geometry

The material in Subsections 3.1, 3.2, and 3.3 is standard. Most of it can be readily found in the book of Nomizu-Sasaki [30]. Other good sources are the papers of Calabi [4, 5] and Cheng-Yau [8], and Blaschke's monograph [3].

Affine differential geometry is concerned with those properties of hypersurfaces in $\mathbb{R}^{n+1}$ which remain invariant under the *unimodular affine group* consisting of affine transformations $x \mapsto Ax + b$ with $\det A = 1$. While much of the formal theory works for any hypersurface which is nondegenerate (i.e. one which can be locally written as the graph of a function with nondegenerate Hessian), we only consider hypersurfaces which are strictly convex. Only in this case is the affine metric a Riemannian metric.

Given a hypersurface immersion $f : H \to \mathbb{R}^{n+1}$, consider a transversal vector field ξ on H. We have the equations:

$$\left\{ \begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + h(X,Y)\xi, \\ D_X \xi &= -f_*(SX) + \tau(X)\xi. \end{aligned} \right. \tag{3.0.1}$$

X and Y are tangent vectors, D is the canonical flat connection induced from $\mathbb{R}^{n+1}$, ∇ is a torsion-free connection, h is a symmetric form on $T_x(H)$, S is an endomorphism of $T_x(M)$, and τ is a one-form. (When it is not confusing, we will drop the f_* and just consider X and Y as vectors in $\mathbb{R}^{n+1}$.)

A good choice of transversal field ξ allows us to study the geometry of $f(M)$. For example, the standard metric on $\mathbb{R}^{n+1}$ allows us to define a normal vector field (at least if H is oriented). If ξ is this normal field, (3.0.1) becomes the familiar Gauss equation in Riemannian geometry. ∇ in this case is the Levi-Civita connection on H with respect to the induced metric and h is the second fundamental form. This choice of ξ, since it respects the metric on $\mathbb{R}^{n+1}$, induces a lot of structure on

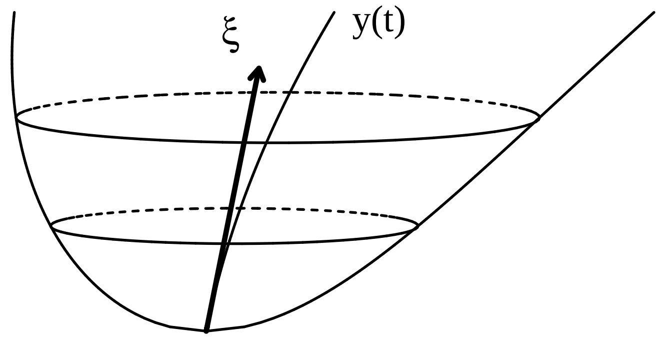

FIGURE 1. The Affine Normal

H which is invariant under the group of transformations $x \mapsto Ax + b$ with $A \in \mathbf{SO}(n+1, \mathbb{R})$. In our case, we want to study properties invariant under the much larger unimodular affine group. Clearly this Riemannian normal field is not invariant under this group, but it turns out that there is a transversal vector field which is invariant, the affine normal.

3.1. **The affine normal.** The *affine normal* is a transversal vector field on H which does remain invariant under the unimodular affine group. Perhaps the easiest way to describe it is the following geometric characterization due to Blaschke [3]. At a point $x \in H$, consider hyperplanes $P(t) \in \mathbb{R}^{n+1}$ displaced a distance t from and parallel to $T_x(M)$. Since we assume M is locally strictly convex, for $t > 0$, $P(t) \cap H$ is the boundary of a convex domain $D(t) \subset P(t)$. Let $y(t) \in D(t) \subset \mathbb{R}^{n+1}$ be the center of gravity of $D(t)$. Define

$$s(t) = \left(\mathrm{Vol} \bigcup_{0<\tau<t} D(\tau) \right)^{\frac{2}{n+2}}, \qquad c_n = 2\left(\frac{V_n}{n+2} \right)^{\frac{2}{n+2}},$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$. The exponent in the definition of s makes s approximately linear as a funtion of t, and c_n is a volume-normalizing factor. Then the affine normal is defined to be $\xi = c_n \left.\frac{dy}{ds}\right|_{s=0}$. Notice that the affine normal points to the convex side of H. See Figure 1.

It is clear that this definition is invariant under the affine unimodular group, because this group preserves volumes. In fact it is invariant under the larger group given by all transformations $x \mapsto Ax + b$ with $A \in \mathbf{SL}^{\pm}(n+1, \mathbb{R})$.

3.2. **Three connections.** For the affine normal ξ, the structure equations for H (3.0.1) become

$$\begin{cases} D_X Y = \nabla_X Y + h(X,Y)\xi, \\ D_X \xi = -SX. \end{cases} \tag{3.2.1}$$

The connection ∇ is called the *Blaschke connection*, or simply the *affine connection*. The bilinear form h is the *affine metric* and the endomorphism S is called the *affine shape operator*. Since H is strictly convex, h is a Riemannian metric on M. We can then also consider $\hat\nabla$ the Levi-Civita connection with respect to h. It is also useful to consider the *conjugate connection* $\bar\nabla$, which is defined to be the connection $2\hat\nabla - \nabla$. As we discuss below in Subsection 3.5, $\bar\nabla$ is always projectively flat.

There is another characterization of the affine normal which will be useful below (see Nomizu-Sasaki [30, p. 45]). If ξ is a transverse vector field for which the structure equations (3.2.1) are satisfied for some ∇, h, and S (this just means that in (3.0.1), the one-form τ vanishes), then ξ must be (up to a choice of sign) the affine normal if and only if the following volume condition is satisfied:

$$\det(X_1, \dots, X_n, \xi) = \pm 1, \tag{3.2.2}$$

with $\{X_i\}$ an orthonormal basis with respect to the metric h. We choose the sign of ξ so that it points to the convex side of H.

Another important invariant is the Pick form. Consider the tensor $\hat\nabla - \nabla$, which is a section of $T \otimes T^* \otimes T^*$, and use the metric h to transform it to a section C of $T^* \otimes T^* \otimes T^*$ (i.e. lower the index). Then C is a totally symmetric form on three indices. Also, we have the *apolarity condition*

$$\sum_i C^i_{ij} = 0 \quad \text{for all} \quad j. \tag{3.2.3}$$

In addition, if C vanishes identically on H, then H must be an open subset of a hyperquadric in $\mathbb{R}^{n+1}$. Hyperquadrics are in a sense the trivial objects in affine differential geometry. There are three strictly convex cases—an ellipsoid, an elliptic paraboloid and one sheet of an elliptic hyperboloid. The affine metrics on these examples have constant curvature, which is respectively positive, zero, and negative.

We have the following curvature formula for the Blaschke connection ∇:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY. \tag{3.2.4}$$

Also, the *affine mean curvature* is defined to be the quantity $\frac{1}{n}\mathrm{tr}S$. Another useful equation is the Codazzi equation for h:

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{3.2.5}$$

3.3. **Affine spheres.** An affine sphere is a hypersurface in $\mathbb{R}^{n+1}$ all of whose affine normals point toward a given point in $\mathbb{RP}^{n+1}$, the *center* of the affine sphere. If the center lies on the convex side of H, the affine sphere is *elliptic*. If it lies on the line at infinity, it is called *parabolic*. If the center lies on the concave side of H, then the affine sphere is *hyperbolic*. We will be concerned exclusively with this case.

