ON SURFACES WITH FINITE TOTAL CURVATURE IN RANK 2

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Abstract. We show that complete maximal surfaces in anti-de Sitter space and hyperbolic affine spheres in $\mathbb{R}^3$ have finite total curvature if and only if they are conformally planar and their embedding data is determined by a holomorphic polynomial differential on the complex plane. Moreover, we prove an analogous result for maximal surfaces in $\mathbb{H}^{2,2}$ under the additional assumption that they belong to the Hitchin section.

Contents

Introduction

Recent works ([Tam19], [DW15]) study the geometry of polynomial maximal surfaces in anti-de Sitter space and hyperbolic affine spheres in $\mathbb{R}^3$ in terms of their boundary at infinity. Precisely, in [DW15] the authors prove that there is a one-to-one correspondence between convex polygons in $\mathbb{RP}^3$ and polynomial hyperbolic affine spheres. Similarly, [Tam19] shows that polynomial maximal surfaces in anti-de Sitter space are in bijection with light-like polygons in the 2-dimensional Einstein Universe. In this short note, we give a different characterization of these surfaces:

Theorem A. *A complete maximal surface in anti-de Sitter space or hyperbolic affine sphere in $\mathbb{R}^3$ is polynomial if and only if it has finite total curvature.*

The proof is identical in both cases and is based on the analysis of the solutions to the vortex equation on the plane ([Li19]), following [HTTW95] closely. Indeed, these surfaces are determined by a Riemannian metric $g_k = e^{u_k}|dz|^2$ on the complex plane and a holomorphic k-differential q_k. Here, $k = 2$ in the case of maximal surfaces in anti-de Sitter space, whereas $k = 3$ for hyperbolic affine spheres. The metric g_k and the holomorphic differential q_k are then related by the vortex equation

$$\|q_k\|^2_{g_k} - 1 = \kappa_{g_k} \tag{1}$$

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which expresses the Gauss equation for maximal surfaces in anti-de Sitter space and the structural equation of affine spheres in $\mathbb{R}^3$. Using the geometry of these surfaces, we will deduce that $u_k - \frac{1}{k}\log(|q_k|^2)$ is a subharmonic function which, together with finiteness of the total curvature, will imply that $|q_k|^{\frac{2}{k}}$ is a complete singular metric on $\mathbb{C}$ with finitely many zeros and hence q_k is a polynomial on the plane. The other implication in Thereom A is instead based on precise estimates on $u_k - \frac{1}{k}\log(|q_k|^2)$, which will allow to compute explicitly the total curvature in the case where q_k is a polynomial and find that it is a rational multiple of π depending on the degree of the polynomial.

In the last part of the paper, we focus on maximal surfaces with finite total curvature in the pseudo-hyperbolic space $\mathbb{H}^{2,2}$. Associated to such surfaces is a holomorphic quartic differential q_4 that partly determines their second fundamental form ([TW20], [CTT19]). Our main result is the following:

Theorem B. *A complete maximal surface in $\mathbb{H}^{2,2}$ with finite total curvature has a polynomial quartic differential q_4. Conversely, if q_4 is a polynomial and the surface belongs to the Hitchin section then it has finite total curvature.*

Maximal surfaces in $\mathbb{H}^{2,2}$ with polynomial quartic differential belonging to Hitchin section were extensively studied by the author, in collaboration with Mike Wolf, in a previous work ([TW20]), where, among the other things, it is proved that these surfaces bound negative light-like polygons in the 3-dimensional Einstein Universe. However, the space of such polygons is not connected and our family of surfaces only fills up one connected component. One expects that the other components correspond to different families of polynomial maximal surfaces. The extra assumption in Theorem B reflects then the fact that the existence of such surfaces is still conjectural and their structural equations have not been studied yet. We conjecture that all maximal surfaces in $\mathbb{H}^{2,2}$ bounding a negative light-like polygon have finite total curvature. Theorem B is a step forward this result.

