ANOSOV REPRESENTATIONS: INFORMAL LECTURE NOTES

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ABSTRACT. "He tried to do his best, but he could not"—Neil Young [224]

CONTENTS

Look what we found
in the park
in the dark.
We will take him home,
we will call him Clark.

He will live at our house,
He will grow and grow.
Will our mother like this?
We don't know.
—————Dr. Suess [195]

Preface

These notes grew out of a graduate course I gave on Anosov representations at the University Michigan in Winter 2020. They became more formal, when our class went on-line, and grew to their present length due to the enforced free time I had that Spring.

The intention of the course, and these notes, was to give an introduction to the theory of Anosov representations, which would be approachable for someone with some background in the study of hyperbolic manifolds, geometric group theory, and/or Teichmüller theory. Basically, I am attempting to write the notes that I would have wanted to be available when I entered the field in 2012.

I will not be assuming that you know any Lie theory (since I don't either), although I may sometimes use aspects of this theory when convenient. Our semisimple Lie groups will almost always be $\mathsf{SL}(n,\mathbb{R})$, $\mathsf{PSL}(n,\mathbb{R})$ or well-known subgroups like $\mathsf{SO}(n,1)$. In this setting, the Lie theory we use will largely be familiar facts from linear algebra rephrased in (intentionally?) confusing ways so that they hold more generally.

Chapter 1 is a (hopefully) motivational introduction. Chapter 2 covers much, but not all, of the basic background theory on hyperbolic groups that we will need later in the notes. If you have some familiarity with hyperbolic groups and/or a willlingness to believe their properties, you can skip this chapter. It exists largely because not all the students in my class had seen the theory of hyperbolic groups before and requested a brief introduction to the theory. Chapter 3 gives a quick review of the theory of convex cocompact hyperbolic manifolds from a viewpoint which should be suggestive of later definitions in the theory of Anosov representations. It can be skipped by readers familiar with the basic theory. In Chapter 4, we give our first higher rank examples, Benoist representations, which are discrete, faithful representations of a hyperbolic group into $\mathsf{PSL}(d,\mathbb{R})$ whose images preserve and act cocompactly on strictly convex subset of $\mathbb{RP}^{d-1}$. We will see many key phenomena which occur for Anosov representations first in this more concrete setting.

In Chapter 5, we define and develop the basic properties of Anosov representations of hyperbolic groups into $\mathsf{SL}(d,\mathbb{R})$. Readers with some background in hyperbolic groups, can begin here if they want to get to the general theory of Anosov representations immediately. In particular, the material in Chapters 5 and 6 does not rely on Chapter 4 (except perhaps for the discussion of

basic linear algebra in $\mathsf{SL}(d, \mathbb{R})$ which occurs there). In Chapter 6, we derive several characterizations of Anosov representations. In particular, we see that convex cocompact representations into rank one Lie groups and Benoist representations are Anosov. We also introduce Hitchin representations. In chapter 7, we prove some of the foundational results in the emerging study of convex cocompact actions on projective spaces. In chapter 8, we give brief introductions, with occasional proof, to some more advanced topics in the theory.

Our approach does have its limitations and here is a (very) partial list of important topics we will not cover, which might naturally be part of an introduction to the field.

- Maximal representations of surface groups into Lie groups of Hermitian type as developed by Burger, Iozzi and Wienhard [52].
- The Higgs bundle approach pioneered by Hitchin [114].
- Labourie's proof [143] that Hitchin representations are Borel Anosov.
- The characterizations of Anosov representations in terms of their actions on the associated symmetric spaces and their boundaries by Kapovich, Leeb and Porti [132, 133]
- The theory of positive representations developed by Fock and Goncharov [97].
- The symplectic structure on character varieties of surface groups introduced by Goldman [103] and the recent work of Sun, Wienhard, and Zhang ([198, 199, 215]) on the symplectic structure of the Hitchin component.
- The compactification of character varieties due to Parreau [174] and more recent work on compactifications of Hitchin components and components of maximal representations by Burger-Iozzi-Parreau-Pozzetti [50].
- Compactifications of the locally symmetric spaces associated to Anosov representations, see Guéritaud-Guichard-Kassel-Wienhard [108] and Kapovich-Leeb [130].
- The collar lemmas for Hitchin representations, discovered by Lee and Zhang [147], and maximal representations, by Burger and Pozzetti [54], see also Martone-Zhang [160].

I don't intend to prepare these notes for publication, but I will try to keep them updated and corrected when necessary. I am sure there are many errors and typos scattered throughout, so please send me your comments, corrections, and criticisms.

Early Reviews

> *You got a way of saying something everyone wants to hear*
> *In a way that no one wants to hear it*
> ————Becky Warren [213]

> *I never knew a man could tell so many lies*
> *He had a different story for every set of eyes*
> *How can he remember who he's talking to?*
> *'Cause I know it ain't me, and I hope it isn't you*
> ——————Neil Young [221]

> *Them that don't know him won't like him and them that do,*
> *Sometimes won't know how to take him.*
> *He ain't wrong, he's just different but his pride won't let him,*
> *Do things to make you think he's right.*
> ————Ed and Patsy Bruce [123]

Part 1. Motivation

> *I'm climbing this ladder*
> *My head in the clouds*
> *I hope that it matters*
> *I'm having my doubts*
> ————Neil Young [223]

Anosov representations were introduced by François Labourie [143] in his study of Hitchin representations. They give a flexible generalization of the theory of convex cocompact representations of hyperbolic groups into rank one Lie groups into the setting of representations of hyperbolic groups into semi-simple Lie groups. Their theory was further developed by Guichard-Wienhard [109], Kapovich-Leeb-Porti [132, 133], Guéritaud-Guichard-Kassel-Wienhard [107], Bochi-Potrie-Sambarino [32], and others. They now serve as an organizing principle for the geometric and dynamical approach to the so-called Higher Teichmüller theory.

The most basic example of a convex cocompact representation into a rank one Lie group is a discrete faithful representation of a surface group into $\mathsf{PSL}(2,\mathbb{R})$. Here we view $\mathsf{PSL}(2,\mathbb{R})$ as the group $\mathrm{Isom}^+(\mathbb{H}^2)$ of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$. So, a discrete faithful representation of $\pi_1(S)$ into $\mathsf{PSL}(2,\mathbb{R})$ gives rise to a hyperbolic structure on the closed surface S.

Recall that the upper half-plane model for the hyperbolic plane is given by

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

with line element

$$ds_{hyp} = \frac{1}{y}\,\sqrt{dx^2 + dy^2}$$

In this metric, the geodesics are lines and semi-circles perpendicular to the real line and the group of Möbius transformations with real co-efficients acts as the group of orientation-preserving isometries of $\mathbb{H}^2$, i.e.

$$\mathrm{Isom}^+(\mathbb{H}^2) = \mathsf{PSL}(2,\mathbb{R}).$$

The classical **Teichmüller space** $\mathcal{T}(S)$ is the space of all discrete, faithful representations of $\pi_1(S)$ into $\mathsf{PSL}(2,\mathbb{R})$ (up to conjugacy in $\mathsf{PGL}(2,\mathbb{R})$). We may also think of Teichmüller space as the space of all (marked) hyperbolic structures on S (up to isotopy). It is well-known that $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$ and is identified with a component of the space of conjugacy classes of representations of $\pi_1(S)$ into $\mathsf{PSL}(2,\mathbb{R})$. Moreover, the mapping class group of (isotopy classes of) orientation-preserving homeomorphisms acts properly discontinuously on $\mathcal{T}(S)$. Teichmüller space is rich with structure and can be studied from a multitude of perspectives. Teichmüller space is the motivating example for Higher Teichmüller theory and Anosov representations.

The first natural generalization of this theory was to the study of higher-dimensional hyperbolic manifolds. We recall that $\mathsf{SO}_0(n,1) \subset \mathsf{SL}(n+1,\mathbb{R})$ is the group of orientation-preserving isometries of hyperbolic space $\mathbb{H}^n$. Here $\mathsf{SO}(n,1)$ is the group of matrices in $\mathsf{SL}(n+1,\mathbb{R})$ preserving the indefinite quadratic form with associated diagonal matrix J with entries $(1,1,\ldots,1,-1)$, i.e. $A \in \mathsf{SO}(n,1)$ if and only if $JA^TJ = A^{-1}$, while $\mathsf{SO}_0(n,1)$ is the index two subgroup of

$\mathsf{SO}(n,1)$ preserving the upper sheet of the hyperboloid

$$H^n = \{\vec{X} \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\}.$$

We may then identify $\mathbb{H}^n$ with the upper sheet of H^n with the metric induced by the indefinite quadratic form. (If this is not familiar to you, we will review this theory in Chapter 3).

If $\rho : \Gamma \to \mathsf{SO}_0(n,1)$ is a representation of a finitely presented torsion-free group Γ, then there is an associated **orbit map**

$$\tau_\rho : \Gamma \to \mathbb{H}^n$$

given by $\gamma \mapsto \rho(\gamma)(x_0)$ where x_0 is some pre-chosen point in $\mathbb{H}^n$. Recall that, given a fixed finite presentation of the group, Γ admits a word metric where $d_\Gamma(\alpha, \beta)$ is the minimal word length of a representative of $\alpha^{-1}\beta$. Then Γ acts by isometries on itself by left multiplication and the orbit map is equivariant in the sense that $\tau_\rho(\alpha\beta) = \rho(\alpha)(\tau_\rho(\beta))$ for all $\alpha, \beta \in \Gamma$. We say that ρ is **convex cocompact** if this orbit map is a quasi-isometric embedding, i.e. is a K-bilpschitz embedding on large enough scale. More formally, τ_ρ is a **quasi-isometric embedding** if there exist K and C so that

$$\frac{1}{K}d_\Gamma(\alpha, \beta) - C \leq d_{\mathbb{H}^n}(\tau_\rho(\alpha), \tau_\rho(\beta)) \leq K d_\Gamma(\alpha, \beta) + C$$

for all $\alpha, \beta \in \Gamma$. Notice that if τ_ρ is a quasi-isometric embedding then ρ is discrete and faithful. Moreover, it immediately follows, since $\mathbb{H}^n$ is Gromov hyperbolic, that Γ is a Gromov hyperbolic group.

On the other hand, if ρ is discrete and faithful and $\rho(\Gamma)$ acts cocompactly on $\mathbb{H}^n$, then the Milnor-Svarc Lemma implies that τ_ρ is a quasi-isometric embedding, so ρ is convex cocompact. However, Mostow proved that if $n \geq 3$ and $\rho(\Gamma)$ acts cocompactly, then any other discrete, faithful representation is conjugate to ρ. So hyperbolic lattices in dimensions above two will not interest us here.

In general, there exist discrete, faithful representations which are not convex cocompact. For example, consider the representation ρ of the free group $F_2 = <a, b>$ into $\mathsf{PSL}(2, \mathbb{R})$ given by $\rho(a) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ while $\rho(b)$ takes the interior of a disk of radius $\frac{1}{8}$ based at $\frac{1}{4}$ to the exterior of a disk of radius $\frac{1}{8}$ based at $\frac{3}{4}$. One may make an elementary ping-pong argument to see that ρ is discrete and faithful. However, a^n has word length n, but $d(\rho(a)^n(i), i) = O(2\log n)$. Therefore, the orbit map cannot be a quasi-isometric embedding. So, in general the space of convex cocompact representations will not constitute an entire component of the character variety. In fact, we can easily see that ρ is a limit of convex cocompact representations but also a limit of indiscrete representations (since $\rho(a)$ can be approximated by irrational rotations).

However, convex cocompact representations are stable, in the sense that if $\rho_0 \in \mathrm{Hom}(\Gamma, \mathsf{SO}_0(n,1))$ is convex cocompact, then it has an open neighborhood U so that if $\rho \in U$, then ρ is also convex cocompact. On the other hand, it is a consequence of a lemma of Margulis, that a limit of discrete, faithful representations is itself discrete and faithful, so being discrete and faithful is a closed condition. Teichmüller space has the special property of being both open and closed, since discrete, faithful representation of closed surface groups into $\mathsf{PSL}(2, \mathbb{R})$ always have images which act cocompactly on $\mathbb{H}^2$.

The key properties we will look for in a generalization of the theory of convex compact representations into the setting of higher rank Lie groups are:

(1) Our representations should induce quasi-isometric orbit maps into the quotient symmetric spaces.
(2) Our representations should form an open subset of the representation variety.

In order to give a quick definition of Anosov representations, it will be useful to recall the **singular value decomposition**. If $A \in \mathsf{SL}(n, \mathbb{R})$, then we may write $A = LDK$ where $L, K \in \mathsf{SO}(n)$ and D is a diagonal matrices with positive entries in descending order along the diagonal, i.e. $d_{11} \geq d_{22} \geq\geq \cdots \geq d_{nn} > 0$ and $d_{11}d_{22}\cdots d_{nn} = 1$. The matrix D depends only on A, but L and K need not be unique when some of the diagonal entries agree. We let $\sigma_i(A) = d_{ii}$ and call it the ith singular value of A. More geometrically, $\sigma_i(A)$ is (half) the length of the ith axis of the ellipsoid $A(S^{n-1})$.

In the hyperboloid model for $\mathbb{H}^n$ we get a nice simple formula for translation distance in terms of singular values:

$$d_{\mathbb{H}^n}(e_{n+1}, A(e_{n+1})) = \log \sigma_1(A) = \log \frac{\sigma_1(A)}{\sigma_2(A)}$$

if $A \in \mathsf{SO}_0(n, 1)$. (The second equality holds, since $\sigma_2(A) = 1$ if $A \in \mathsf{O}(n, 1)$.) Therefore, a representation $\rho : \Gamma \to \mathsf{PSO}(n, 1)$ of finitely generated group is convex cocompact if and only if there exists $K, C > 0$ so that

$$\frac{1}{K} d(id, g) - C \leq \log \left(\frac{\sigma_1(\rho(g))}{\sigma_2(\rho(g))} \right) \leq K \ d(id, g) + C$$

for all $g \in \Gamma$.

This observation hopefully motivates a particularly simple definition of an Anosov representation into $\mathsf{SL}(d, \mathbb{R})$, due to Kapovich-Leeb-Porti [132] and Bochi-Potrie-Sambarino [32].

If $1 \leq k \leq \frac{d}{2}$ and $k \in \mathbb{Z}$, we say that a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ of a finitely generated group into $\mathsf{SL}(n, \mathbb{R})$ is P_k**-Anosov** if there exists K and C so that

$$\frac{1}{K} d(id, g) - C \leq \log \left(\frac{\sigma_k(\rho(g))}{\sigma_{k+1}(\rho(g))} \right) \leq K \ d(id, g) + C$$

for all $g \in \Gamma$. A representation is called **Anosov** if it is P_k-Anosov for some k and is called **Borel Anosov** if it is P_k-Anosov for all $1 \leq k \leq \frac{d}{2}$. Notice that these same definitions apply if the image Lie group is $\mathsf{PSL}(d, \mathbb{R})$, $\mathsf{PGL}(d, \mathbb{R})$, $\mathsf{SL}^\pm(d, \mathbb{R})$ or even $\mathsf{GL}(d, \mathbb{R})$. It is immediate from this definition that the associated orbit map into the symmetric space $X_d = \mathsf{SL}(d, \mathbb{R})/\mathsf{SO}(d)$ is a quasi-isometric embedding. (We will review the structure of X_d in an elementary manner later.) It also turns out that this definition again implies that Γ is a Gromov hyperbolic group.

We note that this definition is simple to state but when we get down to business we will work more directly with Labourie's original more dynamical definition. Labourie's definition more directly indicates that Anosov representations are stable as a simple consequence of the usual stability results for hyperbolic dynamical systems.

One key feature of Anosov representations, which we will establish later, is that they admit limit maps into Grassmanians. In particular, if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k**-Anosov** then there exist continuous, ρ-equivariant maps

$$\xi_\rho^k : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d) \qquad \text{and} \qquad \xi_\rho^{d-k} : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$$

where $\mathrm{Gr}_i(\mathbb{R}^d)$ is the Grassmanian of i-planes in $\mathbb{R}^d$ and $\partial\Gamma$ is the Gromov boundary of the group Γ. Moreover, these limit maps are **transverse**, i.e. if $x \neq y \in \partial\Gamma$, then

$$\xi_\rho^k(x) \oplus \xi_\rho^{d-k}(y) = \mathbb{R}^d,$$

compatible, i.e. if $x \in \partial\Gamma$, then

$$\xi_\rho^k(x) \subset \xi_\rho^{d-k}(x),$$

and **dynamics-preserving**, i.e. if $\gamma^+ \in \partial\Gamma$ is the attracting fixed point of an infinite order element $\gamma \in \Gamma$, then $\xi_\rho^i(\gamma^+)$ is the attracting i-plane of $\rho(\gamma)$. (We will discuss the Gromov boundary of a hyperbolic group in the next chapter, but in the case of the surface group the boundary is just the circle which one can visualize as the boundary of $\mathbb{H}^2$ by embedding $\pi_1(S)$ into $\mathbb{H}^2$ via the orbit map of a Fuchsian representation.)

If Γ is finitely presented and $\rho : \Gamma \to \mathsf{SO}_0(n,1)$ is a representation, then ρ is convex cocompact if and only if ρ is P_1-Anosov. Notice that no representation into $\mathsf{SO}_0(n,1)$ can be P_k-Anosov for any $2 \leq k \leq \frac{n+1}{2}$, since $\sigma_2(A) = \cdots = \sigma_n(A) = 1$ if $A \in \mathsf{SO}_0(n,1)$. So, the theory of Anosov representations is indeed a generalization of the theory of convex cocompact representations.

Anosov representations also arise classically as the holonomy maps of strictly convex projective structures on closed n-manifolds. Concretely, a strictly convex (real) projective n-manifold M is the quotient of a strictly convex domain Ω in $\mathbb{RP}^n$ by a discrete group Γ of projective automorphisms of $\mathbb{RP}^n$ which preserve Ω. The holonomy map is then a representation of $\pi_1(M)$ into the group of projective automorphisms of $\mathbb{RP}^n$, i.e. $\mathsf{PSL}(n+1,\mathbb{R})$. There is a natural metric, typically just Finsler, on Ω, called the **Hilbert metric**, so that the stabilizer of Ω in $\mathsf{PSL}(n+1,\mathbb{R})$ acts as a group of isometries of Ω. Therefore, a convex projective manifold inherits a natural Finsler metric. This generalizes the theory of Fuchsian groups, since $\mathbb{H}^n$ may be embedded as a round disk Δ in $\mathbb{RP}^n$, so that the Hilbert metric is the classical hyperbolic metric and the stabilizer of Δ in $\mathsf{PSL}(n+1,\mathbb{R})$ is $\mathsf{PSO}(n,1)$ which may be identified with $\mathrm{Isom}^+(\mathbb{H}^n)$. We will see that if $\rho : \Gamma \to \mathsf{PSL}(n+1,\mathbb{R})$ is the holonomy map of a closed strictly convex projective manifold, then ρ is P_1-Anosov. We will review some of Benoist's beautiful work on strictly convex projective manifolds in Chapter 4.

Hitchin representations are one of the most prominent classes of Anosov representations. A representation $\rho : \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ is said to be d-**Fuchsian** if it can be written in the form $\tau_d \circ \sigma$ where $\sigma : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ is discrete and faithful and $\tau_d : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ is the irreducible representation. A representation $\rho : \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ is said to be **Hitchin** if it lies in the same component of $\mathrm{Hom}(\pi_1(S), \mathsf{PSL}(d,\mathbb{R}))$ as a d-Fuchsian representation. Hitchin [114] showed that the space of (conjugacy classes of) Hitchin representations of a closed surface of genus g in $\mathsf{PSL}(n,\mathbb{R})$ admits a real analytic diffeomorphism to $\mathbb{R}^{(n^2-1)(2g-2)}$. Labourie [143] showed that Hitchin representations are Borel Anosov. We will introduce Hitchin representations, but Labourie's seminal result will be beyond our purview.

With this definition, it is an easy exercise to check that $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is P_k-Anosov if and only if

$$\Lambda^k \rho : \Gamma \to SL(\Lambda^k \mathbb{R}^d)$$

is P_1-Anosov (where $\Lambda^k \rho$ is the kth exterior power representation.) So P_1-Anosov representations are in some sense the general case. On the other hand, there are very few known Borel Anosov representations, and we now know many restrictions on such representations.

One might naturally ask why we don't simply look at representations into $\mathsf{SL}(n, \mathbb{R})$ whose orbit maps are quasi-isometric embeddings. The basic answer is that quasi-isometric embeddings into non-positively curved spaces are not stable. A very basic example of this phenomenon is a sequence of circles through the origin in the plane with larger and larger radius which converge to a line through the origin. So, there is a sequence of rotations in the plane which converge to a translation. So, there are a sequence of representations of $\mathbb{Z}$ into the isometry group of Euclidean space whose limit is a quasi-isometric embedding while none of the representations in the sequence is a quasi-isometric embedding. We will later give examples of quasi-isometric embeddings of the free group on 2 generators into $\mathsf{SL}(d, \mathbb{R})$ which fail to be Anosov.

A representation $\rho : \Gamma \to \mathsf{SO}_0(n, 1)$ is known to be convex cocompact if and only if there exists a convex $\rho(\Gamma)$-invariant subset C of $\mathbb{H}^n$, so that $C/\rho(\Gamma)$ is compact. (This definition is responsible for the name convex cocompact.) One might attempt to generalize this definition, but Kleiner-Leeb [140] and Quint [181] showed that this does not result in a robust theory, since the only Zariski dense subgroups of $\mathsf{SL}(d, \mathbb{R})$ which are convex cocompact in this sense are uniform lattices, i.e. act cocompactly on X_d (or on $\mathsf{SL}(d, \mathbb{R})$ itself). The basic issue is that the convex hull construction is "too big" in a non-positively curved space. For example, the convex hull of the boundary of the positive quadrant in the plane (which is itself a $\sqrt{2}$-bilipschitz embedding of a line) is the entire positive quadrant, while in the hyperbolic plane the convex hull of the image of any bilipschitz embedding of a line is always contained in a bounded metric neighborhood of the image. However, in Chapter 7, we will see that Danciger-Guéritaud-Kassel [84] and Zimmer [232] have recently pioneered a way to think of Anosov representations as convex cocompact actions on projective spaces.

Part 2. Hyperbolic groups

In this chapter, we give a brief introduction to the portions of the theory of hyperbolic groups which we will use later in the notes. Readers familiar with the basics of hyperbolic groups can skip this section, while those looking for a more complete treatment can consult the expository texts by Bridson-Haefliger [49], Coornaert-Delzant-Papadopoulos [72], Drutu-Kapovich [91], Ghys-de la Harpe [101], or other experts in the field. One can also consult the on-line lecture notes of Panos Papasoglu [14] (as transcribed by Michael Batty), Alessandro Sisto [190] or Karen Vogtmann [212]. Our treatment will be largely inspired by the approach taken by Bridson-Haefliger [49]. I recommend learning this material elsewhere, but if you insist on learning it here, I will give you my amateur perspective on the field.

When convenient I will restrict to the simpler setting of torsion-free groups and try to indicate what is known in the general case. Since every finitely generated subgroup of $\mathsf{SL}(d, \mathbb{R})$ has a finite index subgroup which is torsion-free (Selberg [189], see also Alperin [6]) and a representation is Anosov if and only if its restriction to a finite index subgroup is Anosov (see Corollary 32.3), this will almost always suffice for our purposes.

> *Now if 6 turned out to be 9*
> *I don't mind, I don't mind*
> *Alright, if all the hippies cut off all their hair*
> *I don't care, I don't care*
> *Dig, 'cos I got my own world to live through*
> *And I ain't gonna copy you*
> ————Jimi Hendrix [113]

1. Quasi-isometries and the Milnor-Svarc Lemma

Quasi-isometries and quasi-isometric embeddings are natural classes of mappings in the context of geometric group theory. They are generalizations of bilipschitz homeomorphisms and embeddings which ignore the local structure. However, they need not even be continuous. For example, an infinite line is quasi-isometric to both an infinite Euclidean cylinder and to $\mathbb{Z}$ and all compact metric spaces are quasi-isometric. One justification for working in this looser context, is that the natural geometric structure on a group, given by a word metric associated to some (finite) generating set, is only well-defined up to quasi-isometry.

We will always work in the setting of proper length spaces. A metric space is **proper** if all closed metric balls are compact. A proper metric space X is a **length space** if given any $x, y \in X$, then there exists a rectifiable path joining x to y of length $d(x, y)$. If J is an interval in $\mathbb{R}$ and $\alpha : J \to X$ is a path so that $d(\alpha(s), \alpha(t)) = |t - s|$ for all $s, t \in J$, then we say that α is a **geodesic**. Notice that in this case $\alpha([s, t])$ has length $t - s$ if $t > s$. An action of a group Γ on X is **properly discontinuous** if whenever $K \subset X$ is compact, $\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\}$ is finite. (I include this definition since some standard texts in general topology include the non-standard assumption that the group acts freely to the definition of proper discontinuity.)

A map $f : Y \to Z$ between metric spaces is a **quasi-isometric embedding** if there exists $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K} d_Y(a, b) - C \leq d_Z(f(a), f(b)) \leq K d_Y(a, b) + C$$

for all $a, b \in Y$. If we want to remember the constants, we say that f is a (K, C)-quasi-isometric embedding. We say that $f : X \to Y$ is a **quasi-isometry** if there exists $K \geq 1$ and $C \geq 0$ so that f is a (K, C)-quasi-isometric embedding and if $y \in Y$, then there exists $x \in X$ so that $d(f(x), y) \leq C$, i.e. f is a quasi-isometric embedding which is coarsely surjective. One may think of quasi-isometric embeddings as bilipschitz embeddings "in the large," where you don't care at all what happens on the "scale" of the additive constant C.

If $f : X \to Y$ is a quasi-isometry, one may define a **quasi-inverse** $g : Y \to X$, i.e. a quasi-isometry so that there exists $\hat{C}$ so that $d_X(x, g(f(x)) \leq \hat{C}$ and $d_Y(y, f(g(y)) \leq \hat{C}$ for all $x \in X$ and $y \in Y$. There is only one sensible way to construct g. Given $y \in Y$, there exists some $x \in X$ so that $d(f(x), y)) \leq C$, and we set $g(y) = x$. If you haven't done so before, I recommend checking the claim that g is a quasi-inverse for yourself. Notice that the quasi-inverse is far from canonical.

The Milnor-Svarc lemma assures us that if a group acts properly discontinuously and cocompactly on two spaces, then the spaces are quasi-isometric. This allows one to freely study finitely presented groups by studying their actions on spaces, since any such space is quasi-isometric to the Cayley graph of the group. Moreover, any two Cayley graphs for a group (with respect to different finite generating sets) are quasi-isometric.

Lemma 1.1. (Milnor-Svarc Lemma) *If Γ acts properly discontinuously, cocompactly and by isometries on a proper, length space X, then Γ is finitely generated and the orbit map $\Gamma \to X$ given by $\gamma \mapsto \gamma(x_0)$, for all $\gamma \in \Gamma$ and some $x_0 \in X$, is a quasi-isometry.*

Proof of Milnor-Svarc Lemma: Choose a compact subset $D \subset X$ so that
$$\Gamma(U) = \bigcup_{\gamma \in \Gamma} \gamma(U) = X$$
where U is the interior of D. Let $S = \{\gamma \in \Gamma \mid \gamma(D) \cap D \neq \emptyset\}$. By assumption S is finite.

We first claim that S generates Γ. If not, then let H be the proper subgroup of Γ generated by S and let $V = H(U)$ and $W = (\Gamma \setminus H)(U)$. Since X is connected (as it is a length space) and $X = V \cap W$, there exists $x \in V \cap W$. Then $x = h(p)$ and $x = g(q)$, where $p, q \in U$, $h \in H$ and $g \in \Gamma \setminus H$. But then $g^{-1}h(p) = q$, so $g^{-1}h \in S \subset H$, which implies that $g \in H$, so we have achieved a contradiction.

We now choose x_0 and show that the orbit map $\tau : \Gamma \to X$ given by $\tau(\gamma) = \gamma(x_0)$ is a quasi-isometry with respect to the generating set S chosen above.

Let $K_0 = \max_{s \in S} d(x_0, s(x_0))$. Then $d(x_0, \gamma(x_0)) \leq K_0 d_S(1, \gamma)$ for all $\gamma \in \Gamma$, so, by the equivariance of τ,
$$d(\alpha(x_0), \beta(x_0)) \leq K_0 d_S(\alpha, \beta)$$
for all $\alpha, \beta \in \Gamma$. (Here, the group is acting by left multiplication.)

Choose r so that $D \subset B(r, x_0)$ and let $T = \{\gamma \in \Gamma \mid \gamma(B(3r, x_0)) \cap B(3r, x_0) \neq \emptyset\}$. Again, by assumption, T is finite.

Let $\gamma \in \Gamma$ and let L be a geodesic segment in X joining x_0 to $\gamma(x_0)$. Divide L up into
$$n = \left\lfloor \frac{d(x_0, \gamma(x_0))}{r} \right\rfloor + 1$$
segments of equal length, with endpoints $\{x_0, x_1, \ldots, x_n\}$. Notice that each segment has length less than r. Since $X = \Gamma(U) \subset \Gamma(B(r, x_0))$, there exists, for each i, $\gamma_i \in \Gamma$ so that

$d(x_i, \gamma_i(x_0)) < r$ where we may choose $\gamma_0 = id$ and $\gamma_n = \gamma$. Then, since $d(\gamma_i(x_0), \gamma_{i+1}(x_0)) < 3r$ (by the triangle inequality), $\gamma_i^{-1}\gamma_{i+1} \in T$. Notice that

$$\gamma = \gamma_0(\gamma_0^{-1}\gamma_1)(\gamma_1^{-1}\gamma_2)\cdots(\gamma_{n-1}^{-1}\gamma_n)$$

so $d_T(id, \gamma) \le n$. Notice that since T is finite and S generates Γ, there exists K_1 such that $d_S(1, t) \le K_1$ for all $t \in T$. Therefore,

$$d_S(id, \gamma) \le K_1 n = K_1 \left(\left\lfloor \frac{d(x_0, \gamma(x_0))}{r} \right\rfloor + 1 \right) \le \frac{K_1}{r} d(x_0, \gamma(x_0)) + K_1$$

so

$$\frac{r}{K_1} d_S(id, \gamma) - r \le d(x_0, \gamma(x_0))$$

and, since τ is Γ-equivariant,

$$\frac{r}{K_1} d_S(\alpha, \beta) - r \le d(\alpha(x_0), \beta(x_0))$$

for all $\alpha, \beta \in \Gamma$.

Finally, notice that, by construction, every point in X lies within r of $\tau(\Gamma)$. Therefore, τ is a $\left(\max\{K_0, \frac{K_1}{r}\}, r\right)$-quasi-isometry. $\qquad\square$

2. Gromov hyperbolic spaces

We will say that a proper length space X is (Gromov) **δ-hyperbolic** if whenever T is a geodesic triangle in X with sides s_1, s_2 and s_3 and $y \in s_1$, then $d(y, s_2 \cup s_3) \le \delta$. If X is δ-hyperbolic for some δ, we often simply say that it is **Gromov hyperbolic** or simply **hyperbolic**.

The simplest examples of Gromov hyperbolic spaces are trees, which are 0-hyperbolic. The name is motivated, in part, by the observation that $\mathbb{H}^n$ is hyperbolic.

Lemma 2.1. *Hyperbolic space $\mathbb{H}^n$ is $\cosh^{-1}(2)$-hyperbolic for any n.*

Proof. Let T be a geodesic triangle in $\mathbb{H}^n$ with sides s_1, s_2 and s_3. Since any three points in $\mathbb{H}^n$ are contained in a totally geodesic, isometrically embedded copy of $\mathbb{H}^2$, we may assume that $n = 2$.

By the Gauss-Bonnet Theorem, T has area at most π. If $y \in s_1$ and $r = d(y, s_2 \cup s_3)$, then T contains a half-disk D of hyperbolic radius r. Since D has area $\pi \cosh r - \pi$, we see that

$$\pi \cosh r - \pi \le \pi,$$

so $r \le \cosh^{-1}(2) \approx 1.317$. $\qquad\square$

Remarks: 1) Actually, $\mathbb{H}^n$ is δ-hyperbolic for $\delta = \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.8814$.

2) A stronger notion of negative curvature is given by considering CAT(-1)-spaces. One says that a proper length space is CAT$(-k)$, for some $k \ge 0$, if every geodesic triangle is at least as thin as the triangle with the same lengths in a simply connected, complete Riemannian surface of curvature $-k$. The Comparision Theorem in Riemannian geometry implies that any simply connected Riemannian manifold with sectional curvature $\le -k$ is CAT$(-k)$. The above lemma implies that CAT$(-k)$ spaces are $\cosh^{-1}(2)/k^2$-hyperbolic if $k > 0$.

The key property of Gromov hyperbolic spaces which we will need is the Fellow Traveller Property which tells us that quasi-geodesics remain a bounded distance from actual geodesics in a hyperbolic space. Notice that this is far from true in Euclidean geometry.

Theorem 2.2. (Fellow Traveller Property) *Given (K, C) and δ there exists R so that if X is δ-hyperbolic and $f : [a, b] \to X$ is a (K, C)-quasi-isometric embedding and L is a geodesic joining $f(a)$ to $f(b)$, then the Hausdorff distance between L and $f([a, b])$ is at most R (i.e. if $x \in L$, then $d(x, f([a, b])) \le R$ and if $x \in f([a, b])$, then $d(x, L) \le R$).*

Suppose that C and D are closed subsets of a metric space Y. We say that the **Hausdorff distance** between C and D is at most R if both

(1) $d(c, D) \le R$ for all $c \in C$, and
(2) $d(d, C) \le R$ for all $d \in D$.

Alternatively, one can say that C lies in the (closed) metric neighborhood of radius R of D and D lies in the (closed) metric neighborhood of radius R of C. The Hausdorff distance is symmetric, satisfies the triangle inequality, and equals 0 if and only if $C = D$, but is not truly a distance, since two closed sets can fail to be a finite Hausdorff distance apart.

We first sketch the proof of the Fellow Traveller Property in the case that $X = \mathbb{H}^n$ and f is a K-bilipschitz embedding (i.e. a $(K, 0)$-quasi-isometric embedding). This situation contains all the key ideas of the proof.

The key observation is that it is "exponentially inefficient" for a path to wander far from the geodesic joining the endpoints. One manifestation of this principle is that if β is a path joining the endpoints of a geodesic of length $2A$ in $\mathbb{H}^n$ and lies entirely outside the ball of radius A about the midpoint x_0, then β has length at least $\pi \sinh A$ (which is the length of the shortest such path in the sphere of radius A about x_0).

We first bound how far any point on L can lie from $f([a, b])$. Choose a point $x_0 \in L$ which lies furthest from $f([a, b])$, i.e.

$$D = d\big(x_0, f([a, b])\big) = \sup\{d\big(x, f([a, b])\big) \mid x \in L\}.$$

Choose a point y on L so that y lies between $f(a)$ and x_0 and $d(y, x_0) = 2D$ (or $y = f(a)$ if $d(f(a), y) \le 2D$). Choose $s \in [a, b]$ so that $d(f(s), y) \le D$ (or $s = a$ if $y = f(a)$). Choose a point z on L which lies between x_0 and $f(b)$ and and $d(z, x_0) = 2D$ (or $z = f(b)$ (if $d(f(b), x_0) \le 2D$.) Choose $t \in [a, b]$ so that $d(f(t), y) \le D$ (or $t = b$ if $z = f(b)$). We then concatenate a geodesic joining y to $f(s)$, $f([s, t])$ and the geodesic joining $f(t)$ to z to produce a path γ joining y to z. Since $d(f(s), f(t)) \le 6D$ and f is K-bilipschitz, $\ell(f([s, t])) \le 6DK^2$, so

$$\ell(\gamma) \le 6DK^2 + 2D.$$

Let $\hat{y}$ be the point between x_0 and y so that $d(x_0, \hat{y}) = D$ and let $\hat{z}$ between x_0 and z so that $d(x_0, \hat{z}) = D$, and form a path joining $\hat{y}$ to $\hat{z}$ by appending to γ segments in L joining y to $\hat{y}$ and joining z to $\hat{z}$. Then

$$\ell(\hat{\gamma}) \le 6DK^2 + 4D$$

and $\hat{\gamma}$ lies entirely outside of the ball of radius D about x_0. Therefore,

$$\ell(\hat{\gamma}) \ge \pi \sinh D$$

so

$$D \le \sinh^{-1}\left(\frac{6DK^2 + 4D}{\pi}\right) = D_0.$$

We now bound the distance from any point on $f([a,b])$ to L. Let $f([s,t])$ be maximal subsegment of $f([a,b])$ which stays outside of an open neighborhood of L of radius D_0. Notice that the subset of L consisting of points within D_0 of $f([a,s])$ is closed and the subset of L consisting of points within D_0 of $f([t,b])$ is closed. On the other hand their union is all of L, by the previous paragraph, so, since L is connected, their intersection is non-empty. So, there exists $r \in [a,s]$, $u \in [t,b]$ and $w \in L$ so that $d(w, f(r)) \leq D_0$ and $d(w, f(u)) \leq D_0$.

picture needed here

Since $d(f(r), f(u)) \leq 2D_0$ and f is a K-bilipschitz embedding, $|u - r| \leq 2KD_0$, so

$$\ell(f([u,r])) \leq 2K^2 D_0$$

so if $q \in [s,t] \subset [r,u]$, then

$$d(f(q), L) \leq D_0 + K^2 D_0 = R.$$

Therefore, the Hausdorff distance between $f([a,b])$ and L is at most R.

Our proof in the general case follows the same outline. We first establish a key lemma which again shows that it is "exponentially inefficient" to wander far from the straight path in a hyperbolic space.

Lemma 2.3. *Suppose that X is δ-hyperbolic, $\alpha : [a,b] \to X$ is a rectifiable path and L is a geodesic in X joining $\alpha(a)$ to $\alpha(b)$. If $x \in L$, then*

$$d(x, \alpha([a,b])) \leq \delta |\log_2(\ell(\alpha))| + 1$$

where $\ell(\alpha)$ is the length of $\alpha([a,b])$.

Proof. The lemma is obvious if $\ell(\alpha) \leq 2$, so we may assume that $\ell(\alpha) > 2$. We may also assume that $[a,b] = [0,1]$ and that α is parametrized proportional to arc length. Let $N = \lfloor \log_2(\alpha) \rfloor$.

Let Δ_1 be a geodesic triangle with endpoints $\alpha(0)$, $\alpha(1/2)$ and $\alpha(1)$, one of whose sides is L. Given $x \in L$, we can choose y_1 in another side, say L_1, of Δ_1, so that $d(y_1, x) \leq \delta$. If L_1 joins $\alpha(0)$ to $\alpha(1/2)$, then we choose a triangle Δ_2 with vertices $\alpha(0)$, $\alpha(1/4)$ and $\alpha(1/2)$ one of whose sides is L_1. Then we can choose y_2 lying in a side of Δ_2 other than L_1 so that $d(y_2, y_1) \leq \delta$. (If y_1 is on the side joining $\alpha(1/2)$ to $\alpha(1)$, then we choose Δ_2 to have vertices $\alpha(1/2)$, $\alpha(3/4)$ and $\alpha(1)$ and one side L_1 and proceed similarly.)

Picture needed here

After n stages we have a point y_n on a geodesic L_n joining $\alpha(t_n)$ and $\alpha(t_n + 1/2^n)$ so that $d(y_n, x) \leq n\delta$. We then consider a triangle Δ_{n+1} with endpoints $\alpha(t_n)$, $\alpha(t_n + 1/2^{n+1})$ and $\alpha(t_n + 1/2^n)$ one of whose sides is L_n. We then find a point y_{n+1} on one of the other sides of Δ_{n+1} so that $d(y_n, y_{n+1}) \leq \delta$. After N steps, $d(y_N, x) \leq N\delta$ and $\ell\big(\alpha([t_N, t_N + 1/2^N])\big) \leq 2$, so

$$d\big(y_N, \alpha(t_N) \cup \alpha(t_N + 1/2^N)\big) \leq 1.$$

Therefore,

$$d(x, \alpha([a,b])) \leq N\delta + 1 \leq \delta |\log_2(\ell(\alpha))| + 1$$

as desired. $\square$

We next replace f with a "nearby" piecewise geodesic quasi-isometric embedding.

Lemma 2.4. (Bridson-Haefliger [49, Lemma 11.1]) *If X is δ-hyperbolic and $f : [a,b] \to X$ is a (K,C)-quasi-isometric embedding, then there exists a $(K, \hat{C})$-quasi-isometric embedding $\alpha : [a,b] \to X$ such that*

(1) $\alpha(a) = f(a)$ *and* $\alpha(b) = f(b)$,
(2) $\hat{C} = 2(K + C)$,

(3) $\ell(\alpha([s,t])) \leq K_1 d(\alpha(s), \alpha(t)) + K_2$ *where* $K_1 = K(K + C)$ *and* $K_2 = (K\hat{C} + 3)(K + C)$, *and*

(4) *the Hausdorff distance between* $\alpha([a,b]$ *and* $f([a,b])$ *is at most* $K + C$.

Proof. We choose α so that it agrees with f on $\{a,b\} \cup ([a,b] \cap \mathbb{Z})$ and so that on each complementary interval it is a geodesic (parameterized proportional to arc length). It is not difficult to check that α has the claimed properties, but it is irritating without being instructive, so we will not include the details. $\qquad\square$

Proof of Fellow Traveller Property: We first replace the (K,C)-quasi-isometric embedding f with the piecewise geodesic $(K,\hat{C})$-embedding α given by Lemma 2.4.

We first show that L stays a bounded distance from $\alpha([a,b])$. Let $D = \sup_{x \in L}\{d(x, \alpha([a,b]))\}$ and choose x_0 so that $d(x_0, \alpha([a,b])) = D$. Let y be the point on L which lies to the left of x_0 and $d(y, x_0) = \min\{2D, d(x_0, \alpha(b))\}$ and let z be the point on L which lies to the right of x_0 and $d(y, x_0) = \min\{2D, d(x_0, \alpha(b))\}$. Choose $s, t \in [a, b]$ so that $d(\alpha(s), y) \leq D$ and $d(\alpha(t), z)$. (If $y = \alpha(a)$, we choose $s = a$ and if $z = \alpha(b)$, we choose $t = b$.) Then we can concatenate a geodesic joining y to $\alpha(s)$, $\alpha([s,t])$ and a geodesic joining $\alpha(t)$ to z to produce a path γ joining y to z. Lemma 2.4 implies that

$$\ell(\alpha[s,t]) \leq 6DK_1 + K_2$$

so

$$\ell(\gamma) \leq 2D + 6DK_1 + K_2.$$

Since $d(x_0, \gamma) = D$, Lemma 2.3 implies that

$$D \leq \delta \log_2(\ell(\gamma)) + 1 \leq \delta \log_2(2D + 6DK_1 + K_2) + 1$$

which implies an upper bound D_0 on D which depend only on K_1, K_2, and δ (and hence only on K, C and δ). So, if $x \in L$, then

$$d(x, \alpha([a,b])) \leq D_0.$$

We now show that $\alpha([a,b])$ stays a bounded distance from L. Let $\alpha([s,t])$ be a maximal subsegment of α which stays outside of an open neighborhood of L of radius D_0. Notice that the subset of L consisting of points within D_0 of $\alpha([a,s])$ is closed and the subset of L consisting of points within D_0 of $\alpha([t,b])$ is closed. On the other hand their union is all of L, by the previous paragraph, so, since L is connected, their intersection is non-empty. So, there exists $r \in [a,s]$, $u \in [t,b]$ and $w \in L$ so that $d(w, \alpha(r)) \leq D_0$ and $d(w, \alpha(u)) \leq D_0$.

Since $d(\alpha(r), \alpha(u)) \leq 2D_0$, Lemma 2.4 implies that

$$\ell(\alpha([u,r])) \leq 2D_0K_1 + K_2$$

so if $q \in [s,t] \subset [r,u]$, then

$$d(\alpha(q), L) \leq D_0 + D_0K_1 + \frac{K_2}{2} = R_0$$

where R_0 depends only on K, C and δ. Since $[s,t]$ was chosen arbitrarily, we see that if $q \in [a,b]$, then $d(\alpha(q), L) \leq R_0$. Therefore, the Hausdorff distance between $\alpha([a,b])$ and L is at most R_0. Since the Hausdorff distance between $f([a,b])$ and $\alpha([a,b])$ is at most $K + C$, by Lemma 2.4, we see that the Hausdorff distance between $f([a,b])$ and L is at most $R = R_0 + K + C$ (where R depends only on K, C and δ). $\qquad\square$

As a nearly immediate corollary, we see that any proper length space admitting a quasi-isometric embedding into a Gromov hyperbolic space is itself Gromov hyperbolic.

Corollary 2.5. *Suppose that X and Y are proper length spaces and that there is a quasi-isometric embedding $f : X \to Y$. If Y is Gromov hyperbolic, then X is also Gromov hyperbolic.*

Since any quasi-isometry has a quasi-inverse which is also a quasi-isometry, we see that Gromov hyperbolicity is invariant with respect to quasi-isometry of proper length spaces.

Proof. Suppose that Y is δ-hyperbolic and that f is a (K, C)-quasi-isometry. Let T be a geodesic triangle in T with sides s_1, s_2 and s_3. The Fellow Traveller Property implies that there exists a geodesic triangle $\hat{T}$ in Y with sides $\hat{s}_1$, $\hat{s}_2$ and $\hat{s}_3$ so that $\hat{s}_i$ is a Hausdorff distance at most R from $f(s_i)$ and has the same endpoints. (Here R depends only on K, C and δ.) Therefore, if $y \in s_1$, then there exists $\hat{y} \in \hat{s}_1$ which lies within R of $f(y)$. Since Y is δ-hyperbolic, there exists $\hat{z} \in \hat{s}_2 \cup \hat{s}_3$ so that $d(\hat{y}, \hat{z}) \leq \delta$. So, there exists $z \in s_2 \cup s_3$ so that $d(f(z), \hat{z}) \leq R$. Therefore, $d(f(y), f(z)) \leq 2R + \delta$. Since f is a (K, C)-quasi-isometry, this implies that $d(y, z) \leq K(2R + \delta + C) = \hat{\delta}$, so X is $\hat{\delta}$-hyperbolic. $\qquad\square$

picture
needed?