For an affine sphere, the shape operator satisfies $S = LI$, where the affine mean curvature L is a constant function on H and I is the identity map. L is positive, zero, or negative if H is elliptic, parabolic, or hyperbolic respectively. The center is given by the formula $x+\frac{1}{L}\xi(x)$, where x is any point in H. If H is an elliptic or hyperbolic affine sphere, upon scaling $\mathbb{R}^{n+1}$ away from the center by a constant factor λ, the image of H after this transform remains an affine sphere, but with a different value of affine mean curvature L.

Thus by scaling, we can normalize any hyperbolic affine sphere to have $L = -1$. Also, we can translate so that the center is 0. Then the affine normal $\xi = f$, where f is the embedding of H into $\mathbb{R}^{n+1}$. The structure equations (3.2.1) and the curvature equation (3.2.4) then become

$$\begin{cases} D_X Y = \nabla_X Y + h(X,Y)f \\ D_X f = X \\ R(X,Y)Z = -h(Y,Z)X + h(X,Z)Y \end{cases} \tag{3.3.1}$$

This implies the Ricci curvature of ∇ is given by

$$\mathrm{Ric}(X,Y) = (-n+1)h(X,Y) \tag{3.3.2}$$

The basic examples of affine spheres are the quadratic hypersurfaces introduced above. In fact, for the elliptic and parabolic cases, these are the only examples of affine spheres which are closed subsets of $\mathbb{R}^{n+1}$ [8].

Hyperbolic affine spheres are more complicated, however. The basic result for hyperbolic affine spheres was conjectured by Calabi [4] and proved by Cheng-Yau [7, 8], and Calabi-Nirenberg (with clarifications by Gigena [16] and A.-M. Li [26, 27]):

Theorem 2. *Given a constant $L < 0$ and a convex, bounded domain $\Omega \subset \mathbb{R}^n$, there is a unique properly embedded hyperbolic affine sphere $H \subset \mathbb{R}^{n+1}$ of affine mean curvature L and center 0 asymptotic to the* *boundary of the cone* $\{t\Omega : t > 0\} \subset \mathbb{R}^{n+1}$*. For any immersed hyperbolic affine sphere* $H \to \mathbb{R}^{n+1}$*, properness of the immersion is equivalent to completeness of the affine metric, and any such* H *is a properly embedded hypersurface asymptotic to boundary of the cone given by the convex hull of* H *and its center.*

Outline of proof. By scaling, we may consider just the case $L = -1$.

We first introduce a few facts about the conormal map. See Subsection 3.5 below for more details. Consider a hypersurface H in $\mathbb{R}^{n+1}$ whose position vector f is transverse to H. The conormal map ν maps such a hypersurface H in $\mathbb{R}^{n+1}$ to a dual hypersurface $\bar{H}$ in in the dual vector space $\mathbb{R}_{n+1}$. Define $\nu(x)$ by

$$\nu(X) = 0 \quad \text{for} \quad X \in T_x(H), \quad \nu(f) = 1. \tag{3.3.3}$$

Choose coordinates in $\mathbb{R}^{n+1}$. If H is the (rectilinear) graph of a strictly convex function α, i.e. if

$$H = \{(x, \alpha(x)) : x \in \mathcal{D} \subset \mathbb{R}^n\},$$

and if, with respect to the dual coordinates in $\mathbb{R}_{n+1}$, we write $\bar{H}$ as a radial graph

$$\bar{H} = \{(y\,\zeta(y), \zeta(y)) : y \in \mathcal{E} \subset \mathbb{R}_n\},$$

then $w = -\frac{1}{\zeta}$ is the *Legendre transform* of α.

It is an observation of Calabi [4] that H as above is a hyperbolic affine sphere with center 0 and affine mean curvature -1 if and only if the Legendre transform w satisfies the Monge-Ampère equation

$$\det(w_{ij}) = \left(-\frac{1}{w}\right)^{n+2}.$$

In addition, if H is asymptotic to the boundary of a cone over a region Ω, then w is defined in the projective dual region $\bar{\Omega}$ and approaches 0 at the boundary $\partial\bar{\Omega}$.

In [7] Cheng-Yau prove that for each bounded, convex $\bar{\Omega} \subset \mathbb{R}_n$, the Dirichlet problem

$$\det(w_{ij}) = \left(-\frac{1}{w}\right)^{n+2}, \quad w|_{\partial\Omega} = 0 \tag{3.3.4}$$

has a unique C^∞ convex solution which is continuous to the boundary. (See also Loewner-Nirenberg [28] for earlier work.) The affine metric of the graph of the Legendre transform of w is then $-\frac{1}{w}w_{ij}$ in these coordinates.

We must then assure that, after we take the Legendre transform, H is properly embedded in $\mathbb{R}^{n+1}$. A proof of this fact, using Proposition 3.5.1 below, is found in Gigena's paper [16]. (This proposition is found in the earlier work of Schirokov-Schirokov [31] and was known to experts in the 1970s, when Cheng and Yau completed their work.) In fact, if we solve (3.3.4) on Ω instead of on $\bar{\Omega}$, then the radial graph of $\zeta = -\frac{1}{w}$ is the affine sphere H.

Cheng-Yau in [8] prove (with a small gap) that an affine sphere is properly embedded if and only if its affine metric is complete. Moreover, any such hyperbolic affine sphere H must be asymptotic to the boundary of the cone given by the convex hull of H and its center. (Calabi-Nirenberg, in unpublished work, establish the same results.) In [26, 27], A.-M. Li clarified the proof of Cheng-Yau by using essentially the same estimates developed in [8]. $\square$

Also, we have the following proposition that will be useful later.

Proposition 3.3.1. *The Blaschke connection of a hyperbolic affine sphere is projectively flat. Moreover, if the affine mean curvature is -1, the normal connection associated to ∇ can be realized as the canonical flat connection D induced from $\mathbb{R}^{n+1}$.*

Proof. The curvature equations (3.3.1) and (3.3.2), together with the Codazzi equation (3.2.5), when applied to the characterization of the normal connection in Subsection 2.3, provide the proof.

We also provide a more geometric proof of the first part (see Nomizu-Sasaki [30, pp. 15-18]). Assume for simplicity that the affine mean curvature is -1 and the center is 0. Consider a positive smooth function λ on H and the new hypersurface $\check{H} = \lambda H \subset \mathbb{R}^{n+1}$. Then f is still a transversal vector field on $\check{H}$. Form the connection $\check{\nabla}$ by

$$D_X Y = \check{\nabla}_X Y + \check{h}(X,Y) f. \tag{3.3.5}$$

A simple computation shows

$$\check{\nabla}_X Y = \nabla_X Y + \rho(X) Y + \rho(Y) X, \quad \rho = d(\log \lambda).$$

Now we can take $\check{H}$ to be the intersection of a hyperplane with the cone to which H is asymptotic. Then (3.3.5) implies $\check{\nabla}$ is flat. Furthermore, the ∇-geodesics on H map to straight lines on $\check{H}$ under the map along rays from M to 0. See Figure 2. $\square$

3.4. **Quotients of hyperbolic affine spheres.** Let M be an $\mathbb{RP}^n$ manifold with oriented tautological bundle τ. Then the total space of the positive part of τ is locally a cone in $\mathbb{R}^{n+1}$, and, as in Proposition 2.2.1, the gluing maps from M lift to gluing maps in $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$ to glue these cones together to form the positive part of the total space of τ. We say M *admits an affine sphere structure* if there is a positive section s of τ so that for each coordinate chart $\mathcal{U}$ of M, $s(\mathcal{U})$ is a