The proof of Theorem B follows the same line as the proof of Theorem A. When a maximal surface in $\mathbb{H}^{2,2}$ belongs to the Hitchin section, the estimates for their structural equations in our previous work ([TW20]) allow to explicitly compute the total curvature and show that it is a rational multiple of π that only depends on the degree of the polynomial. For the other implication, we rely on the fact that these surfaces have bounded geometry ([LT20]), to deduce a bound on the number of zeros of the quartic differential based on the total curvature. Then the proof follows the same lines as Theorem A.

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1. Background material

We review in this section the fundamentals of the theory of maximal surfaces in anti-de Sitter space and hyperbolic affine sphere in $\mathbb{R}^3$.

1.1. **Maximal surfaces in anti-de Sitter space.** Consider the vector space $\mathbb{R}^4$ endowed with the bilinear form of signature $(2,2)$

$$\langle x, y\rangle = x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3 \ .$$

Anti-de Sitter is the quadric

$$\mathrm{AdS}_3 = \{x \in \mathbb{R}^4 \mid \langle x, x\rangle = -1\} \ .$$

It can be easily verified that AdS_3 is diffeomorphic to a solid torus and the restriction of the bilinear form to the tangent space at each point endows AdS_3 with a Lorentzian metric of constant sectional curvature -1.

Let $U \subset \mathbb{C}$ be a simply connected domain. We say that $f : U \to \mathrm{AdS}_3$ is a space-like embedding if f is an embedding and the induced metric $g_2 = f^*g_{\mathrm{AdS}}$ is Riemannian. The Fundamental Theorem of surfaces embedded in anti-de Sitter space ensures that such a space-like embedding is uniquely determined, up to post-composition by a global isometry of AdS_3, by its induced metric g_2 and its shape operator $B : TU \to TU$, which satisfy

$$\begin{cases} d^{\nabla}B = 0 & \text{(Codazzi equation)} \\ \kappa_{g_2} = -1 - \det(B) & \text{(Gauss equation)} \end{cases}$$

where ∇ is the Levi-Civita connection and κ_{g_2} is the curvature of the induced metric on $S = f(U)$. We will always assume in this paper that the induced metric g_2 is complete.

We say that $f : U \to \mathrm{AdS}$ is maximal if B is traceless. In this case, the Codazzi equation implies that the second fundamental form $II = g_2(B\cdot, \cdot)$ is the real part of a quadratic differential q_2, which is holomorphic for the complex structure compatible with the induced metric g_2 on S. If we choose a conformal coordinate z on U, there is a function $u_2 : U \to \mathbb{R}$ such that $g_2 = e^{u_2}|dz|^2$ and a holomorphic function $p(z)$ so that $q_2 = p(z)dz^2$. The Gauss equation can then be re-written as

$$\kappa_{g_2} = -1 + \|q_2\|^2_{g_2} \tag{2}$$

or, equivalently,

$$\Delta u_2 = 2e^{u_2} - 2e^{-u_2}|p(z)|^2 \ . \tag{3}$$

Vice versa, given a holomorphic quadratic differential $q_2 = p(z)dz^2$ on U and a solution u_2 to Equation (3) there is a unique maximal embedding $f : U \to \mathrm{AdS}_3$ with induced metric $g_2 = e^{u_2}|dz|^2$ and second fundamental form given by the real part of q_2, up to post-composition by a global isometry.

1.2. **Hyperbolic affine spheres.** Hyperbolic affine spheres arise when studying immersions of surfaces in $\mathbb{R}^3$ up to volume-preserving affine transformations.

Let $U \subset \mathbb{C}$ be a simply connected domain. Consider a strictly convex immersion $f : U \to \mathbb{R}^3$ and choose ξ a vector field transverse to $H = f(U)$. We can split the standard flat connection D into a tangential part ∇ and a transversal part

$$\begin{aligned} D_{f_*X} f_*Y &= f_*(\nabla_X Y) + g_3(X,Y)\xi \\ D_{f_*X}\xi &= -f_*(B(X)) + \tau(X)\xi \ . \end{aligned}$$

One can check that ∇ is a torsion-free connection, g_3 is a symmetric bilinear form, B is an endomorphism of TH and τ is a one-form on H, for any choice of the transverse vector field ξ. We say that ξ is an *affine normal* to f if it satisfies the following requirements:

- g_3 is positive definite;
- $\tau = 0$;
- for any linearly independent vectors X and Y, $\det(X,Y,\xi)^2 = g_3(X,Y)$.