3. The Gromov boundary

> *The borders of reality had reconfigured in such a way that it seemed necessary to map out the patchwork topography. What was needed was a bit of geometric thinking... I taped the tracing paper to the wall attempting to make sense of an impossible scape, but composed nothing more than a fractured diagram containing all the improbable logic of a child's treasure map.*
> ————Patti Smith [194]

Notice that if we fix a basepoint $x_0 \in \mathbb{H}^n$, we may identify $\partial\mathbb{H}^n$ with the set R_{x_0} of (infinite) geodesic rays emanating from x_0 and hence with $T_{x_0}\mathbb{H}^n$. One can then abstractly identify R_{x_0} with R_{x_1} by simply identifying a geodesic ray emanating from x_0 to the unique geodesic ray emanating from x_1 that it remains a bounded distance from throughout its trajectory. Alternatively, one could simply consider the set R of all geodesic rays in $\mathbb{H}^n$ and identify two geodesic rays if they remains a bounded distance from each other throughout their trajectories, i.e. if the Hausdorff distance between the two geodesic rays is finite.

One may generalize this notion to obtain a boundary at infinity for any Gromov hyperbolic space. One must be more careful, since geodesics need not be unique and two distinct geodesic rays emanating from the same point may remain a bounded distance from each other throughout their trajectories. One can see simple examples of this in the Cayley graph of $\pi_1(S)$ or in the Gromov hyperbolic space obtained by removing balls of small radius which are spaced at regular intervals along a geodesic in $\mathbb{H}^2$. (All this would work much more easily if we worked in the simper setting of CAT(-1) spaces, where geodesics are unique, and you can start by imagining this simpler case.)

We will (roughly) follow the treatment in section III.H.3 of Bridson-Haefliger [49]. Another good reference is the survey paper by Benakli and Kapovich [126].

If X is a δ-hyperbolic space, we let

$$G = G(X) = \{\alpha : [0, \infty) \to X \mid \alpha \text{ is an isometric embedding }\}$$

and say that $\alpha \sim \beta$ if $\sup\{d(\alpha(t), \beta(t))\}$ is finite. We then let

$$\partial X = \partial_\infty X = G(X)/\sim .$$

If $\alpha \in G$, then we define $\alpha(\infty) = [\alpha] \in \partial X$.

We encourage the reader to establish the following fact as an exercise in understanding this definition.

Lemma 3.1. *If X is Gromov hyperbolic and $\alpha, \beta \in G(X)$, then $[\alpha] = [\beta]$ if and only if the Hausdorff distance between $\alpha([0, \infty))$ to $\beta([0, \infty))$ is finite.*

We now observe that every point in X is joined to every point in the boundary by a geodesic ray. So if we define $G_{x_0} = \{\alpha \in G \,|\, \alpha(0) = x_0\}$, we could have defined $\partial X = G_{x_0}/\sim$.

Lemma 3.2. *Suppose that X is a δ-hyperbolic space. If $[\alpha] \in \partial X$ and $x_0 \in X$, then there exists $\beta \in [\alpha]$ so that $\beta(0) = x_0$. Moreover, if α and β are any two geodesic rays with $[\alpha] = [\beta]$, then $\alpha([0, \infty))$ and $\beta([0, \infty)$ are a Hausdorff distance at most $\delta + d(\beta(0), \alpha(0))$ apart and*

$$d(\alpha(t), \beta(t)) \leq 3\delta + 2d(\alpha(0), \beta(0))$$

for all $t \in [0, \infty)$.

Proof. Suppose that $[\alpha] \in \partial X$. Let $\beta_n : [0, t_n] \to X$ be a geodesic joining x_0 to $\alpha(n)$. Consider a triangle with vertices $\alpha(0)$, x_0 and $\alpha(n)$ and having a side lying in the image of α and another side lying in the image of β_n. Since X is δ-hyperbolic, every point in $\beta_n([0, t_n])$ lies within δ of a point in the other two sides. Since the side joining $\alpha(0)$ to x_0 has length $d(x_0, \alpha(0))$, every point in $\beta_n([0, t_n])$ lies within $D = \delta + d(\alpha(0), x_0)$ of the image of $\alpha([0, n])$. Notice that $t_n \to \infty$, since $d(x_0, \alpha(n)) \to \infty$. The Arzela-Ascoli Theorem implies that β_n converges, up to subsequence, to a geodesic ray $\beta : [0, \infty) \to X$ and that $\beta([0, \infty))$ lies within Hausdorff distance D of $\alpha([0, \infty))$. Therefore, $[\alpha] = [\beta]$.

Now suppose that α and β are any two geodesic rays so that $[\alpha] = [\beta]$. Since $[\alpha] = [\beta]$, there exists R so that $d(\alpha(t), \beta(t)) \leq R$ for all $t \in [0, \infty)$. Given t, choose $N > \delta + R$. Let L be a geodesic joining $\alpha(0)$ to $\beta(N)$, M be a geodesic joining $\alpha(0)$ to $\beta(0)$ and $\hat{M}$ be a geodesic joining $\alpha(N)$ to $\beta(N)$ Our choice of N implies that $d(\alpha(t), \hat{M}) > \delta$. First consider the geodesic triangle with sides $\alpha([0, n]]$, L and $\hat{M}$. Since X is δ-hyperbolic there exists $x \in L \cup \hat{M}$, so that $d(x, \alpha(t)) \leq \delta$. Since $d(\alpha(t), \hat{M}) > \delta$, we see that x must lie in L. Now consider the geodesic triangle with edges L, M and $\beta[0, n])$ and notice that there exists $y \in \beta([0, N]) \cup M$ so that $d(y, x) \leq \delta$. But since $d(\alpha(t), y)$ and M has length $d(\alpha(0), \beta(0))$ we see that $\alpha(t)$ lies within $\delta + d(\alpha(0), \beta(0))$ of some point $\beta(s)$ in $\beta([0, \infty))$. Since this argument is symmetric we see that the Hausdorff distance between $\alpha([0, \infty))$ and $\beta([0, \infty))$ is at most $\delta + d(\alpha(0), \beta(0))$.

The Triangle inequality implies that $|s - t| \leq d(\alpha(0), \beta(0)) + \delta$, so $d(\alpha(t), \beta(t)) \leq 2d(\alpha(0), \beta(0) + 3\delta$. $\qquad\square$

We can topologize ∂X, by saying that a sequence $\{[\alpha_n]\} \subset \partial X$ converges to $[\alpha] \in \partial X$ if there exists $x_0 \in X$ and (for all n) $\beta_n \in [\alpha_n]$ so that $\beta_n(0) = x_0$ and (every subsequence of) β_n (contains a subsequence which) converges to a geodesic ray β such that $\beta \in [\alpha]$. (This allows one to define closed sets and hence a topology.) We similarly topologize $X \cup \partial X$ by saying that $\{x_n\} \subset X$ converges to $[\alpha] \in \partial X$ if there exists $x_0 \in X$ and (for all n) a geodesic

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$\beta_n : [0, r_n] \to X$ with $\beta_n(0) = x_0$ and $\beta_n(r_n) = x_n$ so that (every subsequence of) β_n (contains a subsequence which) converges to a geodesic ray β such that $\beta \in [\alpha]$.

We will make routine use of the following elementary observation about convergence to the boundary. We leave the proof as an exercise for the intrepid reader.

Lemma 3.3. *Suppose that X is a Gromov hyperbolic and $\{x_n\}$ is a sequence in X which converges to $z \in \partial X$. If $\{y_n\}$ is a sequence in X and there exists R so that $d(x_n, y_n) \leq R$ for all n, then $\{y_n\}$ also converges to z.*

One crucial property of the boundary is that it gives a compactification of X.

Lemma 3.4. *If X is δ-hyperbolic, then ∂X is compact. Moreover, $X \cup \partial X$ is compact.*

Proof. One can check, see the remark below, that the topology on $X \cup \partial X$ above is first countable, so it suffices to show that ∂X and $X \cup \partial X$ are sequentially compact. Let $\{z_n\}$ be a sequence in ∂X (or in $X \cup \partial X$) and fix $x_0 \in X$. Lemma 3.2 implies that we may choose, for all n, a geodesic ray $\beta_n : [0, \infty) \to X$ so that $\beta_n(0) = x_0$ and $[\beta_n] = z_n$ (or a geodesic $\beta_n : [0, r_n] \to X$ so that $\beta_n(0) = x_0$ and $\beta_n(r_n) = z_n$). The Arzela-Ascoli Theorem implies that $\{\beta_n\}$ has a subsequence $\{\beta_{n_k}\}$, so that $\{\beta_{n_k}\}$ converges to a geodesic ray β (or a geodesic $\beta : [0, r] \to \infty$). It follows that $\{z_{n_k}]\}$ converges to $[\beta] \in \partial X$ (or to $\beta(r) \in X$). $\square$

Remark: To check that the topology on ∂X is first countable one can use Lemma 3.2 to check that if we choose $r > 3\delta$ and let

$$V(\beta, n, r) = \{[\alpha] \mid \exists \alpha \in G_{\beta(0)} \text{ such that } d(\alpha(n), \beta(n)) < r\}$$

then $\{V(\beta, n, r)\}_{n \in \mathbb{N}}$ is a countable (not necessarily open) neighborhood system for $[\beta]$ in ∂X. One similarly constructs a countable neighborhood system for $[\beta]$ in $X \cup \partial X$. See Bridson-Haefliger [49, Section III.H.3] for details.

It is then easy to see that a quasi-isometric embedding induces a continuous injection at infinity.

Proposition 3.5. *If X and Y are δ-hyperbolic spaces and $f : X \to Y$ is a quasi-isometric embedding, then there exists a continuous injective map $\partial f : \partial X \to \partial Y$ such that if $\{x_n\} \subset X$ converges to $z \in \partial X$, then $\{f(x_n)\}$ converge to $\partial f(z)$.*

Proof. Suppose that X and Y are δ-hyperbolic and that $f : X \to Y$ is a (K, C)-quasi-isometry. Let $R = R(K, C, \delta)$ be the constant provided by the Fellow Traveller Property. Fix $x_0 \in X$.

Given $[\alpha] \in \partial X$, choose $\beta \in [\alpha]$ so that $\beta(0) = x_0$. Let $\gamma_n : [0, t_n] \to Y$ be a geodesic ray joining $f(x_0)$ to $f(\beta(n))$. Then $t_n \to \infty$ and the image of β_n lies a Hausdorff distance at most R from $f(\beta([0, n]))$. Then, $\{\gamma_n\}$ has a subsequence $\{\gamma_{n_k}\}$ converging to a geodesic ray γ in Y whose image lies within R of the image of $f \circ \beta$. We call γ a straightening of $f \circ \beta$ and define $\partial f([\alpha]) = [\gamma]$.

We first check that ∂f is well-defined. If α and β are geodesic rays in X so that $[\alpha] = [\beta]$, then, by Lemma 3.2, their images in X lie a Hausdorff distance at most $\delta + d(\alpha(0), \beta(0))$ apart. Therefore, $f(\alpha([0, \infty))$ and $f(\beta([0, \infty))$ lie a Hausdorff distance at most $T = K(\delta + d(\alpha(0), \beta(0)) + C$ apart. Therefore, any the image of any straightening γ of $f \circ \alpha$ lies within Hausdorff distance at most $T + 2R$ of any straightening $\hat\gamma$ of $f \circ \beta$. It follows that $[\gamma] = [\hat\gamma]$, so ∂f is well-defined.

Now suppose that a sequence $\{[\alpha_j]\} \subset \partial X$ converges to $[\alpha] \in \partial X$. Without loss of generality $\alpha_j(0) = x_0$ for all j, $\alpha(0) = x_0$ and α_j converges to α. Suppose that γ_j is a straightening of $f \circ \alpha_j$. Then $f([\alpha_j]) = [\gamma_j]$ and $\gamma_j([0, \infty))$ lies within R of $f \circ \alpha_j([0, \infty))$ for all j. If $\{\gamma_{j_k}\}$ is a subsequence of $\{\gamma_j\}$ converging to a geodesic ray γ, then, since α_j converges to α, $\gamma([0, \infty))$ lies witin Hausdorff distance R of $f \circ \alpha([0, \infty))$. Therefore, $\{f([\alpha_j])\}$ converges to $[\gamma] = \partial f([\alpha])$, so ∂f is continuous.

Similarly, suppose that $\{x_n\} \subset X$ converges to $z \in \partial X$. Let $\alpha_n : [0, r_n]$ be a geodesic joining x_0 to x_n. Since $x_n \to x$, we see that $r_n \to \infty$ and if a subsequence of α_n converges to $\alpha : [0, \infty) \to X$, then $[\alpha] = z$. For all n, let γ_n be a geodesic joining $f(x_0)$ to $f(x_n)$. Then the image of γ_n lies within a Hausdorff distance R of the image of $f \circ \alpha_n$. It follows that if a subsequence of $\{\gamma_n\}$ converges to a geodesic ray γ, then the image of γ lies within Hausdorff distance R of the image of $f \circ \alpha$ where α is a limit of a subsequence of $\{\alpha_n\}$. Therefore, $f(z) = f([\alpha]) = [\gamma] = \lim f(x_n)$.

Finally, we check injectivity. Suppose $\partial f([\alpha]) = \partial f([\beta]) = [\gamma]$. We may assume that $\alpha(0) = \beta(0) = x_0$ and $\gamma(0) = f(x_0)$. Then, the images of $f \circ \alpha$ and $f \circ \beta$ lie within $K(2\delta) + C + 2\delta$ of the image of γ. Therefore, the image of α lies within $2K(K - 2\delta + C + 2\delta) - C$ of the image of β, which implies that $[\alpha] = [\beta]$. Therefore, ∂f is injective. $\square$

Recall that any quasi-isometry $f : X \to Y$ has a quasi-inverse $g : Y \to X$. Moreover, if $h : X \to X$ is a bounded distance from the identity on X, then h is a $(1, C)$-quasi-isometry for some C and ∂h is the identity on ∂X. Therefore, $\partial f \circ \partial g = \partial(f \circ g) = id$ and

Proposition 3.5 implies that quasi-isometric spaces have homeomorphic boundaries.

Corollary 3.6. *If X, Y and Z are hyperbolic spaces, and $f : X \to Y$ and $g : Y \to Z$ are quasi-isometric embeddings, then $\partial(g \circ f) = \partial g \circ \partial f$. In particular, if $f : X \to Y$ is a quasi-isometry between hyperbolic spaces, then $\partial f : \partial X \to \partial Y$ is a homeomorphism.*

Proof. First notice that if $[\alpha] \in \partial X$, then

$$\partial(f \circ g)([\alpha]) = [f \circ g \circ \alpha] = \partial f([g \circ \alpha]) = \partial f(\partial g([\alpha]))$$

so $\partial(f \circ g) = \partial f \circ \partial g$.

Recall that any quasi-isometry $f : X \to Y$ has a quasi-inverse $g : Y \to X$, which is a quasi-isometry so that both $f \circ g$ and $g \circ f$ are a bounded distance from the identity. Moreover, if $h : X \to X$ is a bounded distance from the identity on X, then ∂h is the identity on ∂X. Therefore, $\partial f \circ \partial g = \partial(f \circ g) = id_Y$ and $\partial g \circ \partial g = \partial(g \circ f) = id_X$, so ∂f is a homeomorphism. $\square$

Remark: In fact, the topology on ∂X is induced by a metric which is well-defined up to Hölder equivalence and the map ∂f in Proposition 3.5 is Hölder with respect to these metrics on ∂X and ∂Y. This is a generalization of the fact that if $f : \mathbb{H}^2 \to \mathbb{H}^2$ is a quasi-isometry, then ∂f is a bi-Hölder homeomorphism. In fact, ∂f is quasi-symmetric, and the homeomorphism in Corollary 3.6 can be shown to be a generalized quasi-symmetry. (This is discussed, with references to more complete treatments, in Kapovich-Benakli [126].)

If $\alpha : [0, \infty) \to X$ is a (K, C)-quasi-isometric embedding, then, $[0, \infty)$ is Gromov hyperbolic and its Gromov boundary $\partial[0, \infty)$ is a single point, which we denote ∞. Proposition 3.5 implies that we may define $\alpha(\infty) \in \partial X$. (More concretely, we could proceed as in the proof to find a geodesic ray in X which lies a bounded Hausdorff distance from the image of α and define $\alpha(\infty)$

to be the equivalence class of that geodesic ray. For shorthand, we often call α a **quasi-geodesic ray** and say that it **ends** at $\alpha(\infty)$.

If $c : \mathbb{R} \to X$ is a quasi-isometric embedding $c : \mathbb{R} \to X$ into a Gromov hyperbolic space X, we let $c(\infty) = [c|_{[0,\infty)}]$ and $c(-\infty) = [\hat{c}]$ where $\hat{c}(t) = c(-t)$ for all $t \in [0, \infty)$. Alternatively, we could observe that $\mathbb{R}$ is Gromov hyperbolic and its Gromov boundary consists of two points, labelled ∞ and $-\infty$. Proposition 3.5 then implies that $c(\infty) \neq c(-\infty)$. We often call c a **quasi-geodesic** and say that it **joins** $c(-\infty)$ to $c(\infty)$.

We collect together here some useful facts about quasi-geodesics and the Fellow Traveller Property.

Proposition 3.7. *Suppose that X is δ-hyperbolic and $[\alpha]$ and $[\beta]$ are distinct points in ∂X.*

    (1) *there exists a geodesic $\gamma : \mathbb{R} \to X$ joining $[\alpha]$ to $[\beta]$, i.e. $\gamma(-\infty) = [\alpha]$ and $\gamma(\infty) = [\beta]$.*

    (2) *Any two geodesics joining $[\alpha]$ and $[\beta]$ lie a Hausdorff distance at most 2δ apart.*

    (3) *Given $K \geq 1$ and $C \geq 0$ there exists $S = S(\delta, K, C) \geq 0$ so that any two (K, C)-quasigeodesics joining $[\alpha]$ to $[\beta]$ lie a Hausdorff distance at most S apart.*

    (4) *Any two (K, C)-quasigeodesic rays $c : [0, \infty) \to X$ and $\hat{c} : [0, \infty) \to X$ ending at the same point in ∂X lie a Hausdorff distance at most $S + 2d(c(0), \hat{c}(0))$ apart.*

Proof. We may assume, by Lemma 3.2, that $\alpha(0) = \beta(0) = x_0 \in X$. Notice that since $d(\alpha(t), \beta(t)) \to \infty$, there exists N so that if $s, t > N$, then $d(\alpha(t), \beta([0, \infty))) > 2\delta$ and $d(\beta(s), \alpha([0, \infty))) > 2\delta$. For all $n > N$, let $\gamma_n : [a_n, b_n] \to X$ be a geodesic segment joining $\alpha(n)$ to $\beta(n)$.

Since every point in the image of γ_n lies within δ of a point in $\alpha([0, n]) \cup \beta([0, n])$, we see that there must be some point $\alpha(t_n)$ which lies within δ of both $\alpha([0, n])$ and $\beta([0, n])$. We then re-parameterize so that $t_n = 0$ and notice that then $d(\gamma_n(0), x_0) \leq N + \delta$. Moreover, one may check that if $t > N$, then $\gamma_n(t)$ lies within δ of $\beta([0, \infty))$ and if $t < -N$, then $\gamma_n(t)$ lies within δ of $\beta([0, \infty))$.

Then, by the Arzela-Ascoli Theorem, $\{\gamma_n\}$ has a subsequence converging to a geodesic $\gamma : \mathbb{R} \to X$ such that if $t > N + \delta$, then $\gamma(t)$ lies within δ of $\beta([0, \infty))$ and if $t < -N - \delta$, then $\gamma(t)$ lies within δ of $\beta([0, \infty))$. Therefore, $\gamma(-\infty) = [\alpha]$ and $\gamma(\infty) = [\beta]$. This establishes (1).

Suppose $\gamma_1 : \mathbb{R} \to X$ and $\gamma_2 : \mathbb{R} \to X$ are two geodesics joining $[\alpha]$ to $[\beta]$. By definition, there exists D so that $d(\gamma_1(t), \gamma_2(t)) \leq D$ for all $t \in \mathbb{R}$. For all $n \in \mathbb{N}$, let L_n be a geodesic joining $\gamma_1(-n)$ to $\gamma_2(n)$, let T_n^1 be a geodesic triangle with vertices $\gamma_1(-n)$, $\gamma_1(n)$ and $\gamma_2(n)$ and having L_n, $\gamma_1([-n, n])$ and M_n^1 as its sides, and let T_n^1 be a geodesic,triangle with vertices $\gamma_2(-n)$, $\gamma_2(n)$ and $\gamma_1(-n)$ and having L_n, $\gamma_2([-n, n])$ and M_n^2 as its sides. Notice that each M_n^i has length at most D.

Suppose that $t \in \mathbb{R}$ and choose $N > D + 2\delta + |t|$. Then $d(\gamma_1(t), M_N^i) < 2\delta$ for $i = 1, 2$. Since [picture needed] X is δ-hyperbolic, we consider the geodesic triangle T_N^1, and see that there exists $x \in L_N \cup M_N^1$ so that $d(x, \alpha(t)) \leq \delta$. But, since $d(\gamma_1(t), M_N^1) > 2\delta$, x must lie in L_N. Then, considering the triangle T_N^2 we see that there exists $z \in M_N^2 \cup \gamma_2([-N, N])$ so that $d(y, z) \leq \delta$, then $d(\gamma_1(t), z) \leq 2\delta$, so $z \in \gamma_2([-N, N])$. Therefore, every point in $\gamma_1(\mathbb{R})$ lies within 2δ of a point on $\gamma_2(\mathbb{R})$. Since the argument is symmetric, we see that the Hausdorff distance between $\gamma_1(\mathbb{R})$ and $\gamma_2(\mathbb{R})$ is at most 2δ. This establishes (2).

Let $c_1 : \mathbb{R} \to X$ and $c_2 : \mathbb{R} \to X$ be (K, C)-quasi-geodesics joining $[\alpha]$ to $[\beta]$. Let $R = R(K, C, \delta)$ be the constant provided by the Fellow Traveller Property. For all $n \in \mathbb{N}$ and $i = 1, 2$, let L_n^i be a geodesic joining $c_i(-n)$ to $c_i(n)$. Then, the Hausdorff distance between L_n^i

and $c_i(-n, n)$ is at most R. We may pass to a subsequence so that $\{L_n^1\}$ and $\{L_n^2\}$ converge to bi-infinite geodesics L^1 and L^2 joining $[\alpha]$ and $[\beta]$. Then the Hausdorff distance between $c_1(\mathbb{R})$ and L^1 is at most R and the Hausdorff distance between $c_2(\mathbb{R})$ and L^2 is at most R. Since, by part (2), the Hausdorff distance between L^1 and L^2 is at most 2δ, the Hausdorff distance between $c_1(\mathbb{R})$ and $c_2(\mathbb{R})$ is at most $S = 2R + 2\delta$. This establishes (3).

The proofs of (4) follow the same outline as the proof of (3). $\square$

One similarly checks that geodesic triangles with vertices at infinity are 2δ-thin if X is δ-hyperbolic. If a vextex v of a geodesic triangle lies in ∂X, then the edges abutting it both end at v and we do not regard v as a point in the triangle itself. We leave the proof as an exercise.

Lemma 3.8. *Suppose that X is δ-hyperbolic and T is a geodesic triangle with sides s_1, s_2 and s_3 with vertices in $X \cup \partial X$. If $y \in s_1$, then*

$$d(y, s_2 \cup s_3) \leq 2\delta.$$

4. **Hyperbolic groups and their subgroups**

If Γ is a finitely generated group, we consider its Cayley graph C_Γ. The Milnor-Svarc Lemma implies that the Cayley graphs associated to different finite generating systems are quasi-isometric, but one may easily check this directly (since each generator in one generating system can be written as a word in the other generating system.)

We then say that a finitely generated group is **hyperbolic** or **Gromov hyperbolic** or **word hyperbolic** if its Cayley graph is Gromov hyperbolic. (Notice that since the Cayley graph depends on the generating system there is no canonical hyperbolicity constant attached to a group.) We let $\partial \Gamma$ denote the Gromov boundary of C_Γ. Notice that $\partial \Gamma$ is well-defined since all Cayley graphs for Γ are quasi-isometric Moreover, Γ acts naturally as a group of homeomorphisms of Γ, by Corollary 3.6.

The Cayley graph of a finitely generated free group is a tree, so is 0-hyperbolic, so free groups are Gromov hyperbolic. All finite groups are hyperbolic. The Milnor-Svarc lemma implies that any group acting properly discontinuously and cocompactly on a Gromov hyperbolic space is Gromov hyperbolic. So, if $M = \mathbb{H}^n/\Gamma$ is a closed hyperbolic manifold, then Γ is a Gromov hyperbolic group. More generally, the fundamental group of any closed negatively-curved manifold is Gromov hyperbolic. We will later see that if $\rho : \Gamma \to \mathsf{SO}_0(n, 1)$ is convex cocompact, then Γ is Gromov hyperbolic. In fact, these last two examples provide the motivation for much of the basic theory of hyperbolic groups.

Since the Cayley graph of a group Γ is quasi-isometric to the Cayley graph of any finite index subgroup Θ, one sees that Γ is Gromov hyperbolic if and only if Θ is Gromov hyperbolic. (Notice that one may compose a quasi-isometry from C_Θ to Θ with the inclusion map of Θ into C_Γ to obtain a quasi-isometry.)

Since Γ is quasi-isometric to its Cayley graph, Proposition 2.5 immediately implies the following criterion for a group to be hyperbolic.

Proposition 4.1. *If Γ is a finitely generated group, then Γ is hyperbolic if and only if there exists a quasi-isometric embedding of Γ into a Gromov hyperbolic space.*

We say that a finitely generated subgroup Θ of Γ is **quasiconvex** if the inclusion map of Θ into Γ is a quasi-isometric embedding (with respect to some, hence any, finite generating set for Θ). We obtain the following corollary.

Corollary 4.2. *If Γ is a hyperbolic group and Θ is a quasiconvex subgroup, then Θ is hyperbolic. Moreover, the inclusion map induces a Θ-equivariant embedding of $\partial\Theta$ into $\partial\Gamma$.*

Proof. Let $f : \Theta \to \Gamma$ and $g : \Gamma \to C_\Gamma$ and $h : \Theta \to C_\Theta$ be inclusion maps. f is a quasi-isometric embedding, by assumption, and g and h are $(1,1)$-quasi-isometries. Let j be a quasi-inverse for h. Then $\phi : g \circ f \circ j : C_\Theta \to C_\Gamma$ is a quasi-isometric embedding, so Proposition 4.1 implies that C_Θ is hyperbolic, and hence that Θ is hyperbolic. Since ϕ is Θ-equivariant, i.e. $\theta \circ \phi = \phi \circ \theta$ for all $\theta \in \Theta$, $\partial\phi$ is a Θ-equivariant embedding. $\qquad\square$

It is well-known that hyperbolic isometries of $\mathbb{H}^2$ have exactly two fixed points and preserve the geodesic joining them. We will want an analogue of this result in the setting of hyperbolic groups. However, in the setting of hyperbolic groups, there need not be a geodesic preserved by an infinite order element, although there is always a quasi-axis (i.e. a quasi-geodesic preserved by the element). The fundamental observation about cyclic subgroups of hyperbolic groups is that they are quasiconvex.

Proposition 4.3. *Suppose that Γ is a hyperbolic group and γ is an infinite-order element of Γ. Then the the map $\eta = \eta_\gamma : \mathbb{Z} \to C_\Gamma$ given by $n \mapsto \gamma^n$ is a quasi-isometric embedding.*

We will follow the treatment in Batty-Papasoglu [14, Section 3.7]. We first show that the number of iterates it takes to first escape the neighborhood of radius R of the identity grows linearly in R and then do some (somewhat irritating) book-keeping to show that this implies that η is quasi-geodesic.

Proof. Suppose that C_Γ is δ-hyperbolic. Given $R \geq 0$, there exists $k = k(R)$ so that $d(id, \gamma^k) > 8R + 4\delta$. (Notice that this is possible since any metric ball in C_Γ contains only finitely many vertices.) Let L be a geodesic joining id to γ^k and let y be a midpoint of L.

Let $\hat{L}$ be the subsegment of L centered at y and having radius R. We first show that if α and β are vertices in $B(R, id)$ and m is the midpoint of any geodesic M joining α to β, then $d(m, \hat{L}) \leq 4\delta$. First consider the triangle T_1 with vertices α, β and γ^k and having M as an edge. Then, there exists a point y_1 on one of the other edges, say M_1, of T_1 so that $d(m, y_1) \leq \delta$. Since $d(y, \beta) > 3R + 2\delta$ and $d(\beta, \gamma^k) \leq R$, we see that M_1 must join α to γ^k. Let m_1 be the midpoint of M_1. Since $d(m, \beta) = \frac{d(\alpha, \beta)}{2}$ and $d(\beta, \gamma^k) \leq R$ we see that

$$\left| d(m, \gamma^k) - \frac{d(\alpha, \gamma^k)}{2} \right| \leq \frac{R}{2}$$

and since $d(y_1, m) \leq \delta$ and $d(m_1, \gamma^k) = \frac{d(\alpha, \gamma^k)}{2}$ it follows that

$$d(y_1, m_1) \leq \frac{R}{2} + \delta.$$

picture needed

Now consider the triangle T_2 with vertices α, id and g^k and having L and M_1 as edges. There exists a point y_2 on an edge of T_2 other than M_1 so that $d(y_2, y_1) \leq \delta$, so $d(y_2, m) \leq 2\delta$. Since $d(y_2, id) > R + \delta$, y_2 cannot lie on the edge joining id and α, so must lie on L. Since $d(m_1, \gamma^k) = \frac{d(\alpha, \gamma^k)}{2}$ and $d(id, \alpha) \leq R$ we see that

$$\left| d(m_1, \gamma^k) - \frac{d(id, \gamma^k)}{2} \right| \leq \frac{R}{2}$$

and since $d(y_2, m_1) \leq \frac{R}{2} + 2\delta$ and $d(y, \gamma^k) = \frac{d(id, \gamma^k)}{2}$ it follows that

$$d(y_2, y) \leq R + 2\delta.$$

Therefore, $d(m, \hat{L}) \leq 4\delta$.

Let N be the number of vertices (and midpoints of edges) in $B(4\delta, id)$. Then, there are at most $(2R+1)N \leq 3RN$ vertices (and midpoints of edges) in the neighborhood of radius 4δ of L, so at most $3RN$ choices for midpoints of geodesics joining vertices in $B(4\delta, id)$ to vertices in $B(4\delta, \gamma^k)$. Notice that the midpoints of $\gamma^n(L)$ are all disjoint, since each γ^n acts by isometries and without fixed points on C_Γ. Therefore, there exist $P(R)$ such that $1 \leq P(R) \leq 3RN$ and $\gamma^{P(R)}$ does not lie in $B(R, id)$ (Notice that if γ^s lies in $B(R, id)$, then γ^{s+k} lies in $B(R, \gamma^k)$, so $\gamma^s(L)$ joins a point in $B(R, id)$ to a point in $B(\gamma^k, R)$.) The crucial improvement here is that $P(R)$ is linear in R whereas there can be an exponential number of vertices in $B(R, 1)$.)

Now we claim that for all $R \in \mathbb{N}$, $d(1, \gamma^{3NR}) \geq R$. If not, there exists $R_0 \in \mathbb{N}$ so that

$$d(id, \gamma^{3NR_0}) \leq R_0 - \epsilon$$

for some $\epsilon \geq 1$. So, if $s > 3NR_0$ we can write $s = 3nNR_0 + R_1$ with $0 \leq R_1 < 3NR_0$ and $n \in \mathbb{N}$. So,

$$d(1, \gamma^s) \leq d(1, \gamma^{nNR_0}) + d(\gamma^{nNR_0}, \gamma^s) \leq n(R_0 - \epsilon) + R_1 T$$

where $T = d(1, \gamma)$. So, if $n\epsilon > R_1 T$, which always holds if $s > 9N^2 R_0^2 T$, then

$$d(1, \gamma^s) \leq \frac{s}{3N}.$$

Notice that $P(R) \geq \frac{R}{T}$, so we may choose R so that $P(R) > 9N^2 R_0^2 T$. Then, by construction, $d(1, \gamma^{P(R)}) > R$ and $P(R) \leq 3NR$, but, by the previous paragraph, $d(1, \gamma^{P(R)}) \leq \frac{P(R)}{3N} \leq R$, so we have achieved a contradiction and so

$$d(1, \gamma^{3NR}) \geq R$$

for all $R \in \mathbb{N}$. Notice also that

$$d(1, \gamma^s) \leq sT.$$

Therefore, the restriction of η to $3N\mathbb{Z}$ is $\max\{T, 3N\}$-bilipschitz.

It is now fairly clear that the extension to all of $\mathbb{Z}$ must be a quasi-isometry, but let's check the details. If $s > 0$, there exists integers $n \geq 0$ and $m \in [0, 3N)$ so that $s = n(3N) + m$, so

$$d(\eta(0), \eta(s)) \geq d(1, \gamma^s) \geq d(1, \gamma^{n(3N)}) - d(\gamma^{n(3N)}, \gamma^s) \geq n - mT \geq \frac{s}{3N} - 3NT - 1$$

Since η is equivariant (with respect to translation) this shows that η is a $(\max\{T, 3N\}, 3NT+1)$-quasi-isometric embedding. $\qquad \square$

It follows that the action of an infinite order element on C_Γ has North-South dynamics.

Corollary 4.4. *If Γ is a δ-hyperbolic group and $\gamma \in \Gamma$ has infinite order, then there exists a quasi-isometric embedding $c_\gamma : \mathbb{R} \to C_\Gamma$ and $R > 0$, so that $\gamma^n(c_\gamma(t)) = c_\gamma(t + n)$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Moreover, if $x \in C_\Gamma$, then*

$$\lim_{n \to +\infty} \gamma^n(x) = \gamma^+ = c_\gamma(\infty) \qquad \text{and} \qquad \lim_{n \to +\infty} \gamma^{-n}(x) = \gamma^- = c_\gamma(\infty).$$

and γ fixes both γ^+ and γ^-.

We call γ^+ the **attracting** fixed point of γ and γ^- is called the **repelling** fixed point. We can think of the quasi-geodesic $c_\gamma(\mathbb{R})$ as a **quasi-axis** for γ.

Proof. Let $d : [0,1] \to C_\Gamma$ be a path which is a geodesic parametrized proportional to arc length joining id to γ. We then $< \gamma >$-equivariantly extend d to obtain $c = c_\gamma : \mathbb{R} \to C_\Gamma$. Explicitly, if $n = \lfloor t \rfloor$ and $s = t - n \in [0,1)$, then let $c(t) = \gamma^n(d(s))$. By construction, $\gamma^n(c(t)) = c(t+n)$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. In particular, $\gamma(c(\infty)) = [\gamma(c([0,\infty))] = [c([1,\infty))] = c(\infty)$ so γ fixes $c(\infty) = \gamma^+$. Similarly, γ fixes $c(-\infty) = \gamma^-$.

Since $c|_{\mathbb{Z}}$ is a (K,C)-quasi-isometry (for some K and C), by Proposition 4.3, one may easily check that c is a $(K, C + d(id, \gamma))$-quasi-isometry. Therefore, if $t \in \mathbb{R}$, then $\{\gamma^n(c(t))\}_{n \in \mathbb{N}}$ converges to $c(\infty)$ and $\{\gamma^{-n}(c(t))\}_{n \in \mathbb{N}}$ converges to $c(-\infty)$. If $x \in C_\Gamma$, then $d(\gamma^n(x), \gamma^n(id)) = d(x, id)$ and $\gamma^n = c(n) \to \alpha^+$, so, by Lemma 3.3, $\gamma^n(x) \to \alpha^+$. Similarly, $\gamma^{-n}(x) \to \alpha^-$. $\square$

The following lemma places strong restrictions on subgroups of hyperbolic groups.

Lemma 4.5. *Suppose that Γ is hyperbolic and $\gamma \in \Gamma$ has infinite order. If $\beta \in \Gamma$ and $\beta\gamma^n\beta^{-1} = \gamma^m$ then $m = \pm n$.*

Proof. Suppose that $\beta\gamma^n\beta^{-1} = \gamma^m$ and $m \neq \pm n$. Without loss of generality $|m| > |n|$. Then, for all $k \in \mathbb{N}$,

$$\gamma^{m^k} = \beta^k \gamma^{n^k} \beta^{-k}.$$

However, since the orbit map $\eta_\gamma :< \gamma > \to \Gamma$ is a (K,C)-quasi-isometric embedding for some K and C, this implies that

$$\frac{|m|^k}{K} - C \leq d(1, \gamma^{m^k}) \leq 2kd(1,\beta) + d(1, \gamma^{n^k}) \leq 2kd(1,\beta) + K|n|^k + C$$

for all $k \in \mathbb{N}$, which is impossible if $|m| > |n|$. $\square$

Recall that a **Baumslag-Solitar group** has the form

$$BS(m,n) = < a, b \mid ba^m b^{-1} = a^n >$$

for some $m, n \in \mathbb{Z} - \{0\}$, so $BS(m,n)$ cannot appear as a subgroup of a hyperbolic group if $m \neq \pm n$. Moreover, if $m = \pm n$, then $BS(m.n)$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Since, every free abelian subgroup of a hyperbolic group is infinite cylic (as we will prove in the torsion-free setting in the next section), no Baumslag-Solitar group can appear as the subgroup of a hyperbolic group.

It is conjectured that if Γ admits a finite $K(\Gamma, 1)$ (i.e. a finite CW-complex with fundamental group Γ whose universal cover is contractible), then Γ is hyperbolic if and only if it does not contain a subgroup isomorphic to a Baumslag-Solitar group. Brady [40] exhibited a finitely presented subgroup of a hyperbolic group which is not hyperbolic, but does not contain a Baumslag-Solitar subgroup, so the stronger finiteness assumption is necessary.

5. Dynamics on the Gromov boundary

In this section, we further explore the action of a hyperbolic group on its Gromov boundary. These are the results on hyperbolic groups which we will use most often.

We have already seen that cyclic subgroups of hyperbolic groups have North-South dynamics on the Cayley graph. We now show that their action on the boundary has North-South dynamics.

Proposition 5.1. *If Γ is a hyperbolic group and $\gamma \in \Gamma$ has infinite order, then γ has exactly two fixed points $\gamma^+ = \lim_{n \to \infty} \gamma^n$ and $\gamma^- = \lim_{n \to \infty} \gamma^{-n}$ in $\partial\Gamma$. Moreover, if $z \in \partial\Gamma - \{\gamma^-\}$ then $\lim_{n \to \infty} \gamma^n(z) = \gamma^+$.*

Proof. Let $c = c_\gamma : \mathbb{R} \to C_\Gamma$ be the (K, C)-quasi-isometric embedding provided by Corollary 4.4. We have already shown that γ fixes $\gamma^+ = c(\infty)$ and $\gamma^- = c(-\infty)$. There exists δ so that C_Γ is δ-hyperbolic. Let $R = R(K, C, \delta)$ be the constant provided by the Fellow Traveller property.

If $z \in \partial\Gamma - \{\gamma^+, \gamma^-\}$, let β be a geodesic ray based at id, so that $[\beta] = z$. Then $\gamma^n(z) = [\gamma^n \circ \beta]$. Since $z \notin \{\gamma^+, \gamma^-\}$, we see that $d(\gamma(t), \beta([0, \infty))) \to \infty$ as either $t \to \infty$ or $t \to -\infty$, so there exists S so that

$$d(c(s), \beta([0, \infty))) > 8R + 8\delta \text{ if } |s| \geq S.$$

Given $n \in \mathbb{N}$. Let β_n be a geodesic ray joining id to $\gamma^n(z)$ and let $\alpha_n : [0, r_n] \to C_\Gamma$ be a geodesic joining id to γ^n. Since $\gamma^n = c(n)$, we see that $\alpha_n([0, r_n])$ and $c([0, n])$ are a Hausdorff distance at most R apart. We consider the geodesic triangle, with one ideal vertex $\gamma^n(z)$, which has edges $\gamma^n(\beta([0, \infty)))$, $\beta_n([0, \infty))$ and $\alpha_n([0, r_n])$. By Lemma 3.7, every point on $\beta_n([0, \infty))$ lies within 2δ of some point on either $\beta_n([0, \infty))$ and $\alpha_n([0, r_n])$. Let

$$t_n = \inf\{t \in [0, \infty) \mid d(\beta_n(t), \alpha_n([0, r_n])) > 2\delta\}.$$

Notice that we must have $d(\beta_n(t_n), \alpha_n([0, r_n])) = 2\delta$ and $d(\beta_n(t_n), \gamma^n(\beta([0, \infty))) \leq 2\delta$. The following explicit estimate shows that t_n diverges to ∞.

Claim: $t_n \geq w_n = n - S - CK - R - 2\delta$.

Proof of Claim: Choose $a_n \in [0, r_n]$ and $b_n \in [0, \infty)$so that that $d(\beta_n(t_n), \alpha_n(a_n)) \leq 2\delta$ and $d(\beta_n(t_n), \gamma^n(a_n) \leq 2\delta$. By the fellow traveller property, there exists $s_n \in [0, n]$ so that $d(c(s_n), \alpha_n(a_n)) \leq R$. By the triangle inequality,

$$d(c(s_n), \beta_n(t_n)) = d(\gamma^{-n}(c(s_n)), \gamma^{-n}(\gamma^n(\beta(b_n))) = d(c(s_n - n), \beta(t_n)) \leq R + 2\delta.$$

So, by our choice of S, $|s_n - n| \leq S$, so $s_n > n - S$. Since α_n and β_n are geodesics beginning at id and $d(\beta_n(t_n), \alpha_n(a_n)) \leq 2\delta$, we see that $|a_n - t_n| \leq 2\delta$. Similarly, since α_n is a geodesic, c is a (K, C)-quasigeodesic and $d(c(s_n), \alpha_n(a_n)) \leq R$, we see that $b_n \geq \frac{s_n - C}{K} - R$. By combining, we have

$$t_n \geq b_n - 2\delta \geq \frac{s_n - C}{K} - R - 2\delta \geq n - S - CK - R - 2\delta$$

which completes the proof of the claim.   □

Therefore, it follows that $\beta_n([0, t_n])$ lies within the neighborhood of radius 2δ of $\alpha_n([0, r_n])$, and hence within the neighborhood of radius $R + 2\delta$ of $c([0, \infty))$. We pass to a subsequence so that β_n converges to a geodesic ray β. Since $t_n \to \infty$, β lies within a neighborhood of radius R of $c([0, \infty))$. Therefore, $[\beta] = c(\infty) = \gamma^+$. Since, this holds for an arbitrary subsequence of $[\beta_n]$, we see that $\gamma^n(z) = [\beta_n]$ converges to γ^+.

                                                                                    □

For the remainder of the section, we will assume that Γ is torsion-free to avoid proving the simple fact that every infinite hyperbolic group contains an infinite order element (see Bridson-Haefliger [49, Proposition III.Γ.2.22]). Moreover, there are only finitely many conjugacy classes of finite subgroups of a hyperbolic group (see Bridson-Haefliger [49, Theorem III.Γ.3.2]).

As a first application of Proposition 5.1 we see that the stabilizer of any point in $\partial\Gamma$ is cyclic. We recall that the **stabilizer** of z in Γ is the subgroup

$$\mathrm{Stab}(z) = \{\gamma \in \Gamma \mid \gamma(z) = z\}.$$

Corollary 5.2. *If Γ is a torsion-free δ-hyperbolic group and $z \in \partial\Gamma$, then $\mathrm{Stab}_\Gamma(z)$ is cyclic, i.e. either trivial or isomorphic to $\mathbb{Z}$.*

Proof. If $\mathrm{Stab}_\Gamma(z)$ is non-trivial, we fix a non-trivial element $\gamma \in \mathrm{Stab}(z)$ and let $c = c_\gamma : \mathbb{R} \to C_\Gamma$ be the quasi-isometric embedding provided by Corollary 4.4. We may assume, perhaps after replacing γ with γ^{-1}, that $\gamma^+ = c(\infty) = z$. We first show that if $\beta \in \mathrm{Stab}(z)$, then $\{\beta^+, \beta^-\} = \{\gamma^+, \gamma^-\}$. Let $\hat{c} = c_\beta : \mathbb{R} \to C_\Gamma$ be the quasi-isometric embedding provided by Corollary 4.4. We may assume, after perhaps replacing β by β^{-1}, that $\hat{c}(\infty) = z = \beta^+$. By Proposition 3.7 there exists T so that $c([0,\infty))$ and $\hat{c}([0,\infty)$ are Hausdorff distance at most T apart. Therefore, for all $n \in \mathbb{N}$, there exists $k(n) \in \mathbb{N}$ so that $d(\gamma^n, \beta^{k(n)}) \leq T$, so $\beta^{-k(n)}\gamma^n$ has word length at most T. Since there are only finitely many words of length at most T in Γ, there exists $r \neq s \in \mathbb{N}$ so that $\beta^{-k(s)}\gamma^s = \beta^{-k(r)}\gamma^r$, so $\gamma^{s-r} = \beta^{k(s)-k(r)}$. Therefore,

$$\gamma^- = \lim \gamma^{-|r-s|n} = \lim \beta^{-|k(s)-k(r)|n} = \beta^-.$$

(Notice that since $\gamma^+ = \beta^+$, $(r-s)$ is positive if and only if $(k(r) - k(s))$ is positive.) It follows that every element in $\mathrm{Stab}(z)$, fixes both $\gamma^+ = z$ and γ^-.