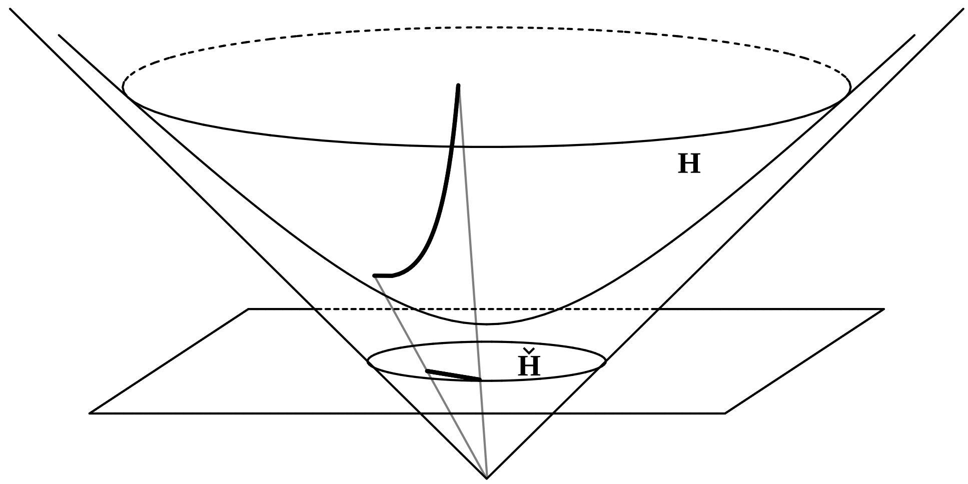

FIGURE 2

hyperbolic affine sphere with center 0 and affine mean curvature -1 in this cone.

Now we show that any properly convex $\mathbb{RP}^n$ manifold M admits an affine sphere structure. Write $M = \Omega/\Gamma$. Proposition 2.2.1 shows that M admits an oriented tautological bundle, and the action of Γ lifts to an action $\Gamma' \subset \mathbf{SL}^{\pm}(n+1,\mathbb{R})$ on the cone $C(\Omega)$. Theorem 2 shows there is a unique affine sphere asymptotic to the boundary of $C(\Omega)$ with center 0 and affine mean curvature -1. By uniqueness and the invariance of the affine normal under $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$, this structure descends to the quotient, and M must admit an affine sphere structure. We record this result and related facts in the following theorem.

Theorem 3. *Let M be an $\mathbb{RP}^n$ manifold with oriented tautological bundle τ. Let τ^* denote the dual line bundle. The following are equivalent:*

1. *M is properly convex.*
2. *M admits a negative strictly convex section w of τ^* satisfying $\det(w_{ij}) = (-\frac{1}{w})^{n+2}$ so that the metric $-\frac{w_{ij}}{w}$ is complete.*
3. *M admits an affine sphere structure whose metric is complete.*

If any of these conditions are satisfied, then the $\mathbb{RP}^n$ structure on M is given by the Blaschke connection ∇. Also, the normal connection of ∇ is exactly the flat connection D on $TM \oplus \tau$ which is induced by the canonical flat connection on $\mathbb{R}^{n+1}$.

Proof. $2 \Leftrightarrow 3$ follows from the outline of the proof of Theorem 2, and $1 \Rightarrow 3$ is proved in the preceeding paragraphs.

We now prove $3 \Rightarrow 1$. An affine sphere structure on M induces one on the universal cover $\tilde{M}$. This is pushed down by the developing map to an affine sphere structure on a spread domain over $\mathbb{RP}^n$, which is equivalent, by looking at the total space of τ, to an immersed affine sphere in $\mathbb{R}^{n+1}$ with complete affine metric. Theorem 2 shows that this is asymptotic to a cone and we have $3 \Rightarrow 1$. □

3.5. **The conormal map.** Recall the definition of the conormal map. Let $H \subset \mathbb{R}^{n+1}$ be a hypersurface transverse to its position vector f. Define $\nu(x) \in \mathbb{R}_{n+1}$ by $\nu(x)(X) = 0$ if $X \in T_x(H)$ and $\nu(x)(f) = 1$. (This construction can be done with any transverse vector field ξ in place of f.)

For any hypersurface $H \subset \mathbb{R}^{n+1}$ with f transverse to H, the image of the conormal map ν, is the dual hypersurface $\bar{H} \subset \mathbb{R}_{n+1}$. We have the following proposition due to Schirokov-Schirokov [31]. (There are also unpublished work of Calabi and the papers of Gigena [15, 16]).

Proposition 3.5.1. *The image of the conormal map of a hyperbolic affine sphere H with center 0 and affine mean curvature -1 is another such hyperbolic affine sphere $\bar{H}$ in the dual space $\mathbb{R}_{n+1}$.*

We call $\bar{H}$ the *dual sphere* of H.

Notice that in this case we have $f = \xi$ the affine normal. For conormal maps with respect to the affine normal, we have [30, p. 57]

$$D_X(\nu_* Y) = \nu_*(\bar{\nabla}_X Y) - h(SX, Y)\nu.$$

This shows, as in Proposition 3.3.1 above, that $\bar{\nabla}$ is always projectively flat. In our case, this equation becomes

$$D_X(\nu_* Y) = \nu_*(\bar{\nabla}_X Y) + h(X, Y)\nu.$$

Therefore, we have the following corollary:

Corollary 3.5.2. *The conormal map ν on H as above is an isometry with respect to the affine metrics. It takes the conjugate connection of H to the Blaschke connection of $\bar{H}$ and vice versa.*

Now consider the cone C formed by the convex hull of H and 0. Consider the *dual cone* $\bar{C} \subset \mathbb{R}_{n+1}$ consisting of all linear functionals y which are positive on C. Then $\bar{H}$ is asymptotic to the boundary of $\bar{C}$. Let C be the space of all rays in some $\Omega \subset \mathbb{RP}^n$. Similarly, projectivize $\bar{C}$ so it is the space of all rays in $\bar{\Omega} \subset \mathbb{RP}_n$. ($\mathbb{RP}_n$, the space of all lines in $\mathbb{R}_{n+1}$, is the *dual projective space*. $\bar{\Omega}$ is called the *projective dual region* to Ω.) Projecting along rays identifies H to Ω, and $\bar{H}$ to $\bar{\Omega}$. Therefore, the conormal map ν induces a map from Ω to $\bar{\Omega}$, which we also refer to as ν.

Now if $\Gamma \subset \mathbf{PGL}(n+1,\mathbb{R})$ acts on Ω, then we have a dual action on $\bar{\Omega}$: First lift the action of Γ to $\Gamma' \subset \mathbf{SL}^{\pm}(n+1,\mathbb{R})$ acting on the cone $C(\Omega)$ as in Proposition 2.2.1 above. Then for $y \in \mathbb{R}_{n+1}$ and $x \in \mathbb{R}^{n+1}$, define

$$(A \cdot y)(x) = y(A^{-1}x)$$

for $A \in \Gamma'$. This induces a projective action of Γ on $\bar{\Omega}$, which we denote by $\bar{\Gamma}$. The uniqueness of the affine sphere, the invariance of the affine normal under $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$, and the definition of ν then clearly show that ν is equivariant with respect to the action of Γ. Therefore, ν descends the the quotient and we have

Proposition 3.5.3. *Given a properly convex $\mathbb{RP}^n$ manifold $M = \Omega/\Gamma$, the conormal map ν with respect to the affine sphere structure induces a map to the dual manifold $\bar{M} = \bar{\Omega}/\bar{\Gamma}$. This map is an isometry of the affine metrics and interchanges the two projectively flat connections ∇ and $\bar{\nabla}$.*

3.6. **Hyperbolic affine spheres in $\mathbb{R}^3$.** C.P. Wang formulates the condition for a two-dimensional surface to be an affine sphere in terms of the conformal geometry given by the affine metric [33]. Since we rely heavily on this work, we give a version of the arguments here for the reader's convenience.