In this case, ∇ is called Blaschke connection and g_3 is the Blaschke metric. Moreover, we say that H is a hyperbolic affine sphere if $B(X) = -X$ for every vector field X. It turns out that, up to post-composition by a volume-preserving affine transformation of $\mathbb{R}^3$, a hyperbolic affine sphere is uniquely determined by the Blaschke metric and the Pick differential, which is defined as follows. We choose coordinates so that the Blaschke metric g_3 is given by $g_3 = e^{u_3}|dz|^2$. This means that the complex tangent vectors $f_z = f_*(\frac{\partial}{\partial z})$ and $f_{\bar{z}} = f_*(\frac{\partial}{\partial \bar{z}})$ satisfy

$$g_3(f_z, f_z) = g_3(f_{\bar{z}}, f_{\bar{z}}) = 0 \quad \text{and} \quad g_3(f_z, f_{\bar{z}}) = \frac{1}{2}e^{u_3} \ .$$

Let $\hat{\theta}$ and θ be the matrices of one-forms expressing the Levi-Civita connection of g_3 and the Blaschke connection, respectively. We define the Pick form C by

$$\hat{\theta}_i^j - \theta_i^j = C_{ik}^j \rho^k$$

where $\rho^1 = dz$ and $\rho^{\bar{1}} = d\bar{z}$ are the dual one-forms. The property of the affine normal, together with the symmetries of the Pick form, implies that

$$\theta = \begin{pmatrix} \theta_1^1 & \theta_{\bar{1}}^1 \\ \theta_{\bar{1}}^1 & \theta_{\bar{1}}^{\bar{1}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\partial u_3 & \bar{q}_3 e^{-u_3} d\bar{z} \\ q_3 e^{-u_3} dz & \sqrt{2}\bar{\partial} u_3 \end{pmatrix}$$

where $q_3 = \sqrt{2} C_{11}^{\bar{1}} e^{u_3}$. The Blaschke metric g_3 and the Pick differential q_3 satisfy the following system of PDEs

$$\begin{cases} \bar{\partial} q_3 = 0 \\ \kappa_{g_3} = -1 + \|q_3\|_{g_3}^2 \ , \end{cases}$$

where we recognize that the first equation simply says that q_3 is a holomorphic cubic differential and the second equation can be re-written as

$$\Delta u_3 = 2e^{u_3} - 2e^{-2u_3}|q_3|^2 \ . \tag{4}$$

Vice versa, given a holomorphic cubic differential q_3 on U and a solution u_3 to Equation (4) there is a unique hyperbolic affine sphere $f : U \to \mathbb{R}^3$ with Blaschke metric $g_3 = e^{u_3}|dz|^2$ and Pick differential q_3, up to post-composition by a volume-preserving affine transformation.

1.3. **Maximal surfaces in $\mathbb{H}^{2,2}$.** Consider the vector space $\mathbb{R}^5$ endowed with the bilinear form of signature $(2,3)$

$$\langle x, y\rangle = x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \ .$$

The pseudo-hyperbolic space $\mathbb{H}^{2,2}$ is the quadric

$$\mathbb{H}^{2,2} = \{x \in \mathbb{R}^5 \mid \langle x, x\rangle = -1\} \ .$$

It can be easily verified that $\mathbb{H}^{2,2}$ is diffeomorphic to $D^2 \times S^2$ and the restriction of the bilinear form to the tangent space at each point endows $\mathbb{H}^{2,2}$ with a pseudo-Riemannian metric of signature $(2,2)$ and constant sectional curvature -1.

