ON THOLOZAN'S VOLUME FORMULA FOR CLOSED ANTI-DE-SITTER 3-MANIFOLDS

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For Olaf, who said yes to all my crazy ideas [1]

I take the opportunity here to warmly thank all the contributors to this volume, all the participants in this birthday conference, not forgetting the organizers, for their amazing contributions.

This short note is meant as an appendix to Nicolas Tholozan's article [13] in the same volume. The point here is to present arguments relating volumes of anti-de-Sitter manifolds of dimension 3 to Chern-Simons invariants, and to give as a corollary Tholozan's volume formula from his thesis [14], later extended to higher dimensions in [15]. After hearing his defence, I presented this proof in a seminar in MSRI in 2015 and was planning to save the writing of it for Nicolas's 60th birthday conference, but finally decided against it.

Following here the convention of [7, 9, 5, 12], an *anti-de-Sitter 3-manifold* (In short AdS manifold) is a manifold of dimension 3, modelled on $\mathsf{PSL}_2(\mathbb{R})$ equipped with its Killing metric normalised so that the projection from $\mathsf{PSL}_2(\mathbb{R})$ to $\mathbf{H}^2$, equipped with its hyperbolic metric, is a metric submersion. Hence, $\mathsf{PSL}_2(\mathbb{R})$ is a Lorentz 3-manifold whose group of isometries is, up to finite covers and quotients, $\mathsf{PSL}_2(\mathbb{R}) \times \mathsf{PSL}_2(\mathbb{R})$ where each factor corresponds respectively to the action on the right and on the left on the group $\mathsf{PSL}_2(\mathbb{R})$. From the homogeneity of the action on timelike vectors, it follows that $\mathsf{PSL}_2(\mathbb{R})$, hence any AdS 3-manifold, has constant curvature.

Recall that by a theorem of Bruno Klingler [6] every closed AdS 3-manifold is complete, and thus a quotient of the universal cover M_0 of $\mathsf{PSL}_2(\mathbb{R})$ by a discrete subgroup of $M_0 \times M_0$.

Ravi Kulkarni and Frank Raymond [7], followed by François Salein [11], have obtained further restrictions on the possible discrete subgroups appearing, and large classes of examples were constructed in [12]. Later Fanny Kassel [5] obtained the full classification. Namely, any closed AdS 3-manifold M is, up to finite covering, a quotient of $\mathsf{PSL}_2(\mathbb{R})$ by a discrete subgroup Γ of $\mathsf{PSL}_2(\mathbb{R}) \times \mathsf{PSL}_2(\mathbb{R})$ isomorphic to the fundamental group $\pi_1(S)$ of a closed surface S of negative Euler characteristic. The group Γ is the image by (ρ, σ) of $\pi_1(S)$ where ρ and σ are representations of $\pi_1(S)$ in $\mathsf{PSL}_2(\mathbb{R})$, respectively Fuchsian a non-Fuchsian and satisfying the following extra condition: there exists a (ρ, σ)-equivariant map from $\mathbf{H}^2$ to $\mathbf{H}^2$ with Lipschitz

[1] Generated by AI

constant less than 1. This easily implies that $M = M(S,k)$ is a circle bundle over S, of Euler number $k \neq 0$ (see [4]).

Nicolas Tholozan [14], answering a question in [1], gave the following striking formula for the volume of the circle bundle $M(S,k)$ of Euler characteristic k over a surface S of Euler characteristic e, with an AdS structure associated to representations ρ and σ of $\pi_1(S)$ in $\mathsf{PSL}_2(\mathbb{R})$ of respective Euler class e and f:

$$\mathrm{Vol}\,(M(S,k)) = \frac{4\pi^2}{k}\left(e^2 - f^2\right)\ . \tag{1}$$

As a corollary of his formula, Tholozan obtains a rigidity result: *the volume is constant under continuous deformation of any AdS 3-manifold.* Tholozan later on generalised this formula and rigidity to higher dimensions.

We will show how this low dimensional case of Tholozan's general result follows from considerations on the Chern–Simons invariant. We will omit important technical details.

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1. Chern–Simons invariant in dimension 3

We present a short and condensed version of the theory and refer to the original article by Shiing Shen Chern and James Simons [2] for a more careful description or [8] for an elementary one in dimension 3.

Let M be a closed oriented 3-manifold equipped with a vector bundle E. We denote by $\mathcal{D}$ the affine space of connections on E and define the tangent space to $\mathcal{D}$ at a connection ∇ as $\mathsf{T}\mathcal{D} \coloneqq \Omega^1(M, \mathsf{End}(E))$.