Choose a local conformal coordinate $z = x + iy$ on the hypersurface. Then the affine metric is given by $h = e^{\psi}|dz|^2$ for some function ψ. Parametrize the surface by $f : \Delta \to \mathbb{R}^3$, with $\Delta \subset \mathbb{C}$. Since $\{e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y\}$ is an orthonormal basis for the tangent space, we have by (3.2.2)

$$\det(e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y, \xi) = 1,$$

which implies

$$\det(f_z, f_{\bar{z}}, \xi) = \tfrac{1}{2} i e^{\psi}, \tag{3.6.1}$$

where ξ is the affine normal.

Now only consider hyperbolic affine spheres. By scaling in $\mathbb{R}^3$, we need only consider spheres with affine mean curvature -1. In this case, we have the following structure equations:

$$\left\{ \begin{array}{c} D_X Y = \nabla_X Y + h(X,Y)\xi \\ D_X \xi = X \end{array} \right. \tag{3.6.2}$$

If the center of the affine sphere is 0, then we also have $\xi = f$.

It is convenient to work with complexified tangent vectors, and we extend ∇, h, D, etc. by complex linearity. Consider the frame for the tangent bundle to the surface $\{e_1 = f_z = f_*(\frac{\partial}{\partial z}), e_{\bar 1} = f_{\bar z} = f_*(\frac{\partial}{\partial \bar z})\}$. Then we have

$$h(f_z, f_z) = h(f_{\bar z}, f_{\bar z}) = 0, \quad h(f_z, f_{\bar z}) = \tfrac{1}{2}e^{\psi}. \tag{3.6.3}$$

Consider θ the matrix of connection one-forms

$$\nabla e_i = \theta_i^j e_j, \quad i, j \in \{1, \bar 1\},$$

and $\hat\theta$ the matrix of connection one-forms for the Levi-Civita connection. We know by (3.6.3) that

$$\hat\theta_{\bar 1}^1 = \hat\theta_1^{\bar 1} = 0, \quad \hat\theta_1^1 = \partial\psi, \quad \hat\theta_{\bar 1}^{\bar 1} = \bar\partial\psi. \tag{3.6.4}$$

The difference $\hat\theta - \theta$ is given by the Pick form. We have

$$\hat\theta_i^j - \theta_i^j = C_{ik}^j \rho^k,$$

where $\{\rho^1 = dz, \rho^{\bar 1} = d\bar z\}$ is the dual frame of one-forms. Now we differentiate (3.6.1) and use the structure equations (3.6.2) to conclude

$$\theta_1^1 + \theta_{\bar 1}^{\bar 1} = d\psi.$$

This implies, together with (3.6.4), the apolarity condition

$$C_{1k}^1 + C_{\bar 1 k}^{\bar 1} = 0, \quad k \in \{1, \bar 1\},$$

as in (3.2.3) above. Then, when we lower the indices, the expression for the metric (3.6.3) implies that

$$C_{\bar 1 1 k} + C_{1 \bar 1 k} = 0.$$

Now by Subsection 3.2 above, C_{ijk} is totally symmetric on three indices. Therefore, the previous equation implies that all the components of C must vanish except C_{111} and $C_{\bar 1 \bar 1 \bar 1} = \overline{C_{111}}$.

This discussion completely determines θ:

$$\begin{pmatrix} \theta_1^1 & \theta_{\bar 1}^1 \\ \theta_1^{\bar 1} & \theta_{\bar 1}^{\bar 1} \end{pmatrix} = \begin{pmatrix} \partial\psi & C_{\bar 1 \bar 1}^1 d\bar z \\ C_{11}^{\bar 1} dz & \bar\partial\psi \end{pmatrix} = \begin{pmatrix} \partial\psi & \bar U e^{-\psi} d\bar z \\ U e^{-\psi} dz & \bar\partial\psi \end{pmatrix}, \tag{3.6.5}$$

where we define $U = C_{11}^{\bar 1} e^{\psi}$.

Recall that D is the canonical flat connection induced from $\mathbb{R}^3$. (Thus, for example, $D_{f_z} f_z = D_{\frac{\partial}{\partial z}} f_z = f_{zz}$.) Using this statement, together with (3.6.3) and (3.6.5), the structure equations (3.6.2) become

$$\begin{cases} f_{zz} = \psi_z f_z + U e^{-\psi} f_{\bar z} \\ f_{\bar z \bar z} = \bar U e^{-\psi} f_z + \psi_{\bar z} f_{\bar z} \\ f_{z\bar z} = \frac{1}{2} e^{\psi} f \end{cases} \tag{3.6.6}$$

Then, together with the equations $f_z = f_z$ and $f_{\bar z} = f_{\bar z}$, these form a linear first-order system of PDEs in f, f_z and $f_{\bar z}$. In order to have a solution of this system, the only condition is that the mixed partials must commute (by the Frobenius theorem). Thus we require

$$\psi_{z\bar z} + |U|^2 e^{-2\psi} - \tfrac{1}{2}e^{\psi} = 0 \qquad (3.6.7)$$

$$\text{and} \quad U_{\bar z} = 0.$$

The system (3.6.6) is an initial-value problem, in that given (1) a base point z_0, (2) initial values $f(z_0)$, $f_z(z_0)$ and $f_{\bar z}(z_0) = \overline{f_z(z_0)}$, and (3) U holomorphic and ψ which satisfy (3.6.7), we have a unique solution f of (3.6.6) as long as the domain of definition Δ is simply connected. We then have that the immersion f satisfies the structure equations (3.6.2). In order for f to be the affine normal of $f(\Delta)$, by (3.2.2) and (3.6.1), we must also have the volume condition $\det(f_z, f_{\bar z}, f) = \frac{1}{2}ie^{\psi}$. We require this at the base point z_0 of course:

$$\det(f_z(z_0), f_{\bar z}(z_0), f(z_0)) = \tfrac{1}{2}ie^{\psi(z_0)}. \qquad (3.6.8)$$

Then use (3.6.6) to show that the derivatives with respect to z and $\bar z$ of $\det(f_z, f_{\bar z}, f)e^{-\psi}$ must vanish. Therefore the volume condition is satisfied everywhere, and $f(\Delta)$ is a hyperbolic affine sphere with affine mean curvature -1 and center 0.

Using (3.6.6), we compute $\det(f_z, f_{zz}, f) = \frac{1}{2}iU$, which implies that U transforms as a section of K^3, and $U_{\bar z} = 0$ means it is holomorphic.

Also, consider two embeddings f and $\tilde f$ from a simply connected Δ to $\mathbb{R}^3$ which satisfy (3.6.6) and the initial value condition (3.6.8) for some z_0 and $\tilde z_0$. Then consider the map $A \in \mathbf{GL}(3,\mathbb{R})$ which takes $\{f(z_0), f_z(z_0), f_{\bar z}(z_0)\}$ to $\{\tilde f(z_0), \tilde f_z(z_0), \tilde f_{\bar z}(z_0)\}$. By the volume condition (3.6.1), $A \in \mathbf{SL}(3,\mathbb{R})$. The uniqueness of solutions to (3.6.6) then shows that $Af = \tilde f$ everywhere.