Let $U \subset \mathbb{C}$ be a simply connected domain. We say that $f : U \to \mathbb{H}^{2,2}$ is a space-like embedding if f is an embedding and the induced metric $g_4 = f^*g_{\mathbb{H}^{2,2}}$ is Riemannian. Note that, if f is space-like, the normal bundle NS to $S = f(U)$ inherits a negative definite Riemannnian metric g_N from $g_{\mathbb{H}^{2,2}}$. A space-like surface is uniquely determined, up to post-composition by a global isometry of $\mathbb{H}^{2,2}$, by its induced metric g_4 and its second fundamental form $\mathrm{II} : TS \times TS \to NS$, which satisfy

$$\begin{cases} d^\nabla \mathrm{II} = 0 & \text{(Codazzi equation)} \\ \kappa_{g_4} = -1 + \frac{1}{2}\|\mathrm{II}\|^2 & \text{(Gauss equation)} \end{cases}$$

where

$$\|\mathrm{II}\|^2 = -\max_{|v|=1}\sum_{i=1}^{2} g_{\mathbb{H}^{2,2}}(\mathrm{II}(v, e_i), \mathrm{II}(v, e_i))$$

with (e_1, e_2) being a local orthonormal frame of TS. We will assume that the induced metric g_4 is complete.

We say that $f : U \to \mathbb{H}^{2,2}$ is maximal if II is traceless, where

$$\mathrm{tr}_{g_4}(\mathrm{II}) = \mathrm{II}(e_1, e_1) + \mathrm{II}(e_2, e_2) \ .$$

In this case, the second fundamental form II is the real part of a holomorphic section $\sigma : T^{1,0}S \times T^{1,0}S \to N^{\mathbb{C}}S$. The tensor $q_4 = g_N(\sigma, \sigma)$ is then a holomorphic quartic differential on S.

In particular, we can define a map Ψ from the space $\mathcal{M}(\mathbb{H}^{2,2})$ of complete maximal surfaces in $\mathbb{H}^{2,2}$ to the space of holomorphic quartic differentials. We say that a maximal surface $S \in \mathcal{M}(\mathbb{H}^{2,2})$ is polynomial if S is conformally planar and the corresponding quartic differential $\Psi(S)$ is a polynomial over $\mathbb{C}$. In [TW20], we defined a section of the map Ψ, using wild Higgs bundles over $\mathbb{CP}^1$, by explicitly constructing a family of maximal surfaces in $\mathbb{H}^{2,2}$ with given polynomial quartic differential. Because the definition of this family resembles those equivariant with respect to $\mathrm{Sp}(4,\mathbb{R})$-Hitchin representations, we refer to the surfaces studied in [TW20] as belonging to the Hitchin section.

Remark 1.1. Maximal surfaces in $\mathbb{H}^{2,2}$ have bounded geometry (see [LT20]), hence the norm of the quartic differential q_4 with respect to the induced metric g_4 is uniformly bounded, In other words, there is a constant $C > 0$ such that

$$\|q_4\|_{g_4} = \frac{q\bar{q}}{g_4^4} \leq C \ .$$

As a consequence, if we denote by ϕ the logarithm of the density of g_4 with respect to the metric $|q_4|^{\frac{1}{2}}$, i.e. $g_4 = e^{\phi}|q_4|^{\frac{1}{2}}$, we deduce a uniform lower bound on ϕ, precisely

$$\phi = u_4 - \frac{1}{4}\log(|q_4|^2) \geq -\frac{1}{4}\log(C)$$

.

2. Proof of the main results

We first start by computing the total curvature of a polynomial maximal surface in AdS_3 and of a polynomial hyperbolic affine sphere in $\mathbb{R}^3$. The main ingredient is the following estimate for the solution of the vortex equation on the plane:

Theorem 2.1 ([DW15], [Li19])**.** *Let $q_k = p(z)dz^k$ be a holomorphic k-differential on $\mathbb{C}$ with $k \geq 2$. Then there is a unique solution $u_k : \mathbb{C} \to \mathbb{R}$ to the equation*

$$\Delta u_k = 2e^{u_k} - 2e^{-(k-1)u_k}|p(z)|^2 \tag{5}$$

such that $g_k = e^{u_k}|dz|^2$ is complete. Moreover, the following estimates hold:

a) $\frac{1}{k}\log(|p(z)|^2) \leq u_k(z)$ *;*

b) if $p(z)$ is a polynomial, then outside of a ball of sufficiently large radius, we have $\|u_k(z) - \frac{1}{k}\log(|p(z)|^2)\|_{C^1} = O(e^{-\sqrt{2k}r}/\sqrt{r})$, as $|z| \to +\infty$, where r is the $|q_k|^{\frac{1}{k}}$-distance from the zeros of $p(z)$.