1.1. **Chern–Simons form.**

Given a connection ∇, we now consider the linear map ω^{CS} from $\mathsf{T}\mathcal{D}$ to $\mathbb{R}$, given by

$$A \mapsto \int_M \mathsf{trace}\ (A \wedge R^\nabla)\ ,$$

where R^∇ is the curvature of ∇ seen as an element of $\Omega^2(M, \mathsf{End}(E))$, and $A \wedge R^\nabla$ the 3-form on M with value in $\mathsf{End}(E)$ given by

$$A \wedge R^\nabla(X_1, X_2, X_3) = \frac{1}{6}\sum_{\sigma\in\mathfrak{S}_3} (-1)^{\varepsilon(\sigma)} A(X_{\sigma(1)}) R^\nabla(X_{\sigma(2)}, X_{\sigma(3)})\ .$$

The starting result of Chern–Simons theory is

Proposition 1.1.1. *Given two gauge equivalent connections ∇_0 and ∇_1 and a path γ of connections joining ∇_0 to ∇_1 then*

$$\frac{1}{8\pi^2}\int_\gamma \omega^{CS} \in \mathbb{Z}\ . \tag{2}$$

Proof. We only sketch the proof: we interpret a path $\{\nabla^t\}_{t\in[0,1]}$ between two gauge equivalent connections as a connection ∇ on the bundle E on $W \coloneqq M \times S^1$ [8, proposition 7.2.6.]. More precisely ∇ is defined by the following procedure: for (m,s) in $M \times S^1$, we write $\mathsf{T}_{(m,s)}W = \mathsf{T}M \oplus \mathbb{R}$ thus writing a generic tangent vector to W as (U, λ) with U in $\mathsf{T}M$ and λ in $\mathbb{R}$; then we identify the space of sections of E on

W, as $C^\infty(S^1, \Gamma(M,E))$, finally we define for a section σ of E its derivative at a point (m, s_0) as

$$\nabla_{(U,\lambda)}\sigma := \nabla_U^{s_0}\sigma + \lambda\left(\frac{\partial\sigma}{\partial s}\right)(s_0)\ .$$

From this interpretation, we get

$$\int_\gamma \omega^{CS} = \int_{M\times S^1} \mathrm{trace}\,\left(R^\nabla \wedge R^\nabla\right) = 8\pi^2 \mathrm{p}_1(E)\ , \tag{3}$$

where $\mathrm{p}_1(E)$ is the first Pontryagin number. Thus

$$\frac{1}{8\pi^2}\int_\gamma \omega^{CS} \in \mathbb{Z}\ .$$

□

However when E has rank 3 and is equipped with a quadratic form of signature $(2,1)$, we have a stronger result:

Proposition 1.1.2. *If E is equipped with a quadratic q form of signature $(2,1)$, if ∇_1 and ∇_2 are two q-connections gauge equivalent (when the gauge group is $\mathsf{SO}_0(2,1)$), then*

$$\int_\gamma \omega^{CS} = 0\ . \tag{4}$$

Proof. We can always choose a Euclidean metric on E and combining with the quadratic form q, we obtain, by finding a common orthogonal basis, a splitting of E as two subbundles

$$E = L \oplus L^\perp\ ,$$

where L is the line bundle generated by the timelike vector of the base. Denoting by p_1 the first Pontryagin class on $M \times S^1$, by additivity of the first Pontryagin class [10] we get

$$\mathrm{p}_1(E) = \mathrm{p}_1(L) + \mathrm{p}_1(L^\perp)\ .$$

Observe now that since L and $L^\perp$ have dimension 1 and 2 respectively, $\mathrm{p}_1(L) = \mathrm{p}_1(L^\perp) = 0$. Thus $\mathrm{p}_1(E) = 0$. The result now follows. □

From equation (2), one deduces that ω^{CS} is exact: the integral of ω^{CS} on a path of connections only depends on the end points of the path. We then define, given two connections ∇_1 and ∇_2, the *Chern–Simons invariant* of the pair (∇_1, ∇_2) by

$$\mathrm{CS}_M\left(\nabla_1, \nabla_2\right) := \frac{1}{8\pi^2}\int_\gamma \omega^{CS},$$

Moreover, if ∇_1 and ∇_2 are two q-flat connections with holonomy ρ_1 and ρ_2 with values in $\mathsf{SO}_0(2,1)$, we define

$$\mathrm{CS}_M\left(\rho_1, \rho_2\right) := \mathrm{CS}_M\left(\nabla_1, \nabla_2\right)\ .$$

The definition is unambiguous: observe that by proposition 1.1.2, $\mathrm{CS}_M\left(\rho_1, \rho_2\right)$ is well defined for representations defined up to conjugacy by the group $\mathsf{SO}_0(2,1)$.