We next show that $<\gamma>$ has finite index in $\mathrm{Stab}(z)$. If $\beta \in \mathrm{Stab}(\gamma)$, then, since β fixes γ^+ and γ^-, both c and $\beta \circ c$ join γ^- to γ^+. Therefore, by Lemma 3.7, there exists S (depending only on δ and the quasi-isometry constants of c), so that the Hausdorff distance between $c(\mathbb{R})$ and $\beta(c(\mathbb{R}))$ is at most S. Therefore, if $T = S + d(id, \gamma)$, there exists $n \in \mathbb{N}$ so that $d(\beta, \gamma^n) \leq T$. It follows that the coset of β in $\mathrm{Stab}(z)/<\gamma>$ has a representative of length at most T. Since there are only finitely many elements of Γ with length at most T, we see that $\mathrm{Stab}(z)/<\gamma>$ is finite.

Since $<\gamma>$ has finite index in $\mathrm{Stab}(z)$ and $\mathrm{Stab}(z)$ is torsion-free, it is a standard exercise in group theory to show that $\mathrm{Stab}(z)$ is infinite cyclic. $\square$

We quickly conclude that nilpotent subgroups of torsion-free hyperbolic groups are cyclic.

Corollary 5.3. *If Γ is a torsion-free hyperbolic group, then any nilpotent subgroup is either trivial or isomorphic to $\mathbb{Z}$.*

Proof. Recall that any non-trivial nilpotent group N has non-trivial center. Choose a non-trivial element γ in the center of N. If $\beta \in N$, then $\beta \in \mathrm{Stab}(\gamma^+)$, since $\beta(\gamma^+) = (\beta\gamma\beta^{-1})^+$ and $\beta\gamma\beta^{-1} = \gamma$. Therefore, $N \subset \mathrm{Stab}(\gamma^+)$ which is cyclic (by Corollary 5.2) and thus N is cyclic. $\square$

We often want to exclude the case where Γ is cyclic or virtually cyclic. We say that a hyperbolic group is **elementary** its Gromov boundary has at most two points.

Corollary 5.4. *If Γ is a torsion-free, elementary hyperbolic group, then Γ is cyclic.*

Proof. If Γ is non-trivial, it contains a non-trivial element γ. So $\gamma^+ \in \partial\Gamma$. Since $\partial\Gamma$ has at most two points (in fact exactly two points since it also contains γ^-),$\mathrm{Stab}(\gamma^+)$ has index at most two. Since $\mathrm{Stab}(\gamma^+)$ is infinite index, has finite index in Γ and Γ is torsion-free, Γ is also cyclic. $\square$

We will need the following fact only in the (optional) proof that limits of discrete faithful representations of torsion-free non-elementary hyperbolic groups into a Lie group are themselves discrete and faithful.

Lemma 5.5. *Let Γ be a torsion-free hyperbolic group. Any two maximal cyclic subgroups of Γ intersect trivially. In particular, every non-trivial element $\gamma \in \Gamma$ is contained in a unique maximal cyclic subgroup $\Theta = \mathrm{Stab}(\gamma^+)$.*

Proof. Suppose that $\gamma \in \Gamma - \{id\}$ is contained in a cyclic subgroup $A = < \alpha >$ of Γ. Then $\gamma = \alpha^r$ and $|r| > 1$. We argue as in the proof of Corollary 5.2 that $\alpha \in \mathrm{Stab}(\gamma^+)$. Therefore, $\mathrm{Stab}(\gamma^+) = \mathrm{Stab}(\gamma^-)$ is the unique maximal cyclic group containing γ.

If M is another maximal cyclic subgroup of Γ and $\alpha \in M$, then $M = \mathrm{Stab}(\alpha^+)$ and α^+ cannot equal γ^- or γ^+, so M and $\mathrm{Stab}(\gamma^+)$ intersect trivially (since, by Proposition 5.1, non-trivial elements of $\mathrm{Stab}(\gamma^+)$ fix only γ^+ and γ^-) ☐

We now use Proposition 5.1 to show that fixed points of infinite order elements are dense in $\partial\Gamma$. Moreover, if Γ is non-elementary, the action of Γ on its boundary is *minimal*. Recall that the action of Γ on $\partial\Gamma$ is **minimal** if $\partial\Gamma$ contains no proper Γ-invariant closed subset.

Proposition 5.6. *If Γ is a torsion-free hyperbolic group, then fixed points of infinite order elements are dense in $\partial\Gamma$. If, in addition, Γ is non-elementary, then Γ acts minimally on $\partial\Gamma$. In particular, if Γ is non-elementary, then $\partial\Gamma$ is perfect, hence uncountable.*

Proof. If Γ is elementary and torsion-free, then it is either trivial or infinite cylic. If Γ is trivial, its boundary is empty, so that statement is vacuously true. If Γ is infinite cyclic, then its boundary consists of two points, each of which is fixed by every element of the group.

So we may assume that Γ is non-elementary. Let A be a closed, non-empty Γ-invariant proper subset of $\partial\Gamma$. If A were a single point, then its stabilizer, which is Γ, would be cyclic, by Corollary 5.2, which is impossible since Γ is non-elementary. Therefore, we may choose $w \neq z \in A$. Suppose that $y \in \partial\Gamma - A$. Let $\{\gamma_n\}$ be a sequence in Γ converging to y and let L be a geodesic in the Cayley graph C_Γ joining w to z. Notice that if x is a point on L, then $\gamma_n(x)$ lies on $\gamma_n(L)$ and $\gamma_n(x) \to y$. Therefore, up to subsequence, $\gamma_n(L)$ either converges to y or to a geodesic with one endpoint at y. Thus, either $\{\gamma_n(w)\}$ or $\{\gamma_n(z)\}$ converges to y. In either case, since A is closed and Γ-invariant, we conclude that $y \in A$ which is a contradiction. Therefore, the action of Γ on $\partial\Gamma$ is minimal.

Notice that, since $\alpha(\beta^+) = (\alpha\beta\alpha^{-1})^+$, the set of fixed points of elements of Γ is Γ-invariant, which implies that its closure is Γ-invariant. Therefore, since the action is minimal, the set of fixed points is dense. Since, similarly, every point in $\partial\Gamma$ lies in the closure of the orbit of any other point in $\partial\Gamma$, we see that $\partial\Gamma$ is perfect and hence uncountable (by a standard fact in point-set topology). ☐

Remarks: If we allow Γ to have torsion, then Proposition 5.6 remains true as stated. In Corollaries 5.2, 5.3 and 5.4 one can only conclude that the group is virtually cyclic. In Lemma 5.5 one sees that any infinite order element is contained in a unique maximal virtually cyclic subgroup and that any two such subgroups agree or have finite intersection.

We will now discuss the more general fact that a hyperbolic group acts on its boundary as a convergence group so that every point is a conical limit point. This will only come up sparingly

in our notes, in Chapter 7, but plays a more important role as one delves further into the theory of Anosov representations.

Suppose that a group Γ acts as a group of homeomorphisms of a compact, perfect metric space X. We say Γ acts as a **convergence group** if any sequence $\{\gamma_n\}$ has a subsequence, $\{\gamma_{n_k}\}$ which either converges to an element of Γ or there exists $a, b \in X$ so that $\{\gamma_{n_k}(x)\}$ converges to a uniformly on compact subsets of $X - \{b\}$. (Notice that a and b need not be distinct.) Notice that this generalizes the notion of North-South dynamics for a cyclic subgroup of a hyperbolic group. Gehring and Martin [100] first observed that the action of a Kleinian group on its limit set is a convergence group action, and more generally that the action of $\mathsf{SO}_0(3,1)$ on $\partial\mathbb{H}^3$ is a convergence group action.

If Γ acts on X as a convergence group, we say that $x \in X$ is a **conical limit point** if there exist distinct points $a \neq b \in X$ and a sequence $\{\gamma_n\}$ so that $\{\gamma_n(x)\}$ converges to b and $\{\gamma_n(y)\}$ converges to a uniformly on compact subsets of $X - \{x\}$. Notice that the fixed points of an element of a hyperbolic group are conical limit points for its action on the boundary of the group. This generalizes a classical notion in Kleinian groups, where Beardon and Maskit [15] proved that a discrete subgroup of $\mathsf{SO}(3,1)$ is convex cocompact if only if every point in its limit set is a conical limit point. Tukia [209] proved that the action of a non-elementary hyperbolic group on its boundary is a convergence group and that all points in the boundary are conical.

Theorem 5.7. (Tukia [209]) *If Γ is a non-elementary hyperbolic group, then the action of Γ on $\partial\Gamma$ is a convergence group action and every point in $\partial\Gamma$ is a conical limit point.*

Proof. Suppose that $\{\gamma_n\}$ is a sequence of elements of Γ. Since $\Gamma \cup \partial\Gamma$ is compact, we may pass to a subsequence, still called $\{\gamma_n\}$, so that either $\{\gamma_n\}$ is constant with value $\hat\gamma$, in which case $\{\gamma_n\}$ converges to $\hat\gamma \in \Gamma$, or $\{\gamma_n\}$ converges to a point $a \in \partial\Gamma$.

So suppose that $\{\gamma_n\}$ converges to $a \in \partial\Gamma$. Let $\{u_n\}$ and $\{v_n\}$ be convergent sequences in $\partial\Gamma$ so that $u = \lim u_n \neq \lim v_n = v$. Let L_n be the geodesic in C_Γ joining u_n to v_n. We may pass to a subsequence so that $\{L_n\}$ converges to a geodesic L joining u to v. There exists $R \geq 0$ and, for all n, a point $x_n \in L_n$, so that $d(x_n, id) \leq R$. Thus, $d(\gamma_n(x_n), \gamma_n) = d(x, id) \leq R$ for all n. So up to subsequence, either $\{\gamma_n(L_n)\}$ converges to a geodesic with one endpoint at a or $\{\gamma_n(L_n)\}$ converges to a. Thus, at least one of the sequences $\{\gamma_n(u_n)\}$ or $\{\gamma_n(v_n)\}$ converges to a. It follows that there is at most one point, say b so that there exists a sequence u_n so that $u_n \to b$ but $\{\gamma_n(u_n)\}$ does not converge to a. (If no such point exists, just choose $a = b$.) Therefore, $\gamma_n(u)$ converges uniformly to a on compact subsets of $\partial\Gamma - \{b\}$. It follows that the action of Γ on $\partial\Gamma$ is a convergence group action.

Now suppose that $z \in \partial\Gamma$. Let $r : [0, \infty) \to C_\Gamma$ be a geodesic ray so that $r(0) = id$ and $r(\infty) = z$. Notice that $r(n) = \gamma_n \in \Gamma$. Then $\gamma_n^{-1} \circ r$ is a geodesic ray starting at γ_n^{-1} passing through id and ending at $\gamma_n^{-1}(z)$. Pass to a subsequence so that γ_n^{-1} converges to $a \in \partial\Gamma$ and $\gamma_n^{-1} \circ r$ converges to a geodesic L passing through id and joining y to $b = \lim \gamma_n^{-1}(z)$. Then, it follows from the previous paragraph $\gamma_n^{-1}(x)$ converges to a uniformly on compact subsets of $\partial\Gamma - \{b\}$. So, z is a conical limit point and our proof is complete. $\square$

The following related fact is also helpful.

Lemma 5.8. *If Γ is a Gromov hyperbolic group and $\{\gamma_n\} \subset \Gamma$ is a sequence such that $\lim \gamma_n = z \in \partial\Gamma$, then $\lim \gamma_n^+ = z$.*

Proof. If the lemma fails, then there exists a sequence $\{\gamma_n\} \subset \Gamma$ so that $\lim \gamma_n = z \in \partial\Gamma$ but $\lim \gamma_n^+ = w \neq z$. We pass to a subsequence so that $\lim \gamma_n^- = v$.

If $v = w$, we see, just as in the proof of Theorem 5.7, that $\{\gamma_n(u)\}$ converges to w uniformly on compact subsets of $\partial\Gamma - \{w\}$. Since id lies on a geodesic joining two points in $\partial\Gamma - \{w\}$, we see that $\lim \gamma_n = w$, which contradicts our assumption that $w \neq z$.

If $v \neq w$, then there exists K so that if L_n is a quasi-axis for γ_n, then $d(id, L_n) \leq K$. Let x_n be a point on L_n so that $d(x_n, id) \leq K$. If $y_n = \gamma_n(x_n)$, then

$$d(y_n, x_n) \geq d(id, \gamma_n) - 2K \to \infty$$

and y_n lies between x_n and γ_n^+. Therefore, $\lim y_n = w$. But, $d(y_n, \gamma_n) \leq K$, so $\lim \gamma_n = w$ (by Lemma 3.3), which implies that $w = z$. This final contradiction completes the proof. $\square$

Bowditch [36] proved that this property characterizes hyperbolic groups in the following strong sense. In characterizations of Anosov representations which do not require that the domain group be hyperbolic, Bowditch's theorem is almost always used to verify that the domain group is in fact hyperbolic.

Theorem 5.9. (Bowditch [36]) *Suppose that Γ acts on a compact, perfect, metric space as a discrete convergence group action so that every point in X is a conical limit point. Then Γ is Gromov hyperbolic and there exists a Γ-equivariant homeomorphism $h : \partial\Gamma \to X$.*

6. **Representations of hyperbolic groups**

In this section, we show that limits of discrete faithful representations of non-elementary hyperbolic groups into Lie groups are themselves discrete and faithful. This result will not be used directly in the remainder of the notes, but it is a crucial piece of the overall picture.

The key tool in the proof is what is often known as the Margulis Lemma, although it first appears in an early paper by Zassenhaus [227]. Kazhdan and Margulis [138] reproved this result and established a number of useful consequences.

Theorem 6.1. (Margulis-Zassenhaus Lemma) *If G is a Lie group, there exists a neighborhood U of the identity, so that if $\Gamma \subset G$ is discrete and $\Gamma \cap U$ generates Γ, then Γ is nilpotent.*

Recall that Γ is **nilpotent** if the lower central series terminates, i.e. if we define $\Gamma_1 = [\Gamma, \Gamma]$ and $\Gamma_{i+1} = [\Gamma_i, \Gamma]$ for all $i \geq 2$, then Γ_r is trivial for some r.

The basic idea of the proof is that the commutator map is contracting near the identity. More concretely, if we regard the commutator as a map from $G \times G \to G$, all the first partial derivatives are 0 at (id, id). Therefore, we can find a neighborhood U of id, so that if $a, b \in U$, then

$$d(id, [a, b]) \leq \frac{1}{2} \min\{d(id, a), d(id, b)\}.$$

So, there exists C so that any r-fold nested commutator of elements of U lies within $\frac{C}{2^r}$ of the identity. So, if Γ is discrete and generated by $\Gamma \cap U$, there exists r so that any r-fold nested commutator of generators is trivial. One then observes that any r-fold nested commutator of elements of Γ can be written as a product of nested commutators of generators (each of which is nested at least r times). Therefore, Γ_r is trivial, so Γ is nilpotent. For a careful proof see Thurston [203, Section 4.1] or Kapovich [127, Section 4.12].

Corollary 6.2. *Suppose that* G *is a Lie group and* Γ *is a non-cyclic, torsion-free, hyperbolic group. If* $\{\rho_n : \Gamma \to \mathsf{G}\}$ *is a sequence of discrete faithful representations converging to* ρ, *then* ρ *is also discrete and faithful.*

Proof. Let U be the neighborhood of the identity in G provided by the Margulis-Zassenhaus Lemma. Recall that if $\Delta \subset G$ is discrete and generated by $\Delta \cap U$, then Δ is nilpotent.

Suppose that ρ is not faithful. Then there exists $g \in \Gamma - \{id\}$ so that $\rho(g) = id$. Choose $h \in \Gamma$, so that h is not contained in the unique maximal cyclic subgroup $\mathrm{Stab}(g^+)$. Then, Corollary 5.5 implies that $\mathrm{Stab}(g^+)$ and $\mathrm{Stab}(h^+)$ intersect trivially. Suppose that $J = <g, hgh^{-1}>$ is nilpotent, so $J \cong \mathbb{Z}$ and $J = <j>$ for some $j \in J$. Then, $g = j^r$ for some r and $hj^r h^{-1} = j^s$ for some s, so, by Lemma 4.5, $r = \pm s$, so $hgh^{-1} = g^{\pm 1}$ and $h^2 g h^{-2} = g$, so $<g, h^2>$ is abelian, and since $<g>$ and $<h>$ intersect trivially, this implies that $<g, h^2>$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which contradicts Corollary 5.3. However, since $\lim \rho_n(g) = id$, $\rho_n(g)$ and $\rho_n(hgh^{-1})$ lie in U for all large enough n, which implies, since $\rho_n(J)$ is discrete, that $\rho_n(J)$ is nilpotent. This contradicts our assumption that ρ_n is faithful.

Suppose that ρ is not discrete. Then there exists a sequence $\{g_n\}$ in $\Gamma - \{id\}$ so that $\rho_n(g_n)$ converges to the identity. Perhaps after passing to a subsequence, we may assume that there exists $h \in \Gamma$ so that h is not contained in $Z(g_n)$ for any n. Then, as before $J_n = <g_n, hg_n h^{-1}>$ fails to be nilpotent for all n. Choose N large enough that $\rho(g_N)$ and $\rho(hg_N h^{-1})$ lie in U. Then, there exists m large enough that $\rho_m(g_N)$ and $\rho_m(hg_N h^{-1})$ also lie in U, which implies, since $\rho_m(J_N)$ is discrete, that $\rho_m(J_N)$ is nilpotent. This again contradicts our assumption that ρ_m is faithful. $\qquad\square$

Remark: This corollary holds whenever the domain group does not contain an infinite normal nilpotent subgroup and the image group is a linear Lie group, see Kapovich [127, Section 8.1].

7. The Tits Alternative

The material in this section will not be used in the rest of the notes, but I couldn't imagine discussing hyperbolic groups without showing that they contain lots of free groups if they aren't virtually cyclic.

Tits [204] proved that every finitely generated linear group is either virtually solvable (i.e. contains a finite index solvable subgroup) or contains a copy of the free group on two generators. This fundamental fact is now know as the Tits Alternative and analogues of it hold in many more general group theoretic settings.

Theorem 7.1. (Tits Alternative) *If* Γ *is a finitely generated subgroup of* $\mathsf{GL}(F)$ *for some field* F, *then either* Γ *is virtually solvable or* Γ *contains a subgroup isomorphic to the free group* F_2 *on two generators.*

The standard tool for producing free subgroups is the Ping Pong Lemma below. We leave its proof as an exercise.

Lemma 7.2. (Ping Pong Lemma) *Let* a *and* b *be bijections of a set* Ω. *If there exist disjoint set* A_1, A_2, B_1 *and* B_2 *such that* $a(\Omega - A_1) \subset A_2$, $a^{-1}(\Omega - A_2) \subset A_1$, $b(\Omega - B_1) \subset B_2$, *and* $b^{-1}(\Omega - B_1) \subset B_2$, *then* $<a, b> \cong F_2$.

We use the Ping Pong Lemma to show that any two infinite order elements in a hyperbolic group with distinct fixed points, give rise to many free subgroups. Again, we roughly follow the treatment in Section III.Γ.3 of Bridson-Haefliger [49].

Theorem 7.3. *Suppose that Γ is a hyperbolic group $\alpha, \beta \in \Gamma - \{id\}$ have infinite order and $\{\alpha^+, \alpha^-\} \neq \{\beta^+, \beta^-\}$, then there exists N so that if $n, m > N$, then $< \alpha^n, \beta^m > \cong F_2$.*

Proof. Notice that, by Proposition 5.2, if $\{\alpha^+, \alpha^-\} \neq \{\beta^+, \beta^-\}$, then $\{\alpha^+, \alpha^-\}$ and $\{\beta^+, \beta^-\}$ are disjoint. Recall that $\hat{\eta}_\alpha : \mathbb{R} \to C_\Gamma$ and $\hat{\eta}_\beta : \mathbb{R} \to C_\Gamma$ are equivariant (K, C)-quasi-isometric embeddings, for some $K, C > 0$, such that $\hat{\eta}_\alpha(n) = \alpha^n$ and $\hat{\eta}_\beta(n) = \beta^n$ for all $n \in \mathbb{N}$, $\alpha^+ = \hat{\eta}_\alpha(\infty)$, $\alpha^- = \hat{\eta}_\alpha(-\infty)$, $\beta^+ = \hat{\eta}_\beta(\infty)$ and $\beta^- = \hat{\eta}_\beta(-\infty)$. For each $x \in C_\Gamma$, let $p_\alpha(x)$ be chosen so that $d(x, \hat{\eta}_\alpha(p_\alpha(x))) = d(x, \hat{\eta}_\alpha(\mathbb{R}))$ and let $p_\beta(x)$ be chosen so that $d(x, \hat{\eta}_\beta(p_\beta(x))) = d(x, \hat{\eta}_\beta(\mathbb{R}))$. Suppose that C_Γ is δ-hyperbolic and that $R = R(K, C, \delta)$ is the constant from the Fellow Traveller Property.

We first claim that if T is sufficiently large, then the sets:

$$A_1(T) = \{x \in C_\Gamma \mid p_\alpha(x) \in \hat{\eta}(-\infty, -T)\},$$
$$A_2(T) = \{x \in C_\Gamma \mid p_\alpha(x) \in \hat{\eta}(T, \infty)\},$$
$$B_1(T) = \{x \in C_\Gamma \mid p_\beta(x) \in \hat{\eta}(-\infty, -T)\}, \quad \text{and}$$
$$B_2(T) = \{x \in C_\Gamma \mid p_\beta(x) \in \hat{\eta}(T, \infty)\}$$

are disjoint. We will use these sets to apply the Ping Pong Lemma. First notice that $A_1(T)$ and $A_2(T)$ are disjoint and that $B_1(T)$ and $B_2(T)$ are disjoint by definition if $T \geq 0$. If x lies in the intersection of any other two of these sets, then $|p_\alpha(x)| \geq T$ and $|p_\beta(x)| \geq T$. Let Δ be a geodesic triangle with vertices $v_1 = id$, $v_2 = x_\alpha = \hat{\eta}_\alpha(p_\alpha(x))$ and $v_3 = x_\beta = \hat{\eta}_\beta(p_\beta(x))$ and let L_i be the edge which does not have v_i as a vertex. Since every point in L_1 lies within δ of a point in $L_2 \cup L_3$, we see that there must be some point $p \in L_1$ which lies within δ of both L_2 and L_3. The Fellow Traveller property implies that L_2 is a Hausdorff distance at most R from $\hat{\eta}_\alpha([0, n])$ and that L_3 is a Hausdorff distance at most R from $\hat{\eta}_\beta([0, n])$. Therefore,

$$d(p, \hat{\eta}_\alpha(\mathbb{R})) \leq R + \delta \quad \text{and} \quad d(p, \hat{\eta}_\beta(\mathbb{R})) \leq R + \delta.$$

Now consider the geodesic triangle $\hat{\Delta}$ with vertices $\hat{v}_1 = x$, $\hat{v}_2 = x_\alpha$, $\hat{v}_3 = x_\beta$ and a side $\hat{L}_1 = L_1$. Then there exists a point $q \in \hat{L}_2 \cup \hat{L}_3$ so that $d(p, q) \leq \delta$. If $q \in \hat{L}_2$, then $d(q, x_\alpha) = d(q, \hat{\eta}_\alpha(\mathbb{R}))$. Then, since $d(p, \hat{\eta}_\alpha(\mathbb{R})) \leq R + \delta$ and $d(p, q) \leq \delta$, we can conclude that $d(p, x_\alpha) \leq R + 3\delta$. Similarly, if $q \in \hat{L}_3$, then $d(x_\beta, p) \leq R + 3\delta$. Now since, $\{\alpha^+, \alpha^-\}$ and $\{\beta^+, \beta^-\}$ are disjoint, we can choose T_0 so that if $|t| > T_0$, then $d(\hat{\eta}_\alpha(t), \hat{\eta}_\beta(\mathbb{R})) > R + 3\delta$ and $d(\hat{\eta}_\beta(t), \hat{\eta}_\alpha(\mathbb{R})) > R + 3\delta$. Therefore, if $T \geq T_0$, then the sets $A_1(T)$, $A_2(T)$, $B_1(T)$ and $B_2(T)$ are disjoint.

We now observe that p_α and p_β are coarsely well-defined. Suppose that

$$d(\hat{\eta}_\alpha(s)), x) = d(\hat{\eta}_\alpha(t), x) = d(x, \hat{\eta}_\alpha(\mathbb{R}))$$

and consider a triangle with vertices x, $\hat{\eta}_\alpha(s)$ and $\hat{\eta}_\alpha(t)$. Let p be the midpoint of the edge joining $\hat{\eta}_\alpha(s)$ and $\hat{\eta}_\alpha(t)$. By the Fellow Traveller Property $d(p, \hat{\eta}_\alpha(\mathbb{R})) \leq R = R(K, C, \delta)$. Moreover, since C_Γ is δ-hyperbolic there exists q lying on one of the other edges so that $d(p, q) \leq \delta$. Without loss of generality q is on the edge joining x to $\hat{\eta}_\alpha(s)$. Since $d(\hat{\eta}_\alpha(s)), q) = d(q, \hat{\eta}_\alpha(\mathbb{R}))$ and $d(q, \hat{\eta}_\alpha(\mathbb{R}) \leq R + \delta$, we see that $d(p, \hat{\eta}_\alpha(s)) \leq R + 2\delta$, which implies that $d(\hat{\eta}_\alpha(s)), \hat{\eta}_\alpha(t)) \leq 2R + 4\delta$, so

$$|s - t| \leq K(2R + 4\delta + C).$$

Similarly, if q lies on the edge joining x to $\hat{\eta}_\alpha(t)$, then $|s - t| \leq K(2R + 4\delta + C)$.

Notice that the previous paragraph implies that, since $\hat{\eta}_\alpha$ is α-equivariant, p_α is coarsely α-equivariant, i.e.

$$|p_\alpha(\alpha^n(x)) - (p_\alpha(x) + n)| \leq K(2R + 4\delta + C)$$

for all $x \in C_\Gamma$ and $n \in \mathbb{Z}$. Similarly,

$$|p_\beta(\beta^n(x)) - (p_\beta(x) + n)| \leq K(2R + 4\delta + C)$$

for all $x \in C_\Gamma$ and $n \in \mathbb{Z}$. Therefore, if $n, m > N = 2T_0 + K(2R + 4\delta + C)$, then $\alpha^n\big(C_\Gamma \setminus A_1(T_0)\big) \subset A_2(T_0)$, $\alpha^{-n}\big(C_\Gamma \setminus A_2(T_0)\big) \subset A_1(T_0)$, $\beta^m\big(C_\Gamma \setminus B_1(T_0)\big) \subset B_2(T_0)$, and $\beta^{-m}\big(C_\Gamma \setminus B_2(T_0)\big) \subset B_1(T_0)$. The Ping Pong Lemma then implies that $< \alpha^n, \beta^m >$ is isomorphic to F_2. $\qquad\square$

By combining Proposition 5.2 and Theorem 7.3 we obtain a strong version of the Tits Alternative for hyperbolic groups.

Corollary 7.4. (Tits Alternative for hyperbolic groups) *If Γ is a torsion-free hyperbolic group, then either Γ is cyclic or Γ contains a subgroup isomorphic to F_2.*

Proof. If Γ is not cyclic, then Proposition 5.2 implies that there exist elements α and β so that $\{\alpha^+, \alpha^-\} \neq \{\beta^+, \beta^-\}$. Theorem 7.3 then implies that Γ contains a subgroup isomorphic to F_2. $\qquad\square$

If a group Γ contains a free subgroup of rank two, then it is immediate that the number of words of length at most R grows at least exponentially in R. However, in every finitely generated group the number of words of length at most R grows at most exponentially in R. So, we see that torsion-free hyperbolic groups have exponential growth if they are not cyclic. In general, all non-elementary hyperbolic groups have exponential growth.

Corollary 7.5. *If Γ is a torsion-free hyperbolic group which is not cyclic, then the number of words of length at most R (with respect to any fixed finite generating set) grows exponentially in R.*

8. Further topics

We first recall a few results which, although not central to our lecture notes, will be used on a handful of occasions later in the notes. All of their proofs build on material we presented and are not significantly more complicated than what we have already done. We may return at a later date to sketch or give proofs.

In the next chapter we will give the proof that small deformations of convex cocompact representations into $\mathrm{Isom}(\mathbb{H}^n)$ are convex cocompact. The proof will use a local-to-global principle which allows one to detect that a infinite path is a quasi-geodesic by observing it only at a scale of some size. We will only use this fact for bi-infinite quasigeodesics in $\mathbb{H}^n$, where the proof is easier, but we state the general fact here. See Coornaert-Delzant-Papadopoulos [72, Thm. 3.1.4] for a complete proof.

Theorem 8.1. (Local to Global Principle) *Given $K \geq 1$, $C \geq 0$ and $\delta \geq 0$, there exists $\hat{K}$, $\hat{C}$ and A so that if J is an interval in $\mathbb{R}$, X is δ-hyperbolic and $h : J \to X$ is a (K, C)-quasi-isometric embedding restricted to every connected subsegment of J with length $\leq A$, then h is a $(\hat{K}, \hat{C})$-quasi-isometric embedding.*

We will give a sketch of the proof in the case when $X = \mathbb{H}^n$ and $J = \mathbb{R}$ which is based on an argument of Minsky [167]. (The assumption that $J = \mathbb{R}$ is simply for convenience, while the restriction to $X = \mathbb{H}^n$ significantly simplifies the proof).

Proof. We will make use of an elementary lemma in hyperbolic geometry which one may prove either using a compactness argument or (presumably) by computation.

Lemma 8.2. *Given $R > 0$, there exists $L > 0$ so that if P and Q are totally geodesic hyperplanes in $\mathbb{H}^n$, $p \in P$, $q \in Q$ and $x \in \mathbb{H}^n$, and $\overline{px}$ and $\overline{qx}$ are geodesic segments perpendicular to P and Q respectively, so that $d(p,q) \geq L$, then $d(P,Q) \geq R$.*

picture needed

Given $K \geq 1$ and $C \geq 0$, let R be the constant provided by the Fellow Traveller property and let L be the constant provided by Lemma 8.2. Choose A large enough so that

$$\frac{2}{K^2}\left(\frac{A}{2K} - 2C - R\right) - C - 2R > L \qquad \text{and} \qquad \frac{A}{2K} - C > 2R.$$

For all $i \in \mathbb{Z}$, let $t_i = iA$ and $y_i = h(t_i)$. Let $G_i = \overline{y_i y_{i+1}}$ be the geodesic segment with vertices y_i and y_{i+1} and midpoint m_i. Notice that $d(y_i, y_{i+1}) \geq \frac{A}{K} - C$. By the Fellow Traveller Property, there exists $s_i \in [t_i, t_{i+1}]$ such that $d(f(s_i), m_i) \leq R$. Thus

$$d(f(s_i), f(t_i)) \geq \frac{A}{2K} - C - R, \quad \text{so} \quad s_i - t_i > \frac{1}{K}\left(\frac{A}{2K} - 2C - R\right).$$

Similarly,

$$t_{i+1} - s_i > \frac{1}{K}\left(\frac{A}{2K} - 2C - R\right).$$

So,

$$d(f(s_i), f(s_{i+1})) \geq \frac{2}{K^2}\left(\frac{A}{2K} - 2C - 2R\right) - C \quad \text{and} \quad d(m_i, m_{i+1}) \geq \frac{2}{K^2}\left(\frac{A}{2K} - 2C - 2R\right) - C - 2R > L.$$

Lemma 8.2 then implies that

$$d(P_i, P_{i+1}) \geq R \qquad \text{for all} \ \ i \in \mathbb{Z}.$$

We next claim that P_{i-1} and P_{i+1} lie on opposite sides of P_i. If not, then y_{i-1} and y_{i+1} lie on the same side of P_i, so the geodesic segment $\overline{y_{i-1} y_{i+1}}$ lies on the opposite side of P_i from y_i, but

$$d(y_i, P_i) = d(y_i, m_i) \geq \frac{A}{2K} - C > 2R, \qquad \text{so} \qquad d\big(f(s_i), \overline{y_{i-1}, y_{i+1}}\big) > R$$

which would contradict the Fellow Traveller Property. It follows that P_{i-1} lies on the opposite side of P_i as P_{i+1}. Therefore, since $d(P_i, P_{i+1}) \geq R$ for all i, we see that

$$d(y_m, y_n) \geq (|m - n| - 1)R \qquad \text{for all} \ \ m, n \in \mathbb{Z},$$

so

$$d(f(a), f(b)) \geq R\left(\frac{|b - a| - 3A}{A}\right) - 2AK - 2C \geq \frac{R}{A}\big(|b-a|\big) - (3 + 2AK + 2C) \qquad \text{for all} \ a, b \in \mathbb{R}$$

Since, by the triangle inequality,

$$d(f(a), f(b)) \leq K|b - a| + \left(\frac{|b - a| + A}{A}\right)C \leq \left(K + \frac{C}{A}\right)|b - a| + C$$

we conclude that f is a $(\hat{K}, \hat{C})$-quasi-isometry where $\hat{K} = \max\left\{\frac{A}{R}, K + \frac{C}{A}\right\}$ and $\hat{C} = 3 + 2AK + 2C$. $\qquad\square$

Fricke proved that the mapping class group acts properly discontinuously on Teichmüller space. Once one has developed the theory of Anosov representations, one may readily generalize a proof of this result to show that the outer automorphism group of a hyperbolic group Γ acts properly discontinuously on the space of Anosov representations of Γ into $\mathsf{SL}(d, \mathbb{R})$. One needs an easy, but somewhat technical, result about automorphisms of hyperbolic groups in the proof. Notice that the result is quite intuitive for surface groups, so is often used implicitly in proofs of Fricke's original theorem. For a complete proof, see [55, Proposition 2.3].

Proposition 8.3. (Canary [55]) *If Γ is a torsion-free hyperbolic group, then there exists a finite collection $\mathcal{B}$ of elements of Γ, so that for any K*

$$\{\phi \in \mathrm{Out}(\Gamma) \mid ||\phi(b)|| \leq K \text{ for all } b \in \mathcal{B}\}$$

is finite.

We will see that if ρ is a Benoist representation, then the ratio of first and last eigenvalues of image elements $\rho(\gamma)$ grow uniformly exponentially in the translation length of γ on C_Γ (or equivalently the minimal word length of an element conjugate to γ.) In the proof, we will use a property of hyperbolic groups which Delzant, Guichard, Labourie, and Mozes call the **U property**. Their proof builds on the ping pong techniques used in Section 7.

Proposition 8.4. (Delzant-Guichard-Labourie-Mozes [87, Proposition 2.2]) *If Γ is a hyperbolic group, then there exist $\alpha, \beta \in \Gamma$ and $K > 0$ so that*

$$d(1, \gamma) \leq 3 \max\{||\gamma||, ||\gamma\alpha||, ||\gamma\beta||\} + K$$

for all $\gamma \in \Gamma$.

We briefly recall a few fundamental facts about hyperbolic groups which will not used in our lecture notes.

One of the original motivations for the study of hyperbolic groups, especially for Cannon [61], is that one may use geometric techniques to show that most decision problems are solvable. For example, the **word problem** is solvable (i.e. there is an algorithm which can decide whether or not an element in the group is trivial) and the **conjugacy problem** is solvable (i.e. there is an algorithm to decide whether two elements of the group are conjugate). (See, for example, Bridson-Haefliger [49, Section III.Γ.2]). A much more difficult fact is that the **isomorphism problem** is solvable in the class of hyperbolic groups (i.e. there is an algorithm to determine whether or not two hyperbolic groups are isomorphic). This was established by Sela [188] for (rigid) torsion-free hyperbolic groups, and, for all hyperbolic groups, by Dahmani and Guirardel [80].

A group Γ is said to have a **linear isoperimetric inequality** if it has a finite presentation $\Gamma = <X \mid R>$ so that there exists a linear function f such that if w is a word in X representing the trivial element, then it can be written as a product of at most $f(\ell(w))$ conjugates of relations in R where $\ell(w)$ is the word length of w. A group is hyperbolic if and only if it has a linear isoperimetric inequality. (See, for example, Bridson-Haefliger [49, Section III.H.2].) Moreover, Papasoglu [172] showed that a group is hyperbolic if it has a sub-quadratic isoperimetric inequality (see also Bowditch [35]).

We will be studying discrete, faithful representations of hyperbolic groups into linear groups. However, there are examples of hyperbolic groups which do not admit faithful representations into any linear group, see Kapovich [128, Section 8]. New examples were recently given by Canary, Stover and Tsouvalas [59].

One can show that all hyperbolic groups are finitely presented (see Bridson-Haefliger [49, Proposition III.Γ.2.2]). However, not all finitely presented subgroups of a hyperbolic group are themselves hyperbolic, see Brady [40]. Moreover, a torsion-free hyperbolic group always has a finite $K(\pi, 1)$, i.e. is the fundamental group of a finite CW-complex with a contractible universal cover. In general, a hyperbolic group Γ it is the fundamental group of a CW-complex whose n-skeleton is finite for all $n \in \mathbb{N}$, so Γ is of type F_∞. (See, for example, Bridson-Haefliger [49, Section III.Γ.3].)

Part 3. Convex cocompact representations in rank one Lie groups

Just us kids hangin' out today
Watchin' our long hair turnin' gray
Not so skinny maybe not so free
Not so many as we used to be
——————James McMurtry [164]

We begin with a quick review of the Teichmüller space of a closed surface. This theory is the motivation for much of what is now known as Higher Teichmüller theory. In particular, Fuchsian representations are the prototypical example of an Anosov representation. The first generalization of the theory of Fuchsian representations is the theory of convex cocompact representations into rank one Lie groups. They are also another class of classical examples of Anosov representations. We will survey this theory with an emphasis on the aspects of the theory which inspire and generalize to the theory of Anosov representations.

We will restrict to the case of $\mathsf{O}_0(n,1)$ which is the isometry group of real hyperbolic space $\mathbb{H}^n$. Everything we do in this section generalizes in some form to the other rank one Lie groups, which are "essentially" $\mathsf{U}(n,1)$, the isometry group of complex hyperbolic space $\mathbb{CH}^n$, $\mathsf{Sp}(n,1)$, the group of orientation-preserving isometries of quaternionic hyperbolic space, and the isometry group of the Cayley plane. See, for example, Bridson-Haefliger [49, Section II.10] or Parker's lecture notes [173] (which are available on his webpage). Our working definition of a rank one Lie group is that it is a semi-simple Lie group whose quotient symmetric space is negatively curved. This turns out to be equivalent to the assumption that the quotient symmetric space does not contain an isometrically embedded Euclidean plane. More generally, the rank of a semi-simple Lie group is the maximal dimension of an isometrically embedded copy of Euclidean space in the quotient symmetric space.

9. Teichmüller space: a refresher

And it's here I see pictures and my madness is clear
And there's no longer logic so therefore no fear
——————Ian Hunter [118]

If you are not already somewhat familiar with Teichmüller theory, I recommend you immediately put down these notes and go read a more complete treatment of this beautiful subject. Farb and Margalit [96] give a nice treatment from a modern geometrical/topological viewpoint. Bers' survey paper [27] is a beautiful treatment of the classical complex analytic approach (and contains an oblique, but poignant, commentary on the tension of working in a subject named after an ardent Nazi). Thurston [203, Section 4.6] gives a concise treatment of the Fenchel-Nielson coordinates. Abikoff [2] gives a treatment of the classical theory with an eye towards the modern viewpoint.

Recall that the upper half-plane model for the hyperbolic plane is given by

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

with Riemannian metric

$$ds_{hyp}^2 = \frac{1}{y^2} \, dxdy.$$

In this metric, the geodesics are lines and semi-circles perpendicular to the real line and the group of Möbius transformations with real co-efficients acts as the group of orientation-preserving isometries of $\mathbb{H}^2$, i.e.

$$\mathrm{Isom}_+(\mathbb{H}^2) = \mathsf{PSL}(2, \mathbb{R}).$$

The hyperbolic plane has constant curvature -1, which is manifested explicitly in the fact that if P is a n-gon in $\mathbb{H}^2$ with internal angles $\{\alpha_1, \ldots, \alpha_n\}$, then

$$\mathrm{Area}(P) = \pi(n-2) - \sum_{i=1}^n \alpha_i.$$

A complete orientable Riemannian surface X is said to be **hyperbolic** if it is locally isometric to $\mathbb{H}^2$. In this case, the universal cover $\tilde{X}$ is a simply connected complete Riemannian manifold locally isometric to $\mathbb{H}^2$ and hence can be identified with $\mathbb{H}^2$. Therefore, $X = \mathbb{H}^2/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^2) \cong \mathsf{PSL}(2, \mathbb{R})$. Notice that Γ is only well-defined up to conjugacy, since the identification of $\tilde{X}$ with $\mathbb{H}^2$ is not canonical.

A **marked hyperbolic structure** on a closed orientable surface S is a pair (X, f) where $f : S \to X$ is a homeomorphism and X is a hyperbolic surface. If $X = \mathbb{H}^2/\Gamma$, then $f_* : \pi_1(S) \to \pi_1(X) \cong \Gamma$ is an isomorphism and hence we obtain a discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{R})$. However, ρ is only well-defined up to conjugation in $\mathsf{PGL}(2, \mathbb{R})$ (which we interpret as the full group of isometries of $\mathbb{H}^2$). In classical Teichmüller theory, both S and X are oriented so one gets a representation which is well-defined up to conjugacy into $\mathsf{PSL}(2, \mathbb{R})$. It will be more convenient for us to ignore orientation. One may view this as a sign of the depravity I have fallen into since entering Higher Teichmüller theory.

> *I used to hate the fool in me, but only in the morning*
> *Now I tolerate him all day long*
> ——————Mike Cooley [90]

One may build a hyperbolic surface of genus two, by starting with a regular hyperbolic octagon, all of whose internal angles are $\frac{\pi}{4}$ and then gluing by the standard gluing pattern. Similarly, one may build a hyperbolic surface of genus g by starting with a regular $(4g-4)$-gon with internal angles $\frac{\pi}{2g}$.

In turn, one can build any hyperbolic surface of genus two from a hyperbolic octagon, all of whose angles add up to 2π. One can see this, by first noticing that the surface is obtained this way topologically, so one has a bouquet of circles on the hyperbolic surface whose complement is an open disk. If we fix the vertex of the bouquet of circles at one point on the surface and pull the edges tight so that they form geodesic arcs, then the complement of the resulting geodesic bouquet of circles is a hyperbolic octagon.

One might then try to guess how big the Teichmüller space of marked hyperbolic structures on S is. The space of hyperbolic octagons is 16-dimensional, since an octagon is determined by its vertices. There are 5 constraints, coming from the fact that the internal angles must add up to 2π and that lengths of edges that are paired must agree, giving a 11-dimensional space of allowed octagons, but $\mathsf{PSL}(2, \mathbb{R})$ acts as congruences of these octagons, so one really has a 8-dimensional space of octagons. Finally, an octagon gives a hyperbolic surface plus a specified point on that surface, so the space of hyperbolic surfaces should be 6-dimensional.

We will choose to formalize Teichmüller space by using representations. Recall that a marked hyperbolic structure on a closed surface S, gives rise to a (conjugacy class of a) discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{R})$. In turn, a discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{R})$ gives rise to a hyperbolic surface $X_\rho = \mathbb{H}^2 / \rho(\pi_1(S))$. Since X_ρ is homotopy equivalent to S, it is homeomorphic to S. Moreover, there is homeomorphism $h_\rho : S \to X_\rho$ so that $(h_\rho)_*$ is conjugate to ρ. (Here, we are using a special property of the topology of closed surfaces. The Nielsen-Baer Theorem, see Farb-Margalit [96, Chapter 8], gives that every homotopy equivalence of a closed orientable surface is homotopic to a homeomorphism.) We then let

$$\mathrm{DF}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R})) = \{\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{R}) \mid \rho \text{ discrete, faithful}\}$$

and the Teichmüller space of S is the quotient

$$\mathcal{T}(S) = \mathrm{DF}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))/\mathsf{PGL}(2, \mathbb{R})$$

where $\mathrm{DF}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$ inherits a topology as a subset of $\mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$, $\mathsf{PGL}(2, \mathbb{R})$ acts by conjugation and $\mathcal{T}(S)$ inherits the quotient topology.

Notice that the Milnor-Svarc Lemma implies that if $\rho \in \mathrm{DF}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$, then the orbit map $\tau_\rho : \pi_1(S) \to \mathbb{H}^2$ is a quasi-isometry and Corollary 3.6 implies that there is a ρ-equivariant homeomorphism $\xi_\rho : \partial \pi_1(S) \to \partial \mathbb{H}^2 \cong S^1$.

Alternatively, one may define $\mathcal{T}(S)$ to be the space of marked hyperbolic structure on S up to the equivalence $(X_1, f_1) \sim (X_2, f_2)$ if and only if $f_2 \circ f_1^{-1}$ is homotopic to an isometry. One may think of X as hyperbolic clothing for the naked topological surface S and f as instructions for how to wear the clothing. The equivalence relation allows one to adjust the clothing, but not to wear it backwards or to stick your head through the hole designated for the arm.

It is a classical theorem, going back to the 19th century, that $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$ if $g \geq 2$ is the genus of S. (Notice that $\pi_1(S)$ has a presentation with $2g$ relations and one relation, one would expect that $\mathrm{DF}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$ has dimension $(2g)3 - 3 = 6g - 3$, so one would predict that Teichmüller space has dimension $6g - 6$.) The *mapping class group* $\mathrm{Mod}(S)$ is the group of (isotopy classes of) self-homeomorphisms of S. Fricke showed that the mapping class group acts properly discontinuously, but not freely, on $\mathcal{T}(S)$ and its quotient is the **moduli space** of unmarked hyperbolic structures on S. We will soon prove a very general version of Fricke's theorem.

There are a variety of metrics on Teichmüller space, all of which have their own advantages and disadvantages. The most prominent are the Teichmuller metric, which is complete, but only a Finsler metric and is not non-positively curved, and the Weil-Petersson metric which is Riemannian and negatively curved, but not complete. Teichmüller showed that Teichmüller space has a natural complex structure (I.e. one invariant under the action of the mapping class group) and Ahlfors [5] showed that the Weil-Petersson metric is Kähler. Wolpert (see [219] for a survey) extensively studied the resulting symplectic structure on Teichmüller space. This hopefully provides a quick taste of the bounty of structure associated to Teichmüller space.