We deduce that the metric g_k is always non-positively curved because

$$\kappa_{g_k} = -1 + \|q_k\|_{g_k}^2 = -1 + \frac{|p(z)|^2}{e^{ku_k}} \leq 0 \ .$$

Proposition 2.2. *Let $q_k = p(z)dz^k$ be a polynomial k-differential and let u_k be the solution to Equation* (5)*. Then the total curvature of the metric $g_k = e^{u_k}|dz|^2$ is $K_{g_k} = -\frac{2\pi}{k}\deg(p(z))$.*

Proof. Let B_R denote the ball of radius R centered at the origin. We compute

$$\begin{aligned}
K_{g_k} = \int_{\mathbb{C}} \kappa_{g_k} \, dA_{g_k} &= \lim_{R\to+\infty}\int_{B_R} -\frac{\Delta u_k}{2e^{u_k}} \, dA_{g_k} \\
&= -\frac{1}{2}\lim_{R\to+\infty}\int_{B_R} \Delta\left(u_k(z) - \frac{1}{k}\log(|p(z)|^2)\right) \, dzd\bar{z} \\
&\quad - \frac{1}{2k}\lim_{R\to+\infty}\int_{B_R} \Delta\log(|p(z)|^2) \, dzd\bar{z} \\
&= -\frac{1}{2}\lim_{R\to+\infty}\int_{\partial B_R} \frac{\partial}{\partial r}\left(u_k(z) - \frac{1}{k}\log(|p(z)|^2)\right) Rd\theta \\
&\quad - \frac{1}{k}\lim_{R\to+\infty}\int_{B_R} \Delta\log(|p(z)|) \, dzd\bar{z}.
\end{aligned}$$

Now, by Theorem 2.1 b), the first integral tends to 0 as R goes to $+\infty$. The second term computes the total curvature of the flat metric $|q_k|^{\frac{2}{k}}$ on $\mathbb{C}$. Indeed, if $z_1, \dots, z_{\deg(p)}$ denote the, possibly coincident, zeros of $p(z)$, we have

$$\frac{1}{k}\Delta \log(|p(z)|) = \sum_{i=1}^{\deg(p)} \frac{2\pi}{k}\delta_{z_i}$$

in the sense of distribution and

$$K_{|q_k|^{\frac{2}{k}}} = -\frac{1}{k}\int_{\mathbb{C}} \Delta \log(|p(z)|) \, dzd\bar{z} = -\frac{2\pi}{k}\deg(p(z)) \ .$$

Hence,

$$K_{g_k} = -\frac{2\pi}{k}\deg(p(z)) \ .$$

□

The computation for the total curvature of a polynomial maximal surface in $\mathbb{H}^{2,2}$ belonging to the Hitchin section relies instead on the following result.

Theorem 2.3 ([TW20], [LM19]). *Let $q_4 = p(z)dz^4$ be a polynomial quartic differential on $\mathbb{C}$. Then there is a unique solution (ψ_1, ψ_2) to the system*

$$\begin{cases} \frac{1}{4}\Delta\psi_1 = e^{\psi_1-\psi_2} - e^{-2\psi_1}|p(z)|^2 \\ \frac{1}{4}\Delta\psi_2 = e^{2\psi_2} - e^{\psi_1-\psi_2} \ . \end{cases} \tag{6}$$

Moreover, the functions $\psi_i : \mathbb{C} \to \mathbb{R}$ satisfy:

a) $\psi_1(z) \geq \frac{3}{8}\log(|p(z)|^2)$ and $\psi_2(z) \geq \frac{1}{8}\log(|p(z)|^2)$;
b) outside a ball of sufficiently large radius

$$\left\| \psi_1(z) - \psi_2(z) - \frac{1}{4}\log(|p(z)|^2) \right\|_{C^1} \leq O(e^{-4r}/\sqrt{r}) \quad \text{as } |z| \to +\infty,$$

where r is the $|q_4|^{\frac{1}{4}}$-distance from the zeros of $p(z)$.