1.2. **Some properties.** As an immediate consequence of the definition, we have the following easy results:

Lemma 1.2.1 (Chasles relation). *We have*

$$\mathsf{CS}_M\left(\nabla_1,\nabla_2\right)=\mathsf{CS}_M\left(\nabla_1,\nabla_3\right)+\mathsf{CS}_M\left(\nabla_3,\nabla_2\right)\ .$$

Lemma 1.2.2. *Let γ be a path of flat connections joining ∇_1 to ∇_2, then*

$$\mathsf{CS}_M\left(\nabla_1,\nabla_2\right)=\int_\gamma \omega^{CS}=0\ .$$

Lemma 1.2.3. *Let M and N be two closed 3-manifolds. Let π be a degree q map from M to N, ∇_1 and ∇_2 two connections on a bundle over N,*

$$\mathsf{CS}_M\left(\pi^*(\nabla_1),\pi^*(\nabla_2)\right)=q\ \ \mathsf{CS}_N\left(\nabla_1,\nabla_2\right)\ .$$

2. Volumes and Chern–Simons

2.1. **Volumes.** Let us consider the bundle $E := \mathsf{T}M_0$ over M_0, where M_0 is the universal cover of $\mathsf{PSL}_2(\mathbb{R})$. The bundle E carries a natural Lie bracket defined fiberwise. Let us now consider the 3-form on M_0 defined by

$$\Omega(X,Y,Z)=\mathsf{trace}(X[Y,Z]))\ ,$$

where the trace is taken on the (adjoint) 3-dimensional representation of $\mathsf{PSL}_2(\mathbb{R})$. Now observe that Ω is invariant by the isometry group. An explicit computation relates Ω to the volume form:

$$\Omega(X,Y,Z)=\frac{1}{2}\,\mathrm{Vol}(X,Y,Z)\ . \tag{5}$$

2.2. **Back to Chern-Simons.** The bundle $E = \mathsf{T}M_0$ can be trivialized in two ways: the *left trivialization* for which left invariant vector fields are constant, the *right trivialization* for which right invariant vector fields are constant. In other words, E carries two flat connections ∇_L and ∇_R which are respectively the left and right invariant connections. These two connections exist on the tangent bundle of any AdS manifold M and the holonomy of ∇_L (on M) is ρ, while the holonomy of ∇_R is σ.

Our first goal is to show the following proposition that initiated [13, Theorem 2.10].

Proposition 2.2.1. *On a closed AdS manifold M :*

$$\mathsf{CS}_M\left(\nabla_L,\nabla_R\right)=-\frac{1}{24\pi^2}\,\mathrm{Vol}(M)\ .$$

One should not take the sign too seriously here, it is a matter of convention.

Proof. We first have

$$\nabla_L-\nabla_R=A\ ,$$

where A, related to the Maurer–Cartan form, is the element of $\Omega^1(X,\mathsf{End}(E))$ given by

$$A(X):Y\mapsto[X,Y]\ .$$

When we compute the curvature we get

$$R^{\nabla_L}=R^{\nabla_R}+\mathrm{d}^{\nabla_R}A+\frac{1}{2}[A\wedge A]\ ,$$

where
$$[A \wedge B](X,Y) = [A(X),B(Y)] - [A(Y),B(X)] \ .$$
and since both ∇_L and ∇_L are trivial hence flat, we obtain the *Maurer–Cartan equation*
$$\mathrm{d}^{\nabla_R} A + \frac{1}{2}[A \wedge A] = 0 \ .$$
Let us now consider the affine path between ∇_R and ∇_L given by
$$\gamma : t \mapsto \nabla^t = t\nabla_L + (1-t)\nabla_R = \nabla_R + tA \ .$$
Then one then sees that
$$R^{\nabla_t} = t \ \mathrm{d}^{\nabla_R} A + t^2\frac{1}{2}[\ A \wedge A] = \frac{(t^2-t)}{2}[A \wedge A] \ .$$
It follows that
$$\int_\gamma \omega^{CS} = -\frac{1}{3}\int_M \mathrm{trace}(A[A \wedge A]) = -\frac{1}{3}\,\mathrm{Vol}(M) \ .$$
The result now follows. □

3. Computing Chern-Simons invariants

We finally need to compute explicitly the Chern–Simons invariant using Guéritaud–Kassel description.