We record all this discussion in the following

Proposition 3.6.1 (Wang [33]). *Let $\Delta \subset \mathbb{C}$ be a simply connected domain. Given U a holomorphic section of K^3 over Δ, ψ a real-valued function on Δ so that U and ψ satisfy (3.6.7), and initial values for f, f_z, $f_{\bar z}$ which satisfy (3.6.8), we can solve (3.6.6) so that $f(\Delta)$ is a hyperbolic affine sphere of affine mean curvature -1 and center 0. Any two such f which satisfy (3.6.6) are related by a motion of $\mathbf{SL}(3,\mathbb{R})$.*

4. The main theorem

Now we prove a result essentially due to C.-P. Wang [33] that will allow us to determine the affine sphere structure on a given Riemann surface. Our proof is simpler than Wang's original one.

Proposition 4.0.2. *Let M be a compact manifold, g be a nonnegative C^∞ function on M, and Δ be the Laplacian with respect to a C^∞ Riemannian metric on M. Then the equation*

$$\Delta u + g(x)e^{-2u} - 2e^u + 2 = 0$$

has a unique C^∞ solution.

Proof. To show existence, by a standard result (see Schoen-Yau [32, Prop. V.1.1]), we only need to find a subsolution and a supersolution for the equation. It is straightforward to check that $s = 0$ is a subsolution. Also, if $G = \max g$, set m to be the positive root of the equation

$$2x^3 - 2x^2 - G = 0.$$

Then $s = \log m$ satisfies

$$\Delta s + g(x)e^{-2s} - 2e^s + 2 = g(x)m^{-2} - 2m + 2 \leq 0,$$

and s is a supersolution. Smoothness follows by standard elliptic theory. Uniqueness follows from a standard maximum-principle argument, since $g(x)e^{-2u} - 2e^u + 2$ is strictly decreasing as a function of u. □

Now we apply this proposition to Wang's formulation of affine spheres. If Σ is a compact Riemann surface of genus $g > 1$, let $h_0 = e^\phi|dz|^2$ be the metric of constant curvature -1. Now write the affine metric as $e^{\phi+u}|dz|^2$. Therefore, u is a globally defined function on Σ and we have $\psi = \phi + u$. The Laplacian $\Delta_0 = 4e^{-\phi}\partial_z\partial_{\bar{z}}$, and the curvature condition is $-\frac{1}{2}\Delta_0\phi = -1$. Therefore, ψ solves (3.6.7) exactly if the following equation in u holds:

$$\begin{aligned} \Delta_0 u &= 4e^{-\phi}(\psi_{z\bar{z}} - \phi_{z\bar{z}}) \\ &= 2e^{-\phi}(-2e^{-2\psi}|U|^2 + e^\psi) - 2 \\ &= -4e^{-2u}\|U\|_0^2 + 2e^u - 2 \end{aligned} \tag{4.0.1}$$

Here $\|\cdot\|_0^2 = |\cdot|^2 e^{-3\phi}$ denotes the metric on K^3 induced by h_0.

This formulation, together with Proposition 4.0.2, provides us with the following characterization of hyperbolic affine spheres which cover surfaces of genus $g \geq 2$.

Proposition 4.0.3 (Wang [33]). *A hyperbolic affine sphere in $\mathbb{R}^3$ with affine mean curvature -1 and center 0 which admits the action of a discrete, properly discontinuous subgroup of $\mathbf{SL}(3, \mathbb{R})$ so that the quotient S has genus $g \geq 2$ is determined by a conformal structure on S and a holomorphic section U of K^3. All such affine spheres are obtained in this way.*

We can now prove Theorem 1.

Proof of Theorem 1. First of all, the previous proposition shows that for any conformal structure and holomorphic section U of K^3, we have an affine sphere structure on S. Since S is compact, Theorem 3 then provides a convex $\mathbb{RP}^2$ structure on S.

Conversely, any convex $\mathbb{RP}^2$ structure on such an S must be properly convex by Proposition 2.4.1. Therefore, Theorem 3 provides an affine sphere structure, which by Wang's Proposition 4.0.3 is equivalent to a conformal structure on S and a holomorphic section U of K^3. □

Proposition 3.5.3 then gives us the following

Corollary 4.0.4. *If we replace the section U of K^3 by $-U$, we recover the projective dual surface (which is made by looking at the dual projective space of lines in $\mathbb{RP}^2$ and taking the dual gluing maps in the construction of the surface). The affine metric h is unchanged and the two projectively flat connections ∇ and $\bar{\nabla}$ are interchanged.*

When the Pick form $U = 0$, the affine sphere structure on the universal cover is the hyperboloid, and the Blaschke connection ∇ is the Levi-Civita connection coming from the metric of constant negative sectional curvature. Therefore, we have the $\mathbb{RP}^2$ structure is given by the hyperbolic structure on the disk, we recover the fact that Teichmüller space is exactly the fixed locus of the action of projective duality on Goldman space.

Remark. We discuss Wang's results for surfaces of genus 1. In this case the curvature of the base metric must be 0 and Wang's equation becomes

$$\Delta_0 u + 4\|U\|_0^2 e^{-2u} - 2e^u = 0. \tag{4.0.2}$$

Then K^3 is a trivial bundle on Σ and $\|U\|_0^2$ must be constant. Then if $U = 0$, it is easy to see there are no solutions. If $U \neq 0$, then we can find a constant solution u. These determine all solutions to (4.0.2) over Σ, and consequently all hyperbolic affine spheres which cover a surface of genus one.

The affine metric given by $e^u h_0$ is flat since u is a constant. Affine spheres in $\mathbb{R}^3$ with flat affine metric have been classified up to affine transformations by Magid-Ryan ([29]; see also Nomizu-Sasaki [30, p. 113]). The only one which is hyperbolic is the surface

$$xyz = c, \quad x, y, z > 0 \quad \text{for some constant} \quad c > 0.$$

This surface is asymptotic to the coordinate planes, which form a cone over a triangle. Thus we have another proof of the fact [25] that any properly convex compact $\mathbb{RP}^2$ surface of genus one must be covered by a triangle.

4.1. **Moduli problems.** In addition to the deformation space considered above, it is also useful to consider the moduli space of convex $\mathbb{RP}^2$ structures. While the deformation space of $\mathbb{RP}^2$ structures is given by projective equivalence classes of torsion-free, projectively flat connections modulo Diff_0, the identity component of the diffeomorphism group, for the *moduli space* of oriented $\mathbb{RP}^2$ structures, we replace Diff_0 by Diff^+, the group of all orientation-preserving diffeomorphisms.

Our main theorem immediately implies this corollary

Corollary 4.1.1. *The moduli space of convex $\mathbb{RP}^2$ structures on an oriented compact surface of genus $g \geq 2$ is equivalent to the moduli space of pairs (Σ, U), where Σ is a Riemann surface of genus g and $U \in H^0(\Sigma, K^3)$.*

Determining some basic facts about this space is an exercise in algebraic curve theory. We have the following proposition

Proposition 4.1.2. *Our moduli space, as a locally finite quotient of Goldman space, is a complex orbifold which is smooth on exactly those convex $\mathbb{RP}^2$ surfaces with no nontrivial automorphisms. The generic convex $\mathbb{RP}^2$ surface for each genus $g \geq 2$ has no nontrivial automorphisms. In fact, for a fixed complex structure on Σ of genus $g \geq 2$, the $\mathbb{RP}^2$ structure corresponding to a generic section in $H^0(\Sigma, K^3)$ has no automorphisms.*

Proof. The first statement follows from standard facts in algebraic curve theory, except for one point. The generic algebraic curve of genus 2 has a nontrivial automorphism, the hyperelliptic involution. We claim that a generic section of K^3 is not fixed by this involution.