Proposition 2.4. *Let $g_4 = e^{u_4}|dz|^2$ be the induced metric on a polynomial maximal surface in $\mathbb{H}^{2,2}$ with holomorphic quartic differential $q_4 = p(z)dz^4$ belonging to the Hitchin section. Then the total curvature of g_4 is $K_{g_4} = -\frac{\pi}{2}\deg(p(z))$.*

Proof. The induced metric g_4 can be written in terms of the solution to the system (6) as $g_4 = 4e^{\psi_1-\psi_2}|dz|^2$. Therefore, if B_R denotes the ball of radius R centered at the origin, we can compute

$$
\begin{aligned}
K_{g_4} &= \int_{\mathbb{C}} \kappa_{g_4}\, dA_{g_4} = \lim_{R\to+\infty} \int_{B_R} -\frac{\Delta(\psi_1-\psi_2)}{8e^{\psi_1-\psi_2}}\, dA_{g_4} \\
&= -\frac{1}{2}\lim_{R\to+\infty}\int_{B_R} \Delta\left(\psi_1-\psi_2-\frac{1}{4}\log(|p(z)|^2\right) dzd\bar{z} \\
&\quad -\frac{1}{8}\lim_{R\to+\infty}\int_{B_R} \Delta\log(|p(z)|^2)\, dzd\bar{z} \\
&= -\frac{1}{2}\lim_{R\to+\infty}\int_{\partial B_R} \frac{\partial}{\partial r}\left(\psi_1-\psi_2-\frac{1}{4}\log(|p(z)|^2\right) Rd\theta \\
&\quad -\frac{1}{8}\lim_{R\to+\infty}\int_{B_R} \Delta\log(|p(z)|^2)\, dzd\bar{z}
\end{aligned}
$$

Now, by Theorem 2.3, the first integral tends to 0 as R goes to $+\infty$. The second term, as we saw in the proof of Proposition 2.2, computes the total curvature of the flat metric with cone singularities $|q_4|^{\frac{1}{4}}$, which is equal to $-\frac{\pi}{2}\deg(p(z))$. Hence, $K_{g_4} = -\frac{\pi}{2}\deg(p(z))$, as claimed. □

We now move to the proof of the other implication in Theorem A and Theorem B. We let S_k be a complete maximal surface in anti-de Sitter space ($k=2$), a complete hyperbolic affine sphere in $\mathbb{R}^3$ ($k=3$), or a complete maximal surface in $\mathbb{H}^{2,2}$ ($k=4$) with finite total curvature. Completeness ensures that S_k is homeomorphic to a disk ([Lof01, Theorem 3], [Tam19, Proposition 3.1], [TW20, Proposition 6.3]). We denote by g_k and q_k the embedding data introduced in the previous section. A general result of Huber ([Hub57, Theorem 15]) shows that an open, complete, non-positively curved Riemannian surface with finite total curvature is parabolic, thus S_k is conformally planar. Choosing conformal coordinates on $\mathbb{C}$ and writing $q_k = p(z)dz^k$, we need to prove that $p(z)$ is a polynomial. We first shows that $p(z)$ has finitely many zeros:

Lemma 2.5. *Assume that the total curvature of S_k is $K_{g_k} = -\frac{2\pi}{k}m$ for some $m \geq 0$. Then $p(z)$ has at most $\lceil m \rceil$ zeros.*