Let us consider the circle bundle $M(S,k)$ of Euler characteristic k, with $k \neq 0$, over a surface S, and let ρ be a representation of $\pi_1(S)$ in $\mathsf{PSL}_2(\mathbb{R})$ of Euler class f.

Our goal is to show

Proposition 3.0.1. *We have for $M = M(S,k)$ with $k \neq 0$*
$$\mathrm{CS}_M\left(\rho, \mathrm{Id}\right) = -\frac{f^2}{6k} \ .$$

Observe that in this formula the Euler characteristic e of S does not appear, except that by Milnor–Wood inequality one has $f \leqslant e$.

We start with a special case. Let us first remark that if S_e is a surface of Euler characteristic e, then the unit tangent bundle $\mathsf{U}S_e$ of S_e identifies with $M(S_e,e)$. Moreover, if S_e is equipped with a hyperbolic metric, and if we denote by ρ_e the associated monodromy, then $M(S_e,e)$ is modelled on $\mathsf{PSL}_2(\mathbb{R})$ and the pair (ρ,σ) in the Kulkarni–Raymond description is (ρ_e, Id) where Id is the trivial representation.

We therefore obtain using this observation

Lemma 3.0.2. *We have on M=$M(S_e,1)$*
$$\mathrm{CS}_M\left(\rho_e, \mathrm{Id}\right) = -\frac{1}{6}e^2 \ .$$

Proof. We first have
$$\mathrm{Vol}\,\mathsf{U}S_e = 4\pi^2 e \ .$$
Thus on $N = M(S_e,e)$ by proposition 2.2.1, we have
$$\mathrm{CS}_N\left(\rho_e, \mathrm{Id}\right) = -\frac{1}{6}e \ .$$
Since we have a degree e map from $M = M(S_e,1)$ to $N = M(S_e,e)$ we get the result from lemma 1.2.3. □

Finally

Proof of proposition 3.0.1. By lemma 1.2.2 and Goldman's theorem [3] that describes connected components of the space of representations of $\pi_1(S)$ in $\mathsf{PSL}_2(\mathbb{R})$, $\mathrm{CS}_M\left(\rho, \mathrm{Id}\right)$ only depends on e.

Let us just choose a degree 1 map π from the surface S Euler characteristic e, to the surface S_f of Euler characteristic f, with $f \leqslant e$, for instance a map obtained by "collapsing handles" as in figure (1)) .

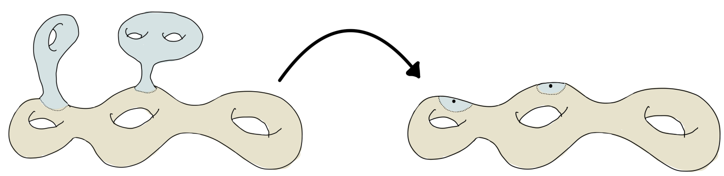

Figure 1. Collapsing handles

Let us equipped S_f with a hyperbolic metric with monodromy ρ_f. We then deduce a degree 1 map from π from $M(S,1)$ to $M(S_f,1)$ and $\pi_*\rho_f$ has Euler class f. Thus using lemma 1.2.3, we have on $M = M(S,1)$

$$\mathrm{CS}_M\left(\rho, \mathrm{Id}\right) = \mathrm{CS}_M\left(\rho_f, \mathrm{Id}\right) = -\frac{1}{6}f^2,$$

where the last equality comes from proposition 3.0.2. Finally since we have a degree k map from $M(S,1)$ to $M(S,k)$ we get the result from lemma 1.2.3. □

3.1. **Tholozan's volume formula.** We can now prove formula (1). Let S be a surface of negative characteristic e. Let $M = M(S,k)$ be equipped with an AdS structure described by the representations ρ of Euler characterstic e and σ of Euler characteristic f. It follows from proposition 2.2.1, that

$$\mathrm{CS}_M\left(\rho, \sigma\right) = -\frac{1}{24\pi^2}\,\mathrm{Vol}(M) \quad .$$

On the other hand, from the definition of the Chern–Simons invariant, we have

$$\mathrm{CS}_M\left(\rho, \sigma\right) = \mathrm{CS}_M\left(\rho, \mathrm{Id}\right) - \mathrm{CS}_M\left(\sigma, \mathrm{Id}\right) = \frac{1}{6k}\left(f^2 - e^2\right) \ .$$

where we used Chasles relation (lemma 1.2.1) in the first equality and the last equality comes from proposition 3.0.1. It follows that

$$\mathrm{Vol}(M) = \frac{4\pi^2}{k}\left(e^2 - f^2\right) \quad .$$

which is what we wanted to prove.

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