We now give a quick sketch of the Fenchel-Nielsen coordinates on Teichmüller space. Suppose that X is a closed orientable hyperbolic surface of genus $g \geq 2$. Recall that, since X is negatively curved, every homotopically non-trivial closed curve is homotopic to a unique closed geodesic. Morever, if two homotopically non-trivial simple closed curves are disjoint and non-parallel, then their geodesic representatives are also disjoint. Let $\alpha = \{\alpha_1, \ldots, \alpha_{3g-3}\}$ be a maximal collection of disjoint simple closed curves and let α^* be their geodesic representatives on X.

The components of $X - \alpha^*$ are a collection of $2g - 2$ hyperbolic pairs of pants with geodesic boundary. (A topological **pair of pants** is a disk with two holes.) Therefore, every closed hyperbolic surface may be built from hyperbolic pairs of pants.

If P is a hyperbolic pair of pants with geodesic boundary and s_1, s_2 and s_3 are the shortest paths joining boundary components (called seams), then $P - \{s_1, s_2, s_3\}$ is a pair of all-right hyperbolic hexagons (i.e. hexagons all of whose interior angles are $\frac{\pi}{2}$). An all-right hexagon is determined by the lengths of any 3 non-consecutive sides. Moreover, any 3 lengths can be achieved. It follows that P is the double of the unique all-right hexagon with alternate sides having lengths agreeing with the lengths of the seams of P. Moreover, we can build a geodesic pair of pants with any collection of boundary lengths and this geodesic pair of pants is entirely determined by its boundary lengths.

pictures needed

So the hyperbolic structure on X is determined, up to isometry, by the lengths of the components of α^* and instructions for gluing the pants together. Since the pants are glued along closed geodesic curves, there is a one-dimensional space of ways to glue them. This suggests more forcefully that the space of hyperbolic structures on X has dimension $6g - 6$.

More formally, we get a map $L : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3}$ where

$$L(X, f) = \left(\ell_X(f(\alpha_i)^*) \right)_{i=1}^{3g-3}.$$

At each element of α we can define a twist coordinate in $\mathbb{R}$ which records how the geodesic pairs of pants are glued along $f(\alpha_i)^*$, so we obtain $\Theta : \mathcal{T}(S) \to \mathbb{R}^{3g-3}$. (I will wave my hands about this in class, but you can read about it carefully elsewhere.) One can then see that

$$(L, \Theta) : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \cong \mathbb{R}^{6g-6}$$

is a homeomorphism. For a careful discussion of twist coordinates see, for example, Thurston [203, Section 4.6], Farb-Margalit [96, Section 10.6] or Martelli [159, Chapter 7].

One of the crucial properties of Teichmüller space is that it is an entire component of the representation variety. In some circles, a Higher Teichmüller space is defined to be a component of the character variety consisting of Anosov representations, but I would argue that this definition is too restrictive.

Theorem 9.1. *If S is a closed oriented surface of genus $g \geq 2$, then $\mathcal{T}(S)$ is a component of*

$$X(\pi_1(S), \mathsf{PSL}(2, \mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))/\mathsf{PGL}(2, \mathbb{R}).$$

We will sketch a simple hands-on proof in our situation.

Proof. It suffices to prove that $\mathcal{T}(S)$ is open and closed in $X(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$, since we have already sketched a proof that it is connected.

Suppose that a sequence $\{[\rho_n]\} \subset \mathcal{T}(S)$ converges to $[\rho] \in X(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$. Then, one may find representatives, $\rho_n \in \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$ which converge to a representative ρ of $[\rho]$. Corollary 6.2 implies that ρ is also discrete and faithful, so $[\rho] \in \mathcal{T}(S)$. Therefore, $\mathcal{T}(S)$ is closed.

In the proof of open-ness we will restrict to the case where S has genus 2, but the proof generalizes. Suppose $[\rho] \in \mathcal{T}(S)$ and $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{R})$ is a representative. We have seen that X_ρ may be obtained by gluing up a hyperbolic octagon. So there is an octagon D which is a fundamental domain for the action of $\rho(\Gamma)$ on $\mathbb{H}^2$. Then we can find a generating set $\{a_1, a_2, a_3, a_4\}$ for $\pi_1(S)$ so that the edges of D occur in the order $E_1, E_2, \rho(a_1)(E_1), \rho(a_2)(E_2), E_3, E_4, \rho(a_3)(E_3), \rho(a_4)(E_4)$.

Let L_i be the bi-infinite geodesic containing E_i. Then, we can choose a neighborhood U of ρ in $\mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$, so that if $\sigma \in U$, then $L_1, L_2, \sigma(a_1)(L_1), \sigma(a_2)(L_2), L_3, L_4, \sigma(a_3)(L_3), \sigma(a_4)(L_4)$ cut out an octagon D_σ close to (and combinatorially equivalent to) D. One can then show that D_σ glues up to give a hyperbolic surface of genus two, and so is a fundamental domain for the action of $\sigma(\pi_1(S))$ on $\mathbb{H}^2$. It follows that $U \subset DF(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ and that the projection $[U]$ of U is an open neighborhood of ρ in $X(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ which is contained in $\mathcal{T}(S)$. (Notice that the pre-image of $[U]$ is the $\mathsf{PGL}(2,\mathbb{R})$ orbit of U which is open in $\mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ and contained in $\mathrm{DF}(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$.) It follows that $\mathcal{T}(S)$ is open in $X(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ which completes the proof. $\qquad\square$

We will later see several facts which allow one to give a more general and more conceptual proof that $\mathcal{T}(S)$ is open. We will see (Theorem 11.4) that every representation into $\mathsf{O}_0(n,1)$ whose orbit map is a quasi-isometric embedding into $\mathbb{H}^n$ has a neighborhood consisting of representations whose orbit maps are quasi-isometric embeddings. Since a representation of a torsion-free group whose orbit map is a quasi-isometric embedding is discrete and faithful and $\mathsf{O}_0(2,1) \cong \mathsf{PSL}(2,\mathbb{R})$, this gives a proof of open-ness.

10. **Hyperbolic geometry in dimension** $n > 2$

You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life....I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind..... I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind. I admit that I expect little from the deviation of your lines. It seems to me that I have been in these regions; that I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time.... For God's sake, I beseech you, give it up. Fear it no less than sensual passions because it too may take all your time and deprive you of your health, peace of mind and happiness in life.

——Farkas Bolyai [165] (advising his son not to work on hyperbolic geometry).[1]

One may again define (real) hyperbolic space of dimension n as the unique simply connected manifold of dimension n with constant sectional curvature -1. There is again an upper half space model for hyperbolic n-space

$$\mathbb{H}^n = \{\vec{x} \in \mathbb{R}^n \mid x_n > 0\}$$

with line element

$$ds = \frac{1}{x_n}\sqrt{dx_1^2 + \cdots + dx_n^2}.$$

It is easy to see that the subset $Y = \{\vec{x} \in \mathbb{H}^n \mid x_2 = \cdots = x_{n-1} = 0\}$ is a totally geodesic copy of $\mathbb{H}^2$ sitting within $\mathbb{H}^n$ (since reflection in the x_1-x_n-plane fixes Y). One can then check that inversions in hemispheres orthogonal to $\{x_n = 0\}$ and reflections in hyperplanes orthogonal to $\{x_n = 0\}$ give isometries of $\mathbb{H}^n$. Since, the group generated by these inversions and reflections

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[1]I found this quote, which I had only seen portions of before, in a delightful blog post by Evelyn Lamb. https://blogs.scientificamerican.com/roots-of-unity/hyperbolic-quotes-about-hyperbolic-geometry/

acts transitively on the orthonormal frame bundle of $\mathbb{H}^n$, we see that this group is the full isometry group of $\mathbb{H}^n$. Therefore, $\mathbb{H}^n$ has constant sectional curvature, and since it contains a totally geodesic copy of $\mathbb{H}^2$, the constant is -1. Moreover, one sees that the geodesics are semi-circles and lines orthogonal to $\{x_n = 0\}$.

In the case $n = 3$, we may identify this group with the group of conformal automorphisms of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \partial \mathbb{H}^3$. So, in this case we can identify $\mathrm{Isom}_+(\mathbb{H}^3)$ with $\mathsf{PSL}(2, \mathbb{C})$. However, in general, this group doesn't have a nice presentation as a matrix group from this viewpoint.

By introducing the hyperboloid model for $\mathbb{H}^n$, we can identify $\mathrm{Isom}_+(\mathbb{H}^n)$ with $\mathsf{PSO}(n, 1)$ (or with $\mathsf{SO}_0(n, 1)$ if one prefers).

Let
$$B(\vec{x}, \vec{y}) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n - x_{n+1} y_{n+1}$$
be the bilinear form associated to the quadratic form
$$Q(\vec{x}) = B(\vec{x}, \vec{x}) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$$
of signature $(n, 1)$. Consider the hyperboloid H^n with two sheets in $\mathbb{R}^{n+1}$ given by
$$H^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid Q(\vec{x}, \vec{x}) = -1\}.$$

Let $\mathsf{O}(n, 1)$ be the set of matrices $A \in \mathsf{GL}(n+1, \mathbb{R})$ preserving the indefinite quadratic form Q. Explicitly,
$$\mathsf{O}(n, 1) = \{A \in \mathsf{GL}(n+1, \mathbb{R}) \mid A^T J A = J\}$$
where J is the diagonal matrix with entries $(1, 1, \ldots, 1, -1)$. Let's check that this is the correct formula. If $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ and $A \in \mathsf{GL}(n+1, \mathbb{R})$, then
$$B(\vec{x}, \vec{y}) = \vec{x}^T J \vec{y} \qquad \text{and} \qquad B(A\vec{x}, A\vec{y}) = (A\vec{x})^T J A\vec{y} = \vec{x}^T A^T J A\vec{y}.$$
Therefore, $A^T J A = J$ if and only if $B(A\vec{x}, A\vec{y}) = B(\vec{x}, \vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$.

Notice that if $U \in \mathsf{O}(n) \subset \mathsf{GL}(n, \mathbb{R})$, then
$$\begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \in \mathsf{O}(n, 1)$$
so $\mathsf{O}(n)$ may be identified with a subgroup of $\mathsf{O}(n, 1)$ which acts transitively on the set of orthonormal frames for $T_{e_{n+1}} H^n$. One may also check that
$$\begin{bmatrix} I_{n-1} & \vec{0} & \vec{0} \\ 0 \cdots 0 & \cosh t & \sinh t \\ 0 \cdots 0 & \sinh t & \cosh t \end{bmatrix} \in \mathsf{O}(n, 1)$$
for all $t \in \mathbb{R}$, where I_{n-1} is the identity matrix in $\mathsf{SL}(n-1, \mathbb{R})$ and $\vec{0}$ is the trivial vector in $\mathbb{R}^{n-1}$. Therefore, $\mathsf{O}(n, 1)$ acts transitively on the intersection of the upper sheet of H^n with the x_n-x_{n+1}-plane. Moreover $J \in \mathsf{O}(n, 1)$ takes e_{n+1} to $-e_{n+1}$. Combining these observations we see that $O(n, 1)$ acts transitively on the orthonormal frame bundle of TH^n.

Notice that $T_{e_{n+1}} H^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$, so B restricts to a positive definite bilinear form on $T_{e_{n+1}} H^n$. Since $\mathsf{O}(n, 1)$ preserves B and acts transitively on H^n, we see that B restricts to a positive definite bilinear form on $T_{\vec{x}} H^n$ for all $\vec{x} \in H^n$. Therefore, B induces a Riemannian metric on H^n.

Since $\mathsf{O}(n,1)$ acts as a group of isometries of H^n and acts transitively on the space of orthonormal frames in TH^n, H^n has constant sectional curvature and $O(n,1)/O(n) = \mathrm{H^n}$.

There is a nice distance formula for points on the upper sheet of H^n given by

$$d(\vec{x}, \vec{y}) = \mathrm{arccosh}\ (-B(\vec{x}, \vec{y})) = \mathrm{arccosh}(-x_1 y_1 - \cdots - x_n y_n + x_{n+1} y_{n+1}).$$

Notice that reflection in the x_n-x_{n+1}-plane lies in $\mathsf{O}(n,1)$ so is an isometry of H^n. Therefore the intersection Z of the x_n-x_{n+1}-plane with H^n is a geodesic in H^n. One then checks that this formula works for $\vec{x} = 0$ and $\vec{y} = (0, \cdots, 0, \sinh t, \cosh t)$ and then observes that the formula is invariant under $\mathsf{O}(n,1)$ and that any two points may be moved into this position by an element of $\mathsf{O}(n,1)$.

Notice that since $\mathsf{O}(n,1)$ acts transitively on the orthonormal frame bundle of TH^n, the Riemannian metric on H^n has constant sectional curvature. In order to evaluate the curvature we can examine the right-angled triangle with vertices e_{n+1}, $(0, \cdots, 0, \sinh t, \cosh t)$ and $(0, \cdots, 0, \sinh t, 0, \cosh t)$, which has two sides of length $|t|$ and one side of length $\mathrm{arccosh}(\cosh^2(t))$. It is immediately clear that this formula is inconsistent with positive or zero curvature, and one may compare with $\mathbb{H}^2$ to check that it is consistent with sectional curvature -1, so in fact H^n must have constant sectional curvature exactly -1.

We can thus identify $\mathbb{H}^n$ with the top sheet of H^n (i.e. the sheet containing e_{n+1}). The isometry group of $\mathbb{H}^n$ is the component $\mathsf{O}_0(n,1)$ of isometries of H^n which preserve the top sheet. Equivalently, we can identify $\mathrm{Isom}(\mathbb{H}^n)$ with $\mathsf{PO}(n,1)$ and identify $\mathrm{Isom}_+(\mathbb{H}^n)$ with $\mathsf{PSO}(n,1)$.

Notice that there is an isomorphism $\tau_3 : \mathsf{PSL}(2, \mathbb{R}) \to \mathsf{PSO}(2,1) \subset \mathsf{PSL}(3, \mathbb{R})$ which is also known as the irreducible representation. There is also an isomorphism between $\mathsf{PSL}(2, \mathbb{C})$ and $\mathsf{PSO}(3,1)$.

Let $\mathcal{K}$ be the copy of $\mathsf{O}(n)$ sitting inside of $\mathsf{O}_0(n,1)$ and fixing the x_{n+1}-axis. Every element of $\mathcal{K}$ has the form

$$\begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}$$

where $U \in \mathsf{O}(n) \subset \mathsf{GL}(n, \mathbb{R})$. Then $\mathcal{K}$ is the stabilizer of the point e_{n+1} within $\mathsf{O}_0(n,1)$. Since $\mathsf{O}_0(n,1)$ acts transitively on $\mathbb{H}^n$, we may identify

$$\mathbb{H}^n = \mathsf{O}_0(n,1)/\mathcal{K}$$

where, if $B \in \mathsf{O}_0(n,1)$, the coset $B\mathcal{K}$ is identified with $B(e_{n+1}) \in \mathbb{H}^n$.

Let $\mathcal{A}$ be the subgroup of all elements of $\mathsf{O}_0(n,1)$ of the form

$$A_t = \begin{bmatrix} I_{n-1} & \vec{0} & \vec{0} \\ 0 \cdots 0 & \cosh t & \sinh t \\ 0 \cdots 0 & \sinh t & \cosh t \end{bmatrix} \in \mathsf{O}(n,1)$$

for some $t \in \mathbb{R}$. Notice that

$$\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

so the eigenvalues and singular values of A_t agree and are equal to $\{e^t, 1, \ldots, 1, e^{-t}\}$ (in that order if $t \geq 0$).

If $T \in \mathsf{O}_0(n,1)$, then we may choose $L \in \mathcal{K}$ so that $L^{-1}T(e_{n+1}) = (0, \cdots, 0, \sinh t, \cosh t)$ where $d_{\mathbb{H}^n}(e_{n+1}, T(e_{n+1})) = t \geq 0$. Then $A_{-t}L^{-1}T(e_{n+1}) = e_{n+1}$ so there exists $K \in \mathcal{K}$ so that $A_{-t}L^{-1}T = K$. So, to conclude, if $T \in \mathsf{O}(n,1)$, then there exists $K, L \in \mathcal{K}$ and $t \geq 0$ so that

$$T = LA_tK \quad \text{and} \quad d_{\mathbb{H}^n}(e_{n+1}, T(e_{n+1})) = t \geq 0.$$

This is the Cartan decomposition of $\mathsf{O}_0(n,1)$, also known as the KAK decomposition. It turns out that $\mathcal{A}$ is a maximal abelian subgroup and $\mathcal{K}$ is a maximal compact subgroup of $\mathsf{O}_0(n,1)$.

Since the singular values, but not necessarily the eigenvalues, of T agree with those of A. We see that $\sigma_1(T) = e^t$, $\sigma_2(T) = \cdots = \sigma_n(T) = 1$ and $\sigma_{n+1}(T) = e^{-t}$. It follows that, if $T \in \mathsf{O}_0(n,1)$, then

$$d_{\mathbb{H}^n}(e_{n+1}, T(e_{n+1})) = \log \sigma_1(T) = \log \frac{\sigma_1(T)}{\sigma_2(T)} = \frac{1}{2} \log \frac{\sigma_1(T)}{\sigma_{n+1}(T)}.$$

One can read more about the hyperboloid model for $\mathbb{H}^n$ in Thurston [203, Chapter 2] or Martelli [159, Chapter 2].

11. Convex cocompact representations into $\mathrm{Isom}(\mathbb{H}^n)$

We've been doing this longer than you've been alive
Propelled by some mysterious drive
And they still let me do it as weird as that seems
And I do it most nights and then again in my dreams
–Ken Bethea, Murry Hammond, Rhett Miller, and Philip Peoples [171]

Given a representation $\rho : \Gamma \to \mathsf{O}_0(n,1)$ of a finitely generated group Γ and $x_0 \in \mathbb{H}^n$, there is an **orbit map** $\tau_\rho : \Gamma \to \mathbb{H}^n$ given by

$$\tau_\rho(\gamma) = \rho(\gamma)(x_0)$$

for all $\gamma \in \Gamma$. We say that ρ is **convex cocompact** if and only if Γ is finitely generated and τ_ρ is a quasi-isometric embedding. One may check that this definition does not depend on the choice of (finite) generating set for Γ or the choice of the basepoint x_0 for the orbit map.

It is immediate from this definition that $\rho(\Gamma)$ is discrete and ρ is **almost faithful**, i.e. the kernel of ρ is finite. If Γ is torsion-free, then ρ must be faithful. Since Γ is quasi-isometric to its Cayley graph, Corollary 2.5 implies that Γ is a Gromov hyperbolic group. Theorem 3.5 implies that there exists a continuous, injective map $\xi_\rho : \partial\Gamma \to \partial\mathbb{H}^n$ so that if $\{\gamma_n\} \subset \Gamma$ converges to $z \in \partial\Gamma$, then $\{\tau_\rho(\gamma_n))\} = \{\rho(\gamma_n)(x_0)\}$ converges to $\xi_\rho(z)$. Since τ_ρ is ρ-equivariant, it follows that ξ_ρ is also ρ-equivariant. Since $\rho(\gamma)$ fixes $\xi_\rho(\gamma^+)$ and $\xi_\rho(\gamma^+) = \lim \gamma^n(x_0)$, it is easy to check that ξ_ρ is **dynamics-preserving**, i.e. that $\xi_\rho(\gamma^+)$ is the attracting fixed point of $\rho(\gamma)$. We will almost always assume that Γ is non-elementary, i.e. that Γ does not contain a finite index cyclic group.

We collect all these observations in the following result.

Theorem 11.1. *Suppose that Γ is a finitely generated group and $\rho : \Gamma \to \mathsf{O}_0(n,1)$ is a convex cocompact representation.*

(1) *ρ is discrete and almost faithful.*
(2) *Γ is Gromov hyperbolic.*

(3) *There exists a continuous, dynamics-preserving, injective ρ-equivariant map $\xi_\rho : \partial\Gamma \to \partial\mathbb{H}^n$.*

Notice that, by the equivariance of the orbit map, the orbit map τ_ρ (with respect to some point $x_0 \in \mathbb{H}^n$) of a representation $\rho : \Gamma \to \mathsf{O}_0(n,1)$ is a (K,C)-quasi-isometric embedding if and only if

$$\frac{1}{K}d(1,\gamma) - C \leq d(x_0, \rho(\gamma)(x_0)) \leq Kd(1,\gamma) + C$$

for all $\gamma \in \Gamma$. Therefore, if we choose $x_0 = e_{n+1}$ and recall that $d_{\mathbb{H}^n}(e_{n+1}, T(e_{n+1})) = \log\frac{\sigma_1(T)}{\sigma_2(T)}$ for all $T \in \mathsf{O}(n,1)$, we obtain the following more Lie-theoretic characterization of convex co-compact representations into $\mathsf{O}_0(n,1)$. (This should remind you of our definition of Anosov representations in Chapter 1.)

Lemma 11.2. *A representation $\rho : \Gamma \to \mathsf{O}_0(n,1)$ is convex cocompact if and only if there exists K and C so that*

$$\frac{1}{K}d(1,\gamma) - C \leq \log\frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \leq Kd(1,\gamma) + C$$

for all $\gamma \in \Gamma$.

If ρ is discrete and almost faithful, we define its **limit set** $\Lambda(\rho) \subset \partial\mathbb{H}^n$ to be the set of accumulation points of an orbit, i.e.

$$\Lambda(\rho) = \overline{\rho(\Gamma)(x_0)} - \rho(\Gamma)(x_0) \subset \partial\mathbb{H}^n$$

for some $x_0 \in \mathbb{H}^n$. One may check that $\Lambda(\rho)$ does not depend on the choice of x_0. In the upper half-space model, $\partial\mathbb{H}^n$ is identified with $\mathbb{R}^{n-1} \cup \{\infty\}$ where $\mathbb{R}^{n-1}$ is identified with the hyperplane $\{x_n = 0\}$. In the hyperboloid model we let L be the "light cone," i.e.

$$L = \{\vec{x} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 0\}$$

and then $\partial\mathbb{H}^n = \mathbb{P}(L)$ where a sequence $\{x_n\}$ in $\mathbb{H}^n$ converges to $[z] \in \partial\mathbb{H}^n$ if and only if the sequence of lines $\{\overleftrightarrow{0x_n}\}$ converges to $\overleftrightarrow{0z}$. The limit set may be defined more dynamically as the minimal Γ-invariant subset of $\partial\mathbb{H}^n$.

The name convex cocompact arises because if ρ is convex cocompact, then it acts cocompactly on the convex hull $\mathrm{CH}(\Lambda(\rho))$ of its limit set. The quotient $C(N_\rho) = \mathrm{CH}(\Lambda(\rho))/\rho(\Gamma)$ is called the **convex core** of $N_\rho = \mathbb{H}^n/\rho(\Gamma))$.

Proposition 11.3. *A discrete faithful representation $\rho : \Gamma \to \mathsf{PO}(n,1)$ is convex cocompact if and only if there exists a convex subset Ω of $\mathbb{H}^n$ so that $\rho(\Gamma)$ preserves and acts cocompactly on Ω. Moreover, if ρ is convex cocompact, $\xi_\rho(\partial\Gamma) = \Lambda(\rho)$ and $\rho(\Gamma)$ acts cocompactly on $\mathrm{CH}(\Lambda(\rho))$.*

Proof. If $\rho(\Gamma)$ preserves and acts cocompactly on Ω, then τ_ρ gives a quasi-isometry from Γ to Ω (by the Milnor-Svarc Lemma). Since Ω is convex, it isometrically embeds in $\mathbb{H}^n$. Therefore, τ_ρ is a quasi-isometric embedding into $\mathbb{H}^n$, so ρ is convex cocompact.

Now suppose ρ is convex cocompact, so τ_ρ is a quasi-isometric embedding into $\mathbb{H}^n$. We choose $\Omega = \mathrm{CH}(\Lambda(\rho))$, which may be formed as the union of all ideal polyhedral n-simplices in $\mathbb{H}^n$ with endpoints in the limit set. We may assume that x_0 has been chosen to lie in Ω, which implies, since Ω is Γ-invariant, that $\tau_\rho(\Gamma) \subset \Omega$. Notice that $\rho(\Gamma)$ acts cocompactly on Ω if and only if $\tau_\rho(\Gamma)$ is coarsely dense in Ω, i.e. there exists $A > 0$ so that if $x \in \Omega$, then there exists $\gamma \in \Gamma$ so that $d(x, \rho(\gamma)(x_0)) \leq A$.

Recall that τ_ρ extends to a (K, C)-quasi-isometric embedding $\hat{\tau}_\rho : C_\Gamma \to \mathbb{H}^n$ where C_Γ is the Cayley graph of Γ. Notice that the image of every edge of C_Γ has uniformly bounded length, so it suffices to show that $\hat{\tau}_\rho(C_\Gamma)$ is coarsely dense in Ω. The Fellow Traveller Property implies that there exists R so that if $\alpha : [a, b] \to \mathbb{H}^n$ is a (K, C)-quasi-isometric embedding, then $\alpha([a.b])$ is a Hausdorff distance at most R apart from the geodesic $\overline{\alpha(a)\alpha(b)}$ joining $\alpha(a)$ to $\alpha(b)$.

If $z \neq w \in \Lambda(\rho)$, then there exists $\{x_n\}$ and $\{y_n\}$ in $\tau_\rho(\Gamma)$ so that $x_n \to z$ and $y_n \to w$. Then $\overline{x_n y_n}$ lies in the (closed) neighborhood $\mathcal{N}_R(\hat{\tau}_\rho(C_\Gamma))$ of $\hat{\tau}_\rho(C_\Gamma)$ of radius R, for all n, and $\overline{x_n y_n} \to \overline{zw}$, so $\overline{zw} \subset \mathcal{N}_R(\hat{\tau}_\rho(C_\Gamma))$. Notice that there exists B_n so that if T is an ideal polyhedral n-simplex in $\mathbb{H}^n$, then every point in T lies within B_n of an edge of T. Therefore, every point in Ω lies with $R + B_n$ of a point in $\hat{\tau}_\rho(\Gamma)$, so $\rho(\Gamma)$ preserves and acts cocompactly on the convex set Ω.

Finally, we check that $\xi_\rho(\partial\Gamma) = \Lambda(\rho)$. If $z \in \partial\Gamma$, then there exists a sequence $\{\gamma_n\} \subset \Gamma$ which converges to c. Then, $\xi_\rho(z) = \lim \gamma_n(x_0)$ lies in the limit set by definition. On the other hand, if $w \in \Lambda(\rho)$, then there exists a sequence $\{\rho(\gamma_n)\} \subset \rho(\Gamma)$, so that $\lim \rho(\gamma_n(x_0)) = w$. If $\{\gamma_{n_k}\}$ is a convergent subsequence of $\{\gamma_n\}$ with $\lim \gamma_{n_k} = z$, then $\xi_\rho(z) = w$. Therefore, $\xi_\rho(\partial\Gamma) = \Lambda(\rho)$. $\qquad\square$

Let

$$CC(\Gamma, \mathsf{O}_0(n, 1)) \subset \mathrm{Hom}(\Gamma, \mathsf{O}_0(n, 1))$$

be the set of convex cocompact representations and let $\widehat{CC}(\Gamma, \mathsf{O}(n, 1))$ be its image in the quotient space

$$X(\Gamma, \mathsf{O}_0(n, 1)) = \mathrm{Hom}(\Gamma, \mathsf{O}_0(n, 1)) / \mathsf{O}_0(n, 1).$$

It is a crucial property of convex cocompact representations, known as stability, that $CC(\Gamma, \mathsf{O}_0(n, 1))$ is open in $\mathrm{Hom}(\Gamma, \mathsf{O}_0(n, 1))$. Informally, if you wiggle a convex cocompact representation a little bit it remains convex cocompact.

Theorem 11.4. *If Γ is a finitely generated group and $\rho : \Gamma \to \mathsf{O}_0(n, 1)$ is convex cocompact, then there exists a neighborhood U of ρ in $\mathrm{Hom}(\Gamma, \mathsf{PSL}(2, \mathbb{R}))$ such that if $\sigma \in U$, then σ is convex cocompact. Moreover, we may choose U so that there exists $\hat{K}$ and $\hat{C}$ so that if $\sigma \in U$, then τ_σ is a $(\hat{K}, \hat{C})$-quasi-isometric embedding (with respect to a fixed generating set for Γ and some fixed basepoint $x_0 \in \mathbb{H}^n$).*

Theorem 11.4 was establshed by Marden [154, Theorem 10.1] when $n = 3$ and by Thurston [202, Proposition 8.3.3] in the general case, see also Bowditch [38, Theorem 1.5] or Canary-Epstein-Green [56, Section I.2.5].

Proof. Suppose that the orbit map τ_ρ is a (K, C)-quasi-isometric embedding with respect to a finite generating set S and $x_0 \in \mathbb{H}^n$. The local-to-global principle, Theorem 8.1, implies that there exists A, $\hat{K}$, and $\hat{C}$ so that if $f : J \to \mathbb{H}^n$ (where J is an interval in $\mathbb{R}$) is a $(K + C + 1, C + 1)$-quasi-isometry on all segments of length at most A, then f is a $(\hat{K}, \hat{C})$-quasi-isometry.

Let U be an open neighborhood of ρ in $\mathrm{Hom}(\Gamma, \mathsf{PSL}(2, \mathbb{R}))$ so that if $\sigma \in U$, $\gamma \in \Gamma$ and $d_S(1, \gamma) \leq A + 1$, then $d(\rho(\gamma(x_0)), \sigma(\gamma(x_0))) < 1$. (We may do so since there are only finitely many elements of γ within $A + 1$ of id.)

If $\sigma \in U$, let τ_σ be the orbit map of σ. We see that if $d_S(1,\gamma) \leq A+1$, then

$$\frac{1}{K}d_S(id,\gamma) - C - 1 \leq d(\tau_\rho(id), \tau_\rho(\gamma)) \leq Kd_S(id,\gamma) + C + 1$$

so by the equivariance of τ_ρ

$$\frac{1}{K}d_S(\alpha,\beta) - C - 1 \leq d(\tau_\rho(\alpha), \tau_\rho(\beta)) \leq Kd_S(\alpha,\beta) + C + 1$$

whenever $d_S(\alpha,\beta) \leq A+1$.

Notice that if we let C_Γ be the Cayley graph of Γ with respect to S, then we may extend τ_σ to a map with domain C_Γ, by simply mapping all edges to geodesic segments in $\mathbb{H}^2$. The resulting map is a $(K+C+1, C+1)$-quasi-isometry on all geodesic segments in C_Γ of length at most A. Therefore, τ_σ is a $(\hat{K}, \hat{C})$-quasi-isometric embedding on all geodesic segments in C_Γ, which implies that τ_σ is a $(\hat{K}, \hat{C})$-quasi-isometric embedding. $\qquad\square$

Since the set $CC(\pi_1(S), \mathsf{O}_0(n,1))$ is invariant under conjugation, we immediately see that both $CC(\pi_1(S), \mathsf{O}_0(n,1))$ and its quotient are open.

Corollary 11.5. *If Γ is a finitely generated group, then $CC(\Gamma, \mathsf{O}_0(n,1))$ is open in $\mathrm{Hom}(\Gamma, \mathsf{O}_0(n,1))$ and $\widehat{CC}(\Gamma, \mathsf{O}_0(n,1))$ is open in $X(\Gamma, \mathsf{O}_0(n,1))$.*

Similarly, let

$$\mathrm{DF}(\Gamma, \mathsf{O}_0(n,1)) \subset \mathrm{Hom}(\Gamma, \mathsf{O}_0(n,1))$$

be the set of discrete, almost faithful, representations and let $\mathrm{AH}(\Gamma, \mathsf{PSL}(2,\mathbb{R}))$ be its image in $X(\Gamma, \mathsf{O}_0(n,1))$. Corollary 6.2 implies immediately that $\mathrm{DF}(\Gamma, \mathsf{O}_0(n,1))$ is closed in $\mathrm{Hom}(\Gamma, \mathsf{O}_0(n,1))$.

Corollary 11.6. *If Γ is a finitely generated group which is not virtually cyclic, then $\mathrm{DF}(\Gamma, \mathsf{O}_0(n,1))$ is closed in $\mathrm{Hom}(\Gamma, \mathsf{O}_0(n,1))$ and $\mathrm{AH}(\Gamma, \mathsf{O}_0(n,1))$ is closed in $X(\Gamma, \mathsf{O}_0(n,1))$.*

If Γ is the fundamental group of a closed hyperbolic n-manifold N, one may again use the Milnor-Svarc Lemma to show that $\rho : \Gamma \to \mathsf{PO}(n,1)$ is convex cocompact if and only if ρ is discrete and faithful. Therefore, as in the case of Teichmüller space, $\widehat{CC}(\Gamma, \mathsf{O}_0(n,1)) = \mathrm{AH}(\Gamma, \mathsf{O}_0(n,1))$ is a component of $X(\Gamma, \mathsf{O}_0(n,1))$. However, if $n \geq 3$, then Mostow's Rigidity Theorem (see Mostow [169]) implies that $\widehat{CC}(\Gamma, \mathsf{PO}(n,1))$ is exactly one point, so we will not be interested in this situation.

However, in general, $CC(\Gamma, \mathsf{O}_0(n,1))$ is not closed in $\mathrm{Hom}(\Gamma, \mathsf{O}_0(n,1))$ and $\mathrm{AH}(\Gamma, \mathsf{O}_0(n,1))$ is not open in $X(\Gamma, \mathsf{O}_0(n,1))$. We will give examples when $\Gamma = F_2$ is the free group on two generators and $n = 2$. It will be more convenient to work in $\mathsf{PSL}(2,\mathbb{R}) = \mathsf{SO}(2,1)$.

We first describe the classical Schottky construction of convex cocompact representations of free groups. If $\{C_1, C_2, \ldots, C_{2n-1}, C_{2n}\}$ is a family of disjoint geodesics in $\mathbb{H}^2$ bounding disjoint (closed) half-spaces $\{D_1, D_2, \ldots, D_{2n-1}, D_{2n}\}$ (whose closures are disjoint in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$), then we may construct a convex cocompact representation $\rho : F_n \to \mathsf{PSL}(2,\mathbb{R})$ by letting $\rho(a_i)$ be a Möbius transformation taking D_{2i-1} to $\mathbb{H}^2 - \mathrm{int}(D_{2i})$ for all i, where $F_n = <a_1, \ldots, a_n>$. If $P = \mathbb{H}^2 - \bigcup \mathrm{int}(D_i)$, then one may form a complete hyperbolic surface from P by identifying C_{2i-1} to C_{2i} by $\rho(a_i)$ for all i. Covering space theory then allows us to conclude that P is a fundamental domain for the action of $\rho(F_n)$ and that orbits of P tesselate $\mathbb{H}^2$. (One may also verify these facts, using the Ping Pong Lemma, see Section [204].)

If we choose $x_0 \in \text{int}(P)$ and let $\delta = \min\{d(C_i, C_j) \mid i \neq j\}$, then one may easily check that

$$d(x_0, \gamma(x_0)) \geq \delta d(1, \gamma).$$

On the other hand, if $K = \max\{d(x_0, \rho(a_i)(x_0))\}$, then

$$d(x_0, \gamma(x_0)) \leq K d(1, \gamma).$$

Therefore, τ_ρ is a quasi-isometric embedding, so ρ is convex cocompact. Notice that, in this case, one may easily see that all representations near to ρ are also convex cocompact, since wiggling the representation, just amounts to wiggling the C_i.

We now observe that not all discrete, faithful representations of F_2 are convex cocompact. Suppose that C_1 is the x-axis, C_2 is the line $Re(z) = 1$, C_3 is a semi-circle based at $1/4$ with radius $1/8$ and C_4 is a semi-circle based at $3/4$ with radius $1/8$. Let $\rho_0(a_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and let $\rho_0(a_2)$ be a Möbius transformation taking the half-space "below" C_3 to the half-space "above" C_4 and preserving the height of points on C_3. Let P be the (closure of the) region between C_1 and C_2 and above C_3 and C_4. Consider the hyperbolic surface X obtained from identifying C_1 with C_2 by $\rho_0(a_1)$ and identifying C_3 with C_4 by $\rho_0(a_2)$ and the sequence of regions X_n given by the quotient of $\{z \in P \mid e^{-n} \leq Im(z) \leq e^n\}$. Notice that X_n contains the ball of radius n about the quotient of $i + 1/2$ and exhausts X. It follows that X is complete. Covering space theory, then guarantees that $\rho_0 : F_2 \to \mathsf{PSL}(2, \mathbb{R})$ is discrete and faithful and that P is a fundamental domain for the action of $\rho_0(F_2)$ on $\mathbb{H}^2$. However, τ_ρ is not a quasi-isometric embedding, since if we choose $x_0 = 1 + 1/2$, then

$$d(x_0, \rho_0(a^n)(x_0)) = 2 \log \frac{n + \sqrt{n^2 + 4}}{2} \sim 2 \log n$$

picture needed   so ρ_0 is not convex cocompact.

Notice that it is important to be careful in checking completeness. Suppose that we choose C_1, C_2, C_3 and C_4 as for ρ_0 but then let $\hat{\rho}_0(a_1)$ to be given by $z \mapsto \frac{1}{2}z + 1$ and $\hat{\rho}_0(a_2) = \rho_0(a_2)$. The region X_n is not preserved by the gluings $\hat{\rho}(a_i)$, the quotient of P is not complete and, in fact, $\hat{\rho}_0$ is convex cocompact. One may see that the quotient of P is not complete, by considering the path in the quotient which is the union of horizontal segments in P of height 2^n for all n. This path has finite length but leaves every compact subset of the quotient of P. Notice that the lines $\{\hat{\rho}(a_1^n)(C_0)\}$ accumulates at the line $Re(z) = \sum_{i=0}^n \frac{1}{2^n} = 2$, so the translates of P do not tesselate $\mathbb{H}^2$. A fundamental domain for the action of $\hat{\rho}(F_2)$ is given by looking at the region below the circle of radius 2 about $z = 2$ and above the circle of radius 1 about $z = 2$ and above C_3 and C_4. One may then use this picture, just as above, to show that $\hat{\rho}_0$ is convex cocompact.

We now observe that ρ is a limit of a sequence $\{\rho_n\}$ of representations whose image is not discrete and faithful. For all $n \geq 2$, let $\rho_n(a)$ be an element of $\mathsf{PSL}(n, \mathbb{R})$ which fixes $ni + \frac{1}{2}$ and takes i to $i + 1$ and let $\rho_n(b) = \rho(b)$. It is then easy to check that $\{\rho_n\}$ converges to ρ and that $\rho_n(F_2)$ is either indiscrete or not faithful (since either $\rho_n(a)$ has finite order, or $< \rho_n(a) >$ is indiscrete). Similarly, we choose $\hat{\rho}_n(a) \in \mathsf{PSL}(2, \mathbb{R})$ to take the interior of the circle R_{-n} of radius n about $-n$ to the exterior of the circle R_n of radius n about $n + 1$, so that the "height" (i.e. the imaginary component) of points on R_{-n} is preserved and let $\hat{\rho}_n(b) = \rho(b)$. Then $\hat{\rho}_n$ is convex cocompact for all n and $\lim \hat{\rho}_n = \rho$.

We summarize in the following proposition:

Proposition 11.7. *If F_2 is the free group with 2 generators, then $CC(F_2, \mathsf{PSL}(2, \mathbb{R}))$ is open, but not closed, in $\mathrm{Hom}(F_2, \mathsf{PSL}(2, \mathbb{R}))$. Moreover, $DF(F_2, \mathsf{PSL}(2, \mathbb{R}))$ is closed, but not open, in $\mathrm{Hom}(F_2, \mathsf{PSL}(2, \mathbb{R}))$.*

12. Fricke's Theorem

Fricke proved that the mapping class group $\mathrm{Mod}(S)$ of a closed surface acts properly discontinuously, but not freely, on its Teichmüller space. In this section, we will generalize this by showing that $\mathrm{Out}(\Gamma)$ acts properly discontinuously on $\widehat{CC}(\Gamma, \mathsf{O}_0(n, 1))$.

Recall that $\mathrm{Mod}(S)$ of a closed, orientable surface S is the group of (isotopy classes of) orientation-preserving homeomorphisms of S. It is classical that two homeomorphisms of S are isotopic if and only if they are homotopic and that every homotopy equivalence of S is homotopic to a homeomorphism. Therefore, $\mathrm{Mod}(S)$ is identified with an index two subgroup of $\mathrm{Out}(\pi_1(S))$. Recall that if Γ is a group, then $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ and $\mathrm{Inn}(\Gamma)$ is the group of inner automorphisms of Γ. (All these facts are covered carefully in Farb and Margalit's *A primer on mapping class groups*.) The mapping class group $\mathrm{Mod}(S)$ acts naturally on $\mathcal{T}(S)$, by

$$\phi([\rho]) = [\rho \circ (\phi_*)^{-1}].$$

Theorem 12.1. (Fricke's Theorem) *If S is a closed oriented surface of genus $g \geq 2$, then $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathcal{T}(S)$.*

The quotient $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$ is the **moduli space** of isometry classes of (unmarked) hyperbolic structures on S. By the Uniformization Theorem, it may be identified with the space of conformal (or complex) structures on S. In algebraic geometry, it occurs as the space of smooth algebraic curves of genus g. It admits a natural geometric compactification which has the structure of a projective algebraic variety. Since $\mathrm{Mod}(S)$ does not act freely, moduli space has the structure of an orbifold rather than of a manifold.

We begin with an elementary observation about translation lengths. Recall that if Γ is a group, then $||\gamma||$ denotes the minimal translation length of the action of γ on Γ. Equivalently, $||\gamma||$ is the minimal length of a word conjugate to γ. We let ℓ_X denote the translation length of the action of Γ on X, i.e.

$$\ell_X(\gamma) = \inf_{x \in X} d_X(x, \gamma(x)).$$

The following fact is really part of the discussion surrounding the Milnor-Svarc Lemma.

Lemma 12.2. *If Γ acts properly discontinuously, cocompactly and by isometries on proper length space X, then there exists $J, B > 0$ so that*

$$\frac{1}{J}||\gamma|| - B \leq \ell_X(\gamma) \leq J||\gamma|| + B$$

for all $\gamma \in \Gamma$.

Moreover, if the orbit map $\tau : \Gamma \to X$ is a (K, C)-quasi-isometric embedding, for some choice of $x_0 \in X$, then we may choose $J = K$ and $B = 3C$.

One may easily check that if $X = \mathbb{H}^n$ and $A \in \mathsf{O}_0(n, 1)$, then

$$\ell_{\mathbb{H}^n}(A) = 2 \log \lambda_1(A) = 2 \log \frac{\lambda_1(A)}{\lambda_2(A)} = \log \frac{\lambda_1(A)}{\lambda_{n+1}(A)}.$$

Proof. Fix $x_0 \in X$. The Milnor-Svarc Lemma guarantees that there exists K and C so that the orbit map $\tau : \Gamma \to X$, given by $\gamma \mapsto \gamma(x_0)$ is a (K, C)-quasi-isometry. Let $\alpha \in \Gamma$ be an element so that

$$d(\alpha, \gamma\alpha) = d(id, \alpha^{-1}\gamma\alpha) = ||\gamma||.$$

Then,

$$d(x_0, \alpha^{-1}\gamma\alpha(x_0)) \le K||\gamma|| + C$$

so

$$\ell_X(\alpha^{-1}\gamma\alpha) = \ell_X(\gamma) \le K||\gamma|| + C.$$

On the other hand, if $x \in X$, then there exists $\alpha \in \Gamma$ so that

$$d(\alpha(x_0), x) \le C.$$

Moreover,

$$d(\alpha(x_0), \gamma\alpha(x_0))) \ge \frac{1}{K}d(\alpha, \gamma\alpha) - C \ge \frac{1}{K}||\gamma|| - C$$

and

$$d(\gamma\alpha(x_0), \gamma(x)) \le C$$

so

$$d(x, \gamma(x)) \ge K||\gamma|| - 3C.$$

Therefore,

$$\ell_X(\gamma) \ge K||\gamma|| - 3C$$

and we may take $J = K$ and $B = 3C$ $\square$

We will also use the following group-theoretic fact, first discussed in Section 8.

Proposition 8.3. *If Γ is a torsion-free hyperbolic group, then there exists a finite collection $\mathcal{B}$ of elements of $\pi_1(S)$, so that for any K*

$$\{\phi \in \mathrm{Out}(\Gamma) \mid ||\phi(b)|| \le K \text{ for all } b \in \mathcal{B}\}$$

is finite.

If $\Gamma = \pi_1(S)$, then this is equivalent to the claim that there is a finite collection F of curves on S, so that any self-homeomorphism of S is determined, up to homotopy, by the homotopy classes of the images of curves in F. This is not hard to check using surface topology.

We are now ready to establish our generalization of Fricke's Theorem.

Theorem 12.3. *If Γ is a finitely generated torsion-free group and $\widehat{CC}(\Gamma, \mathsf{O}_0(n, 1))$ is non-empty, then $\mathrm{Out}(\Gamma)$ acts properly discontinuously on $\widehat{CC}(\Gamma, \mathsf{O}_0(n, 1))$.*

Proof. To warm up, we first show that the action of $\mathrm{Out}(\Gamma)$ on $\widehat{CC}(\Gamma, \mathsf{O}_0(n, 1))$ has discrete orbits. If $\rho \in \widehat{CC}(\Gamma, \mathsf{O}_0(n, 1))$, then Lemma 12.2 implies that there exists J and B so that

$$\frac{1}{J}||g|| - B \le \ell(\rho(g)) \le J||g|| + B$$

Let $\mathcal{B}$ be the finite collection of elements of Γ provided by Proposition 8.3. If $\{\phi_n\}$ is a sequence of distinct elements of $\mathrm{Out}(\Gamma)$, then, up to subsequence, there exists $b \in \mathcal{B}$ so that $||\phi_n^{-1}(b)|| \to \infty$. Therefore, by the above inequality, $\ell(\rho(\phi_n^{-1}(\beta)) \to \infty$, which implies that

$\phi_n(\rho) = \rho \circ \phi_n^{-1} \to \infty$ in $X(\Gamma, \mathsf{O}_0(n,1))$. Therefore, the action of $\mathrm{Out}(\Gamma)$ on $\widehat{CC}(\Gamma, \mathsf{O}_0(n,1))$ has discrete orbits.