In order to prove this claim, we use the Riemann-Hurwitz and Riemann-Roch formulas to calculate the dimension of the subspace of $H^0(\Sigma, K^3)$ fixed by a given automorphism of Σ. Let σ be an automorphism of Σ of order d. Then we consider the quotient Ξ, which is another smooth Riemann surface. The quotient map $Q : \Sigma \to \Xi$ has degree d, and it is branched exactly at the fixed points of powers of σ.

Consider a point p where Q is branched to order n. Then a simple local calculation shows that sections in $H^0(\Sigma, K^3)$ fixed by Q near p are exactly sections in $H^0(\Xi, K^3)$ with a certain pole order allowed at $q = Q(p)$. If $n = 2$, then we allow poles of order 1, and if $n > 2$, we allow poles of order at most 2.

If Q has degree d and is branched over points p_i with branching order n_i, then we can use Riemann-Hurwitz to determine the genus g' of Ξ:

$$2g - 2 = d(2g' - 2) + \sum_i (n_i - 1).$$

Also, if q_j are the images of the p_i under Q, m_j the allowed pole order at q_j, and D is line bundle determined by the divisor $\prod_j [q_j]^{m_j}$, Riemann-Roch gives us

$$\dim_{\mathbb{C}} H^0(\Xi, K^3D) - \dim_{\mathbb{C}} H^0(\Xi, K^{-2}D^{-1}) = 5g' - 5 + \sum_j m_j.$$

These and other similar numerical statements (see e.g. [1, p. 45]) can be used to prove that if $g \geq 2$ then

$$\dim_{\mathbb{C}} H^0(\Xi, K^3D) \leq \max\{\tfrac{5}{2}(g-1), \tfrac{7}{3}(g-1) + 2\}.$$

Notice that this bound is always less than $5g - 5$; therefore, a generic section U of K^3 is not fixed by any automorphism σ of Σ.

In the particular case where σ is the hyperelliptic involution of a curve of genus 2, the map Q is a double cover of $\mathbb{CP}^1$ branched over 6 points. Then $K^3 = \mathcal{O}(-6)$, $D = \mathcal{O}(6)$, and

$$\dim_{\mathbb{C}} H^0(\Xi, K^3D) = \dim_{\mathbb{C}} H^0(\mathbb{CP}^1, \mathcal{O}) = 1.$$

Thus there is only a 1-dimensional subspace of the 5-dimensional space $H^0(\Sigma, K^3)$ which is fixed by σ. □

4.2. **Results on noncompact surfaces.** The moduli space of Riemann surfaces of genus g can be compactified by including surfaces with nodal singularities. The natural hyperbolic metric is complete in the noncompact surface obtained by removing the nodes. Such surfaces are of *finite type*; i.e. they are of the form $\Sigma = \bar{\Sigma} \setminus E$, where $\bar{\Sigma}$ is a (possibly disconnected) compact Riemann surface and $E = \{p_i\}$ is a finite set.

Therefore, if we want to study the boundary of the moduli space of oriented, convex $\mathbb{RP}^2$ structures on a surface of genus g, we must understand Wang's equation on such a noncompact surface Σ. By Theorem 3 above, the affine sphere metric must be complete in order to give the universal cover the structure of a hyperbolic affine sphere. Also, a natural condition we impose is that Σ have finite area with respect to the affine sphere metric. With that in mind, we formulate the following conjecture and provide some propositions toward a proof of it.

Conjecture 1. *Let $\Sigma = \bar{\Sigma} \setminus \{p_i\}$ be a hyperbolic Riemann surface. Let U be a meromorphic section of K^3 over $\bar{\Sigma}$ with poles of order at most 2 allowed only at the points p_i. Then there is a unique finite-area, complete affine sphere metric on Σ associated to U. Conversely, given any affine sphere structure on a noncompact oriented surface Σ which is complete and has finite area, we must have Σ is hyperbolic and of* *finite type, and the Pick form U can have poles of order at most 2 at the punctures p_i.*

Remark. Compare this to the following fact from Riemann surface theory. Any noncompact hyperbolic Riemann surface for which the canonical complete hyperbolic metric has finite area must be of finite type.

Proposition 4.2.1. *Let $\Sigma = \bar{\Sigma} \setminus \{p_i\}$ have complete hyperbolic metric h_0, and let U be a meromorphic section of K^3 with poles of order at most 2 at the punctures. Then there is a unique bounded function u satisfying Wang's equation*

$$\Delta_0 u + 4\|U\|_0^2 e^{-2u} - 2e^u + 2 = 0.$$

Therefore, $h = e^u h_0$ is complete and has finite area.

Proof. Choose a local coordinate z of $\bar{\Sigma}$ centered at a puncture p_i, and let $h_0 = e^\phi |dz|^2$. We have

$$e^\phi = O(|z|^{-2}(\log|z|^2)^{-2}). \tag{4.2.1}$$

Therefore, since U has pole order at most 2,

$$\|U\|_0^2 = e^{-3\phi}|U|^2 \le C|z|^2(\log|z|^2)^6.$$

In particular, $\|U\|_0^2$ is bounded.

We proceed by approximating the metric h_0 on Σ by smooth metrics on $\bar{\Sigma}$. Choose local coordinates z_i centered at each p_i. We can easily choose a sequence of metrics h_ϵ which are smooth and equal h_0 outside the balls $\{|z_i| < \epsilon\}$. Let Δ_ϵ denote the Laplacian with respect to h_ϵ.

Now by Proposition 4.0.2, the equation

$$\Delta_\epsilon u + 4\|U\|_0^2 e^{-2u} - 2e^u + 2 = 0 \tag{4.2.2}$$

has a unique solution u_ϵ on $\bar{\Sigma}$. ($\|U\|_0^2$ is not smooth at p_i, but it can be approximated by smooth functions as we did for h_0.) The subsolution and supersolution of (4.2.2) show, since $\|U\|_0^2$ is bounded, that the u_ϵ are uniformly bounded $C \ge u_\epsilon \ge 0$.

Now we let $\epsilon \to 0$, and we claim there is a subsequence of u_ϵ converging to a smooth u in C^∞_{loc}. This u will be a bounded solution of Wang's equation (4.0.1) on Σ. By the C^0 bounds on u_ϵ, the equation (4.2.2) can be written as $\Delta_\epsilon u_\epsilon = f_\epsilon$ with $f_\epsilon \in L^p_{\mathrm{loc}}$. Then, using the fact that on any compact subset of Σ, $\Delta_\epsilon = \Delta_0$ for small ϵ, standard estimates [17] show that the u_ϵ are locally uniformly bounded with respect to the $W^{2,p}$ norm. Higher regularity is standard and Ascoli-Arzela gives convergence of a subsequence. Therefore, the claim is proved.

In order to prove uniqueness, we use the following maximum principle of Cheng-Yau [6]:

Lemma 4.2.2. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded below. Let Δ be the Laplacian on M and w be a bounded C^2 function. Then there is a sequence of points q_i such that* $\lim w(q_i) = \sup_M w$ *and* $\limsup(\Delta w)(q_i) \leq 0$.

If u and v are bounded functions which both satisfy (4.0.1) on Σ, then we claim $w = u - v$ must be identically 0. Now w satisfies the equation

$$\Delta_0 w + f w = 0, \tag{4.2.3}$$

$$f = \int_0^1 (-8\|U\|^2 e^{-2u_\theta} - 2e^{u_\theta})d\theta \leq -2e^m,$$

where m is a lower bound for u and v, and $u_\theta = \theta u + (1-\theta)v$ for $\theta \in [0,1]$.