Proof. Assume by contradiction that $p(z)$ has more than $\lceil m \rceil$ zeros. Then

$$
\begin{aligned}
0 &\leq -\frac{2\pi}{k}m + \frac{2\pi}{k}\lceil m \rceil < \lim_{R\to+\infty}\int_{B_R} \kappa_{g_k}\, dA_{g_k} + \frac{1}{k}\lim_{R\to+\infty}\int_{B_R} \Delta\log(|p(z)|)\, dzd\bar{z} \\
&= -\frac{1}{2}\lim_{R\to+\infty}\int_{B_R} \Delta u_k\, dzd\bar{z} + \frac{1}{2k}\lim_{R\to+\infty}\int_{B_R} \Delta\log(|p(z)|^2)\, dzd\bar{z} \\
&= -\frac{1}{2}\lim_{R\to+\infty}\int_{\partial B_R} \frac{\partial}{\partial r}\left(u_k(z) - \frac{1}{k}\log(|p(z)|^2)\right) Rd\theta \\
&= \lim_{R\to+\infty}\left(-\pi R\frac{d}{dR}\fint_{\partial B_R}\left(u_k(z) - \frac{1}{k}\log(|p(z)|^2)\right)\right) Rd\theta\ .
\end{aligned}
$$

Hence, since the zeros of the holomorphic function $p(z)$ are discrete, there exists $\epsilon > 0$ such that the inequality

$$
\frac{d}{dR}\fint_{\partial B_R}\left(u_k(z) - \frac{1}{k}\log(|p(z)|^2)\right) Rd\theta < -\frac{\epsilon}{R}
$$

holds for all large R except possibly countably many values. Integrating in R, we get

$$\fint_{\partial B_R} \left(u_k(z) - \frac{1}{k}\log(|p(z)|^2) \right) Rd\theta < -\epsilon \log(R) + C$$

for some R sufficiently large, which contradicts Theorem 2.1 part a) when $k = 2, 3$ and Remark 1.1 if $k = 4$. □

We can now conclude the proof of Theorem A and Theorem B. Repeating the same estimates as in Lemma 2.5, there is a constant $C > 0$ such that

$$\fint_{\partial B_R} \left(u_k(z) - \frac{1}{k}\log(|p(z)|^2) \right) Rd\theta < \frac{2m}{k}\log(R) + C$$

for all R sufficiently large, except for at most countably many values. Because the metric g_k is negatively curved, the function $u_k(z) - \frac{1}{k}\log(|p(z)|^2)$ which represents the logarithmic density of the metric g_k with respect to $|q_k|^{\frac{2}{k}}$ is subharmonic. Therefore, for any z sufficiently far from the origin, we have

$$\begin{aligned}
u_k(z) - \frac{1}{k}\log(|p(z)|^2) &\leq \fint_{B_{\frac{|z|}{2}}(z)} \left(u_k(w) - \frac{1}{k}\log(|p(w)|^2) \right) \, dzd\bar{z} \\
&= \frac{4}{\pi|z|^2} \int_{B_{\frac{|z|}{2}}(z)} \left(u_k(w) - \frac{1}{k}\log(|p(w)|^2) \right) \, dzd\bar{z} \\
&\leq \frac{4}{\pi|z|^2} \int_{B_{\frac{3|z|}{2}}(z)\setminus B_{\frac{|z|}{2}}(z)} \left(u_k(w) - \frac{1}{k}\log(|p(w)|^2) \right) \, dzd\bar{z} \\
&= \frac{4}{\pi|z|^2} \int_{\frac{|z|}{2}}^{\frac{3|z|}{2}} \int_0^{2\pi} \left(u_k(Re^{i\theta}) - \frac{1}{k}\log(|p(Re^{i\theta})|^2) \right) Rd\theta dR \\
&\leq \frac{4}{\pi|z|^2} \int_{\frac{|z|}{2}}^{\frac{3|z|}{2}} \left(2\pi R \left(\frac{2m}{k}\log(R) + C \right) \right) dR \\
&\leq a_1 + b_1 \log(|z|)
\end{aligned}$$

for some $a_1, b_1 > 0$. Exponentiating both sides, we deduce that, for $|z|$ sufficiently large,

$$e^{u_k} \leq e^{a_1}|z|^{b_1}|p(z)|^{\frac{2}{k}},$$

hence the metric $|z|^{b_1}|p(z)|^{\frac{2}{k}}$ is complete, because g_k is complete, and has finitely many zeros. It is well-known that this implies that $p(z)$ is a polynomial. See for instance [Oss86, Lemma 9.6].

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