Let R be a compact subset of $\widehat{CC}(\Gamma, \mathsf{O}_0(n,1))$. Theorem 11.4 implies that if $\rho \in R$, then there exist K_U, C_U and an open neighborhood U of ρ so that if $\sigma \in U$, then τ_σ is a (K_U, C_U)-quasi-isometric embeddding (for some choice of $x_0 \in \mathbb{H}^n$). Since R is compact, it can be covered by finitely many such neighborhoods. So, there exist K and C so that if $\rho \in R$, then τ_ρ is a (K, C)-quasi-isometric embedding (for some choice of basepoint). Lemma 12.2 then implies that if $J = K$ and $B = 3C$, then

$$\frac{1}{J}||\gamma|| - B \le \ell(\rho(\gamma)) \le J||\gamma|| + B$$

for all $\gamma \in \Gamma$ and all $\rho \in R$.

So if $\{\phi_n\}$ is a sequence of distinct elements of $\mathrm{Out}(\Gamma)$, then, by Proposition 8.3, there exists $b \in \mathcal{B}$ so that, after perhaps passing to a subsequence, $||\phi_n^{-1}(b)|| \to \infty$. So, if $\rho \in R$, then

$$\ell(\rho(\phi_n^{-1}(\beta))) \ge \frac{1}{J}||\phi_n^{-1}(b)|| - B \to \infty$$

so $\{\phi_n(R)\}$ exits every compact subset of $X(\pi_1(F), \mathsf{PSL}(2,\mathbb{R}))$. Therefore, $\mathrm{Out}(\Gamma)$ acts properly discontinuously on $\widehat{CC}(\Gamma, \mathsf{O}_0(n,1))$. $\square$

Remarks: 1) The mapping class group $\mathrm{Mod}(S)$ does not act properly discontinuously on $\mathrm{AH}(\pi_1(S), \mathsf{PO}(3,1))$. We say that $[\phi] \in \mathrm{Mod}(S)$ is **pseudo-Anosov** if whenever α is a homotopically non-trivial simple closed curve α on S and $n \in \mathbb{N}$, then $\phi^n(\alpha)$ is not (freely) homotopic to α. Then ϕ has infinite order in $\mathrm{Mod}(S)$ and fixes a point in $\mathrm{AH}(\pi_1(S), \mathsf{PO}(3,1))$.

Thurston proved that the mapping torus $M_\phi = S \times [0,1]/(x,0) \sim (\phi(x),0)$ admits a hyperbolic metric. So there exists a convex cocompact representation $\hat{\rho}_\phi : \pi_1(M_\phi) \to \mathsf{PO}(3,1)$. The fixed point is then given by $[\rho_\phi]$ where $\rho_\phi = \hat{\rho}_\phi|_{\pi_1(S)}$.

2) Goldman [102] proved that $X(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ has $2g-1$ components (indexed by the absolute value of the Euler number of representations in the component). Goldman [105] conjectured that if the genus of S is at least 3, then the mapping class group acts ergodically on all components other than the component which is Teichmüller space. If S has genus 2, Marché and Wolff [152, 153] proved that the mapping class group acts ergodically on the component of $X(\pi_1(S), \mathsf{PSL}(2,\mathbb{R}))$ consisting of representations whose Euler number has modulus 1 and the the component consisting of representations of modulus 0 splits into two open sets which are preserved and acted on ergodically by the mapping class group. In 2014, Souto announced a proof of Goldman's conjecture for components consisting of representations with Euler number 0.

3) Goldman also conjectured that the mapping class group acts ergodically on

$$X_0(\pi_1(S), \mathsf{PSL}(2,\mathbb{C})) - CC(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))$$

where $X_0(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))$ is the component of $X_0(\pi_1(S), \mathsf{PSL}(2,C))$ containing $CC(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))$. Minsky [167], Canary-Storm [58] and Lee [148] showed that for many 3-manifolds M, the domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M), \mathsf{OSL}(2,\mathbb{C}))$ on $X(\pi_1(S), \mathsf{PSL}(2,C))$ is strictly larger than $CC(\pi_1(M), \mathsf{PSL}(2,\mathbb{C}))$, see [55] for a more detailed survey of these results.

13. **Further topics: Hyperbolic 3-manifolds**

I left the four lane highway took a blacktop seven miles
Down by the old country school I went to as a child
Two miles down a gravel road I could see the proud old home
A tribute to a way of life that's almost come and gone.

The roots of my raising run deep
I come back for the strength that I need
And hope comes no matter how far down I sink
The roots of my raising run deep.
—————————Tommy Collins [110]

One may view the theory of Kleinian groups as the lowest of all Higher Teichmüller theories, or perhaps as "only somewhat higher Teichmüller theory." However, this theory has progressed rather dramatically, beginning with Thurston's groundbreaking work in the 1970s and 1980s. We will review some of this work in the hopes that it could provide inspiration for future directions in "truly" higher Teichmüller theory. (This hope was one of the motivations for my attempts to begin working in the field, although the main motivation was my enjoyment talking math with the collaborators I found there.)

Part 4. **Convex projective manifolds**

From the geometry of his heart he mapped it out
He saw the King rise, fitted with armor
Set upon a white horse
An immaculate cross in his right hand.
He advanced toward the enemy
And the symmetry, the perfection of his mathematics
Caused the scattering of the enemy
Agitated, broken, they fled
————Patti Smith [193]

Our first examples of Anosov representations into higher rank Lie group will be the Benoist representations. We say that a discrete, faithful representation $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation if there exists a strictly convex domain in $\mathbb{RP}^n$ so that $\rho(\Gamma)$ acts cocompactly on Ω. If Γ is torsion-free, then the quotient $M = \Omega/\rho(\Gamma)$ is a strictly convex real projective manifold. We will first see that in this case Γ is a hyperbolic group and Ω admits a natural Γ-invariant metric, called the Hilbert metric, which has unique geodesics and is Gromov hyperbolic. Moreover, $\partial\Omega$ is C^1 and there exists a ρ-equivariant homeomorphim $\xi_\rho : \partial\Gamma \to \partial\Omega$.

The most basic examples are provided by projective bending of cocompact representations into $\mathsf{PO}(n, 1) \subset \mathsf{PGL}(n+1, \mathbb{R})$. Benoist proved that the set $\mathrm{Ben}(\Gamma, \mathsf{PGL}(n+1, \mathbb{R}))$ of Benoist representations of Γ is always a collection of components of $\mathrm{Hom}(\Gamma, \mathsf{PGL}(n+1, \mathbb{R}))$. So, this situation is a natural generalization of the classical Teichmüller theory. You may find it easier when first encountering this material to always assume that we are working in $\mathsf{PSL}(n+1, \mathbb{R})$ and that all our groups are torsion-free.

Our main resources for the material in this chapter are the article "Convex divisibles I" by Yves Benoist [20] and the survey article "Around groups in Hilbert geometry" by Ludovic Marquis [158].

14. **Basic definitions**

Despite all the computations
You could just dance to that rock'n'roll station
—————Lou Reed [210]

Recall that $\mathbb{RP}^n = \mathbb{R}^{n+1}/(\mathbb{R} - \{0\})$ and that the action of $\mathsf{PGL}(n+1, \mathbb{R})$ on $\mathbb{RP}^n$ is its group of *projective automorphisms*. If P is a 2-plane through the origin, $L = \mathbb{P}(P)$ is a **projective line**. More generally, if Q^{d+1} is a $(d+1)$-plane which passes through the-origin, then $\mathbb{P}(Q)$ is a *projective d-plane*. Projective automorphism of $\mathbb{RP}^n$ naturally take projective d-planes to projective d-planes and restrict to projective automorphisms.

Any hyperplane A in $\mathbb{R}^{n+1}$ which does not pass through the origin gives rise to an *affine chart* for $\mathbb{RP}^n$ such that $\mathbb{RP}^n - A$ is the projective hyperplane of lines through the origin parallel to A. We say that $E \subset \mathbb{RP}^n$ is a (projective) **ellipsoid** if it is an ellipsoid in some affine chart. Notice that if E is an ellipsoid in some affine chart, then it is an ellipsoid in every chart containing it. All ellipsoids are projectively equivalent to a round disk.

We say that a domain $\Omega \subset \mathbb{RP}^n$ is **properly convex**, if it is a bounded convex subset of some affine chart. Notice that this is equivalent to saying that the closure of Ω is disjoint from some projective hyperplane and every two points in Ω are joined by a projective line segment contained in Ω. We say that a domain $\Omega \subset \mathbb{RP}^n$ is **strictly convex** if it is properly convex and there are no projective line segments contained in $\partial\Omega$.

If Ω is properly convex, then we let $\mathrm{Aut}(\Omega)$ denote the set of projective automorphisms which preserve Ω. A (properly) **convex projective manifold** $M = \Omega/\Gamma$ is the quotient of a properly convex domain Ω by a subgroup Γ of $\mathrm{Aut}(\Omega)$ acting freely and properly discontinuously on Ω. If Ω is strictly convex, we say that $M = \Omega/\Gamma$ is a **strictly convex projective manifold**.

The first examples of strictly convex projective manifolds are provided by hyperbolic manifolds. Let $H^n \subset \mathbb{R}^{n+1}$ be the two-sheeted hyperboloid whose upper sheet gives the hyperboloid model for $\mathbb{H}^n$, i.e. $H^n = \{\vec{x} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\}$. Then $\mathbb{P}(H^n)$ is the unit disc in the affine chart given by $A = \{\vec{x} \mid x_{n+1} = 1\}$ and $\mathrm{Aut}(\mathbb{P}(H^n)) = \mathsf{PO}(n,1)$. (Notice that $\mathbb{P}(H^n)$ is an ellipsoid in any affine chart containing it.) Then if Γ is any discrete torsion-free subgroup of $\mathsf{PO}(n,1)$, $M = \mathbb{P}(H^n)/\Gamma$ is a strictly convex projective n-manifold.

If $\hat{\Delta}$ is the positive octant in $\mathbb{R}^3$, i.e. $\hat{\Delta} = \{\vec{x} \mid x_1 > 0, \ x_2 > 0, \ x_3 > 0\}$, then $\Delta = P(\hat{\Delta})$ is a simplex in the affine chart $A = \{\vec{x} \mid x_1 + x_2 + x_3 = 1\}$. So Δ is properly convex, but not strictly convex. Let $\Gamma \subset \mathsf{SL}(3,\mathbb{R}) = \mathsf{PSL}(3,\mathbb{R})$ be generated by the diagonal matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Notice that A and B commute, $\Gamma \subset \mathrm{Aut}(\Delta)$ and if $m, n \in \mathbb{Z}$, then

$$A^n B^m = \begin{bmatrix} 4^n 2^m & 0 & 0 \\ 0 & \frac{1}{2^n 4^m} & 0 \\ 0 & 0 & 2^{m-n} \end{bmatrix}$$

so $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$, Γ is discrete and Γ acts freely on Δ. (We will see later that this implies that Γ acts properly discontinuously on Δ). Therefore, $M = \Delta/\Gamma$ is a convex projective manifold and $\pi_1(M) \cong \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$, so M is homeomorphic to a torus.

We say that a group $\Gamma \subset \mathsf{PGL}(n+1,\mathbb{R})$ **divides** a properly convex domain Ω if $\Gamma \subset \mathrm{Aut}(\Omega)$ and Γ acts properly discontinuously and cocompactly on Ω. In an abuse of notation, we say that a discrete faithful representation $\rho : \Gamma \to \mathsf{PGL}(n+1,\mathbb{R})$ **divides** a properly convex domain Ω if $\rho(\Gamma)$ divides Ω. We will call a representation which divides a strictly convex domain a **Benoist representation**. All Benoist representations into $\mathsf{PSL}(3,\mathbb{R})$ are deformations of Fuchsian representations into $\mathsf{PO}(2,1) \subset \mathsf{PGL}(3,\mathbb{R})$. The only known Benoist representations which are not deformations of cocompact representations into $\mathsf{PO}(n,1) \subset \mathsf{PGL}(n+1,\mathbb{R})$ are certain representations of Coxeter groups in $\mathsf{PSL}(5,\mathbb{R})$ by Benoist [24] and representations of fundamental groups of Gromov-Thurston manifolds of dimension $n \geq 4$ by Kapovich [129].

We now introduce the Hilbert metric on a properly convex domain. It is a projectively invariant Finsler metric, so descends to a Finsler metric on any quotient convex projective manifold. We first need to introduce the cross-ratio.

Given 4 distinct points x, y, z and w in $\mathbb{RP}^n$ which lie on a projective line L, we can choose an affine chart A containing the points and define their **cross-ratio** to be given by

$$[w, x, y, z] = \frac{|w - y| \cdot |x - z|}{|w - x| \cdot |y - z|}.$$

Notice that $[w, x, y, z] > 1$ if the points w, x, y, and z appear in that order on the line L (either from left to right or right to left, so this statement is independent of the ordering of the line).

Lemma 14.1. *The cross-ratio of four collinear points in $\mathbb{RP}^n$ is well-defined (i.e. independent of the choice of affine chart A) and invariant under projective automorphisms, i.e. if $T \in$ $\mathsf{PGL}(n + 1, \mathbb{R})$ and x, y, z and w lie on a projective line then*

$$[T(w), T(x), T(y), T(z)] = [x, y, z, w].$$

Proof. Given two affine charts A and B containing x, y, z and w, let $L_A = L \cap A$ and $L_B = L \cap B$. Then the identity map on L is a projective automorphism from $L_A \cup \{\infty\}$ to $L_B \cup \{\infty\}$. Now notice that every projective automorphism may be written as a product of some combination of the inversion $x \to \frac{1}{x}$, dilations $x \to \lambda x$ and translations $x \to x + a$. It is clear that the cross-ratio is invariant under dilations and translations and it is an easy calculation to check that it is invariant under inversion since

$$\frac{\left|\frac{1}{w} - \frac{1}{y}\right| \cdot \left|\frac{1}{z} - \frac{1}{x}\right|}{\left|\frac{1}{w} - \frac{1}{x}\right| \cdot \left|\frac{1}{z} - \frac{1}{y}\right|} = \frac{|w - y| \cdot |x - z|}{|w - x| \cdot |y - z|}.$$

Now suppose that $T \in \mathsf{PGL}(n + 1, \mathbb{R})$ and A is an affine chart containing w, x, y and z. Then $T(A)$ is an affine chart containing $T(w)$, $T(x)$, $T(y)$ and $T(z)$. Moreover, T restricts to a projective automorphism from L to $T(L)$. The result then follows as above. $\square$

Suppose that Ω is a properly convex domain and that A is an affine chart so that Ω is a bounded convex subset of A. If $x, y \in \Omega$, let $L_{x,y}$ be the line in A containing x and y and let w and z be the endpoints of the line segment $L_{x,y} \cap \bar{\Omega}$, so that w, x, y and z occur in that order along the line $L_{x,y}$. We then define the **Hilbert distance** between x and y in Ω to be

picture would help

$$d_\Omega^H(x, y) = \frac{1}{2} \log\left([w, x, y, z]\right).$$

It follows from Lemma 14.1 that d_H is well-defined and invariant under $\mathrm{Aut}(\Omega)$. We further claim that it is a metric, which we call the **Hilbert metric** and that projective lines are geodesics for this metric. We first notice the following obvious monotonicity property of the Hilbert metric.

Lemma 14.2. *If Ω and Ω' are properly convex domains, $\Omega \subset \Omega'$, and $x, y \in \Omega$, then*

$$d_\Omega^H(x, y) \geq d_{\Omega'}^H(x, y).$$

Moreover, if $L_{x,y} \cap \Omega \neq L_{x,y} \cap \Omega'$, then

$$d_\Omega^H(x, y) > d_{\Omega'}^H(x, y).$$

Proof. Notice that if $a \geq b > 0$ and $c \geq 0$, then $\frac{a}{b} \geq \frac{a+c}{b+c}$. Moreover, if $a > b > 0$ and $c > 0$, then $\frac{a}{b} > \frac{a+c}{b+c}$. Let w' and z' be the endpoints of $L_{x,y} \cap \Omega'$, labelled so that w', x, y and z' occur in

that order on the line $L_{x,y}$. Then notice that $|w - y| \geq |w - x|$ and $|w' - x| = |w - x| + |w - w'|$ and $|w' - y| = |w - y| + |w - w'|$, so $\frac{|w-y|}{|w-x|} \geq \frac{|w'-y|}{|w'-x|}$. Similarly, $\frac{|x-z|}{|y-z|} \geq \frac{|x-z'|}{|x-z|}$, so

$$d_\Omega^H(x, y) = \frac{1}{2} \log\left([w, x, y, z]\right) \geq \frac{1}{2} \log\left([w', x, y, z']\right) = d_{\Omega'}^H(x, y).$$

Notice that this inequality is strict if $w \neq w'$ or $z \neq z'$ and that this occurs exactly when $L_{x,y} \cap \Omega \neq L_{x,y} \cap \Omega'$. $\square$

We use this monotonicity property in the proof that the Hilbert metric is indeed a metric.

Lemma 14.3. *If Ω is a properly convex domain in $\mathbb{RP}^n$, then d_Ω^H is a complete metric, $\mathrm{Aut}(\Omega) \subset \mathrm{Isom}(\Omega, d_\Omega^H)$ and intersections of projective lines with Ω are geodesics (Ω, d_Ω^H).*
Moreover, if $p, x, y \in \Omega$ and

$$d_\Omega^H(x, y) = d_\Omega^H(x, p) + d_\Omega^H(p, y),$$

then there exists line segments (possibly degenerate) $[a, c]$ and $[b, d]$ in $\partial\Omega$ so that the projective lines $L_{p,x}$ (through p and x) and $L_{p,y}$ (through p and y) each have one endpoints in $[a, c]$ and the other in $[b, d]$. In particular, if Ω is strictly convex, then projective line segments are the only geodesics in (Ω, d_Ω^H).

Proof. Choose an affine chart A so that Ω is a bounded convex subset of A. Notice that $d_H^\Omega(x, y) \geq 0$ (since, by our choice of ordering, $[w, x, y, z] \geq 1$) and that $d_H^\Omega(x, y) = 0$ if and only if $x = y$ (since $[w, x, y, z] = 1$ if and only if $x = y$) and $d_\Omega^H(x, y) = d_\Omega^H(y, x)$ (since $[w, x, y, z] = [z, y, x, w]$). The main difficulty is to check the triangle inequality.

We first check the case that p lies on $L_{x,y}$ between x and y. It is calculation to verify that

$$[w, x, p, z] \cdot [w, p, y, z] = \frac{|w - p| \cdot |x - z|}{|w - x| \cdot |z - p|} \cdot \frac{|w - y| \cdot |z - p|}{|w - p| \cdot |z - y|} = \frac{|w - y| \cdot |x - z|}{|w - x| \cdot |y - z|} = [w, x, y, z]$$

so

$$d_\Omega^H(x, p) + d_\Omega^H(p, y) = d_\Omega^H(x, y)$$

which verifies the triangle inequality for collinear points. Moreover, once we check that d_Ω^H is a metric, it implies that projective line segments are geodesics in (Ω, d_Ω^H).

Now suppose that p does not lie on $L_{x,y}$. Let a and b be the endpoints of $L_{x,p} \cap \bar{\Omega}$ and let c and d be the endpoints of $L_{p,y} \cap \bar{\Omega}$ (with consistent orderings a, x, p, b and c, p, y, d). Let Q be the quadrilateral spanned by $\{a, b, c, d\}$ and let $e = L_{a,c} \cap L_{x,y}$ and $f = L_{x,y} \cap L_{b,d}$. Notice that

e lies between w and x on $L_{x,y}$ and that f lies between y and z.

Let $r = L_{a,c} \cap L_{b,d}$ and let $q = L_{p,r} \cap L_{x,y}$. Notice that q lies between x and y on $L_{x,y}$. Since a and e lie on a line through r, q and p lie on a line through r, and b and f lie on a line through r, the projective invariance of the cross-ratio implies that $[e, x, q, f] = [a, x, p, b]$. Therefore, $d_Q^H(x, q) = d_Q^H(x, p)$. Similarly, $d_Q^H(q, y) = d_Q^H(p, z)$, so

$$d_Q^H(x, y) = d_Q^H(x, p) + d_Q^H(p, y).$$

However, by construction, $d_Q^H(x, p) = d_\Omega^H(x, p)$ and $d_Q^H(p, y) = d_\Omega^H(p, y)$, and Lemma 14.2 implies that $d_\Omega^H(x, y) \geq d_Q(x, y)$. Therefore,

$$d_\Omega^H(x, y) \geq d_\Omega^H(x, p) + d_\Omega^H(p, y)$$

which completes the verification of the triangle inequality and hence the proof that d_Ω^H is a metric.

Lemma 14.2 also implies that the inequality above is strict if $w \neq e$ or $z \neq f$. However, if $w = e$ and $z = f$, then e and f lie in $\partial\Omega$, so the line segments $[a, c]$ and $[b, d]$ must lie in $\partial\Omega$. However, if Ω is strictly convex, then $\partial\Omega$ contains no line segments, so

$$d_\Omega^H(x, y) > d_\Omega^H(x, p) + d_\Omega^H(p, y)$$

unless p lies in the line segment between x and y. It follows that if Ω is strictly convex, then projective line segments are the only geodesics.

Finally, notice that if $x \in \Omega$, $\{y_n\} \subset \Omega$, $d(y_n, \partial\Omega) \to 0$, and w_n and z_n are the endpoints of $L_{x,y_n} \cap \bar{\Omega}$, with the ordering w_n, x, y_n, z_n, then, up to subsequence $\{y_n\}$ converges to a point $y \in \partial\Omega$, $\{w_n\}$ converges to a point $w \in \partial\Omega$ and $\{z_n\}$ converges to a point $z \in \partial\Omega$. Notice that $y = z$ since otherwise $\partial\Omega$ contains the line segment $\overline{yz}$, which would imply, by convexity, that $L_{y,z}$ is disjoint from Ω which would contradict the fact that $x \in L_{y,z} \cap \Omega$. Moreover, $w \neq y$ since $x \in \overline{wz}$. It follows that $|y_n - z_n| \to 0$ and $|w_n - x|$, $|w_n - z_n|$ and $|z_n - x|$ are all bounded away from zero, so $[w_n, x, y_n, z_n] \to \infty$ which implies that $d_\Omega^H(x, y_n) \to \infty$. Therefore, d_Ω^H is complete. $\qquad\square$

Remarks: (1) Notice that the proof also demonstrates that if Q is a quadrilateral, then geodesics are not unique, since one can easily check that both $\overline{xy}$ and $\overline{xp} \cup \overline{py}$ are geodesics joining x to y in the quadrilateral Q that we construct in the proof.

(2) Pierre de la Harpe [111] showed that if Ω is properly convex and has the unique geodesic property, then $\mathrm{Aut}(\Omega) = \mathrm{Isom}(\Omega, d_\Omega^H)$. In particular, if Ω is strictly convex, then $\mathrm{Aut}(\Omega) = \mathrm{Isom}(\Omega, d_\Omega^H)$. He further shows that Ω has the unique geodesic property if and only if Ω has at most one support plane which intersects $\partial\Omega$ in more than one point.

We record here the elementary observation that discrete subgroups of $\mathrm{Aut}(\Omega)$ acts properly discontinuously on Ω. We will use this observation, usually without comment, for the remainder of the notes.

Lemma 14.4. *If Ω is properly convex and Γ is a discrete subgroup of $\mathrm{Aut}(\Omega)$, then Γ acts properly discontinuously on Ω. If Γ is also torsion-free, it acts freely, so Ω/Γ is a convex projective manifold.*

Proof. If Γ does not act properly discontinuously on Ω, there exists a sequence $\{\gamma_n\}$ in Γ and a compact set K so that $\gamma_n(K) \cap K$ is non-empty for all n. Since Γ acts as a group of isometries of Ω in the Hilbert metric, there exists a subsequence of $\{\gamma_n\}$ converging to an isometry of Ω (by the Arzela-Ascoli Theorem). Therefore there exists a sequence $\{\beta_i\}$ in Γ (each of the form $\gamma_n \gamma_m^{-1}$) which converges to an isometry which is the identity on Ω. Thus, as a sequence of elements of $\mathsf{PGL}(d, \mathbb{R})$, $\{\beta_i\}$ converges to the identity in $\mathsf{PGL}(d, \mathbb{R})$, which contradicts discreteness.

If Γ is torsion-free, then every non-trivial element Γ of Γ has infinite order, so cannot fix any point in Ω (since every power of γ would fix the point, violating proper discontinuity). Therefore, if Γ is discrete and torsion-free, Ω/Γ is a manifold. $\qquad\square$

We now observe that the Hilbert metric is Finsler. We will only use the Finsler property in our discussion of the Hilbert geodesic flow in Section 22. Recall that a **Finsler metric** on a manifold M is a continuous family of norms on the tangent bundle of M. From a Finsler metric

one obtains a distance function (an actual metric) just as in the Riemannian case by using the norm to define the length of a smooth curve in M and taking the distance between two points to be the infimum of the length of smooth curves joining the two points.

If $x \in \Omega$ and $\vec{v} \in T_x\Omega$, let p_+ and p_- be the endpoints of the line segment $\{x + t\vec{v} \mid t \in \mathbb{R}\} \cap \bar{\Omega}$. We define the Finsler norm associated to the Hilbert metric by letting

$$
\begin{aligned}
F_\Omega^H(x, \vec{v}) &= \left. \frac{d}{dt} \right|_{t=0} d_\Omega^H(x, x + t\vec{v}) \\
&= \frac{1}{2} \left. \frac{d}{dt} \right|_{t=0} \left(\log \frac{(|p_- - x| + t|\vec{v}|)(|x - p_+|)}{|p_- x|(|p_+ - x| - t|\vec{v}|)} \right) \\
&= \frac{|\vec{v}|}{2} \left(\frac{1}{|x - p_-|} + \frac{1}{|x - p_+|} \right)
\end{aligned}
$$

We observe that if $\Omega = \mathbb{P}(H^n)$ then $\mathsf{PO}(n, 1) = \mathrm{Aut}(\Omega) \subset \mathrm{Isom}(\Omega, d_\Omega^H)$, so the isometries act transitively on the space of orthonormal frames in $T\Omega$. It follows that the Hilbert metric must be a metric of constant negative sectional curvature. (Notice that the Finsler metric is Riemannian at the origin and hence everywhere, by transitivity.) It is then a calculation to show that we have chosen the correct normalization to get the actual hyperbolic metric of constant sectional curvature -1. The round disk $\mathbb{P}(H^n)$ with the Hilbert metric is often called the **Beltrami-Klein model** (or Klein or Cayley-Klein model) for hyperbolic geometry.

Remark: Kay [137] proved that a Hilbert metric is Riemannian if and only if it is an ellipsoid (i.e. projectively equivalent to $\mathbb{P}(H^n)$). For example, you can see that in the case of the simplex, metric ϵ-balls are hexagons, so are not asymptotically round. Therefore, the Finsler metric cannot be Riemannian (since metric ϵ-balls are always asymptotically elliptical as $\epsilon \to 0$ in a Riemannian manifold).

15. **Geometry of properly convex domains**

Since our first explicit example of a Hilbert metric is the hyperbolic metric on the round disk it is natural to ask when the Hilbert metric is Gromov hyperbolic. Notice that the simplex Δ is quasi-isometric to the Euclidean plane, since it has a quotient which is a torus, so the simplex is not Gromov hyperbolic. We first observe that if the Hilbert metric is Gromov hyperbolic, then the domain must be strictly convex.

Proposition 15.1. *If Ω is properly convex and its Hilbert metric is Gromov hyperbolic, then Ω is strictly convex.*

Proof. Suppose that Ω is Gromov hyperbolic, but not strictly convex. Let $[x, y]$ be a maximal line segment in $\partial\Omega$. Choose $z \in \Omega$ and sequences $\{x_n\} \subset \overline{xz}$ and $\{y_n\} \subset \overline{yz}$ so that $x_n \to x$, $y_n \to y$ and $\overline{x_n y_n}$ is parallel to $\overline{xy}$ for all n. Let u_n be the midpoint of $\overline{x_n y_n}$, for all n, so $\{u_n\}$ converges to the midpoint u of $\overline{xy}$.

If Ω is δ-hyperbolic, then, for all n, there exists $z_n \in \overline{x_n z} \cup \overline{y_n z}$ so that $d(u_n, z_n) \leq \delta$. Without loss of generality $z_n \in \overline{xz}$ for all n. Since $d(u_n, z) \to \infty$, we must have $d(z_n, z) \to \infty$, so $z_n \to x$. Suppose that $\overline{p_n q_n} = L_{u_n, z_n} \cap \bar{\Omega}$. Since $\overline{u_n z_n} \to \overline{ux}$, we see that $\overline{p_n q_n}$ converges to a line segment in $\partial\Omega$ containing $\overline{ux}$. Since $\overline{xy}$ is a maximal line segment in $\partial\Omega$, we see that $p_n \to x$ and $\{q_n\}$ converges up to subsequence to a point in $\overline{uy}$.

picture needed

But then $|p_n - z_n| \to 0$, but $|p_n - u_n|$ and $|q_n - z_n|$ are bounded away from zero, so $[p_n, z_n, u_n, q_n] \to \infty$ which implies that $d(z_n, u_n) \to \infty$ which is a contradiction. So, Ω must not be Gromov hyperbolic. $\qquad\square$

We next observe that when Ω is Gromov hyperbolic, then its Gromov boundary agrees with its topological boundary.

Proposition 15.2. *If Ω is a strictly convex domain and (Ω, d_Ω^H) is hyperbolic, then the Gromov boundary $\partial_\infty \Omega$ of Ω may be identified with the topological boundary $\partial \Omega$ of Ω. More precisely, there exists a homeomorphism from $\bar{\Omega}$ to $\Omega \cup \partial_\infty \Omega$ which is the identity on Ω.*

Proposition 15.2 will be a consequence of the following more general fact which we will use on several other occasions.

Lemma 15.3. *Suppose that Ω is a properly convex domain, $x_0 \in \Omega$, $w \neq z \in \partial\Omega$ and $L_w = [x_0, w)$ and $L_z = [x_0, z)$ are projective line segments in Ω joining x_0 to w and z. Then L_w and L_z are a finite Hausdorff distance apart (in the Hilbert metric on Ω) if and only z and w lie in the interior of a projective line segment in $\partial\Omega$.*

Proof. First suppose that w and z lie in the interior of a (maximal) projective line segment $[x, y]$ in $\partial\Omega$ and assume that they occur in the order x, w, z, y. If L_w does not lie in a bounded metric neighborhood of L_z, then there exists a sequence $w_n \in L_w$ so that $d(w_n, L_z) \to \infty$. Choose a point z_n on L_z so that $[w_n, z_n]$ is parallel to $[w, z]$. If x_n and y_n are the points of intersection of the projective line $\overleftrightarrow{w_n z_n}$ with $\partial\Omega$ (in the order x_n, w_n, z_n, y_n), then $[x_n, y_n]$ converges to $[x, y]$, so $[x_n, w_n, z_n, y_n]$ converges to $[x, y, z, w]$ which is finite. Therefore, $d(w_n, z_n)$ is bounded and we have achieved a contradiction. Thus, L_w lies in a bounded metric neighborhood of L_z. Symmetrically, L_z lies in a bounded metric neighborhood of L_w, so L_w and L_z are a finite Hausdorff distance apart.

On the other hand, suppose that L_w and L_z are a Hausdorff distance at most K apart. Choose sequences w_n in L_w and $z_n \in L_z$ so that $w_n \to w$ and $d(w_n, z_n) \leq K$. Let x_n and y_n be the points of intersection of the projective line $\overleftrightarrow{w_n z_n}$ with $\partial\Omega$ (in the order x_n, w_n, z_n, y_n), and pass to a subsequence so that $x_n \to x \in \partial\Omega$ and $y_n \to y \in \partial\Omega$. Notice that if either $x = w$ or $y = z$, then $[x_n, w_n, z_n, y_n] \to \infty$, so $d(w_n, z_n) \to \infty$, which is a contradiction. Therefore, z and w lie in the interior of the line segment $\overline{xy}$ contained in $\partial\Omega$. $\qquad\square$

Proof of Proposition 15.2. Fix $p \in \Omega$. Every geodesic ray emanating from p is a projective line segment which intersects $\partial\Omega$ in a unique point. Moreover, every point in $\partial\Omega$ determines a geodesic ray emanating from p. Since Ω is strictly convex, Lemma 15.3 implies that two geodesic rays emanating from p lie a bounded Hausdorff distance apart if and only they agree. So, we may identify $\partial_\infty \Omega$ with $\partial\Omega$. We see that this identification is a homeomorphism by noting that if $\{z_n\} \subset \bar{\Omega}$, then $\{\overline{pz_n}\}$ converges to $\overline{pz}$ if and only if $\{z_n\}$ converge to z. $\qquad\square$

Every properly convex domain Ω admits a **dual domain** $\Omega^* \subset \mathbb{P}\left((\mathbb{R}^{n+1})^*\right)$ where Ω^* denote the set of (projective classes of) linear functionals $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\phi(\vec{v}) \neq 0$ if $\vec{v} \neq \vec{0}$ and $[\vec{v}] \in \bar{\Omega}$. Alternatively, we may equivariantly identify $\mathbb{P}\left((\mathbb{R}^{n+1})^*\right)$ with the Grassmanian $\mathrm{Gr}_n(\mathbb{R}^{n+1})$ of hyperplanes in $\mathbb{R}^{n+1}$, by identifying a projective class of linear functional with its kernel. Then, Ω^* may be identified with the set of projective hyperplanes disjoint from $\bar{\Omega}$ and $\partial\Omega^*$ is the set of support planes to Ω.

Similarly if $\rho : \Gamma \to \mathsf{GL}(n+1, \mathbb{R})$ is a representation, then we may define **dual representation** $\rho^* : \Gamma \to \mathsf{GL}\left((\mathbb{R}^{n+1})^*\right)$. If $\phi \in (\mathbb{R}^{n+1})^*$ and $\gamma \in \Gamma$, then $\rho^*(\gamma)(\phi) = \phi \circ \rho(\gamma)^{-1}$. Notice that $\ker(\rho^*(\gamma)(\phi))) = \rho(\gamma)(\ker(\phi))$. If one identifies $\mathsf{GL}\left((\mathbb{R}^{n+1})^*\right)$ with $\mathsf{GL}(n+1, \mathbb{R})$ by choosing the standard basis $\{e_1^\perp, \ldots, e_{n+1}^\perp\}$ for $(\mathbb{R}^{n+1})^*$, then A acts by $(A^T)^{-1}$ on $\mathbb{P}\left((\mathbb{R}^{n+1})^*\right)$. So with this identification $\rho^*(\gamma) = (\rho(\gamma)^{-1})^T$. Notice that this whole construction commutes with projection, so if $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a representation, then we obtain a dual representation $\rho^* : \Gamma \to \mathsf{PGL}\left((\mathbb{R}^{n+1})^*\right)$.

We record a few elementary observations in a lemma for future use. The proof uses the standard fact that if Ω is properly convex, then $\partial\Omega$ is C^1 at a point x if and only if there is a unique support plane to Ω through the point x. (In fact, this is the key property of C^1 points that we will use, so one may simply take this as the definition for our purposes.)

Lemma 15.4. *Suppose that $\Omega \subset \mathbb{RP}^n$ is a properly convex domain. Then*

(1) *The dual domain Ω^* is properly convex.*
(2) *If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a representation and $\rho(\Gamma) \subset \mathrm{Aut}(\Omega)$, then $\rho^*(\Gamma) \subset \mathrm{Aut}(\Omega^*)$.*
(3) *The boundary $\partial\Omega$ of Ω is C^1 if and only if Ω^* is strictly convex. More precisely, if $x \in \partial\Omega$ is not a C^1 point, then there exists a line segment in $\partial\Omega^*$ consisting of support planes to Ω at x.*

Proof. Suppose that Ω is properly convex. If $x \in \mathbb{RP}^n$, let $P_x^* \subset ((\mathbb{R}^{n+1})^*)$ be the projective hyperplane

$$P_x^* = \mathbb{P}\Big(\big\{\phi \in (\mathbb{R}^{n+1})^*\big) \mid \phi(\vec{u}) = 0 \text{ if } [\vec{u}] = x\big\}\Big)$$

consisting of (projective classes of) linear functional with the line x in their kernel. Notice that if $x \in \Omega$, then, by definition, Ω^* is disjoint from P_x^*, so Ω^* is a bounded subset of an affine chart for $\mathbb{P}\left((\mathbb{R}^{n+1})^*\right)$. If $[\phi], [\psi] \in \Omega^*$, then we may choose a component Ω_0 of the pre-image of Ω in S^n and linear functionals $\phi_0 \in [\phi]$ and $\psi_0 \in [\psi]$ so that if $v \in \Omega_0$, then $\phi_0(v) > 0$ and $\psi_0(v) > 0$. Then $(t\phi_0 + (1-t)\psi_0)(v) > 0$ for all $t \in [0,1]$ and $v \in \Omega_0$. It follows that $\{[t\phi_0 + (1-t)\psi_0] \mid t \in [0,1]\}$ is a projective line segment in Ω^* joining $[\psi]$ to $[\phi]$. Therefore, Ω^* is properly convex, so (1) holds.

Consider a representation $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ so that $\rho(\Gamma)$ preserves Ω. If $\gamma \in \Gamma$ and H is a hyperplane disjoint from $\bar{\Omega}$, then $\rho(\gamma)(H)$ is also disjoint from $\bar{\Omega}$. Therefore, $\rho^*(\gamma)$ preserves Ω^*, so (2) holds.

The set of support planes to x is a non-trivial convex subset C_x of the projective hyperplane $\mathcal{P}_x^*$ and C_x is contained in $\partial\Omega^*$. Notice that $\partial\Omega$ is C^1 at x if and only if C_x is a single point, Therefore, if $\partial\Omega$ is not C^1, then Ω^* is not strictly convex. So, if Ω^* is strictly convex, then Ω is C^1.

Conversely, if Ω^* is not strictly convex, then there exists a non-trivial line segment $[a, b]$ in $\partial\Omega^*$. Each point $y \in [a, b]$ is associated to a support plane H_y to Ω. Suppose that H_a is a support plane at $c \in \partial\Omega$ and H_b is a support plane at $d \in \partial\Omega$. If $c \neq d$, then $[c, d] \in \bar{\Omega}$ and if $y \in (a, b)$, then H_y intersects $[c, d]$ transversely. Hence, H_y must intersect Ω which contradicts the fact that it is a support plane to Ω. Therefore, $c = d$, so c is not a C^1 point of $\partial\Omega$ (since it admits two distinct support planes). Thus, if Ω is not strictly convex, then $\partial\Omega$ is not C^1. So, if $\partial\Omega$ is C^1, then Ω^* is strictly convex, which completes the proof of (3). $\qquad\square$

The following result, concerning endpoints of geodesics in properly convex domains, will only be used in Section 39, so can be ignored for now if you want to move on immediately to the study of Benoist representations. Notice that this result is obvious for strictly convex domains, since geodesics in strictly convex domains are projective line segments.

Proposition 15.5. (Foertsch-Karlsson [98, Theorem 3], see also [83, Lemma 2.6]) *Suppose that* $\Omega \subset \mathbb{RP}^n$ *is a properly convex domain.*

    (1) *If* $r : [0, \infty) \to \Omega$ *is a geodesic ray, then* $\lim_{t \to \infty} r(t) = r(\infty)$ *exists and* $r(\infty) \in \partial\Omega$.
    (2) *If* $s : \mathbb{R} \to \Omega$ *is a bi-infinite geodesic, then*

$$s(\infty) = \lim_{t \to \infty} s(t) \neq \lim_{t \to -\infty} s(t) = s(-\infty).$$

Proof. Let $r : [0, \infty) \to \Omega$ be a geodesic ray in a properly convex domain Ω. For all $t > 0$, let $z_t = \overrightarrow{r(0)r(t)} \cap \partial\Omega$ and let P_t be the facet of $\partial\Omega$ containing z_t (i.e. the intersection of a supporting hyperplane to $\partial\Omega$ through z_t with $\partial\Omega$). Lemma 14.3 implies that if $0 < s < t$, then $z_s \subset P_t$. Therefore, $P_t = P_s$, so z_t lies in a single facet P for all $t > 0$.

Let $w_t = \overrightarrow{r(t)r(0)} \cap \partial\Omega$ and Q_t be the facet of $\partial\Omega$ containing w_t. We argue just as above to show that $Q_t = Q_s$ if $0 < s < t$, so w_t lies in a single facet Q for all $t > 0$. Moreover, P and Q are disjoint.

Let C be the set of accumulation points of $r([0, \infty))$ in $\partial\Omega$. Since any accumulation in C is also an accumulation point of $\{z_t\}$, we see that $C \subset P$. Suppose that C contains more than one point. Then, there exist sequence $\{s_n\}$ and $\{t_n\}$ so that $\lim r(s_n) = v \neq w = \lim r(t_n)$. Notice that the line L_n joining $r(s_n)$ to $r(t_n)$ converges to the line L containing $[v, w]$. It follows that L intersects $\partial\Omega$ only at P.

Lemma 14.3 implies that if $0 < s_n < t_n$, then z_{t_n} and $\overrightarrow{r(s_n)r(t_n)} \cap \partial\Omega$ span a line segment in $\partial\Omega$ and w_{t_n} and $\overrightarrow{r(t_n)r(s_n)} \cap \partial\Omega$ span a line segment in $\partial\Omega$, so $L_n \cap \bar{\Omega}$ has endpoints in P and Q. If $0 < t_n < s_n$, we argue similarly that $L_n \cap \bar{\Omega}$ has endpoints in P and Q. Since P and Q are closed and disjoint this implies that $L \cap \bar{\Omega}$ has endpoints in P and Q. We have achieved a contradiction, since we previously saw that L intersected $\partial\Omega$ only in P. Therefore, C is a single point $\{c\}$ and $\lim \lim_{t \to \infty} r(t) = c$.

Now suppose that $s : \mathbb{R} \to \Omega$ is a geodesic, the previous argument implies that $c = \lim_{t \to \infty} s(t)$ and $d = \lim_{t \to -\infty} s(t)$ exist. Moreover, we may check, just as above that c lies in the same facet of $\partial\Omega$ as $\overrightarrow{s(-1)s(1)} \cap \partial\Omega$ and that d lies in the same facet of $\partial\Omega$ as $\overrightarrow{s(1)s(-1)} \cap \partial\Omega$. Since $\overleftrightarrow{s(-1)s(1)}$ must intersect $\partial\Omega$ in distinct facets, c and d must be distinct. $\square$

Karlsson and Noskov proved that that Gromov hyperbolic properly convex domains have C^1 boundary. We will not use this fact in these notes, but I feel like it illuminates the general picture. Feel free to skip it if you prefer.

Proposition 15.6. (Karlsson-Noskov [135]) *If* $\Omega \subset \mathbb{RP}^n$ *is strictly convex and Gromov hyperbolic, then* $\partial\Omega$ *is a* C^1-*submanifold of* $\mathbb{RP}^n$.

Remarks: (1) If Ω is properly convex and has at least two line segments in its boundary, then the Hilbert metric is not even CAT(0) since geodesics are not unique. However, they still exhibit some of the geometric flavor of non-positively curved manifolds. Ludovic Marquis [158] refers to them as "damaged non-positively curved manifolds."

(2) Benoist [19] showed that a properly convex domain is Gromov hyperbolic in its Hilbert metric if and only if it is quasisymmetrically convex. Colbois, Vernicos and Verovic [67] showed that Ω is Gromov hyperbolic if and only if there is an upper bound on the area of all ideal triangles.

(3) Benzecri [26] proved that if $\Omega \subset \mathbb{RP}^2$ is properly convex and divisible, but not strictly convex, then Ω is projectively equivalent to the simplex Δ. Benoist [23] produced and studied more interesting examples of properly convex divisible domains in $\mathbb{RP}^3$ which are not strictly convex (see also Ballas-Danciger-Lee [10])).

16. **Benoist's characterizations of strictly convex divisible domains**

Yves Benoist established the following beautiful characterization of strictly convex divisible domains.

Theorem 16.1. (Benoist [20]) *If a discrete group $\Gamma \subset \mathsf{PGL}(n+1, \mathbb{R})$ acts cocompactly on a properly convex domain $\Omega \subset \mathbb{RP}^n$, then the following are equivalent.*

(1) (Ω, d_Ω^H) *is Gromov hyperbolic.*
(2) Γ *is Gromov hyperbolic.*
(3) Ω *is strictly convex.*
(4) $\partial \Omega$ *is a C^1 submanifold of $\mathbb{RP}^n$.*

Proof. Notice that the equivalence of (1) and (2) is an immediate consequence of the Milnor-Svarc Lemma and Proposition 4.1.

Proposition 15.1 asserts that if Ω is not strictly convex, then Ω is not Gromov hyperbolic, so (1) implies (3). We next show that (3) implies (2).

Proposition 16.2. (Benoist [20]) *If $\Gamma \subset \mathsf{PGL}(n+1, \mathbb{R})$ divides a strictly convex domain Ω, then (Ω, d_Ω^H) is Gromov hyperbolic.*

Proof. Recall that if Ω is strictly convex, then all geodesics are projective line segments, see Lemma 14.3. If Ω is not Gromov hyperbolic, then there exists sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ of points in Ω such that $u_n \in \overline{x_n y_n}$ for all n, and $d(u_n, \overline{x_n z_n} \cup \overline{y_n z_n}) \geq n$ for all n. (For the remainder of this chapter d will be the relevant Hilbert metric unless we say otherwise.)