Then (4.2.3), together with Cheng-Yau's maximum principle, gives a contradiction if w has a positive supremum. Therefore, $w \leq 0$. A similar argument shows $w \geq 0$. Thus Proposition 4.2.1 is proved. □

Also we have the following partial converse toward proving the conjecture.

Proposition 4.2.3. *Let Σ, h_0 be as above, U have a pole of order ≥ 3 at some puncture. Then Wang's equation has no solution u so that $h = e^u h_0$ is complete and has finite area.*

Proof. C will denote any constant that does not depend on u. For any complete hyperbolic affine sphere, Cheng-Yau show that the norm of the Pick form with respect to the affine metric is bounded [8]. In our case, this means $e^{-3u}\|U\|_0^2 \leq C$. Since U has pole order at least 3 at a local coordinate $z = 0$, we see as above that $\|U\|_0^2 \geq C(\log|z|^2)^6$. Therefore, we must have $e^u \geq C(\log|z|^2)^2$. Then the area of Σ with respect to the affine metric is

$$\begin{aligned}
\int_\Sigma e^u dV_0 &\geq \int_{\{|z|<\epsilon\}} e^u dV_0 \\
&\geq C\int_0^{2\pi}\int_0^\epsilon (\log r^2)^2 \frac{1}{r^2(\log r^2)^2} r\,dr\,d\theta \\
&= \infty.
\end{aligned}$$

Here we use (4.2.1) to provide the bound for dV_0, the volume form for the metric h_0. □

4.3. **A Kähler structure on Goldman space.** Our main theorem gives us a complex structure on Goldman space. Also, there is the natural symplectic form ω given by equation (2.5.1). It is natural to ask, therefore, whether these two structures fit together to make a Kähler structure.

Remark. Darvishzadeh-Goldman in [14] have used different invariant structures on sharp cones to construct an almost complex structure on Goldman space which fits with the symplectic form to make an almost Kähler structure. It seems unlikely that this almost complex structure is integrable, however, and it is unclear how it is related to the complex structure induced by Theorem 1.

Conjecture 2. *The tensor $g(V,W) = \omega(V, iW)$ is a Riemannian metric on Goldman space.*

We present some partial results toward this conjecture.
First of all, we remark that Goldman has shown

Proposition 4.3.1 (Goldman [20])**.** *$\omega|_{\mathcal{T}}$ is 8 times the Weil-Petersson Kähler form on Teichmüller space $\mathcal{T}$.*

Also, we have the following proposition:

Proposition 4.3.2. *Consider the fiber subspace $\mathcal{F}$ of Goldman space defined by $\mathcal{F} = H^0(\Sigma, K^3)$ for a fixed conformal structure on Σ a surface of genus $g > 1$. Then with respect to the natural complex structure on $\mathcal{F}$, ω is a Kähler form and the induced metric is complete.*

Remark. We know that the Weil-Petersson metric on the moduli space is incomplete, as the nodal curves at the boundary of the moduli space are at a finite distance from the interior. These two propositions together show that if the tensor g is a metric on all of Goldman space, it is incomplete in the directions of the nodal curves at the boundary of Teichmüller space but it should be complete on some complementary directions.

Proof. Recall the formula for ω

$$\omega(\sigma_1, \sigma_2) = \int_S \tfrac{1}{3} \operatorname{tr} \sigma_1 \wedge \operatorname{tr} \sigma_2 - \operatorname{tr}(\sigma_1 \wedge \sigma_2).$$

The σ_i are tangent vectors to the space of connections. In our case we have that they are deformations of the canonical projectively flat connection (the Blaschke connection) that the affine sphere formulation gives us.

With respect to the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$ of the (complexified) tangent space of Σ, the connection form for the Blaschke connection associated to $U \in \mathcal{F}$ is

$$\theta = \begin{pmatrix} \partial\psi & \bar{\beta} \\ \beta & \bar{\partial}\psi \end{pmatrix},$$

where $\psi = \phi + u$ is as above and $\beta = Ue^{-\psi}dz$.

Now consider V in the tangent space $T(\mathcal{F}) = \mathcal{F}$ and the deformation of the connection form

$$\delta_V\theta = \frac{d}{d\epsilon}\theta(U + \epsilon V)|_{\epsilon=0} = \begin{pmatrix} \partial(\delta_V u) & \delta_V\bar{\beta} \\ \delta_V\beta & \bar{\partial}(\delta_V u) \end{pmatrix}.$$

Here we have $\delta_V\beta = (V - U\,\delta_V u)e^{-\psi}dz$ and by (4.0.1), $\delta_V u$ satisfies

$$\Delta_0\,\delta_V u - (8e^{-2u}\|U\|_0^2 + 2e^u)\delta_V u + 8\,\mathrm{Re}(U\bar{V})e^{-3\phi-2u} = 0.$$

Therefore, if we multiply by e^{-u} and consider $\Delta = e^{-u}\Delta_0$ the Laplacian with respect to the affine metric $h = e^u h_0$, $\langle U, V\rangle = e^{-3\psi}U\bar{V}$ the induced Hermitian inner product on sections of K^3, and $\|U\|^2 = \langle U, U\rangle$, we have

$$\Delta\,\delta_V u - (8\|U\|^2 + 2)\delta_V u + 8\,\mathrm{Re}\langle U, V\rangle = 0. \tag{4.3.1}$$

In order to check that $\omega|_{\mathcal{F}}$ is a Kähler form with respect to the complex structure, we must check

$$\omega(V, W) = \omega(iV, iW) \quad \text{for all} \quad V, W \in \mathcal{F}$$
$$\text{and} \quad \omega(V, iV) > 0 \quad \text{for} \quad V \neq 0.$$

Compute

$$\begin{aligned}
\omega(V, W) &= \int_\Sigma \tfrac{1}{3}\,\mathrm{tr}\,\delta_V\theta \wedge \mathrm{tr}\,\delta_W\theta - \mathrm{tr}(\delta_V\theta \wedge \delta_W\theta) \\
&= \int_\Sigma \tfrac{1}{3}\,d(\delta_V u) \wedge d(\delta_W u) - \mathrm{tr}\begin{pmatrix} \delta_V\bar{\beta} \wedge \delta_W\beta & \cdot \\ \cdot & \delta_V\beta \wedge \delta_W\bar{\beta} \end{pmatrix} \\
&= -2\,\mathrm{Re}\int_\Sigma \delta_V\beta \wedge \delta_W\bar{\beta} \\
&= -2\,\mathrm{Re}\int_\Sigma (V - U\,\delta_V u)(\bar{W} - \bar{U}\,\delta_W u)e^{-2\psi}dz \wedge d\bar{z} \\
&= -2\,\mathrm{Re}\int_\Sigma (\langle V, W\rangle - \langle U, W\rangle\delta_V u - \langle V, U\rangle\delta_W u \\
&\qquad + \|U\|^2\delta_V u\,\delta_W u)e^{\psi}dz \wedge d\bar{z} \\
&= -4\,\mathrm{Im}\int_\Sigma (\langle V, W\rangle - \langle U, W\rangle\delta_V u + \langle U, V\rangle\delta_W u)dV,
\end{aligned}$$

where $dV = \frac{1}{2} i e^{\psi} dz \wedge d\bar{z}$ is the volume form for h. Also, we have

$$\omega(iV, iW) = -4 \int_{\Sigma} (\mathrm{Im}\langle V, W\rangle + \mathrm{Re}\langle U, W\rangle \delta_{iV} u - \mathrm{Re}\langle U, V\rangle \delta_{iW} u) dV$$

and

$$\begin{aligned}\omega(V, W) - \omega(iV, iW) &= -4 \int_{\Sigma} (-\mathrm{Im}\langle U, W\rangle \delta_V u - \mathrm{Re}\langle U, W\rangle \delta_{iV} u \\ &\quad + \mathrm{Im}\langle U, V\rangle \delta_W u + \mathrm{Re}\langle U, V\rangle \delta_{iW} u) dV.\end{aligned}$$