Since Γ acts cocompactly on Ω, there is a compact set K such that $\Gamma(K) = \Omega$. So, we can always choose $\gamma_n \in \Gamma_n$ so that $\gamma_n(u_n) \in K$. So, after replacing x_n, y_n, z_n, u_n with $\gamma(x_n), \gamma(y_n), \gamma(z_n), \gamma(u_n)$, we may assume that $u_n \in K$ for all n. We may then pass to a subsequence so that $u_n \to u$, $x_n \to x$, $y_n \to y$ and $z_n \to z$. Since $d(u_n, \overline{x_n z_n} \cup \overline{y_n z_n}) \to \infty$, we must have $x, y, z \in \partial \Omega$. Since $u \in \overline{xy}$, $x \neq y$. If $x = z$, then $\overline{y_n z_n} \to \overline{yz}$, so $d(u_n, \overline{y_n z_n}) \to 0$, which is a contradiction, so $x \neq z$. Similarly $y \neq z$. But then, $d(u, \overline{xz}) < \infty$ since Ω is strictly convex. (This is the only usage of strict convexity in the proof.) But $d(u_n, \overline{x_n z_n}) \to d(u, \overline{xz})$, so we have again achieved a contradiction. Therefore, Ω is Gromov hyperbolic. $\square$

The following result shows that (3) holds if and only if (4) holds and completes the proof of Theorem 16.1. $\square$

Proposition 16.3. (Benoist [20]) *If $\Gamma \subset \mathsf{PGL}(n+1, \mathbb{R})$ divides a properly convex domain $\Omega \subset \mathbb{RP}^n$, then Ω is strictly convex if and only $\partial \Omega$ is a C^1 submanifold of $\mathbb{RP}^n$.*

Proof. Suppose that Ω is strictly convex. Then, by Proposition 16.2, Ω, and hence Γ, is Gromov hyperbolic. Let $\Gamma^* = \{(\gamma^{-1})^T \mid \gamma \in \Gamma\}$ (so Γ^* is the image of the dual to the inclusion map of Γ into $\mathsf{PGL}(n+1, \mathbb{R})$). Lemma 15.4 implies that Ω^* is properly convex and $\Gamma^* \subset \mathrm{Aut}(\Omega^*)$. Since Γ^* is discrete, it acts properly discontinuously on Ω^*. Since $M = \Omega/\Gamma$ is a closed manifold and is homotopy equivalent to the manifold $M^* = \Omega^*/\Gamma^*$, M^* is also closed, so Γ^* divides Ω^*. Since Γ^*, and hence Ω^* is Gromov hyperbolic, Proposition 15.1 implies that Ω^* is strictly convex. Lemma 15.4 then implies that Ω is a C^1 submanifold of $\mathbb{RP}^n$. (If Γ has torsion, we first pass to a finite index torsion-free subgroup of Γ and then apply the above argument.)

On the other hand, if Γ divides a properly convex domain Ω so that $\partial\Omega$ is a C^1 submanifold of $\mathbb{RP}^n$, then we argue as above to show that Γ^* divides the strictly convex domain Ω^*. It follows, from Proposition 16.2, that Γ^*, and hence Γ and Ω, is Gromov hyperbolic. Proposition 15.1 then implies that Ω is strictly convex. $\qquad\square$

Remarks: Benzecri [26] proved that if a group Γ divides a strictly convex domain and $\partial\Omega$ is a C^2-submanifold, then Ω is an ellipsoid. Benoist [20] proved that $\partial\Omega$ is always $C^{1+\alpha}$ for some $\alpha > 0$ if Ω is a strictly convex divisible domain.

17. **Linear algebra in** $\mathsf{GL}(d, \mathbb{R})$

> *The country I come from*
> *Is called the Midwest*
> *I was taught and brought up there*
> *The laws to abide*
> ————Bob Dylan [92]

We say that $A \in \mathsf{GL}(d, \mathbb{R})$ is **proximal** if it has an attracting fixed point for its action on $\mathbb{RP}^n$, i.e. there exists a point $x \in \mathbb{RP}^n$ such that $A(x) = x$ and an open neighborhood U of x so that if $u \in U$, then $A^n(u) \to x$ uniformly on compact subsets of U. We say that A is **biproximal** if both A and A^{-1} are proximal.

We will see how the linear algebra reflects this property. Consider the (real) Jordan canonical form for $A \in \mathsf{GL}(d, \mathbb{R})$, i.e. $A = BJB^{-1}$ for some $B \in \mathsf{GL}(n+1, \mathbb{R})$ and J is a block diagonal matrix

$$
\begin{bmatrix}
J_1 & 0 & \cdots & 0 \\
0 & J_2 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & J_m
\end{bmatrix}
$$

where each J_i is either a single real entry j_i or a (real) Jordan block of the form

$$
J_i = \begin{bmatrix}
C_i & 1 & 0 & \cdots & 0 \\
0 & C_i & 1 & \cdots & 0 \\
0 & 0 & C_i & \cdots & 0 \\
\vdots & \vdots & & & \vdots \\
0 & 0 & 0 & \cdots & C_i
\end{bmatrix}
$$

and C_i is either a single (real) entry j_i, or $C_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$, in which case $j_i = \pm\sqrt{|\det(C_i)|}e^{i\arccos(a_i)}$.

We will call j_i a **generalized eigenvalue**, and we call the subspace preserved by J_i its **generalized eigenspace** E_i. We may assume that

$$|j_1| \geq |j_2| \geq \cdots \geq |j_m|.$$

If J_i is a Jordan block, then its generalized eigenspace contains a vector $\vec{v}_i$ so that $\|J^n(\vec{v}_i)\| = |j_i|^n\|\vec{v}_i\|\|$ for all n. If J_i is upper-triangular, but not a single entry, then E_i contains an eigenvector $\vec{w}_i$ so that if $\vec{u} \in E_i$, then $J^n([\vec{u}]) \to [w_i]$ in $\mathbb{RP}^n$. However, the convergence will not be uniform. For example, if $J = J_i = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, then $< e_1 >$ is the only eigenline, but if $v_n = \left[\begin{smallmatrix} n \\ -1 \end{smallmatrix}\right]$, then $\{< v_n >\}$ converges to $< e_1 >$ and $J^n(v_n) = \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]$ for all n. If J_i is Jordan block which is not upper triangular, then its generalized eigenspace contains no eigenlines.

Notice that J is proximal if and only if $|j_1| > |j_2|$ and J_1 is a single entry j_1. In this case, the eigenspace E_1 is called the **attracting eigenline**, and is an attracting fixed point for the action of J on $\mathbb{RP}^{d-1}$. Moreover, $J^n(v) \to [e_1]$ if and only if v does not lie in $< e_2, \ldots, e_n, e_{n+1} >$, so we call $< e_2, \ldots, e_n, e_{n+1} >$ the **repelling hyperplane** of J.

If $A = BJB^{-1}$, then A is proximal if and only if J is proximal. Then, $B(< e_1 >)$ is the attracting eigenline of A and $B(< e_2, \ldots, e_{n+1} >)$ is the repelling hyperplane of A. If A is biproximal, the repelling hyperplane of A^{-1} is sometimes called the **attracting hyperplane** of A. In this case, in our notation, J_m is a single real number and the attracting hyperplane has the form $B(< e_1, \ldots, e_n >)$.

We may also rephrase this in terms of the eigenvalues of A. If we let $\{\lambda_i(A)\}_{i=1}^{n+1}$ denote the complex eigenvalues of A, then we may order them so that

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_d(A)|.$$

Notice that each block J_i gives rise to $\dim(J_i)$ eigenvalues of modulus $|j_i|$. Therefore, A is proximal if and only if $|\lambda_1(A)| > |\lambda_2(A)|$.

Notice that the notions of proximality, biproximality, attracting eigenline, repelling hyperplane all make sense in $\mathsf{PGL}(n+1, \mathbb{R})$ since they are unaffected by the choice of element in the projective class. Moreover, the tuple $(|\lambda_1(A)|, \ldots, |\lambda_d(A)|)$ is well-defined.

We also recall the **singular value decomposition**, which we discussed briefly in the introduction. If $B \in \mathsf{GL}(d, \mathbb{R})$, then we may write $B = LAK$ where $L, K \in \mathsf{O}(d)$ and A is a diagonal matrices with positive entries in descending order along the diagonal, i.e. $a_{11} \geq a_{22} \geq \cdots \geq a_{nn} > 0$. The matrix A depends only on B, but L and K need not be unique when some of the diagonal entries agree. We let $\sigma_i(B) = a_{ii}$ and call it the ith **singular value** of B. It is (half) the length of the ith minor of the ellipse $B(S^{n-1})$. In the literature, you will often see references to the **Cartan projection** which is given by

$$\mu : \mathsf{SL}_{\pm}(d, \mathbb{R}) \to \vec{\mathfrak{a}}^+$$

where

$$\vec{\mathfrak{a}}^+ = \{\vec{x} \in \mathbb{R}^d \mid x_1 \geq x_2 \geq \cdots \geq x_d \text{ and } x_1 + \cdots + x_n = 0\}$$

so that

$$\mu(B) = (\log \sigma_1(B), \ldots, \log \sigma_d(B)).$$

We will probably not use this language ourselves.

It is important to remember that singular values and eigenvalues are rather different animals (as I have learned to my dismay in the past). For example,

$$\sigma_1 \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \sim n$$

However, one does have the relationship

$$\log |\lambda_i(B)| = \lim \frac{\log \sigma_i(B^n)}{n}.$$

Notice that since A and $-A$ have the same singular values it makes sense to talk about $\sigma_i(X)$ when $X \in \mathsf{PSL}_\pm(n, \mathbb{R})$, which may be identified with $\mathsf{PGL}(d, \mathbb{R})$. Moreover, if $i \neq j$ and $A \in \mathsf{PGL}(d, \mathbb{R})$, then both $\frac{\sigma_i(A)}{\sigma_j(A)}$ and $\frac{|\lambda_i(A)|}{|\lambda_j(A)|}$ are well-defined.

18. Limit maps

Told my little Pollyanna
There's a place for you and me
We'll go down to Transverse City
Life is cheap, and death is free
————Warren Zevon [229]

In this section, we produce ρ-equivariant limit maps $\xi_\rho : \partial\Gamma \to \partial\Omega$ and $\theta_\rho : \partial\Gamma \to \mathbb{P}\left((\mathbb{R}^{n+1})^*\right)$ associated to a Benoist representation $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$. These limit maps generalize the limit maps we saw in the study of convex cocompact representations into $\mathsf{O}_0(n, 1)$ and foreshadow the limit maps which will play a crucial role in the theory of Anosov representations.

Suppose that Γ is a hyperbolic group and $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a representation and that

$$A_\rho : \partial\Gamma \to \mathbb{RP}^n \qquad \text{and} \qquad B_\rho : \partial\Gamma \to \mathrm{Gr}_n(\mathbb{R}^{n+1}) \cong \mathbb{P}\left((\mathbb{R}^{n+1}))^*\right)$$

are continuous ρ-equivariant maps. We say that A_ρ and B_ρ are **compatible** if $A_\rho(z) \subset B_\rho(z)$ for all $z \in \partial\Gamma$. They are said to be **transverse** if $A_\rho(w) \oplus B_\rho(z) = \mathbb{R}^{n+1}$ for all $z \neq w \in \partial\Gamma$.

Proposition 18.1. *If a Benoist representation $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ divides a strictly convex domain Ω, then it admits ρ-equivariant maps $\xi_\rho : \partial\Gamma \to \mathbb{RP}^m$ and $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_n(\mathbb{R}^{n+1})$ which are compatible and transverse.*

Proof. Theorem 16.1 implies that Γ and Ω are Gromov hyperbolic and that Ω is strictly convex and has C^1 boundary. The Milnor-Svarc Lemma implies that the orbit map $\tau_\rho : \Gamma \to \Omega$ is a quasi-isometry. Proposition 15.2 allows us to identify the Gromov boundary $\partial_\infty\Omega$ with the topological boundary $\partial\Omega$. Proposition 3.5 and Corollary 3.6 then show that τ_ρ extends to a homeomorphism $\xi_\rho : \partial\Gamma \to \partial\Omega$ so that if $\{\gamma_n\} \subset \Gamma$, $\lim \gamma_n = z \in \partial\Gamma$ and $x \in \Omega$, then $\lim \rho(\gamma_n)(x) = \xi_\rho(z)$.

We define $\theta_\rho : \partial\Gamma \to \mathbb{P}\left((\mathbb{R}^{n+1})^*\right)$ by letting $\theta_\rho(z)$ be the tangent plane (unique support plane) to $\partial\Omega$ at $\xi_\rho(z)$. By construction, θ_ρ is a ρ-equivariant embedding. Notice that ξ_ρ and θ_ρ are compatible by definition. Since $\theta_\rho(\gamma^+)$ is invariant under $\rho(\gamma)$ and does not contain the repelling eigenline of $\rho(\gamma)$ it must be the attracting n-plane of $\rho(\gamma)$.

Since Ω is strictly convex, if $z \in \partial\Gamma$, then $\theta_\rho(z)$ is the tangent plane to $\partial\Omega$ at $\xi_\rho(z)$, so it cannot intersect $\partial\Omega$ at any other point. In particular, if $w \neq z$, then the line $\xi_\rho(w) \in \partial\Omega - \{\xi_\rho(z)\}$ is

not contained in the hyperplane $\theta_\rho(z)$. Therefore, $\xi_\rho(w) \oplus \theta_\rho(z) = \mathbb{R}^{n+1}$ if $z \neq w \in \partial\Gamma$, so ξ_ρ and θ_ρ are transverse. $\square$

We next show that if ρ is a Benoist representation, then all images of infinite order elements are biproximal and the limit maps are dynamics-preserving.

Suppose that Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a representation and

$$A_\rho : \partial\Gamma \to \mathbb{RP}^n \qquad \text{and} \qquad B_\rho : \partial\Gamma \to \mathrm{Gr}_n(\mathbb{R}^{n+1}) \cong \mathbb{P}\left((\mathbb{R}^{n+1}))^*\right)$$

are continuous ρ-equivariant maps. The map A_ρ is said to be **dynamics preserving** if whenever γ is an infinite order element of Γ, then $\rho(\gamma)$ is biproximal and $A_\rho(\gamma^+)$ is the attracting eigenline of $\rho(\gamma)$. Similarly, $B_\rho(\gamma^+)$ is said to be dynamics preserving if whenever γ is an infinite order element of Γ, then $\rho(\gamma)$ is biproximal and $B_\rho(\gamma^+)$ is the attracting hyperplane of $\rho(\gamma)$.

Proposition 18.2. *If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation, then ξ_ρ and θ_ρ are dynamics preserving. In particular, if $\gamma \in \Gamma$ has infinite order, then $\rho(\gamma)$ is biproximal.*

We derive Proposition 18.2 from a more general statement which will be useful in Section 42.

Lemma 18.3. *Suppose that $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a discrete, almost faithful representation, $\rho(\Gamma)$ preserves a properly convex domain Ω and there exist continuous, transverse ρ-equivariant maps $\xi : \partial\Gamma \to \mathbb{RP}^n$ and $\theta : \partial\Gamma \to \mathrm{Gr}_n(\mathbb{R}^{n+})$. If $\gamma \in \Gamma$ has infinite order and $\{\rho(\gamma)^n(x)\}_{n\in\mathbb{N}}$ converges to $\xi(\gamma^+)$ if $x \in \Omega$, then $\rho(\gamma)$ is proximal, $\xi(\gamma^+) = \rho(\gamma)^+$ and $\theta_\rho(\gamma^-)$ is the repelling hyperplane of $\rho(\gamma)$.*

Proof. We may assume without loss of generality that $J = \rho(\gamma)$ is in Jordan canonical form. Notice that $\xi(\gamma^+)$ and $\theta(\gamma^-)$ are transverse and invariant under $\rho(\gamma)$, so every generalized eigenspace of a Jordan block of J is contained in either $\xi(\gamma^+)$ or $\theta(\gamma^-)$. Therefore, the Jordan block J_+ whose generalized eigenspace contains the line $\xi(\gamma^+)$ must be one-dimensional. Let j_+ be the eigenvalue of J_+ and let j_- be the largest generalized eigenvalue of a Jordan block contained in $\theta(\gamma^-)$. Let $\vec{v}_1$ be a vector in J_+ and let $\vec{v}_2$ be a vector in $\theta(\gamma^-)$ so that $||J^n(\vec{v}_2)|| = (j_-)^n ||\vec{v}_2||$. Let $p_+ : \mathbb{R}^{n+1} \to J_+$, $p_- : \mathbb{R}^{n+1}$ and $p_2 : \mathbb{R}^{n+1} \to <\vec{v}_2>$ be projection maps. Since Ω is open, there exists $x = [\vec{v}] \in \Omega$, so that $p_2(\vec{v}) \neq 0$. If $|j_-| \geq |j_+|$, then $\left\{ \frac{||p_-(J^n(\vec{v}))||}{||p_+(J^n(\vec{v}))||} \right\}_{n\in\mathbb{N}}$ does not converge to 0, so $\rho(\gamma)^n(x)$ does not converge to $\xi(\gamma^+)$. This violates our assumption, so $|j_+| > |j_-|$ which implies that $\rho(\gamma)$ is proximal and $\xi(\gamma^+) = \rho(\gamma)^+$. $\square$

Proof of Proposition 18.2. Proposition 18.1 implies that ξ_ρ and θ_ρ are continuous, transverse ρ-equivariant maps, such that if γ has infinite order then $\{\rho(\gamma)^n(x)\}_{n\in\mathbb{N}}$ converges to $\xi(\gamma^+)$. Lemma 18.3 then implies that if γ has infinite order, then $\rho(\gamma)$ is proximal and $\xi(\gamma^+) = \rho(\gamma^+)$. Similarly, $\rho(\gamma^{-1})$ is proximal, so $\rho(\gamma)$ is biproximal and $\theta_\rho((\gamma^{-1})^-)$ is the repelling hyperplane of $\rho(\gamma^{-1})$. Since $(\gamma^{-1})^- = \gamma^+$, this implies that $\theta_\rho(\gamma^+)$ is the attracting hyperplane of $\rho(\gamma)$. This completes the proof that ξ_ρ and θ_ρ are dynamics-preserving. $\square$

We may use the analysis above to check that Benoist representations are strongly irreducible, which is a special case of a result of Vey [211]. Recall that a representation $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is said to be **irreducible** if every subspace of $\mathbb{R}^d$ invariant under every element of $\rho(\Gamma)$ is either trivial or all of $\mathbb{R}^d$. It is called **strongly irreducible** if the restriction of ρ to any finite index subgroup is still irreducible.

Proposition 18.4. (Vey [211]) *If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation, then ρ is strongly irreducible.*

Proof. Suppose that a proper linear subspace V is invariant under $\rho(\Gamma)$. If γ is any infinite order element of Γ, then the translates $\Gamma(\gamma^+)$ of γ^+ are dense in $\partial\Gamma$, by Proposition 5.6. Therefore, the attracting eigenlines of conjugates of $\rho(\gamma)$ are dense in $\partial\Omega$ and hence span $\mathbb{R}^{n+1}$. It follows that there exists an infinite order element $\gamma \in \Gamma$ so that V does not contain the attracting eigenline $\xi_\rho(\gamma^+)$ of $\rho(\gamma)$. Thus, V must be contained in the repelling hyperplane of $\rho(\gamma)$, since otherwise there exists $v \in V$, so that $\rho(\gamma)^n(v) \to \xi_\rho(\gamma^+)$. So V is contained in the hyperplane P tangent to $\partial\Omega$ at $\xi_\rho(\gamma^-)$. If $\alpha \in \Gamma$, then $\alpha(P)$ is the tangent plane to $\partial\Omega$. If $\alpha \in \Gamma$, then $\alpha(P)$ is the tangent plane to $\partial\Omega$ at $\alpha(\xi_\rho(\gamma^-))$, and since V is Γ-invariant, we see that $V \subset \alpha(P)$. Since translates of γ^- are dense in $\partial\Gamma$, again by Proposition 5.6, we see that V is contained in the tangent plane to every point of $\partial\Omega$.

Now notice that every hyperplane in the affine chart is parallel to exactly two distinct tangent planes to Ω, since Ω is strictly convex and $\partial\Omega$ is C^1. Therefore, the intersection of all the projective hyperplanes tangent to $\partial\Omega$ is trivial, so V must be trivial, which is again a contradiction. Therefore, ρ must be irreducible.

Since the restriction of every Benoist representation to a finite index subgroup is a Benoist representation, we see immediately that Benoist representations are strongly irreducible. $\square$

Remarks: 1) Vey [211] further proved that if a discrete subgroup Γ of $\mathsf{PGL}(n+1, \mathbb{R})$ divides a properly convex, irreducible domain in $\mathbb{RP}^n$, then Γ is strongly irreducible.

2) Benoist [18] proved that the Zariski closure of the image of a Benoist representation into $\mathsf{PSL}(n+1, \mathbb{R})$ is either $\mathsf{PSO}(n, 1)$ or $\mathsf{PSL}(n+1, \mathbb{R})$. Benoist [21] proved, more generally, that if a discrete group $\Gamma \subset \mathsf{PGL}(n+1, \mathbb{R})$ divides a properly convex domain Ω, then Γ is Zariski dense unless Ω is a product or symmetric cone. (The symmetric cones have been completely classified and are all associated to semi-simple Lie groups, and $\mathsf{PO}(n, 1)$ is the only rank one Lie group which arises. We will later see the symmetric domain associated to $\mathsf{PO}(n, n)$ in Section 41.)

3) If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation and $\rho(\Gamma)$ preserves Ω, then, by Lemma 15.4, $\rho^*(\Gamma)$ preserves Ω^*. We saw, in the proof of Proposition 16.3, that $\rho^*(\Gamma)$ acts cocompactly on Ω^* and that Ω^* is strictly convex, so ρ^* is also a Benoist representation. If $z \in \partial\Gamma$, then $\theta_\rho(z)$ is the unique support plane to Ω at $\xi_\rho(z)$, so $\theta_\rho(z)$ lies in $\partial\Omega^*$. So, θ_ρ is a ρ^*-equivariant homeomorphism from $\partial\Gamma$ to $\partial\Omega^*$ which suggest that $\theta_\rho = \xi_{\rho^*}$. If $\gamma \in \Gamma$ has infinite order, then $\theta_\rho(\gamma^+)$ is the attracting n-plane of $\rho(\gamma)$, so it is the attracting eigenline of $\rho^*(\gamma)$. Therefore, $\xi_{\rho^*}(\gamma^+) = \theta_\rho(\gamma^+)$ and since both θ_ρ and ξ_{ρ^*} are continuous, we see that $\theta_\rho = \xi_{\rho^*}$. We will later establish a generalization of this observation in the context of Anosov representations, see Corollary 32.4.

19. Translation length, eigenvalues and singular values

If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation dividing a domain Ω, we define the **translation length** of an element $\rho(\gamma)$ by

$$\ell_\rho(\gamma) = \inf \left\{ d_\Omega^H(x, \rho(\gamma)(x)) \mid x \in \Omega \right\}.$$

Since $\Omega/\rho(\Gamma)$ is compact, the translation length is always achieved.

Lemma 19.1. *If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation dividing a strictly convex domain Ω and γ is an infinite order element of Γ, then*

$$\ell_\rho(\gamma) = \frac{1}{2} \log \left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_{n+1}(\rho(\gamma))|} \right)$$

and the translation distance is achieved exactly on the geodesic A_γ in Ω joining $\xi_\rho(\gamma^+)$ to $\xi_\rho(\gamma^-)$.

Proof. Notice that A_γ is preserved by $\rho(\gamma)$, since $\rho(\gamma)$ fixes its endpoints and all geodesics in Ω are projective line segments. If p does not lie on A_γ, then $\overline{p\rho(\gamma)(p)} \cup \overline{\rho(\gamma)(p)\rho(\gamma)^2(p)}$ is not a geodesic, since geodesics are projective line segments in a strictly convex domain. If q is the midpoint of $\overline{p\rho(\gamma)(p)}$, then

$$d(q, \rho(\gamma)(q)) < d(p, \rho(\gamma)(p)) = d(q, \gamma(p)) + d(\gamma(p), \gamma(q))$$

so translation distance is not minimized at p. Therefore, the translation distance is minimized on A_γ. Since $\rho(\gamma)$ acts as an isometry of A_γ the translation distance is constant on A_γ.

It only remains to compute the translation distance on A_γ. We may assume that $\rho(\gamma)$ is in Jordan canonical form, so A_γ joins $[e_1] = \xi_\rho(\gamma^+)$ to $[e_{n+1}] = \xi_\rho(\gamma^-)$. We may conjugate so that $A_\gamma = \{[t, 0, \ldots, 0, 1-t] \mid t \in (0,1)\} \subset \Omega$ (since either $\{[t, 0, \ldots, 0, 1-t] \mid t \in (0,1)\}$ or $\{[-t, 0, \ldots, 0, 1-t] \mid t \in (0,1)\}$ is contained in Ω.) Notice that $\lambda_1(\rho(\gamma))$ and $\lambda_{n+1}(\rho(\gamma))$ must have the same sign, since otherwise $\rho(\gamma)$ does not preserve A_γ,

If $x_0 = [\frac{1}{2}, 0, \cdots, 0, \frac{1}{2}]$, then

$$
\begin{aligned}
\rho(\gamma)(x_0) &= \left[\frac{1}{2}\lambda_1(\rho(\gamma)), 0, \cdots, 0, \frac{1}{2}\lambda_{n+1}(\rho(\gamma)) \right] \\
&= \left[\frac{\lambda_1(\rho(\gamma))}{\lambda_1(\rho(\gamma)) + \lambda_{n+1}(\rho(\gamma))}, 0, \cdots, 0, \frac{\lambda_{n+1}(\rho(\gamma))}{\lambda_1(\rho(\gamma)) + \lambda_{n+1}(\rho(\gamma))} \right]
\end{aligned}
$$

so

$$
\begin{aligned}
d_\Omega^H(x, \rho(\gamma)(x)) &= \frac{1}{2} \log \left(\left[[e_{n+1}], x_0, \rho(\gamma)(x_0), [e_1] \right] \right) \\
&= \frac{1}{2} \log \left(\left[e_{n+1}, (\frac{1}{2}, 0, \cdots, \frac{1}{2}), \left(\frac{\lambda_1(\rho(\gamma))}{\lambda_1(\rho(\gamma)) + \lambda_{n+1}(\rho(\gamma))}, 0, \cdots, \frac{\lambda_{n+1}(\rho(\gamma))}{\lambda_1(\rho(\gamma)) + \lambda_{n+1}(\rho(\gamma))} \right), e_1 \right] \right) \\
&= \frac{1}{2} \log \left(\frac{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}\lambda_1(\rho(\gamma))}{\lambda_1(\rho(\gamma)) + \lambda_{n+1}(\rho(\gamma))}}{\frac{\sqrt{2}\lambda_{n+1}(\rho(\gamma))}{\lambda_1(\rho(\gamma)) + \lambda_{n+1}(\rho(\gamma))} \cdot \frac{1}{\sqrt{2}}} \right) \\
&= \frac{1}{2} \log \left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_{n+1}(\rho(\gamma))|} \right)
\end{aligned}
$$

$\square$

Let $||\gamma||$ denote the translation length of γ on the Cayley graph C_Γ. (Recall that $||\gamma||$ is the minimal word length of an element conjugate to γ.) Then since the orbit map from C_Γ to Ω is a quasi-isometry, we immediately see that $\log \left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_{n+1}(\rho(\gamma))|} \right)$ grows linearly in $||\gamma||$. Thus, Benoist representations are **well displacing**, in the language of Delzant-Guichard-Labourie-Mozes [87]).

Corollary 19.2. *If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation, then there exists J and B so that*

$$J||\gamma|| + B \geq \log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_{n+1}(\rho(\gamma))|}\right) \geq \frac{1}{J}||\gamma|| - B$$

for all $\gamma \in \Gamma$.

Proof. The Milnor-Svarc Lemma implies that the orbit map $\tau_\rho : \Gamma \to \Omega$ (which is given by $\tau_\rho(\gamma) = \rho(\gamma)(x_0)$ for some fixed $x_0 \in \Omega$) is a (K, C)-quasi-isometry for some K and C.

Suppose that $\gamma \in \Gamma$. Then there exists $\alpha \in \Gamma$ so that $d(\gamma\alpha, \alpha) = ||\gamma||$. Therefore,

$$K||\gamma|| + C \geq d(\tau_\rho(\gamma\alpha), \tau_\rho(\alpha)) = d(\rho(\gamma)(\tau_\rho(\alpha)), \tau_\rho(\alpha)) \geq \ell_\rho(\gamma)) = \frac{1}{2}\log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_{n+1}(\rho(\gamma))|}\right)$$

On the other hand, there exists $\beta \in \Gamma$ and a point x on the axis A_γ of γ so that $d(\tau_\rho(\beta), y) \leq C$. Since τ_ρ is a (K, C)-quasi-isometry,

$$d(\rho(\gamma)(\tau_\rho(\beta)), \tau_\rho(\beta)) = d(\tau_\rho(\gamma\beta), \tau_\rho(\beta)) \geq \frac{1}{K}||\gamma|| - C.$$

Applying the triangle inequality, we see that

$$\frac{1}{2}\log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_{n+1}(\rho(\gamma))|}\right) = \ell_\rho(\gamma) = d(\rho(\gamma)(x)), x \geq \frac{1}{K}||\gamma|| - 3C.$$

Therefore, the desired inequality holds with $J = 2K$ and $B = 6C$. $\qquad\square$

We would also like to show, in analogy with Lemma 11.2, that $\log\left(\frac{|\sigma_1(\rho(\gamma))|}{|\sigma_{n+1}(\rho(\gamma))|}\right)$ grows linearly in the word length of γ. In order to do so, we use the U property of hyperbolic groups.

Corollary 19.3. *If $\rho : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})$ is a Benoist representation, then there exists L and D so that*

$$Ld(1, \gamma) \geq \log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_{n+1}(\rho(\gamma))}\right) \geq \frac{1}{L}d(1, \gamma) - D$$

for all $\gamma \in \Gamma$.

Proof. Let S be the finite generating set for Γ, which we assume is symmetric, i.e. $s \in S$ if and only if $s^{-1} \in S$, which we have been using in the background. Let

$$M = \max\{\log\sigma_1(\rho(s)) \mid s \in S\}.$$

Since Γ has the U property, see Proposition 8.4, there exists $\alpha, \beta \in \Gamma$ and $K \geq 0$, so that if $\gamma \in \Gamma$, then there exists $\eta \in \{1, \alpha, \beta\}$ such that $d(1, \gamma) \leq 3||\gamma\eta|| + K$.

If $\gamma \in \Gamma$, then, since $\sigma_1(AB) \leq \sigma_1(A)\sigma_1(B)$ and $\sigma_{n+1}(A) = \sigma_1(A^{-1})^{-1}$, we see that

$$\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_{n+1}(\rho(\gamma))}\right) \leq M^2 d(1, \gamma).$$

Now choose $\eta \in \{1, \alpha, \beta\}$ so that $d(1, \gamma) \leq 3||\gamma\eta|| + K$. If J and B are the constants from Corollary 19.2, then

$$\log\left(\frac{|\lambda_1(\rho(\gamma\eta))|}{|\lambda_{n+1}(\rho(\gamma\eta))|}\right) \geq \frac{1}{J}||\gamma\eta|| - B \geq \frac{1}{3J}d(1, \gamma) - \frac{K}{J} - B.$$

But, since $\sigma_1(A) \geq |\lambda_1(A)|$, $\sigma_1(AB) \leq \sigma_1(A)\sigma_1(B)$ and $\sigma_{n+1}(A) = \sigma_1(A^{-1})^{-1}$,

$$\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_{n+1}(\rho(\gamma))}\right) \geq \frac{1}{\max\{1, 2M^2 d(1,\eta)\}}\log\left(\frac{\sigma_1(\rho(\gamma\eta))}{\sigma_{n+1}(\rho(\gamma\eta))}\right) \geq \frac{1}{\max\{1, 2M^2 d(1,\eta)\}}\log\left(\frac{|\lambda_1(\rho(\gamma\eta))|}{|\lambda_{n+1}(\rho(\gamma\eta))|}\right)$$

so, if we set $G = \max\{d(1,\alpha), d(1,\beta)\}$, we may combine the last two inequalities to see that

$$\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_{n+1}(\rho(\gamma))}\right) \geq \frac{1}{2M^2 GJ}d(1,\gamma) - \frac{K}{2JM^2 G} - B$$

so our result holds with $L = M^2 JG$ and $D = \frac{K}{JM^2 G} + B$.

$\square$

A more honest way to prove Corollary 19.3 would be to first relate the singular values to the translation length of the point. Once one has done this, Corollary 19.3 follows immediately from the Milnor-Svarc Lemma. The key result here is due to Danciger, Guéritaud and Kassel:

Proposition 19.4. (Danciger-Guéritaud-Kassel [84, Proposition 10.1]) *If Ω is a properly convex domain and $x_0 \in \Omega$, then there exists $\kappa > 0$ so that if $A \in \mathrm{Aut}(\Omega)$, then*

$$\log\left(\frac{\sigma_1(A)}{\sigma_{n+1}(A)}\right) \geq 2d(x_0, \gamma(x_0)) - \kappa.$$

Remarks: (1) The converses to Corollaries 19.2 and 19.3 fail, since if ρ is a Benoist representation and $H \subset \Gamma$ is a quasiconvex subgroup, then the conclusions are satisfied by $\rho|_H$, but $\rho|_H$ is Benoist if and only if H has finite index in Γ. However, the converses hold if $n \geq 3$ and we assume that Γ is hyperbolic and has (virtual) cohomological dimension n, see Corollary 23.3.

(2) If $X_{n+1} = \mathsf{PGL}(n+1,\mathbb{R})/\mathsf{PO}(n+1)$ is the symmetric space associated to $\mathsf{PGL}(n+1,\mathbb{R})$, then Corollary 19.3 implies that the orbit map $\tau_\rho : \Gamma \to X_{n+1}$ is a quasi-isometric embedding if ρ is a Benoist representation. One need only recall that if $x_0 = [\mathsf{PO}(n+1)]$, then

$$d(x_0, A(x_0)) = ||(\log\sigma_1(A), \ldots, \log\sigma_{n+1}(A))||.$$

See Section 26 for a discussion of X_{n+1} and a proof of the distance formula above.

(3) See Cooper-Long-Tillman [70, Section 2] for a more general treatment of projective isometries of properly convex domains.

20. Benoist components

If Γ is a hyperbolic group, we consider the set

$$\mathsf{Ben}(\Gamma, n) \subset \mathrm{Hom}(\Gamma, \mathsf{PGL}(n+1,\mathbb{R}))$$

of Benoist representations of Γ into $\mathsf{PGL}(n+1,\mathbb{R})$.

Koszul used the technology of (G, X)-structures to show that $\mathsf{Ben}(\Gamma, n)$ is open. We will later be able to give a simple proof that uses the stability of Anosov representations and a little convex geometry, see Theorem 33.6.

Theorem 20.1. (Koszul [141]) *If Γ is a hyperbolic group, then $\mathsf{Ben}(\Gamma, n)$ is an open subset of $\mathrm{Hom}(\Gamma, \mathsf{PGL}(n+1,\mathbb{R}))$.*

Choi and Goldman [66] showed that $\mathsf{Ben}(\Gamma, 2)$ is closed, and Benoist [22] proved $\mathsf{Ben}(\Gamma, n)$ is closed for all n.

Theorem 20.2. (Choi-Goldman [66], Benoist [22]) *If Γ is a hyperbolic group, then* $\mathsf{Ben}(\Gamma, n)$ *is a closed subset of* $\mathrm{Hom}(\Gamma, \mathsf{PGL}(n+1, \mathbb{R}))$.

As a corollary of Theorems 20.1 and 20.2 we see that $\mathsf{Ben}(\Gamma, n)$ is a collection of components of $\mathrm{Hom}(\Gamma, \mathsf{PGL}(n+1, \mathbb{R}))$.

Corollary 20.3. *If Γ is a hyperbolic group, then* $\mathsf{Ben}(\Gamma, n)$ *is a collection of components of* $\mathrm{Hom}(\Gamma, \mathsf{PGL}(n+1, \mathbb{R}))$.

We will give a somewhat sketchy proof of the following weaker fact which gives some indication why Corollary 20.3 is true. This is essentially the argument given by Choi and Goldman [66], see also Marquis [158, Lemma 7.4].

Lemma 20.4. *If $\{\rho_k : \Gamma \to \mathsf{PGL}(n+1, \mathbb{R})\}$ is a sequence in $\mathsf{Ben}(\Gamma, n)$ which converges to $\rho \in \mathrm{Hom}(\Gamma, \mathsf{SL}(n+1, \mathbb{R}))$ and ρ is strongly irreducible, then $\rho \in \mathsf{Ben}(\Gamma, n)$.*

Proof. Let Ω_k be the strictly convex domain divided by $\rho_k(\Gamma)$. We may pass to a subsequence so that $\{\bar{\Omega}_k\}$ converges to a closed subset K of $\mathbb{RP}^n$ in the Hausdorff topology on the set of closed subsets of $\mathbb{RP}^n$. (We say that $\{\Omega_k\}$ converges to K in the Hausdorff topology if the Hausdorff distance between Ω_k and K converges to 0. One may easily check that the Hausdorff topology is compact.) Notice that K must be convex, since any two points in K are approximated by points in $\bar{\Omega}_k$ and hence by a projective line segment in $\bar{\Omega}_k$.

If K is properly convex and has non-empty interior Ω, then $\rho(\Gamma) \subset \mathrm{Aut}(\Omega)$. Corollary 6.2 implies that ρ is discrete and faithful. Since $\rho(\Gamma)$ is discrete and faithful, the quotient $M = \Omega/\Gamma$ is a manifold (or orbifold if Γ has torsion) homotopy equivalent to $M_k = \Omega_k/\rho_k(\Gamma)$. Since M_n is compact, it follows that M is compact. Since Γ is hyperbolic, Ω is strictly convex. Therefore, we would see that ρ is a Benoist representation.

If K has empty interior, then since it is convex, it lies in a proper projective sub-plane. Since $\rho(\Gamma)$ preserves the proper subspace spanned by K, ρ is reducible, which contradicts our assumptions.

Now suppose that K is not properly convex. The pre-image of each Ω_k in S^n is a pair of copies $\Omega_k^{\pm}$ of Ω_k. Since each Ω_k is properly convex, the closure of the two is separated by a disjoint hyperplane P_k. We may pass to a subsequence of $\{P_k\}$ which converges to a hyperplane P separating S^n into two hemispheres H^+ and H^-. Then $\{\Omega_k^{\pm}\}$ converges to $K^{\pm}$, where each $K^{\pm}$ is a copy of K. If we label consistently throughout, $K^+ \subset \bar{H}^+$ and $K^- \subset \bar{H}^-$. Notice that, by construction, $K^+ \cap P$ is the set of points which are antipodal to points in $K^- \cap P$. If K^+ and K^- are disjoint, we may find $v \in S^n - (K^+ \cup K^-) \cap P$ and a plane $\hat{P}$ through $v \cup -v$ which is disjoint from K^+ and K^-, so K would be properly convex. Therefore, $K^+ \cap K^-$ must be non-empty.

Let $p_n : \mathsf{GL}(n+1, \mathbb{R}) \to \mathsf{PGL}(n+1, \mathbb{R})$ be the obvious projection map and let Θ_k be the intersection of all index two subgroups of $p_n^{-1}(\rho_k(\Gamma)) \cap \mathsf{SL}(n+1, \mathbb{R})$. Then Θ_k has finite index in $p_n^{-1}(\rho_k(\Gamma))$ and preserves Ω_k^+ and Ω_k^-. Therefore, the intersection Θ of all index two subgroups of $p_n^{-1}(\rho(\Gamma))$ preserves both K^+ and K^-, and hence preserves $K^+ \cap K^-$. Therefore, Θ is reducible, which contradicts the strong irreducibility of ρ. $\qquad\square$

Choi and Goldman complete the argument in dimension $n = 2$, by using special properties of low-dimensional representations to show that any limit of Benoist representation is strongly irreducible. This portion of the proof does not generalize, so we will not go into details here.

Remarks: Koszul [141] and Benoist [22] further showed that if Γ does not contain an infinite normal nilpotent sugbroup, then the space of representations of Γ which divide a properly convex set is a collection of components of $\mathrm{Hom}(\Gamma, \mathsf{PGL}(n+1, \mathbb{R}))$.

21. **Projective bending**

The most general known way of constructing deformations of Benoist representations is the procedure known as projective bending. We begin with an algebraic description of projective bending, our "proof" that the projective bending of a lattice in $\mathsf{PSO}(n,1)$ remains a Benoist representation will hopefully give some intuition for the geometry of this construction.

Suppose that $\rho_0 : \Gamma \to \mathsf{PSO}(n,1)$ is a discrete, faithful representation, so that Γ is torsion-free and $M = \mathbb{H}^n/\rho(\Gamma)$ is a closed manifold which contains a separating, connected, totally geodesic submanifold X. If we let M_1 and M_2 be the result of cutting the manifold M along X, then

$$\Gamma = \pi_1(M) = \pi_1(M_1) *_{\pi_1(X)} \pi_1(M_2) = \Gamma_1 *_{\Gamma_X} \Gamma_2$$

and we may assume that

$$\rho_0(\Gamma_X) \subset \mathsf{PSO}(n-1,1) \subset \mathsf{PSO}(n,1)$$

where $\mathsf{PSO}(n-1,1)$ is identified with a subgroup of $\mathsf{PSO}(n,1)$ by identifying A with $\begin{bmatrix} 1 & \vec{0}^{\;T} \\ \vec{0} & A \end{bmatrix}$
(i.e. by sticking A in the lower righthand corner). Notice that if $B_t \in \mathsf{PSL}(n+1, \mathbb{R})$ is the diagonal matrix with diagonal entries $(e^{nt}, e^{-t}, \ldots, e^{-t})$, then each B_t centralizes $\mathsf{PSO}(n-1,1)$ within $\mathsf{PSO}(n,1)$. We then define, for all $t \in \mathbb{R}$, the *projective bending* $\rho_t : \Gamma \to \mathsf{PSL}(n, \mathbb{R})$ of ρ_0 along X, by letting $\rho_t(\gamma_1) = \rho_0(\gamma_1)$ for all $\gamma_1 \in \Gamma_1$ and letting $\rho_t(\gamma_2) = B_t \circ \rho_0(\gamma_2) \circ B_{-t}$ for all $\gamma_2 \in \Gamma_2$. There is a similar construction, involving the associated HNN decomposition, when X is a totally geodesic, connected non-separating codimension one submanifold of M. Corollary 20.3 implies that the result of projective bending is always a Benoist representation, but we will later sketch a geometric proof.

The motivation for projective bending was provided by the bending construction in hyperbolic geometry. We will describe the geometry of the hyperbolic construction first, since it is easier to visualize and hopefully will provide some intuition for the projective setting. We will restrict to the setting of surface groups (although the construction generalizes to higher dimensions). Let S be a closed surface and let $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{R})$ be a Fuchsian representation. Suppose that a simple closed curve C separates S into two components, S_1 and S_2, and consider the associated group-theoretic decomposition

$$\Gamma = \pi_1(S) = \pi_1(S_1) *_{\pi_1(C)} \pi_1(S_2) = \Gamma_1 *_{\Gamma_C} \Gamma_2.$$

Let β be a generator of Γ_C. We may assume that

$$\rho(\beta) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

(where $\lambda > 1$). Notice that $\rho(\beta)$ commutes with

$$B_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \in \mathsf{PSL}(2, \mathbb{C})$$

and define $\rho_\theta : \Gamma \to \mathsf{PSL}(2, \mathbb{C})$ by letting $\rho_\theta(\gamma_1) = \rho(\gamma_1)$ for all $\gamma_1 \in \Gamma_1$ and letting $\rho_\theta(\gamma_2) = B_\theta \circ \rho(\gamma_2) \circ B_{-\theta}$ for all $\gamma_2 \in \Gamma_2$. Theorem 11.4 implies that ρ_θ is convex cocompact for small

values of θ, but it will not be convex cocompact for all θ. (One may see this by noticing that $\rho_\pi(\pi_1(S)) \subset \mathsf{PSL}(2, \mathbb{R})$, but is clearly not Fuchsian.)

There are pretty pictures of the result of bending a Fuchsian representation available on the web. See, for example, the bottom two pictures here:

https://gauss.math.yale.edu/~yhm3/research/limset/pictures.html

or here:

https://www.dumas.io/poster/

We now give a geometric description of bending in the hyperbolic setting. (Here I really need a GIF of me waving my hands appropriately.) First notice that if $S_\rho = \mathbb{H}^2/\rho(\Gamma)$ and β^* is the geodesic representative of C on S_ρ, then the pre-image of β^* is an infinite collection of geodesics (each of which is the axis of an element of $\rho(\alpha)$ where α is conjugate to β) which divides $\mathbb{H}^2$ into infinite-sided polygons. Let C_1 and C_2 be components of the complement stabilized by $\rho(\Gamma_1)$ and $\rho(\Gamma_2)$. We will describe a ρ_θ-equivariant map $h_\theta : \mathbb{H}^2 \to \mathbb{H}^3$. Recall that $\mathbb{H}^2$ sits inside $\mathbb{H}^3$, call this copy of $\mathbb{H}^2$ by the name P_1. We first map C_1 into P_1 by the identity map. If C_1 and C_2 intersect along the geodesic g, then let P_2 be a totally geodesic copy of $\mathbb{H}^2$ which intersects P_1 along g and makes an angle θ with P_1, and let h_θ map C_2 isometrically into P_2. The rest of the construction is forced by the fact that we require that h_θ be ρ_θ-equivariant. Every time we extend h_θ over a new copy C_N of C_1 or C_2, we assume that h_θ has already been defined on an adjacent copy C_A of C_2 or C_1 which lies in a hyperbolic plane P_A. One then finds a hyperbolic plane P_N meeting P_A along the geodesic $h_\theta(C_A \cap C_N)$ at an angle θ. We then map C_N isometrically into P_N. Continuing this process indefinitely produces h_θ. Notice that since we have taken care to alway bend in the same direction, that $h_\theta(\mathbb{H}^2)$ bounds a convex region in $\mathbb{H}^3$, if h_θ is an embedding.

Let L be the minimum distance between two copies of the pre-image of β^*. One may use hyperbolic geometry to show that there exists θ_L so that any piecewise geodesic in $\mathbb{H}^3$ so that each segment has length at least L and two adjacent segments make an angle at most θ, then the piecewise geodesic is a quasi-isometric embedding. It then immediately follows that if $|\theta| < \theta_L$, then h_θ is a quasi-isometric embedding, so ρ_θ is convex cocompact. (With a little more work one can show that h_θ is an embedding).

Armed with this intuition, we attempt to sketch a proof of the fact that the result of projective bending is always a Benoist representation. Our sketch should probably be viewed as an invitation to work out the proof for yourself and draw the pictures necessary to convince yourself.

Proposition 21.1. *If $\rho_t : \Gamma \to \mathsf{PSL}(n+1, \mathbb{R})$ is obtained as a result of projective bending of a discrete, faithful, cocompact representation $\rho_0 : \Gamma \to \mathsf{PSO}(n, 1)$, then ρ_t is a Benoist representation.*

We loosely follow the treatment in Goldman [104]. Goldman works in the case $n = 2$ and I encourage you to focus on this case first as well. (Our discussion will be somewhat awkward since we have not developed the natural language of developing maps.) We regard $\mathbb{H}^n$ as the unit disk D^n in an affine chart. Then, if $\pi : \mathbb{H}^n \to M$ is the covering map, $\pi^{-1}(X)$ is a discrete collection of totally geodesic hyperplanes in $\mathbb{H}^n$, which appear in our model as a collection of disks in projective hyperplanes which accumulate only at $\partial \mathbb{H}^n$. Each component of $\mathbb{H}^n - \pi^{-1}(X)$ is (the interior of) a component of either $\pi^{-1}(M_1)$ or $\pi^{-1}(M_2)$, each component has infinitely

many boundary components, each of which is a component of $\pi^{-1}(X)$, and no two components of $\pi^{-1}(M_i)$ intersect along a component of $\pi^{-1}(X)$.

We now describe the domain Ω_t so that $\rho_t(\Gamma)$ divides Ω_t. Let C_1 and C_2 be the components of $\pi^{-1}(M_1)$ and $\pi^{-1}(M_2)$ which intersect along the component P_0 of $\pi^{-1}(X)$ stabilized by Γ_X. Notice that $B_t(C_2)$ is entirely contained in the crescent $R_0 = B_t(R_0)$ bounded by the tangent planes to D^n at ∂P_0 which contains D^n. If we work in the traditional affine chart $A = \{\vec{x} \mid x_{n+1}\} = 1$, then P_0 is the unit disk in the x_2-$x_3 \cdots x_{n-1}$-x_n-plane and the crescent

$$R_0 = \{(x_1, x_2, \ldots, x_n) \mid (0, x_2, \ldots, x_n) \in P_0\}.$$

It follows that $C_1 \cup B_t(C_2)$ is properly convex since C_1 and $B_t(C_2)$ are convex and any line segment joining a point in C_1 to a point in $B_t(C_2)$ must pass through P_0. Moreover, C_1 is preserved by Γ_1 and C_2 is preserved by $B_t\Gamma_2 B_{-t}$.

Pictures would help a lot here

The rest of the construction is determined equivariantly. We must show that the procedure generates a properly convex domain tesselated by copies of C_1 and C_2. Each component P of ∂C_2 has the form $\gamma_2(P_0)$ for some $\gamma_2 \in \Gamma$. The equivariance of our construction forces us to attach $B_t\gamma_2 B_t^{-1}(C_1)$ to $C_1 \cup B_t(C_2)$ along $B_t(\gamma_2(P_0))$. Notice that $B_t\gamma_2 B_t^{-1}(C_1)$ is contained in the crescent R_P^t spanned by the tangent lines to $B_t(D^n)$ at $B_t(\partial P)$ since $C_1 \subset R_P^0$ and $B_t\gamma_2 B_t^{-1}(R_P^0) = R_P^t$. Let

$$X_1 = C_1 \cup B_t(C_2) \bigcup_{\gamma_2 \in \Gamma_2} B_t\gamma_2 B_t^{-1}(C_1)$$

be the result of attaching copies of C_1 along all components of $B_t(\partial C_2)$. Notice that if $P = \gamma_2(P_0)$ is a component of ∂C_2, then $B_t(\gamma_2(C_1))$ is a properly convex subset of R_P^t, X_1 is contained in R_P^T and P separates $B_t\gamma_2 B_t^{-1}(C_2)$ from the remainder of X_1. Therefore, any line segment joining a point in $B_t\gamma_2 B_t^{-1}(C_2)$ to a point in the remainder of X_1 must pass through P. So, the line segment joining any two points in X_1 may be divided up into segments joining copies of P which lie on the boundary of some translate of C_1 and C_2, and hence lies in X_1. So, X_1 is convex. Notice that if Q_t^P denote the portion of R_P^t which lies on the same side of P as $B_t(C_2)$, then $Q_t^P \subset R_0$. Moreover, if we choose some $\hat{\gamma}_2 \in \Gamma_2 - \{id\}$ and let $\hat{P}_2 = \hat{\gamma}_2(P_0)$, then $X_1 - C_1$ is contained in the portion of $R_{\hat{P}_2}^t$ which lies on the same side of P_0 as C_2, which is bounded. Therefore, X_1 is a bounded subset of A and hence properly convex.

Similarly, each component P of ∂C_1 has the form $\gamma_1(P_0)$ for some $\gamma_1 \in \Gamma$. The equivariance of our construction forces us to attach $\gamma_1(B_t(C_1))$ to $C_1 \cup B_t(C_2)$ along $\gamma_1(P_0)$. Notice that each $\gamma_1(B_t(C_1))$ is contained in the crescent $R_P^t = R_P^0$ spanned by the tangent lines to D^n at ∂P. Let

$$X_2 = C_1 \cup B_t(C_2) \bigcup_{\gamma_2 \in \Gamma_2} B_t\gamma_2 B_t^{-1}(C_1)) \bigcup_{\gamma_1 \in \Gamma_1} \gamma_1(B_t(C_1))$$

be the result of attaching copies of C_2 along all components of ∂C_1. We check, just as before, that X_2 is convex and if we choose $\hat{\gamma}_1 \in \Gamma_1 - \{id\}$ and let $\hat{P}_1 = \hat{\gamma}_1(P_0)$, then X_2 is contained in $R_{\hat{P}_1}^t \cap R_{\hat{P}_2}^t$, which is a bounded subset of A. Therefore X_2 is properly convex.

Notice that Ω_t is constructed by successively adding translates of C_1 and C_2. Each time we attach a new copy of C_i along $\alpha(P_0)$, for some $\alpha \in \rho_t(\Gamma)$, then there is a crescent $R_{\alpha(P_0)}^t$, bounded by tangent planes to $\alpha(\partial\Omega)$ passing through $\alpha(\partial P_0)$, which contains the previous copies of C_1 and C_2 and is separated in $R_{\alpha(P_0)}^t$ from the previous copies by $\alpha(P_0)$. Therefore,

the domain remains convex after adding the new copy of C_i. One also checks, via the nesting of the crescents, that Ω_t is contained in $R_{\hat{P}_1}^t \cup R_{\hat{P}_2}^t$, which is a bounded subset of A. Thus, Ω_t is properly convex. Since Ω_t is divided by the hyperbolic group $\rho_t(\Gamma)$ it is strictly convex, so ρ_t is a Benoist representation.

Remarks: (1) Johnson and Millson [124] introduced projective bending, which generalizes the bending construction of Thurston [202] in the hyperbolic setting (see also Apanasov [9] and Kouroniotis [142]). Goldman [104], in dimension $n = 2$, and Benoist [22] were the first to notice that projective bending always produces Benoist representations.

(2) Goldman used a more general version of Proposition 21.1, when $n = 2$, in his construction of a parametrization of the space of Benoist representations of $\pi_1(S)$ into $\mathsf{PSL}(3, \mathbb{R})$. This space is homeomorphic to $\mathbb{R}^{16g-16}$ and his coordinates are natural generalizations of the Fenchel-Nielsen coordinates. In this setting, Goldman refers to projective bending as bulging. Kapovich [129] and Marquis [157] produced much more general versions which work in all dimensions.

(3) Johnson and Millson [124] used projective bending to show that if Γ is a cocompact lattice in $\mathsf{PSO}(n, 1)$, then $\mathrm{Hom}(\Gamma, \mathsf{PSL}(n + 1, \mathbb{R})$ can have singularites at the identity when $n \geq 3$. The singularities they discover arise when $\mathbb{H}^3/\Gamma$ contains intersecting totally geodesic codimension one submanifolds.

(4) One may use a similar construction to define the twist flow on Teichmüller space. Suppose that C is a separating curve on a closed surface S determining the decomposition $\pi_1(S) = \Gamma_1 *_{\Gamma_C} \Gamma_2$ and that β generates Γ_C. If $\rho \in \mathcal{T}(S)$, we may assume that $\rho(\beta) = \begin{bmatrix} \lambda_\rho & 0 \\ 0 & \lambda_\rho \end{bmatrix}$ where $\lambda_\rho > 1$. We then define the time t twist of ρ about C as ρ_t where $\rho_t|_{C_1} = \rho$ and $\rho_t = B_t \rho B_{-t}$ where

$$B_t = \begin{bmatrix} e^{\frac{t}{\log \lambda_\rho}} & 0 \\ 0 & e^{\frac{-t}{\log \lambda_\rho}} \end{bmatrix}.$$

22. The Hilbert geodesic flow

We recall, from section 14, that the Hilbert metric on a properly convex domain is induced by a Finsler norm F_Ω^H. If $\vec{v} \in T_x\Omega$. let $p^+ = p_+(x, , \vec{v})$ and $p^- = p_-(x, \vec{v})$ are the endpoints of the line segment $\{x + t\vec{v} \mid t \in \mathbb{R}\} \cap \bar{\Omega}$, oriented so that $\vec{v}$ points toward p_+. Then

$$F_\Omega^H(x, \vec{v}) = \frac{|\vec{v}|}{2} \left(\frac{1}{|x - p_-|} + \frac{1}{|x - p_+|} \right)$$

If Ω is strictly convex, then $\partial\Omega$ is C^1, so F_Ω^H is also C^1.

We can then consider the unit tangent bundle $T^1\Omega$ associated to the Hilbert metric on Ω and define a geodesic flow $\{\phi_t\}_{t \in \mathbb{R}}$ on $T^1\Omega$. Explicitly, if $c_{x,\vec{v}} : \mathbb{R} \to \Omega$ is a unit speed geodesic so that $c_{x,\vec{v}}(0) = x$ and $c'_{x,\vec{v}}(0) = \vec{v}$, then

$$\phi_t(x, \vec{v}) = (c_{x,\vec{v}}(t), c'_{x,\vec{v}}(t)).$$

Following Benoist [20, Sec. 3.2.2], one may calculate that

$$\phi_t(x, \vec{v}) = \left(x + \left(\frac{e^{t/2} - 1}{\sigma^+(x, \vec{v})e^{t/2} + \sigma^-(x, \vec{v})} \right) \vec{v}, \left(\frac{e^{t/2}}{\left(\sigma^+(x, \vec{v})e^{t/2} + \sigma^-(x, \vec{v}) \right)^2} \right) \vec{v} \right)$$

where
$$\vec{v} = \sigma^+(x,\vec{v})\left(p^+(x,\vec{v}) - x\right) = \sigma^-(x,\vec{v})\left(x - p^-(x,\vec{v})\right).$$

In fact, if you would like a more complete and better written account of the Hilbert geodesic flow, I recommend you put down these lecture notes and read Benoist's beautiful paper [20].

We will assume for the remainder of the section that Ω is strictly convex and $\partial\Omega$ is C^1. We define the **stable leaf** through $(x,\vec{v})$ by

$$\mathcal{F}^s(x,\vec{v}) = \left\{(y,\vec{w}) \in T^1\Omega \mid p^+(y,\vec{w}) = p^+(x,\vec{v}) \text{ and } \overleftrightarrow{xy} \cap \overleftrightarrow{p^-(x,\vec{v})p^-(y,\vec{w})} \subset T_{p^+(x,\vec{v})}\Omega\right\} \cup \{(x,\vec{v})\}.$$

The key observation is then:

Lemma 22.1. (Benoist [20, Lemma 3.4]) *If Ω is strictly convex and $\partial\Omega$ is C^1, then*
$$\mathcal{F}^s(x,\vec{v}) = \{(y,\vec{w}) \in T^1\Omega \mid \lim_{t\to\infty} d_\Omega^H(\pi(\phi_t(y,\vec{w})), \pi(\phi_t(x,\vec{v})) = 0\}$$

where $\pi : T^1\Omega \to \Omega$ is the projection onto the first factor.

Proof. We first notice that $\lim_{t\to\infty} \pi(\phi_t(x,\vec{v})) = p^+(x,\vec{v})$ and $\lim_{t\to\infty} \pi(\phi_t(y,\vec{w})) = p^+(y,\vec{w})$, so if $p^+(y,\vec{w}) \neq p^+(x,\vec{v})$, then $\lim_{t\to\infty} d_\Omega^H(\pi(\phi_t(y,\vec{w})), \pi(\phi_t(x,\vec{v})) = \infty$.

If $p^+(y,\vec{w}) = p^+(x,\vec{v})$, then let $q = \overleftrightarrow{xy} \cap \overleftrightarrow{p^-(x,\vec{v})p^-(y,\vec{w})}$. Notice that q exists, since both $\overleftrightarrow{xy}$ and $\overleftrightarrow{p^-(x,\vec{v})p^-(y,\vec{w})}$ lie in the projective plane spanned by x, y and $p^+(x,\vec{v})$. Let $x_t = \pi(\phi_t(x,\vec{v}))$ and $y_t = \pi(\phi_t(y,\vec{w}))$ Then, since
$$[p^-(x,\vec{v}), x, x_t, p^+(x,\vec{v})] = [p^-(y,\vec{v}), y, y_t, p^+(x,\vec{v})] = e^{2t},$$

we see, by considering 4 lines through q as in the figure, that
$$y_t = \overleftrightarrow{qx_t} \cap \overleftrightarrow{yp^+(x,\vec{v})}.$$

(See Figure **??**.) Suppose that
$$\overline{w_t z_t} = \overleftrightarrow{x_t y_t} \cap \bar{\Omega}$$

and assume that the points occur in the order w_t, x_t, y_t, z_t.

Our result is then equivalent to the claim that
$$\lim_{t\to\infty} d_\Omega^H(\pi(\phi_t(y,\vec{w})), \pi(\phi_t(x,\vec{v})) = 0 \quad \text{if and only if} \quad if q \in T_{p^+(x,v)}\partial\Omega.$$

We first check that if q does not lie in $T_{p^+(x,v)}\partial\Omega$, then
$$\lim_{t\to\infty} \frac{|w_t - y_t|}{|w_t - x_t|} > 1,$$

which implies that $[w_t, x_t, y_t, z_t]$ does not converge to 1, so $d_\Omega^H(\pi(\phi_t(y,\vec{w})), \pi(\phi_t(x,\vec{v}))$ does not converge to 0 as $t \to \infty$ (and in fact is bounded below as $t \to \infty$.) We normalize so that in our affine chart $p^+(x,\vec{v}) = \vec{0}$, $x = (1,0,\dots,0)$ and y lies in the x_1-x_2 plane, so we may assume that $n = 2$. We may also assume that $T_{p^+(x,\vec{v})}\partial\Omega$ is the x_1-axis. Let θ be the angle between the line joining $p^+(x,v)$ to q and the line $T_{p^+(x,\vec{v})}\partial\Omega$ and let η be the angle between $\overline{p^+(x,\vec{v})x}$ and $\overline{p^+(x,\vec{v})y}$. (See Figure 22.) We will assume that q and y lie on the same side of the x_2-axis. If $\delta + t = |x_t - p^+(x,\vec{v})|$ and θ_t is the angle between $\overleftrightarrow{x_t q}$ and $T_{p^+(x,v)}\partial\Omega$,

$$|w_t - x_t| \sim \frac{\delta_t}{\sin\theta_t} \qquad \text{and} \qquad |x_t - y_t| = \delta_t \frac{\sin\eta}{\cos(\theta_t + \eta)},$$
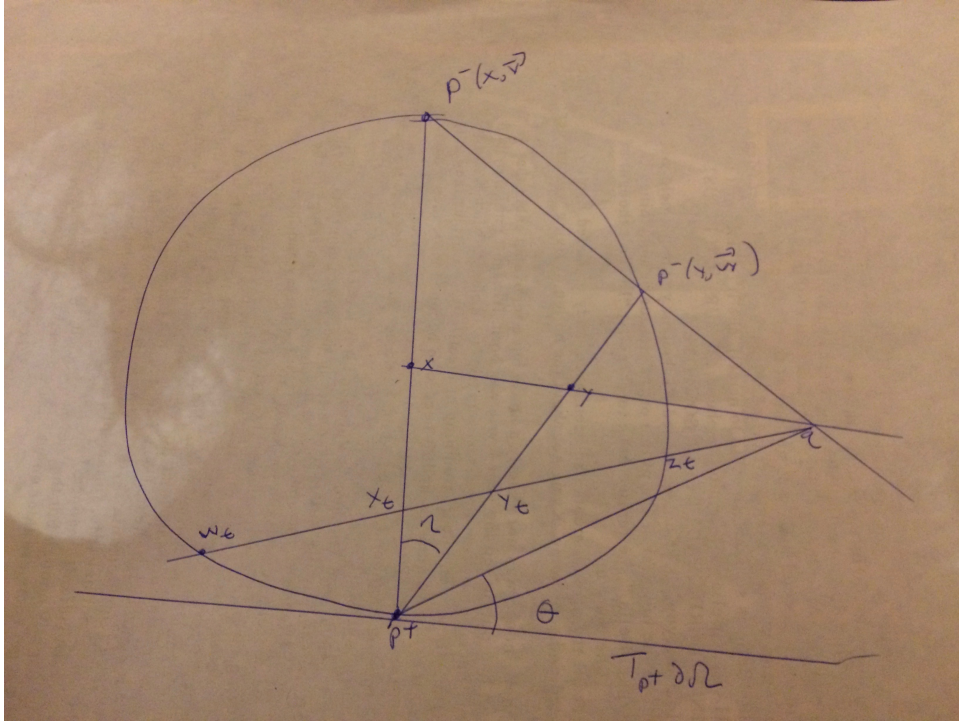
FIGURE 1.

so, since $\lim_{t\to\infty}\theta_t = \theta$ and $\lim_{t\to\infty}\delta_t$,

$$\lim_{t\to\infty}\frac{|w_t - x_t|}{|w_t - y_t|} = 1 + \frac{\sin\eta\sin\theta}{\cos(\theta+\eta)} > 1.$$

On the other hand, if $q \in T_{p^+(x,v)}\partial\Omega$, we may similarly check that $\lim_{t\to\infty} d_\Omega^H(\pi(\phi_t(y,\vec{w})), \pi(\phi_t(x,\vec{v})) = 0$, which completes the proof. $\qquad\square$

We call $\mathcal{F}^s(x,\vec{v})$ the stable leaf, since it is easy to check that the collection of stable leaves gives a foliation of $T^1\Omega$ by leaves of dimension $n-1$ which is invariant under the geodesic flow, called the **stable foliation**.

Consider the involution $\iota : T^1\Omega \to T^1\Omega$ given by $\iota(x,v) = (x,-v)$. Then $\iota \circ \phi_t = \phi_{-t} \circ \iota$ for all t. We may then define the **unstable leaf** $\mathcal{F}^u(x,\vec{v})$ through $(x,\vec{v})$ by setting $\mathcal{F}^u(x,\vec{v}) = \iota(\mathcal{F}^s_{\iota(x,\vec{v})})$. Then, by definition and Lemma 22.1, we see that $(y,\vec{w}) \in \mathcal{F}^u(x,\vec{v})$ if and only if $\lim_{t\to-\infty} d_\Omega^H(\pi(\phi_t(y,\vec{w})), \pi(\phi_t(x,\vec{v})) = 0$.

If Γ is torsion-free and $\rho : \Gamma \to \mathsf{PGL}(n+1,\mathbb{R})$ is a Benoist representation dividing a strictly convex domain Ω, then the geodesic flow on Ω descends to the geodesic flow $\{\hat\phi_t\}$ on $T^1 M_\rho$ where $M_\rho = \Omega/\rho(\Gamma)$. Let $\hat\pi : T^1 M_\rho \to M_\rho$ be the obvious projection map. The stable and unstable foliations descend to foliations $\widehat{\mathcal{F}}^s$ and $\widehat{\mathcal{F}}^s$ of $T^1(M_\rho)$, so that x and y lie in the same leaf of $\widehat{\mathcal{F}}^s$ if and only if $\lim_{t\to\infty} d(\hat\pi(\hat\phi_t(x)), \hat\pi(\hat\phi(y))) = 0$ and x and y lie in the same leaf of $\widehat{\mathcal{F}}^u$ if
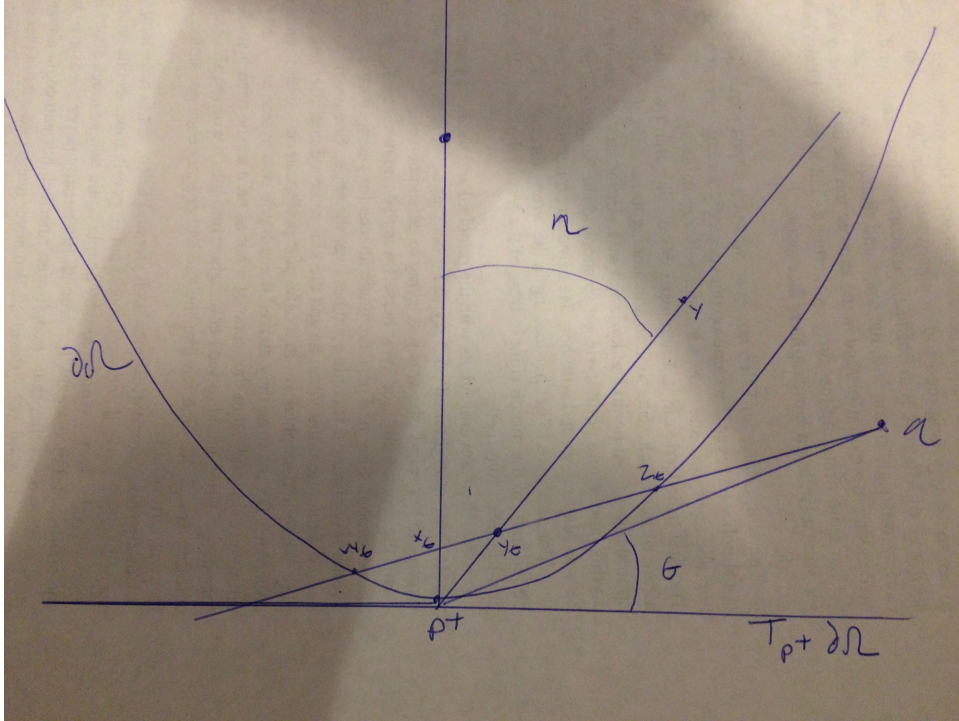
FIGURE 2.

and only if $\lim_{t \to -\infty} d(\hat{\pi}(\hat{\phi}_t(x)), \hat{\pi}(\hat{\phi}_t(y))) = 0$. Moreover, these foliations determine a splitting

$$T(T^1 M_\rho) = E_\rho^s \oplus F_\rho \oplus E_\rho^u$$

where $E_\rho^s|_z$ is the tangent plane to the leaf of $\widehat{\mathcal{F}}^s$ containing z, $F_\rho|_z$ is the tangent line to the flow line through z, and $E_\rho^s|_z$ is the tangent plane to the leaf of $\widehat{\mathcal{F}}^s$ containing z. This splitting lifts to a splitting

$$T(T^1 \Omega) = \tilde{E}_\rho^s \oplus \tilde{F}_\rho \oplus \tilde{E}_\rho^u.$$

We say that a flow $\{\phi_t\}$ on a closed manifold N is **Anosov** if there exists a flow-invariant splitting

$$TN = E^+ \oplus F \oplus E^-$$

such that F is tangent to the flow line at each point and there exists constants C and a so that if $v^+ \in E^+$ or $v^- \in E^-$ and $t > 0$, then

$$||D\phi_{-t}(v^+)|| \le Ce^{-at} \qquad \text{and} \qquad ||D\phi_t(v^-)|| \le Ce^{-at}.$$

(Notice that if a flow is Anosov with respect to one continuous family of norms on TN, then it is Anosov with respect to any other, since N is compact.)

We place an equivariant continuous Finsler metric on $T^1 \Omega$. If $w \in T_{(x,\vec{v})}\Omega$, we decompose

$$w = w^u + w^f + w^s \quad \text{where} \quad w^u \in \tilde{E}_\rho^u, \ w^f \in \tilde{F}_\rho, \ \text{and} \ w^u \in \tilde{E}_\rho^s$$

and let
$$||w|| = \left(||D\pi(w^s)||^2 + ||D\pi(w^f)||^2 + ||D\pi(w^u)||^2 \right)^{\frac{1}{2}}.$$

(Notice that this norm is $\rho(\Gamma)$-equivariant by construction.)

We now check that if $u^s \in E_\rho^s$, then $\lim_{t \to -\infty} ||D\phi_t(u^s)|| = 0$. (The moral of the story is that the map ϕ_t takes the horoball $\pi(\mathcal{F}^s(x, \vec{v}))$ to the horoball $\pi(\mathcal{F}^s(\phi_t(x, \vec{v})))$ and contracts uniformly.) Let $u^s = (x, \vec{u})$. We first normalize so that in our affine chart $p^+(x, \vec{v}) = \vec{0}$, $x = (1, 0, \ldots, 0)$, $p^-(x, \vec{v}) = (2, 0, \ldots, 0)$, and $T_{p^+(x, \vec{v})} \partial\Omega$ and $T_{p^-(x, \vec{v})} \partial\Omega$ are both parallel to the x_2-$x_3 \cdots$-x_n plane, $\vec{u}$ also lies in the x_2-$x_3 \cdots$-x_n plane, and we may normalize so that $\vec{u} = ce_2$ for some $c \neq 0$. One may then calculate that $D\phi_t(u^s) = |x_t| u^s$ where $\pi(\phi_t(x, \vec{v})) = x_t$ and so

$$||D\phi_t(u^s)|| = F_\Omega^H(x_t, |x_t|\vec{u}) = \frac{c|x_t|}{2} \left(\frac{1}{|x_t - w_t|} + \frac{1}{|x_t - z_t|} \right)$$

where $\overline{w_t z_t}$ is the intersection of the line through x_t in the direction $\vec{u} = ce_2$ with $\bar{\Omega}$. One then easily checks that $\lim_{t \to -\infty} ||D\phi_t(u^s)|| = 0$. (See Benoist [20, Section 3.2.6] or Crampon [75, Corollary 4.5] for details.) Similarly, if $w^u \in E_\rho^u$, then $\lim_{t \to \infty} ||D\phi_t(w^u)|| = 0$.

Since $T^1 M_\rho$ is closed, there exists $t_0 > 0$ such that for all $z \in T^1 M_\rho$, there exists $t_z \in [0, t_0]$ such that if $v^u \in E_\rho^u$, then $||D\phi_t(v^u)|| \leq \frac{1}{2}||v^u||$ and if $v^s \in E_\rho^s$, then $||D\phi_t(v^s)|| \leq \frac{1}{2}||v^s||$. Therefore, we may choose $a = \frac{\log 2}{t_0}$ and $C = \sup\{ \frac{||D\phi_t(v)||}{||v||} \mid v \in T(T^1 M_\rho),\ |t| \leq t_0,\ ||v|| \neq 0 \}$ and check that the splitting $T(T^1 M_\rho) = E_\rho^s \oplus F \oplus E_\rho^u$ is an Anosov splitting with constants C and a.

Theorem 22.2. (Benoist [20]) *If Γ is torsion-free and $\rho : \Gamma \to \mathsf{PL}(n + 1, \mathbb{R})$ is a Benoist representation, then its geodesic flow is Anosov.*

This generalizes the classical fact that the geodesic flow of a closed hyperbolic manifold is Anosov.

Another natural part of the picture here, which we will not elaborate on, is the **Busemann function** of Ω (see Crampon [75, Section 4.1]). Suppose that Ω is strictly convex and $\partial\Omega$ is C_1. If $z \in \partial\Omega$ and $x, y \in \Omega$, let $w = \overrightarrow{xy} \cap \partial\Omega$ and q be the point of intersection of $T_w \partial\Omega$ and $T_z \partial\Omega$. Then $B_z(x, y)$ is the (signed) distance between y and $r = \overrightarrow{yz} \cap \overrightarrow{qx}$ where the sign is positive if y lies between r and z and negative otherwise. If $(x, v) \in T^1(\Omega)$, then we define the generalized **horoball**

$$\mathcal{H}_{(x, \vec{v})} = \{y \in \Omega \mid B_{p^+(x, \vec{v})}(x, y) = 0\}$$

and notice that

$$\mathcal{F}^u(x, \vec{v}) = \{(y, \vec{w}) \mid y \in \mathcal{H}_{(x, \vec{v})} \text{ and } p^+(x, \vec{v}) = p^+(y, \vec{w})\}$$

and

$$\mathcal{F}^s(x, \vec{v}) = \{(y, \vec{w}) \mid y \in \mathcal{H}_{(x, -\vec{v})} \text{ and } p^-(x, \vec{v}) = p^-(y, \vec{w})\}.$$

Moreover, one may check that

$$B_z(x, y) = \lim_{z_n \to z} d(x, z_n) - d(y, z_n)$$

where $\{z_n\}$ is any sequence in Ω converging to z.

Remark: Benoist [20] further proves that if Γ divides a properly convex domain Ω, then Ω is strictly convex if and only if the geodesic flow on Ω/Γ is Anosov.

23. **Further topics: Convex projective manifolds**

I went walking in the wasted city
Started thinking about entropy
Smelled the wind from the ruined river
Went home to watch TV
——————Warren Zevon [228]

The geodesic flow and entropy. Benoist [20] further proves that if Ω is strictly convex and divisible, then $\partial\Omega$ is $C^{1+\alpha}$ for some $\alpha > 0$. Recall that a function $f : U \to \mathbb{R}$ is $C^{1+\alpha}$ on an open subset U of $\mathbb{R}^d$ if it is C^1 and there exists $C > 0$ so that

$$|f(y) - f(x) - Df_x(y - x)| \le C|x - y|^{1+\alpha}$$

for all $x, y \in U$. We say that $\partial\Omega$ is $C^{1+\alpha}$ if it is locally the graph of a $C^{1+\alpha}$ function. It follows that if ρ is a Benoist representation, then the geodesic flow on $T^1 M_\rho$ is $C^{1+\alpha}$ for some $\alpha > 0$.

Since $\{(\gamma^+, \gamma^-) \mid \gamma \in \Gamma \text{ infinite order}\}$ is dense in $\partial\Gamma \times \partial\Gamma$ whenever Γ is a hyperbolic group (see Benakli-Kapovich [126]), $\{(\xi_\rho(\gamma^+), \xi_\rho(\gamma^-))\}$ is dense in $\partial\Omega \times \partial\Omega$ if ρ is a Benoist representation dividing Ω. It follows that closed geodesics are dense in $T^1 M_\rho$. Anosov [8] showed, as a consequence of the Anosov Closing Lemma, that an Anosov flow is topologically transitive if periodic orbits are dense. Recall that a flow is topologically transitive, if U and V are two open subsets of $T^1\Omega$, then $\phi_t(U) \cap V$ is non-empty for some t. Benoist [20] further shows that the geodesic flow is topologically mixing, i.e. if U and V are open subsets of $T^1 M_\rho$, then there exists T so that if $t \ge T$, then $\phi_t(U) \cap V$ is non-empty. (Benoist uses his classification of the Zariski closures of Benoist representations and the main result of Benoist [17] to show that the subgroup of $\mathbb{R}_{>0}$ generated by lengths of closed geodesics is dense. The fact that $T^1 M_\rho$ is topologically mixing then follows from standard results, see for example Hasselblatt-Katok [112, Exercise 18.3.4].)

Once we know that the geodesic flow is $C^{1+\alpha}$ and topologically transitive, we may start applying the powerful methods of the Thermodynamic formalism. In particular, one may define the **topological entropy** of a Benoist representation to be the exponential growth rate of the number of closed orbits of the geodesic flow on $T^1 M_\rho$ with period at most T, i.e.

$$h(\rho) = \lim_{T\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] \mid \ell_\rho(\gamma) \le T\}}{T}$$

where $[\Gamma]$ is the collection of conjugacy classes of elements of Γ. Moreover, Benoist [20] shows that

$$\lim_{T\to\infty} \frac{h(\rho)T \left(\#\{[\gamma] \in [\Gamma] \mid \ell_\rho(\gamma) \le T\}\right)}{e^{h(\rho)T}} = 1.$$

(See Sambarino [185, 186] for generalizations of Benoist's result.)

Crampon [74] proved the following remarkable rigidity result.

Theorem 23.1. (Crampon [74]) *If $\rho : \Gamma \to \mathsf{PGL}(n + 1, \mathbb{R})$ is a Benoist representation, then*

$$h(\rho) \le n - 1$$

with equality if and only if $\rho(\Gamma)$ is conjugate into $\mathsf{PO}(n, 1)$.

Crampon [74] also proves that $h(\rho)$ is the exponential growth rate of the volume of balls of radius T on the universal cover of M_ρ, so the same result holds for volume growth entropy. Potrie and Sambarino [179] give a very different proof of Theorem 23.1.

Crampon [75] notes that entropy varies continuously on $\mathsf{Ben}(\Gamma, n)$. Pollicott and Sharp [177] show that entropy varies analytically on $\mathsf{Ben}(\Gamma, 2)$. Bridgeman, Canary, Labourie and Sambarino [45] show that entropy varies analytically over the smooth points of $\mathsf{Ben}(\Gamma, n)$ for all n. (The key new tool here is a proof that the limit map varies analytically over smooth points of $\mathsf{Ben}(\Gamma, n)$.) If $n = 2$, Nie [170] and Zhang [230] show that the entropy achieves every value between 0 and 1 on $\mathsf{Ben}(\Gamma, 2)$.

Kim [139] and Cooper-Delp [68] showed that if $\rho, \sigma \in \mathsf{Ben}(\Gamma, n)$, then ρ is conjugate to either σ or σ^* if and only if $\ell_\sigma(\gamma) = \ell_\rho(\gamma)$ for all $\gamma \in \Gamma$. Bridgeman, Canary, Labourie and Sambarino [45] show that ρ is conjugate to σ if and only if $|\lambda_1(\sigma(\gamma))| = |\lambda_1(\rho(\gamma))|$ for all $\gamma \in \Gamma$. Bridgeman, Canary and Labourie [47] show that if $n = 2$ and Γ is a closed surface group of genus at least 3, then it suffices to consider elements representing simple closed curves in both of the previous results. We will discuss rigidity properties of Benoist and Anosov representations more fully in Section 47.

A characterization of Benoist representations. Canary and Tsouvalas [60] give a characterization of Benoist representations when $n \geq 3$. Recall that Γ has **cohomological dimension** m, if m is the minimal dimension so that if R is any $\mathbb{Z}\Gamma$-module, then $H^r(\Gamma, R) = 0$ if $r > m$. For example, if X is a closed n-manifold with contractible universal cover, then $\pi_1(M)$ has cohomological dimension n.

Theorem 23.2. (Canary-Tsouvalas [60]) *Suppose that $n \geq 3$, Γ is a torsion-free hyperbolic group of cohomological dimension n, and $\rho : \Gamma \to \mathsf{PSL}(n + 1, \mathbb{R})$ is a representation. If there exists a continuous, non-constant ρ-equivariant map $\xi : \partial\Gamma \to \mathbb{RP}^n$, then ρ is a Benoist representation.*

The direct product of a Fuchsian representation into $\mathsf{SL}(2, \mathbb{R})$ and a trivial one-dimensional representation, provides a counterexample to this characterization when $n = 2$.

We may combine Theorem 23.2 with Theorems 35.1 and 36.1 to obtain the following partial converses to Corollaries 19.2 and 19.3.

Corollary 23.3. *Suppose that Γ is a torsion-free hyperbolic group of cohomological dimension n and $\rho : \Gamma \to \mathsf{SL}(n + 1, \mathbb{R})$ is a representation. If* **either**

(1) *There exists J and B so that*

$$J||\gamma|| + B \geq \log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_{n+1}(\rho(\gamma))|}\right) \geq \frac{1}{J}||\gamma|| - B$$

for all $\gamma \in \Gamma$, **or**
(2) *There exists L and D so that*

$$Ld(1, \gamma) \geq \log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_{n+1}(\rho(\gamma))}\right) \geq \frac{1}{L}d(1, \gamma) - D$$

for all $\gamma \in \Gamma$,

then ρ is a Benoist representation.

Dynamics on Benoist components. Fricke's Theorem generalizes to the setting of Benoist components.

Theorem 23.4. *If* $\mathrm{Ben}(\Gamma, n)$ *is non-empty, then* $\mathrm{Out}(\Gamma)$ *acts properly discontinuously on* $\mathrm{Ben}(\Gamma, n)/\mathsf{PGL}(n+1, \mathbb{R})$.

Theorem 23.4 is a special case of a result of Guichard and Wienhard [109, Theorem 5.4] (see also [55, Theorem 6.2]) concerning actions of outer automorphism groups on deformation spaces of Anosov representations. We will discuss Guichard and Wienhard's result more fully in Section 5, see Theorem 28.3.

If $n \geq 3$, then $\mathrm{Out}(\Gamma)$ will be finite, so this result is only relevant when $n = 2$, where it was first proved by Goldman [104], see also Labourie [145, Theorem 1.0.2].

Finite volume strictly convex projective manifolds. Cooper-Long -Tillmann [70] made an extensive study of finite volume convex projective manifolds. In particular, they extend Benoist's characterization of strictly convex closed projective manifolds, see Theorem 16.1, to this setting.

Theorem 23.5. (Cooper-Long-Tillmann [70, Theorem 0.15]) *Suppose that* $\Omega \subset \mathbb{RP}^n$ *is properly convex,* $\Gamma \subset \mathrm{Aut}(\Gamma)$ *is discrete and torsion-free, and* $N = \Omega/\Gamma$ *has finite volume and is homeomorphic to the interior of a compact manifold* M *Then the following are equivalent:*

(1) Ω *is strictly convex.*
(2) $\partial\Omega$ *is* C^1
(3) Γ *is hyperbolic relative to the subgroups associated to boundary components of* M.

Among the structural tools they develop in this setting are an analogue of the Margulis Lemma ([70, Theorem 0.1], see also Crampon-Marquis [76]), a thick-thin decomposition ([70, Theorem 0.2]), see Choi [65] for the surface case) and a proof that cusps of finite volume strictly convex manifolds are projectively equivalent to cusps of finite volume hyperbolic manifolds ([70, Theorem 0.5]). In a sequel paper [71], they prove an analogue of Koszul's open-ness theorem for strictly convex projective manifolds of finite volume. Marquis [155, 156] had earlier studied deformation spaces of finite area properly convex surfaces.

Barthelmé, Marquis and Zimmer proved a generalization of Crampon's rigidity theorem for finite volume properly convex projective manifolds. We recall that the **volume growth entropy** of a properly convex manifold $N = \Omega/\Gamma$ is given by

$$h_{\mathrm{vol}}(N) = \lim_{T \to \infty} \frac{\log\big(\mathrm{vol}(B(x_0, T))\big)}{T}$$

where $\mathrm{vol}(B(x_0, T))$ is the (Hilbert) volume of the (metric) ball of radius T about a fixed point $x_0 \in \Omega$.

Theorem 23.6. (Barthelmé-Marquis-Zimmer [13]) *Suppose that* $\Omega \subset \mathbb{RP}^n$ *is properly convex,* $\Gamma \subset \mathrm{Aut}(\Gamma)$ *is discrete and torsion-free, and* $N = \Omega/\Gamma$ *has finite volume. Then*

$$h_{\mathrm{vol}}(N) \leq n - 1$$

with equality if and only if Γ *is conjugate into* $\mathsf{O}_0(n, 1)$.

Manning [151] showed that if N is a compact Riemannian manifold with non-positive sectional curvature, then the topological entropy of its geodesic flow and its volume growth entropy coincide. Crampon [74] established the same result in the setting of closed strictly convex projective manifolds and Crampon-Marquis [78] extended it to the setting of finitely volume strictly convex projective manifolds. It is unknown whether it extends to finite volume properly convex projective manifolds.

Adeboye, Bray and Constantine [4, 43] have also studied the relationship between volume and entropy for finite volume striclty convex projective manifolds.

Closed properly convex projective manifolds. Benoist actually develops much of his theory in the setting of discrete cocompact actions of groups of projective automorphisms of a properly convex domain Ω. A key role in this enlarged theory are **properly embedded triangles**, i.e. Euclidean triangles whose edges are in $\partial\Omega$ and whose interior lies in Ω. In dimension 3, Benoist [23] showed that the JSJ decomposition of a properly convex projective manifold is realized by a disjoint collection of properly embedded triangles. Benoist also exhibited examples but it is a subject of current interest to ascertain which irreducible closed 3-manifolds admit properly convex projective structures, see, for example, Ballas-Danciger-Lee [10]. Bobb [30] and Islam-Zimmer [120, 121] have successfully generalized portions of Benoist's 3-dimensional results to higher dimensions. Bray [41, 42] has studied dynamics on closed properly convex projective 3-manifolds, proving that their geodesic flows are topologically mixing and are ergodic with respect to their Bowen-Margulis measures.

Part 5. Anosov representations: Basic properties

> *You can have your buildings and your heavy 'rithmetic*
> *I don't need no crowded streets or city slicker tricks*
> *I just need to be someplace where I can move around*
> *Look down at my toes and I can still see the ground*
> ————Rick Hirsch [214]

In this chapter we develop the basic theory of Anosov representations into $\mathsf{SL}(d,\mathbb{R})$, which we will view as a common generalization of convex cocompact representations into rank one Lie groups and Benoist representations. Anosov representations are discrete, faithful representations and their orbit maps into the associated symmetric spaces

$$X_d = \mathsf{SL}(d,\mathbb{R})/\mathsf{SO}(d)$$

are quasi-isometric embeddings. Moreover, they are stable in the sense that any small deformation of an Anosov representation remains an Anosov representation. They arose in Labourie's study [143] of Hitchin representations, but are now an organizing principle for the field of Higher Teichmüller theory.

Anosov representations have many flavors. Let $1 \le k \le \frac{d}{2}$ be an integer. One simple definition, which follows work of Kapovich-Leeb-Porti [133] and Bochi-Potrie-Sambarino [32] is that $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is P_k-**Anosov** if there exists K and C so that if $\gamma \in \Gamma$, then

$$Kd(1,\gamma) \ge \log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \ge \frac{1}{K}d(1,\gamma) - C.$$

This definition immediately implies that ρ is discrete and faithful, and its associated orbit map is a quasi-isometric embedding. Moreover, one can show that it implies that Γ is a hyperbolic group. However, in our discussion we will begin with Labourie's dynamical definition which allows us more quickly to establish stability. Notice that ρ is P_k-Anosov if and only if the k-fold exterior product $\Lambda^k\rho : \Gamma \to \mathsf{SL}(\Lambda^k\mathbb{R}^d)$ is P_1-Anosov, so the P_1-Anosov representations are the most general class of Anosov representations. More generally, associated to any parabolic subgroup P of a semi-simple Lie group G, there is a notion of a P-Anosov representation $\rho : \Gamma \to \mathsf{G}$ and there is an irreducible representation $\tau : \mathsf{G} \to \mathsf{SL}(d,\mathbb{R})$ (for some d) so that $\rho : \Gamma \to \mathsf{G}$ is P-Anosov if and only if $\tau \circ \rho$ is P_1-Anosov.

One might naturally ask why one doesn't simply generalize one of the definitions of a convex cococompact representation in rank one Lie groups into the higher rank setting. Notice that quasi-isometric embedding into non-positively curved spaces are not stable, e.g. a line in the plane is a limit of circles of larger and larger radius. More convincingly, Guichard [107, Appendix A] exhibits a representation of a free group F_2 of rank two into $\mathsf{SL}(4,\mathbb{R})$ whose orbit map is a quasi-isometric embedding, but is approximated by representations which aren't even discrete. (We will discuss a variation of Guichard's example at the end of Section 27.) So, the class of representations of a hyperbolic group into $\mathsf{SL}(d,\mathbb{R})$ whose orbit maps are quasi-isometric embeddings is not stable.

One might instead consider faithful representations whose images act properly discontinuously and cocompactly on convex subsets of X_d. However, Kleiner-Leeb [140] and Quint [181] show that any Zariski dense subgroup of $\mathsf{SL}(d,\mathbb{R})$ which acts properly discontinuously and cocompactly on a convex subset of X_d is a lattice, so there will not be many examples. For

example, no Benoist representation, which does not have image conjugate into $\mathsf{PO}(d-1,1)$, acts properly discontinuously and cocompactly on a convex subset of X_d. Moreover, any Benoist representation with image in $\mathsf{PO}(d-1,1)$ does act properly discontinuously and cocompactly on a convex subset of X_d (which is the totally geodesic copy of $\mathbb{H}^{d-1}$ in X_d given by $\mathsf{SO}(d-1,1)/(\mathsf{SO}(d) \cap \mathsf{SO}(d-1,1))$), so the phenomenon of projective bending demonstrates that this notion of convex cocompactness would also fail to be stable. In Chapter 7, we will discuss a notion of convex cocompactness for actions on projective spaces and its relationship with Anosov representations.

24. Geodesic flows and flat bundles

We have already studied the geodesic flow on $T^1\mathbb{H}^n$, since we may regard $\mathbb{H}^n$ as a divisible strictly convex domain in $\mathbb{RP}^n$ with its associated Hilbert metric. In particular, the geodesic flow on $T^1\mathbb{H}^n$ is Anosov and the (quotient) geodesic flow on the unit tangent bundle of any closed hyperbolic manifold is Anosov and topologically transitive.