This difference being 0 is equivalent to

(4.3.2)
$$\mathrm{Im} \int_{\Sigma} \langle U, V\rangle (\delta_W u + i\, \delta_{iW} u) dV = \mathrm{Im} \int_{\Sigma} \langle U, W\rangle (\delta_V u + i\, \delta_{iV} u) dV.$$

First add (4.3.1) and i times the corresponding equation for iV to get

$$(4.3.3) \quad \langle U, V\rangle = \tfrac{1}{8}[-\Delta(\delta_V u + i\, \delta_{iV} u) + (8\|U\|^2 + 2)(\delta_V u + i\, \delta_{iV} u)].$$

Therefore,

$$\int_{\Sigma} \langle U, V\rangle (\delta_W u + i\, \delta_{iW} u) dV = \tfrac{1}{8} \int_{\Sigma} -\Delta(\delta_V u + i\, \delta_{iV} u)(\delta_W u + i\, \delta_{iW} u) dV$$

$$+ \tfrac{1}{8} \int_{\Sigma} (8\|U\|^2 + 2)(\delta_V u + i\, \delta_{iV} u)(\delta_W u + i\, \delta_{iW} u) dV.$$

Integrating by parts twice in the first integral on the right hand side gives us that the expression is symmetric in V and W. Therefore, by (4.3.2), we have $\omega(V, W) = \omega(iV, iW)$.

Now we consider

$$\begin{aligned}\omega(V, iV) &= -4 \int_{\Sigma} (\mathrm{Im}\langle V, iV\rangle + \mathrm{Im}\langle U, V\rangle \delta_V u - \mathrm{Im}\langle U, iV\rangle \delta_{iV} u) dV \\ (4.3.4) \qquad &= 4 \int_{\Sigma} \|V\|^2 dV - 4 \int_{\Sigma} \mathrm{Re}[\langle U, V\rangle (\delta_V u - i\, \delta_{iV} u)] dV.\end{aligned}$$

As above, we use (4.3.3) to substitute in for $\langle U, V\rangle$ and then integrate by parts. The second term on the right is then

$$\begin{aligned} P &= 4\int_\Sigma \mathrm{Re}[\langle U, V\rangle(\delta_V u - i\,\delta_{iV}u)]dV \\ &= \mathrm{Re}\,4\int_\Sigma -\tfrac{1}{8}\Delta(\delta_V u + i\,\delta_{iV}u)(\delta_V u - i\,\delta_{iV}u)dV \\ &\quad + \mathrm{Re}\,4\int_\Sigma \tfrac{1}{8}(8\|U\|^2+2)(\delta_V u + i\,\delta_{iV}u)(\delta_V u - i\,\delta_{iV}u)dV \\ &= \int_\Sigma \tfrac{1}{2}(|\hat\nabla\delta_V u|^2 + |\hat\nabla\delta_{iV}u|^2)dV \\ &\quad + \int_\Sigma (4\|U\|^2+1)[(\delta_V u)^2 + (\delta_{iV}u)^2]dV. \qquad (4.3.5) \end{aligned}$$

Here $|\hat\nabla\cdot|^2$ is the squared norm of the gradient with respect to the metric. This equation clearly shows $P \geq 0$.

In order to proceed, we need the following estimate:

Lemma 4.3.3. $\|U\|^2 \leq \frac{1}{2}$.

Proof. At a maximum point of $\|U\|^2$, we have

$$0 \geq \Delta_0(\log\|U\|^2) = \Delta_0(\log\|U\|_0^2 - 3u) = \Delta_0(-3\phi - 3u).$$

Then using the fact $\Delta_0\phi = 2$ and Wang's equation (4.0.1), we have

$$-3(2) - 3(-4\|U\|_0^2 e^{-2u} + 2e^u - 2) \leq 0,$$

which implies $\|U\|^2 \leq \frac{1}{2}$. □

Now apply the Schwarz inequality to the defining equation of P:

$$P \leq \left(4\int_\Sigma \|V\|^2 dV\right)^{\frac{1}{2}}\left(4\int_\Sigma \|U\|^2[(\delta_V u)^2 + (\delta_{iV}u)^2]dV\right)^{\frac{1}{2}}. \qquad (4.3.6)$$

Therefore, if we let $G = 4\|U\|^2[(\delta_V u)^2 + (\delta_{iV}u)^2]$, Lemma 4.3.3 gives us that

$$\tfrac{3}{2}G \leq (4\|U\|^2+1)[(\delta_V u)^2 + (\delta_{iV}u)^2].$$

(4.3.5) and (4.3.6) then imply

$$\tfrac{3}{2}\int_\Sigma G\,dV \leq \left(4\int_\Sigma \|V\|^2 dV\right)^{\frac{1}{2}}\left(\int_\Sigma G\,dV\right)^{\frac{1}{2}}.$$

We then have $(\int_\Sigma G\,dV)^{\frac{1}{2}} \leq \frac{2}{3}(\int_\Sigma 4\|V\|^2 dV)^{\frac{1}{2}}$, which implies by (4.3.6) that $0 \leq P \leq \frac{2}{3}\int_\Sigma 4\|V\|^2 dV$. Therefore, we have by (4.3.4)

$$\tfrac{1}{3}\int_\Sigma 4\|V\|^2 dV \leq \omega(V, iV) \leq \int_\Sigma 4\|V\|^2 dV.$$

This clearly shows ω is a Kähler form. In other words the tensor g, when restricted to $\mathcal{F}$, is a Riemannian metric.

Now we must estimate $\int_\Sigma \|V\|^2 dV$ to determine that g is complete. By the proof of Proposition 4.0.2, we have $e^u \leq m$ where m is the positive root of

$$2x^3 - 2x^2 - 4\max\|U\|_0^2 = 0.$$

Therefore, if U lies on the unit sphere $\mathcal{S}$ of $\mathcal{F}$ (with respect to some norm), and u is the solution to (4.0.1) for the Pick form λU, then $e^u \leq C\lambda^{\frac{2}{3}}$, where C will denote any uniform constant. Therefore,

$$\|V\|^2 dV = e^{-2u}\|V\|_0^2 dV_0 \geq C\lambda^{-\frac{4}{3}}\|V\|_0^2 dV_0.$$

(Here dV_0 is the volume form for the hyperbolic metric h_0.) This estimate shows that g is complete: Consider any path $p(t)$ going to ∞ in $\mathcal{F}$. Parametrize it so that $\frac{dp}{dt} = \dot{p}$ always lies in $\mathcal{S}$. The length of the path is given by

$$\begin{aligned} \int_p g(\dot{p},\dot{p})^{\frac{1}{2}} dt &\geq C\int_p \left(\int_\Sigma \lambda^{-\frac{4}{3}}\|\dot{p}\|_0^2 dV_0\right)^{\frac{1}{2}} dt \\ &\geq C\int_p \lambda(t)^{-\frac{2}{3}} dt \end{aligned}$$

This last integral is infinite since λ must go to ∞ along $p(t)$. Therefore, g is complete. □

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