We may identify $T^1\mathbb{H}^n$ with $(\partial\mathbb{H}^n \times \partial\mathbb{H}^n - \Delta) \times \mathbb{R}$ where $\Delta = \{(z,z) \mid z \in \partial\mathbb{H}^n\}$ is the diagonal in $\mathbb{H}^n \times \mathbb{H}^n$. First notice that a point $(x, \vec{v}) \in T^1\mathbb{H}^n$ determines a point on an oriented geodesic, by considering the oriented geodesic through x in the direction $\vec{v}$. The space of oriented geodesics is identified with $\partial\mathbb{H}^n \times \partial\mathbb{H}^n - \Delta$ by identifying a geodesic with its forward and backward endpoints. We then identify each oriented geodesic with $\mathbb{R}$ by an orientation-preserving isometry which takes 0 to the point on the geodesic closest to $\vec{0}$ (in the ball model). The resulting parametrization is known as the **Hopf parametrization**. Notice that in these coordinates the geodesic flow has the simple form

$$\phi_t(x,y,s) = (x,y,s+t)$$

for all t. We will choose the convention that x is the forward endpoint and that y is the backward endpoint. Notice that if $M = \mathbb{H}^n/\Gamma$ is a closed hyperbolic n-manifold, Γ acts on $T^1\mathbb{H}^n$ and $T^1M = T^1\mathbb{H}^n/\Gamma$.

If M is a closed negatively curved manifold with universal cover $\widetilde{M}$, then $\widetilde{M}$ is Gromov hyperbolic and one obtains a Hopf parameterization $T^1\widetilde{M}$ as $(\partial\widetilde{M} \times \partial\widetilde{M} - \Delta) \times \mathbb{R}$. Anosov [8] showed that the geodesic flow on T^1M is topologically transitive and Anosov.

If Γ is a torsion-free convex cocompact subgroup of $\mathsf{PO}(n,1)$ then we can consider the flow-invariant subset

$$U(\Gamma) = \{(z,w,s) \in T^1\mathbb{H}^n \mid z \neq w \in \Lambda(\Gamma) \ s \in \mathbb{R}\} \cong (\partial\Gamma \times \partial\Gamma - \Delta) \times \mathbb{R}$$

of $T^1\mathbb{H}^n$. Then the geodesic flow $\{\hat{\phi}_t\}$ on $T^1\mathbb{H}^n/\Gamma$ descends to a flow on the compact space

$$\widehat{U}(\Gamma) = U(\Gamma)/\Gamma$$

and we may regard $\widehat{U}(\Gamma)$ as the geodesic flow of the group Γ.

More generally, Gromov [106] showed that every hyperbolic group Γ has an associated geodesic flow $\widehat{U}(\Gamma)$. We first consider the space $\mathcal{G}(\Gamma)$ of isometric embeddings $c : \mathbb{R} \to C_\Gamma$ of $\mathbb{R}$ into the Cayley graph C_Γ. (Notice that the space of isometric embeddings of $\mathbb{R}$ into $\mathbb{H}^n$ is simply $T^1\mathbb{H}^n$, so this is a natural generalization to consider). If there is a unique geodesic in C_Γ joining any two points in $\partial\Gamma$, for example if Γ is a free group, then $\mathcal{G}(\Gamma)$ is identified with $(\partial\Gamma \times \partial\Gamma - \Delta) \times \mathbb{R})$ and we can define $\widehat{U}(\Gamma) = \mathcal{G}(\Gamma)/\Gamma$. However, in general this will

not be the case, so one must "collapse" $\mathcal{G}(\Gamma)$ by identifying geodesics with the same endpoints to (somehow) obtain a flow space $U(\Gamma)$ which admits an action by Γ, which commutes with the flow, and then define $\widehat{U}(\Gamma) = U(\Gamma)/\Gamma$. The details of this construction are worked out by Champetier [64] and Mineyev [166].

Theorem 24.1. (Gromov, Champetier, Mineyev) *If Γ is a hyperbolic group, then there exists a complete metric on $U(\Gamma) = (\partial\Gamma \times \partial\Gamma - \Delta) \times \mathbb{R}$, a flow $\{\phi_t : U(\Gamma) \to U(\Gamma)\}_{t\in\mathbb{R}}$, and a properly discontinuous cocompact action of Γ on $U(\Gamma)$ by isometries such that*

- (1) *There exists K and C so that, for all $z \ne w \in \partial\Gamma$, the map $t \to (z, w, t)$ is a (K, C)-quasi-isometric embedding.*
- (2) *The flow $\phi_t(z, w, s) = (z, w, s + t)$ for all $z \ne w \in \partial\Gamma$ and $s, t \in \mathbb{R}$.*
- (3) *Each ϕ_t is biLipschitz.*
- (4) *The action of Γ commutes with ϕ_t for all t and $\{\phi_t\}$ descends to a topologically transitive flow $\{\hat{\phi}_t\}$ on $\widehat{U}(\Gamma) = U(\Gamma)/\Gamma$.*
- (5) *For all $\gamma \in \Gamma$, there exists a function $c_\gamma : \partial\Gamma \times \partial\Gamma - \Delta \to \mathbb{R}$ so that $\gamma(x, y, t) = (\gamma(x), \gamma(y), c_\gamma(x, y) + t)$.*

For our purposes, it is easiest to assume that Γ is just the fundamental group of a closed negatively curved manifold M and $\widehat{U}(\Gamma) = T^1 M$. This will not impact most of our proofs. The one crucial difference is that it is not known whether geodesic flows of general hyperbolic groups are metric Anosov. However, it is known that geodesic flows of groups which admit Anosov representations are metric Anosov (see [45, Section 5] and [69]).

We assume throughout the next several sections that Γ **is a hyperbolic group.** I encourage you to also assume that Γ is torsion-free.

There are various bundles over $\widehat{U}(\Gamma)$ which are naturally associated with linear representations. The most classical is the flat vector bundle $\widehat{E}_\rho$ over $\widehat{U}(\Gamma)$ with fiber $\mathbb{R}^d$ determined by a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$. We first consider the flat vector bundle $E(\Gamma)$ over $U(\Gamma)$ given by

$$E(\Gamma) = U(\Gamma) \times \mathbb{R}^d.$$

Notice that the flow $\{\phi_t\}$ lifts to an "obvious" flow on $E(\Gamma)$, given by

$$\psi_t((z, w, s), v) = ((z, w, s + t), v) \qquad \text{or} \qquad \psi_t(Z, v) = (\phi_t(Z), v)$$

for all $z \ne w \in \partial\Gamma$ and $s, t \in \mathbb{R}$ or $Z \in U(\Gamma)$. This flow is (confusingly) called the flow **parallel to the flat connection**.

Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation. Let Γ act on $E(\Gamma)$ as the group of covering transformations of $U(\Gamma)$ in the first factor and as $\rho(\Gamma)$ in the second factor, i.e. if $\gamma \in \Gamma$, then $\gamma(Z, v) = (\gamma(Z), \rho(\gamma)(v))$ for all $(Z, v) \in E(\Gamma)$. Then

$$\widehat{E}_\rho = E(\Gamma)/\Gamma$$

is the **flat vector bundle** associated to ρ. Notice that the action of Γ on $E(\Gamma)$ commutes with the flow $\{\psi_t\}$, so $\{\psi_t\}$ descends to a flow $\{\hat{\psi}_t\}$ on $\widehat{E}_\rho$.

Suppose that $1 \le k \le \frac{d}{2}$ is a positive integer and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation. We say that a pair

$$\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d) \qquad \text{and} \qquad \theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$$

of continuous, ρ-equivariant maps are P_k**-transverse limit maps** for ρ if

$$\xi_\rho(x) \oplus \theta_\rho(y) = \mathbb{R}^d$$

whenever $x \neq y \in \partial\Gamma$. (Notice that the limit maps of a Benoist representation are a pair of P_1-transverse limit maps.)

A pair of transverse P_k-limit maps gives rise to an Γ-equivariant, flow-invariant splitting

$$E(\Gamma) = \Xi \oplus \Theta$$

where $\Xi|_{(x,y,s)} = \xi_\rho(x)$ and $\Theta_\rho|_{(x,y,s)} = \theta_\rho(y)$. Hence, the splitting descends to a flow-invariant splitting

$$\widehat{E}_\rho = \widehat{\Xi} \oplus \widehat{\Theta}.$$

If $1 \leq k \leq \frac{d}{2}$ is a positive integer, we may perform this entire construction replacing $\mathbb{R}^d$ with $\mathrm{Gr}_k(\mathbb{R}^d) \times \mathrm{Gr}_{d-k}(\mathbb{R}^d)$. Let

$$E^k(\Gamma) = U(\Gamma) \times \mathrm{Gr}_k(\mathbb{R}^d) \times \mathrm{Gr}_{d-k}(\mathbb{R}^d)$$

and notice that the geodesic flow again lifts to an obvious flow ψ_t^k on $E^k(\Gamma)$. If $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a representation, we let Γ act on the first factor by the group of covering transformations of $U(\Gamma)$ over $\widehat{U}(\Gamma)$ and let it act on the second and third factors by the action of $\rho(\Gamma)$ on the Grassmanian. We then form the k**-Grassmannian bundle** associated to ρ by letting

$$\widehat{E}_\rho^k = E^k(\Gamma)/\Gamma.$$

Notice that the action of Γ commutes with the flow $\{\psi_t^k\}$, so $\{\psi_t^k\}$ descends to a flow $\{\hat{\psi}_t^k\}$ on $\widehat{E}_\rho^k$.

We can then define the vector bundle $V^k(\Gamma)$ over $E^k(\Gamma)$ so that $V^k(\Gamma)|_{(Z,P,Q)} = T_P\mathrm{Gr}_k(\mathbb{R}^d)$ and the vector bundle $V^{d-k}(\Gamma)$ over $E^k(\Gamma)$ with fiber $V^{d-k}(\Gamma)|_{(Z,P,Q)} = T_Q\mathrm{Gr}_{d-k}(\mathbb{R}^d)$. The flow ψ_t^k lifts to flows η_t^k and η_t^{d-k} on $V^k(\Gamma)$ and $V^{d-k}(\Gamma)$. The action of Γ extends to actions on $V^k(\Gamma)$ and $V^{d-k}(\Gamma)$ which commute with the flows, so we get flows on the quotient vector bundles

$$\widehat{V}_\rho^k = V^k(\Gamma)/\Gamma \qquad \text{and} \qquad \widehat{V}_\rho^{d-k} = V^{d-k}(\Gamma)/\Gamma$$

over $\widehat{E}_\rho^k$.

A pair of (ξ_ρ, θ_ρ) of transverse P_k-limit maps for a representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ gives rise to a Γ-equivariant flow-equivariant section $\sigma : U(\Gamma) \to E^k(\Gamma)$ given by

$$\sigma(x,y,s) = ((x,y,s), \xi_\rho(x), \theta_\rho(y))$$

which descends to a flow-equivariant section $\hat{\sigma} : \widehat{U}(\Gamma) \to \widehat{E}_\rho^k$. We can then consider the natural pull-back vector bundles $\hat{\sigma}^*(\widehat{V}_\rho^k)$ and $\hat{\sigma}^*(\widehat{V}_\rho^{d-k})$ over $\widehat{U}(\Gamma)$, each of which admits a flow which "lifts" the flow on $\widehat{U}(\Gamma)$.

Notice that if $P \in \mathrm{Gr}_k(\mathbb{R}^d)$ and $Q \in \mathrm{Gr}^{d-k}(\mathbb{R}^d)$ are transverse, i.e. $P \oplus Q = \mathbb{R}^d$, then we may identify $T_P\mathrm{Gr}_k(\mathbb{R}^d)$ with $\mathrm{Hom}(P,Q)$. Basically, $\phi \in \mathrm{Hom}(P,Q)$ is identified with the tangent vector $c'(0)$ to the path $c(t) = \mathrm{graph}(t\phi)$ for all $t \in \mathbb{R}$, where $\mathrm{graph}(t\phi)$ is the k-plane $\{v + t\phi(v) \mid v \in P\}$. Similarly, $T_Q\mathrm{Gr}_{d-k}(\mathbb{R}^d)$ may be identified with $\mathrm{Hom}(Q,P)$. Therefore, if (ξ_ρ, θ_ρ) is a pair of transverse k-limit maps for a representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$, then we

may identify $\hat{\sigma}^*(V_\rho^k)$ with the bundle $\mathrm{Hom}(\widehat{\Xi}, \widehat{\Theta})$ and the bundle $\sigma^*(V_\rho^{d-k})$ with the bundle $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$.

Remark: If we let P_k^+ be the stabilizer of the standard k-plane $<e_1, \ldots, e_k>$ and let P_k^- be the stabilizer of the standard complementary $(d-k)$-plane $<e_{k+1}, \ldots, e_d>$, then P_k^+ and P_k^- are opposite parabolic subgroups of $\mathrm{SL}(d, \mathbb{R})$. Notice that

$$\mathrm{Gr}_k(\mathbb{R}^d) = \mathrm{SL}(d, \mathbb{R})/P_k^+ \qquad \text{and} \qquad \mathrm{Gr}_{d-k}(\mathbb{R}^d) = \mathrm{SL}(d, \mathbb{R})/P_k^-.$$

If we let $\left(\mathrm{Gr}_k(\mathbb{R}^d) \times \mathrm{Gr}_{d-k}(\mathbb{R}^d)\right)^T$ denote the collection of transverse pairs of k-planes and $(d-k)$-planes, then $\left(\mathrm{Gr}_k(\mathbb{R}^d) \times \mathrm{Gr}_{d-k}(\mathbb{R}^d)\right)^T$ may be identified with an open subset of $\mathsf{SL}(d, \mathbb{R})/P_k^+ \cap P_k^-$. So, if we let $E^k(\Gamma)^T$ be the bundle over $U(\Gamma)$ with fiber $\left(\mathrm{Gr}_k(\mathbb{R}^d) \times \mathrm{Gr}_{d-k}(\mathbb{R}^d)\right)^T$, then $E^k(\Gamma)^T$ is contained in the bundle over $U(\Gamma)$ with fiber $\mathsf{SL}(d, \mathbb{R})/P_k^+ \cap P_k^-$. We can then pass to a quotient $(\widehat{E}_\rho^k)^T$ which is contained in a bundle over $\widehat{U}(\Gamma)$ with fibre $\mathsf{SL}(d, \mathbb{R})/P_k^+ \cap P_k^-$. The theory of Anosov representations is sometimes formalized with this convention in place.

25. **Definitions and first principles**

We say that a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-**Anosov** if there exists a pair

$$\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d) \qquad \text{and} \qquad \theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$$

of transverse P_k-limit maps which determine a section $\hat{\sigma} : \widehat{U}(\Gamma) \to E_\rho^k$ so that the flow on $\hat{\sigma}^*(\widehat{V}_\rho^k)$ is expanding and the flow on $\hat{\sigma}^*(\widehat{V}_\rho^{d-k})$ is contracting, i.e. given continuously varying family of norms on $\hat{\sigma}^*(\widehat{V}_\rho^k)$ and $\hat{\sigma}^*(\widehat{V}_\rho^{d-k})$ (both denoted by $||\cdot||$), there exists $C > 0$ and $a > 0$ so that if $\hat{v} \in \hat{\sigma}^*(\widehat{V}_\rho^k)$ and $\hat{w} \in \hat{\sigma}^*(\widehat{V}_\rho^{d-k})$, then

$$||\hat{\eta}_{-t}^k(\hat{v})|| \leq Ce^{-at}||\hat{v}|| \qquad \text{and} \qquad ||\hat{\eta}_t^{d-k}(\hat{w})|| \leq Ce^{-at}||\hat{w}||$$

for all $t > 0$. We will sometimes refer to a P_1-Anosov representation as **projective Anosov** (just to make the whole thing seem less like jargon).

Equivalently, we may require that the flow on $\mathrm{Hom}(\widehat{\Xi}, \widehat{\Theta}) = \widehat{\Theta} \otimes \widehat{\Xi}^*$ (where $\widehat{\Xi}^* = \mathrm{Hom}(\widehat{\Xi}, \mathbb{R})$) is expanding and that the flow on $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi}) = \Xi \otimes \Theta^*$ is contracting. Notice that since $\widehat{\Xi} \otimes \widehat{\Theta}^*$ is dual to $\widehat{\Theta} \otimes \widehat{\Xi}^*$, then the flow on $\widehat{\Xi} \otimes \widehat{\Theta}^*$ is contracting if and only if the flow on $\widehat{\Theta} \otimes \widehat{\Xi}^*$ is expanding, so it suffices to assume that the flow on $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$ is contracting.

Notice that the fact that $\widehat{\Xi} \otimes \widehat{\Theta}^*$ is contracting means "roughly" that all vectors in $\widehat{\Xi}$ are "contracted uniformly more" than vectors in $\widehat{\Theta}$. Suppose that $\gamma \in \Gamma$ has infinite order. Since $\xi_\rho(\gamma^+)$ and $\theta_\rho(\gamma^-)$ are both preserved by $\rho(\gamma)$, this suggests that each is a product of eigenspaces and that the moduli of the eigenvalues of the generalized eigenspaces making up $\xi_\rho(\gamma^+)$ should be strictly greater than the moduli of the eigenvalues of the generalized eigenspaces making up $\theta_\rho(\gamma^-)$. Moreover, the logarithm of the ratio of the eigenvalues should be roughly comparable to the length of the periodic orbit of $\widehat{U}(\Gamma)$ associated to γ, which is itself roughly comparable to $||\gamma||$ (by the Milnor-Svarc Lemma). The following lemma begins to make this intuition more precise.

Lemma 25.1. ([45, Proposition 2.3]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation with transverse P_k-limit maps ξ_ρ and θ_ρ, then $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$ is contracting if and only if given any continuous norm on $\widehat{E}_\rho$, there exists $a, C > 0$ such that if $\hat{Z} \in \widehat{U}(\Gamma)$, $\hat{v} \in \widehat{\Xi}|_{\hat{Z}}$, $\hat{w} \in \widehat{\Theta}|_{\hat{Z}}$, and $t > 0$ then*

$$\frac{||\hat{\psi}_t(\hat{v})||}{||\hat{\psi}_t(\hat{w})||} \leq C e^{-at} \frac{||\hat{v}||}{||\hat{w}||}.$$

Recall that $\{\hat{\psi}_t\}$ is the flow on the flat bundle $\widehat{E}_\rho$ associated to ρ.

Proof. Let $|| \cdot ||$ be a continuous family of norms on $\widehat{E}_\rho$, which induces a continuous family of norms on $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$. Notice that, by construction, the flow $\{\hat{\eta}_t^{d-k}\}$ on $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$ is consistent with the flow $\{\hat{\psi}_t\}$ on $\widehat{E}_\rho$.

Suppose that $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$ is contracting, so there exists $a, C > 0$ so that if $\hat{A} \in \mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$ and $t > 0$, then

$$||\eta_t^{d-k}(\hat{A})|| \leq C e^{-at} ||\hat{A}||.$$

Given $\hat{Z} \in \widehat{U}(\Gamma)$, $\hat{v} \in \widehat{\Xi}_Z$, and $\hat{w} \in \widehat{\Theta}_Z$, choose $\hat{A} \in \mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})_{\hat{Z}}$ such that

$$\hat{A}(\hat{w}) = \hat{v} \quad \text{and} \quad ||\hat{A}|| = \frac{||\hat{v}||}{||\hat{w}||}.$$

(One may do so, by composing orthogonal projection of $\Theta|_{\hat{Z}}$ onto $< w >$, with respect to $||\cdot||_{\hat{Z}}$, with a linear map taking w to v.) We may lift this whole picture, including the norms $||\cdot||$, up to the bundle $\mathrm{Hom}(\Theta, \Xi)$ over $U(\Gamma)$, to obtain $Z \in U(\Gamma)$, $v \in \Xi|_Z$, $w \in \Theta|_Z$ and $A \in \mathrm{Hom}(\Theta, \Xi)|_Z$. Then, since $\psi_t(v) = \psi_t(A(w)) = \eta_t^{d-k}(A)(\psi_t(w))$, we see that

$$\frac{||\hat{\psi}_t(\hat{v})||}{||\hat{\psi}_t(\hat{w})||} = \frac{||\psi_t(v)||}{||\psi_t(w)||} = \frac{||\eta_t^{d-k}(A)(\psi_t(w)))||}{||\psi_t(w)||} \leq \max_{u \in \Theta|_{\phi_t(Z)}} \left\{ \frac{||\eta_t^{d-k}(A)(u)||}{||u||} \right\} = ||\eta_t^{d-k}(A)|| = ||\hat{\eta}_t^{d-k}(\hat{A})||$$

so

$$\frac{||\hat{\psi}_t(\hat{v})||}{||\hat{\psi}_t(\hat{w})||} \leq ||\hat{\eta}_t(\hat{A})|| \leq C e^{-at} ||\hat{A}|| = C e^{-at} \frac{||v||}{||w||}.$$

This establishes the forward direction of our claim.

On the other hand, suppose that there exists $C, a > 0$ so that if $\hat{Z} \in \widehat{U}(\Gamma)$, $\hat{v} \in \widehat{\Xi}|_{\hat{Z}}$, $\hat{w} \in \widehat{\Theta}|_{\hat{Z}}$, and $t > 0$ then

$$\frac{||\hat{\psi}_t(\hat{v})||}{||\hat{\psi}_t(\hat{w})||} \leq C e^{-at} \frac{||\hat{v}||}{||\hat{w}||}.$$

Let $\hat{A} \in \mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$ and suppose that $t > 0$, then there exists $\hat{w} \in \widehat{\Theta}|_{\psi_t(Z)}$ so that

$$||\hat{\eta}_t^{d-k}(\hat{A})|| = \frac{||\hat{\eta}_t^{d-k}(\hat{A})(\hat{w})||}{||\hat{w}||}.$$

Since $\hat{\eta}_t^{d-k}(\hat{A})(\hat{w}) = \psi_t(\hat{A}(\hat{\psi}_{-t}(\hat{w}))$, we see that

$$\frac{||\hat{\eta}_t^{d-k}(\hat{A})(\hat{w})||}{||\hat{w}||} \leq C e^{at} \frac{\hat{A}(\hat{\psi}_{-t}(\hat{w}))||}{||\hat{\psi}_{-t}(w)||} \leq C e^{-at} ||\hat{A}||.$$

which establishes the reverse direction.

$\square$

Remark: In Labourie's definition [143], see also Guichard-Wienhard [109], a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov, if there exists a continuous ρ-equivariant map

$$\alpha : \partial\Gamma \times \partial\Gamma - \Delta \to \left(\mathrm{Gr}_k(\mathbb{R}^d) \times \mathrm{Gr}_{d-k}(\mathbb{R}^d) \right)^T$$

which gives rise to a flow-invariant section $\hat{\delta} : \widehat{U}(\Gamma) \to \widehat{E}^k_\rho$ so that so that the flow on $\hat{\delta}^*(\widehat{V}^k_\rho)$ is expanding and the flow on $\hat{\delta}^*(\widehat{V}^{d-k}_\rho)$ is contracting. He then observes that if this is the case, then α must have the form $\alpha = \xi_\rho \times \theta_\rho$ where ξ_ρ and θ_ρ are a pair of transverse P_k-limit maps for ρ. So, his definition is equivalent to the one we gave above.

We can generalize the discussion in Section 17 to the setting of P_k-proximal matrices. We say that $A \in \mathsf{GL}(d, \mathbb{R})$ is P_k-**proximal** if $|\lambda_k(A)| > |\lambda_{k+1}(A)|$. In this case, there is well-defined **attracting k-plane** and **repelling $(d-k)$-plane**, such that v does not lie in the repelling $(d-k)$-plane, then $A^n(v)$ converges to the attracting k-plane (i.e. all accumulation points of $\{A^n(v)\}_{n \in \mathbb{N}}$ lie in the attracting k-plane). Moreover, the attracting k-plane is an attracting fixed point for the action of A on $\mathrm{Gr}_k(\mathbb{R}^d)$ and any k-plane disjoint from the repelling $(d-k)$-plane will be attracted to the attracting k-plane. Similarly, the repelling $(d-k)$-plane is an attracting fixed point for the action of A^{-1} on $\mathrm{Gr}_{d-k}(\mathbb{R}^d)$. We say that A is P_k-biproximal if both A and A^{-1} are P_k-proximal. In this language, proximal elements are exactly the P_1-proximal elements.

Suppose that Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation, $1 \leq k \leq \frac{d}{2}$, and $\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$ are continuous ρ-equivariant maps. We say

   (1) ξ_ρ and θ_ρ are P_k-**compatible** if

$$\xi_\rho(x) \subset \theta_\rho(x)$$

   for all $x \in \partial\Gamma$, and
   (2) ξ_ρ and θ_ρ are P_k-**dynamics preserving** if whenever $\gamma \in \Gamma$ has infinite order, then
      a) $\rho(\gamma)$ is P_k-biproximal,
      b) $\xi_\rho(\gamma^+)$ is the attracting k-plane of $\rho(\gamma)$, and
      c) $\theta_\rho(\gamma^-)$ is the repelling $(d-k)$-plane of $\rho(\gamma)$.

> *Although such an approach involves more familiar methods,*
> *the author brutally chose to develop extra structure.*
> ————William Thurston [203]

We can now use Lemma 25.1 to show that the limit maps of an Anosov representation are well-behaved. We have chosen a brutally concrete argument, which I hope will allow you to get a hands on sense of what the Anosov condition actually means. Slicker, but less direct, proofs are available.

Proposition 25.2. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a P_k-Anosov representation with transverse P_k-limit maps $\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$, then ξ_ρ and θ_ρ are P_k-compatible and P_k-dynamics preserving.*

Moreover, there exists $J, B > 0$, so that

$$J||\gamma|| \geq \log \left(\frac{|\lambda_k(\rho(\gamma))|}{|\lambda_{k+1}(\rho(\gamma))|} \right) \geq \frac{1}{J}||\gamma|| - B$$

for any element $\gamma \in \Gamma$. In particular, ρ has finite kernel.

Since fixed points of infinite order elements are dense in $\partial\Gamma$, by Proposition 5.6, Proposition 25.2 implies that limit maps for P_k-Anosov representations are unique.

Proof. Suppose that $\gamma \in \Gamma$ has infinite order. The flow line $(\gamma^-, \gamma^+) \times \mathbb{R}$ of $U(\Gamma)$ projects to a periodic orbit Q_γ of $\widehat{U}(\Gamma)$. Let t_γ be the length of Q_γ. Consider a continuous family of norms $||\cdot||$ on $\widehat{E}_\rho$ which induces a continuous family of norms on $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$. These norms lift to equivariant norms on $E(\Gamma)$ and $\mathrm{Hom}(\Theta, \Xi)$. Therefore, if $\vec{v} \in \mathbb{R}^d$ and $Z_0 = (\gamma^-, \gamma^+, 0) \in U(\Gamma)$, then

$$||\psi_{nt_\gamma}(Z_0, \vec{v})|| = ||\left(Z_0, \rho(\gamma)^{-n}(\vec{v})\right)||$$

for all $n \in \mathbb{Z}$. In particular, if $\vec{v}$ is an eigenvector of $\rho(\gamma)$ with eigenvalue μ, then

$$||\psi_{nt_\gamma}(Z_0, \vec{v})|| = |\mu|^{-n}||(Z_0, \vec{v})||.$$

We begin with the simplest case when $k = 1$. In this case, $\xi_\rho(\gamma^+)$ is an eigenline, so there exists μ so that if $\vec{v} \in \xi_\rho(\gamma^+)$, then $A(\vec{v}) = \mu\vec{v}$, so $||\psi_{nt_\gamma}(Z_0, \vec{v})|| = |\mu|^{-n}||(Z_0, \vec{v})||$. We claim that

$$|\mu| = |\lambda_1(\rho(\gamma))| > |\lambda_2(\rho(\gamma))|.$$

First notice that the Jordan block J of $\rho(\gamma)$ whose generalized eigenspace E contains $\vec{v}$ must be one-dimensional. Notice that J has an eigenline, so it must be upper triangular and if $[\vec{v}] \in \mathbb{P}(E)$, then $\lim \rho(\gamma)^n([\vec{v}]) = \xi_\rho(\gamma^+)$. However, if E is not one-dimensional then $E \cap \theta_\rho(\gamma^-)$ is non-empty, which would contradict our assumptions that $\theta_\rho(\gamma^-)$ is $\rho(\gamma)$-invariant and transverse to $\xi_\rho(\gamma^+)$.

Now suppose that the generalized eigenspace E_2 of another Jordan block has eigenvalue μ_2 and $|\mu_2| \geq |\mu|$. If E_2 contains an eigenline $< \vec{w} >$, then $U = < \vec{v}, \vec{w} >$ must intersect $\theta_\rho(\gamma^-)$ in a non-trivial vector $\vec{u} = a\vec{v} + b\vec{w}$ (with $b \neq 0$). Then,

$$||\psi_{nt_\gamma}(Z_0, \vec{u})|| = ||\left(Z_0, \frac{a}{\mu^n}\vec{v} + \frac{b}{\mu_2^n}\vec{w}\right)||.$$

So, if $|\mu| < |\mu_2|$, then $||\psi_{nt_\gamma}(Z_0, \vec{u})|| \sim \frac{b}{|\mu_2|^n}||(Z_0, \vec{w})||$, while if $|\mu| = |\mu_2|$, then $||\psi_{nt_\gamma}(Z_0, \vec{u})|| = \frac{1}{|\mu|^n}||(Z_0, \vec{u})||$. So, in either case

$$\liminf \frac{||\psi_{nt_\gamma}(Z_0, \vec{v})||}{||\psi_{nt_\gamma}(Z_0, \vec{u})||} \geq \frac{||(Z_0, \vec{v})||}{||(Z_0, \vec{u})||}$$

which contradicts Lemma 25.1, since $(Z_0, \vec{v}) \in \Xi$ and $(Z_0, \vec{u}) \in \Theta$.

If E_2 does not contain an eigenline, then either there exists $s > 0$ so that E_2 contains an eigenline for $\rho(\gamma)^s$ (in which case, the argument above, shows that $|\mu|^s > |\mu_2|^s$ and hence that $|\mu| > |\mu_2|$) or E_2 contains a plane W so that $\rho(\gamma)(W) = W$ and if $w \in W - \{\vec{0}\}$, then $\{\rho(\gamma)^n(< w >)\}$ is dense in $\mathbb{P}(W)$. Notice that $W \cap \theta_\rho(\gamma^+)$ must be non-empty, so, since $\theta_\rho(\gamma^+)$ is invariant under $\rho(\gamma)$, $W \subset \theta_\rho(\gamma^+)$. We can then choose a sequence $n_r \to \infty$ so that $\rho(\gamma)^{n_r}(< \vec{w} >)$ converges to $< \vec{w} >$ in $\mathbb{P}(V)$. Therefore,

$$\lim \frac{||(Z_0, \rho(\gamma)^{n_r}(\vec{u}))||}{|\mu_2|^{n_r}||(Z_0, \vec{u})||} = 1$$

which again contradicts Lemma 25.1 if $|\mu_2| \geq |\mu|$.

We have established that $|\mu| = |\lambda_1(\rho(\gamma))| > |\lambda_2(\rho(\gamma))|$, so $\rho(\gamma)$ is P_1-proximal and $\xi_\rho(\gamma)^+$ is the attracting eigenline. Notice that $\theta_\rho(\gamma^-)$ is a $(d-1)$-plane invariant under $\rho(\gamma)$ and transverse to the attracting eigenline of $\rho(\gamma)$, so it must be the repelling hyperplane of $\rho(\gamma)$. Notice that $\xi_\rho(\gamma^+) \subset \theta_\rho(\gamma^+)$, since $\xi_\rho(\gamma^+)$ is the repelling eigenline of $\rho(\gamma^{-1}$ and $\theta_\rho(\gamma^+)$ is the repelling hyperplane of $\rho(\gamma^1)$. Since ξ_ρ and θ_ρ are continuous and attracting fixed points of infinite order elements are dense in $\partial\Gamma$, it follows that $\xi_\rho(x) \subset \theta_\rho(x)$ for all $x \in \partial\Gamma$. Therefore, ξ_ρ and θ_ρ are P_1-dynamics preserving and P_1-compatible.

Let C and a be the constants provided by Lemma 25.1. If $\theta_\rho(\gamma^-)$ contains an eigenline $<\vec{w}>$ with eigenvalue $\lambda_2(\rho(\gamma))$, then,

$$\frac{||\psi_{t_\gamma}(Z_0, \vec{v})||}{||\psi_{t_\gamma}(Z_0, \vec{w})||} = \frac{|\lambda_2(\rho(\gamma))|}{|\lambda_1(\rho(\gamma))|} \frac{||(Z_0, \vec{v})||}{||(Z_0, \vec{w})||} \leq Ce^{-at_\gamma} \frac{||(Z_0, \vec{v})||}{||(Z_0, \vec{w})||}$$

so

$$\log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_2(\rho(\gamma))|}\right) \geq at_\gamma - \log(C).$$

If not, there exists $\vec{u} \in \theta_\rho(\gamma^-) - \{0\}$ and $n_r \to \infty$ so that

$$\lim\left(\frac{||(Z_0, \rho(\gamma)^{n_r}(\vec{u}))||}{|\lambda_2(\rho(\gamma)|)^{n_r}||(Z_0, \vec{u})||}\right) = 1$$

which allows us to obtain the same estimate. Notice that this lower bound implies that every element in the kernel of ρ has finite order (since if γ has infinite order, $t_{\gamma^n} = nt_\gamma \to \infty$), so ρ has finite kernel, since every subgroup of a hyperbolic group consisting of finite order elements is finite, see [49, Prop. 2.22].

We now observe that t_γ is comparable to $||\gamma||$. The Milnor-Svarc Lemma shows that the orbit map $\tau_\rho : C_\Gamma \to U(\Gamma)$ is a (K, C')-quasi-isometry, so, just as in the proof of Corollary 19.2, we see that, if $Z \in U(\Gamma)$, then

$$d(Z, \gamma(Z)) \geq \frac{1}{K}||\gamma|| - 3C'$$

so

$$t_\gamma \geq d(Z_0, \gamma(Z_0)) \geq \frac{1}{K}||\gamma|| - 3C'$$

and

$$\log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_2(\rho(\gamma))|}\right) \geq \frac{a||\gamma||}{K} - 3C'$$

which gives the lower bound we want. (It suffices to establish the lower bound for infinite order elements, since every hyperbolic group contains only finitely many conjugacy classes of finite order elements, see Bridson-Haefliger [49, Thm. 3.Γ.3.2], so we may assume that B is greater than $\frac{a||\alpha||}{K}$ for any finite order element $\alpha \in \Gamma$.)

To get the upper bound, let $M = \sup\{\log\sigma_1(\rho(s)) \mid s \in S\}$ and notice that there exists $\hat{\gamma}$ which is conjugate to γ so that $d(1, \hat{\gamma}) = ||\gamma||$. Then,

$$2M||\gamma|| = 2Md(1, \hat{\gamma}) \geq 2\log\left(\frac{\sigma_1(\rho(\hat{\gamma}))}{\sigma_d(\rho(\hat{\gamma}))}\right) \geq 2\log\left(\frac{|\lambda_1(\rho(\hat{\gamma}))|}{|\lambda_d(\rho(\hat{\gamma}))|}\right) = 2\log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_d(\rho(\gamma))|}\right) \geq 2\log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_2(\rho(\gamma))|}\right).$$

(Notice that the exact same argument shows that

$$2M||\gamma|| \geq 2\log\left(\frac{|\lambda_k(\rho(\gamma))|}{|\lambda_{k+1}(\rho(\gamma))|}\right)$$

for all k.) This upper bound completes the proof when $k = 1$.

We now discuss the alterations needed for the case of general k. (It would probably be fine to skip this portion of the argument when reading through the notes for the first time.) We could also handle the general case by first proving that ρ is P_k-Anosov if and only if $\Lambda^k\rho$ is P_1-Anosov and applying the above argument to $\Lambda^k\rho$ and translating the results back to the setting of ρ.

Let A be the subspace of $\mathbb{R}^d$, spanned by all the generalized eigenspaces with eigenvalue of modulus at least $|\lambda_k(\rho(\gamma))|$ and let B be the subspace of $\mathbb{R}^d$, spanned by all the generalized eigenspaces with eigenvalue of modulus at most $|\lambda_{k+1}(\rho(\gamma))|$.

If $\rho(\gamma)$ is not P_k-proximal, then $|\lambda_k(\rho(\gamma))| = |\lambda_{k+1}(\rho(\gamma))|$, so A has dimension at least $k+1$ and B has dimension at least $d - k + 1$. Therefore, there exist non-trivial vectors in $\vec{a} \in A \cap \theta_\rho(\gamma^-)$ and $\vec{b} \in B \cap \xi_\rho(\gamma^+)$. Let $<\vec{u}>$ be an accumulation point of $\{<\rho(\gamma)^n(\vec{a})>\}_{n\in\mathbb{N}}$ in $\mathbb{P}(A \cap \theta_\rho(\gamma^-))$, then, by considering the Jordan normal form, one can see that there exists $j \geq k + 1$ and $n_k \to \infty$ so that

$$\lim\left(\frac{||(Z_0, \rho(\gamma)^{n_r}(\vec{u}))||}{|\lambda_j(\rho(\gamma))|^{n_r}||(Z_0, \vec{u})||}\right) = 1.$$

Similarly, if $<\vec{v}>$ be an accumulation point of $\{<\rho(\gamma)^{n_r}(\vec{b})>\}$ in $\mathbb{P}(B \cap \xi_\rho(\gamma^+))$ then there exists $m \leq k$ and a subsequence $\{n_\ell\}$ of $\{n_k\}$ such that

$$\lim\left(\frac{||(Z_0, \rho(\gamma)^{n_\ell}(\vec{u}))||}{|\lambda_m(\rho(\gamma))|^{n_\ell}||(Z_0, \vec{u})||}\right) = 1.$$

Then, Lemma 25.1, implies that

$$\lim\frac{||\phi_{n_\ell t_\gamma}(Z_0, \vec{u})||}{||\phi_{n_\ell t_\gamma}(Z_0, \vec{v})||} = \left(\frac{\lambda_m(\rho(\gamma))^{n_\ell}}{\lambda_j(\rho(\gamma))^{n_\ell}}\right)\frac{||(Z_0, \vec{u})||}{||(Z_0, \vec{v})||} = 0$$

which contradicts the face that $m > k$. Therefore, $\rho(\gamma)$ is P_k-proximal, A is the attracting k-plane and B is the repelling $(d - k)$-plane.

The same argument gives, more generally, that at least one of $A \cap \theta_\rho(\gamma^-)$ or $B \cap \xi_\rho(\gamma^+)$ is trivial. First suppose that $A \cap \theta_\rho(\gamma^-)$ is trivial, then we see that if $\vec{v} \in \xi_\rho(\gamma^+) - A$, then $\{\rho(\gamma)^{-n}(\vec{v})\}_{n\in\mathbb{N}}$ accumulates at $\theta_\rho(\gamma^-)$. Since A and $\theta_\rho(\gamma^-)$ are $\rho(\gamma)$-invariant and transverse, this is impossible. Thus, $\xi_\rho(\gamma^+) = A$ is the attracting k-plane for $\rho(\gamma)$. If $\vec{w} \in \theta_\rho(\gamma^-) - B$, then $\{\rho^n(w)\}$ accumulates on $A = \xi_\rho(\gamma^+)$ which is again impossible, since $\xi_\rho(\gamma^+)$ and $\theta_\rho(\gamma^-)$ are transverse, so $\theta_\rho(\gamma^-) = B$ is the repelling $(d - k)$-plane.

We argue similarly in the case that $B \cap \xi_\rho(\gamma^+)$ is trivial. If $\vec{v} \in \theta_\rho(\gamma^-) - B$, then $\{\rho(\gamma)^n(\vec{v})\}$ accumulates at $\xi_\rho(\gamma^+)$, which is impossible, so $\theta_\rho(\gamma^-) = B$. On the other hand, if $\vec{w} \in \xi_\rho(\gamma^+) - A$, then $\{\rho(\gamma)^{-n}(\vec{w})\}$ accumulates at $B = \theta_\rho(\gamma^-)$ which is impossible, so $\xi_\rho(\gamma^+) = A$ as required.

We complete the remainder of the argument, just as in the case where $k = 1$.

$\square$

26. **The symmetric space of** $\mathsf{SL}(d, \mathbb{R})$

Ink mathematics, grey mass ecstatics
Noggin elastics, cerebral tactics
Cranium classics, brainium domics
Denizen omics, grey massmatistics

Quantum puree, it's plain to feel, hard to see
Fission antics, abombastics
Death antiques, wrong deductions
Poor instructions, mass destructions
——————Don Van Vliet [63]

In analogy with the action of $\mathsf{PO}(n, 1)$ on $\mathbb{H}^n = \mathsf{PO}(n, 1)/\mathsf{O}(n)$, the Lie group $\mathsf{SL}(d, \mathbb{R})$ naturally acts on its associated symmetric space

$$X_d = \mathsf{SL}(d, \mathbb{R})/\mathsf{SO}(d).$$

We will recall the basic theory of this symmetric space and observe that the orbit map of a P_k-Anosov representation is a quasi-isometric embedding. (We will base our discussion on lecture notes by Rich Schwartz [187]. One can also look at the treatment of Bridson and Haefliger [49, Section II.10] which begins with $\mathsf{GL}(d, \mathbb{R})/\mathsf{O}(d)$ which turns out to be a metric product $X_d \times \mathbb{R}$.)

It is convenient to regard X_d as the space of symmetric, positive definite, matrices in $\mathsf{SL}(d, \mathbb{R})$. (Recall that $A \in \mathsf{SL}(d, \mathbb{R})$ is positive definite if $A(\vec{v}) \cdot \vec{v} > 0$ for every non-trivial vector in $\mathbb{R}^d$.) It is then obvious that X_d is a submanifold of $\mathbb{R}^{d^2}$.

The Lie group $\mathsf{SL}_\pm(d, \mathbb{R})$ (of linear transformations of $\mathbb{R}^d$ with determinant ± 1) acts on X_d by letting $A(M) = AMA^T$ for all $A \in \mathsf{SL}_\pm(d, \mathbb{R})$ and $M \in X_d$. One may check that AMA^T is positive definite, by noticing that if $\vec{v}$ is non-trivial

$$AMA^T(\vec{v}) \cdot \vec{v} = MA^T\vec{v} \cdot A^T\vec{v} = M(A^T\vec{v}) \cdot (A^T\vec{v}) > 0$$

where the last inequality follows from the fact M is positive definite. Since it is obvious that AMA^T is symmetric and has determinant 1, $AMA^T \in X_d$. One may also check that this a group action by noticing that if $A, B \in \mathsf{SL}_\pm(d, \mathbb{R})$ and $M \in X_d$, then

$$AB(M) = (AB)M(AB)^T = ABMB^TA^T = A(BMB^T) = A(B(M)).$$

We notice that $A \in \mathsf{SL}_\pm(d, \mathbb{R})$ fixes $\mathrm{I} \in X_d$ if and only if $AA^T = I$ which occurs if and only if $A \in \mathsf{O}(d)$. Every positive-definite symmetric matrix M can be written as $M = gDg^{-1}$ where $g \in \mathsf{SO}(d)$ and D is a diagonal matrix with positive entries and determinant 1. If we let $\sqrt{D}$ be the diagonal matrix whose entries are the square roots of the corresponding entries of D, then $A = g\sqrt{D}g^{-1} \in \mathsf{SL}(d, \mathbb{R})$ and $A(I) = AA^T = M$. Therefore, $\mathsf{SL}(d, \mathbb{R})$ acts transitively on X_d. So, we see that,

$$X_d = \mathsf{SL}(d, \mathbb{R})/\mathsf{SO}(d) = \mathsf{SL}_\pm(d, \mathbb{R})/\mathsf{O}(d).$$

The tangent space $T_I(X_d)$ is naturally identified with the space x_d of symmetric matrices with trace zero. One can see this concretely by noting that if $c : (-\epsilon, \epsilon) \to X_d$ is a smooth path

and $c(0) = I$, then we may write $c(s) = g_s D_s g_s^{-1}$ where $g_s \in \mathsf{SO}(d)$ and the diagonal matrices D_s both vary smoothly, then

$$c'(o) = g_0 \left(\frac{d}{ds}\Big|_0 D_s \right) g_0^T + \left(\frac{d}{ds}\Big|_0 g_s \right) g_0^T + g_0 \left(\frac{d}{ds}\Big|_0 g_s^T \right).$$

We then notice that the last two terms cancel since

$$0 = \frac{d}{ds}\Big|_0 I = \frac{d}{ds}\Big|_0 (g_s g_s^T) = \left(\frac{d}{ds}\Big|_0 g_s \right) g_0^T + g_0 \left(\frac{d}{ds}\Big|_0 g_s^T \right)$$

and that $E = \frac{d}{ds}\big|_0 D_s$ is a diagonal matrix with trace zero, so $c'(0) = g_0 E g_0^T \in \mathsf{x}_d$. (We will use, mostly without saying so, the fact that if $g \in \mathsf{SO}(d)$, then $g^{-1} = g^T$.)

So we define a symmetric blinear from on x_d,

$$< V, W >= \frac{1}{4}\mathrm{trace}(AB)$$

which is called the **Killing form**. Notice that the Killing form is positive definite, since if $V = gEg^{-1} \in \mathsf{x}_d$ is non-zero, then $4 < V, V >= \mathrm{trace}(E^2) > 0$ since trace is conjugacy invariant and E is a non-zero diagonal matrix. Moreover, the Killing form is invariant under the action of $\mathsf{O}(d)$, since if $g \in \mathsf{O}(d)$ and $V, W \in \mathsf{x}_d$, then

$$4 < g(V), g(W) >= 4 < gVg^T, gWg^T >= \mathrm{trace}(gVg^T gWg^T) = \mathrm{trace}(gVWg^{-1}) = \mathrm{trace}(VW) = 4 < V, W >.$$

(Notice that since g is a linear transformation $dg = g$.)

If $M \in X_d$, then $T_M X_d = dM(\mathsf{x}_d) = M\mathsf{x}_d$, so it is natural to transport the Killing form on x_d to $T_M X_d$ by letting

$$< V, W >_M =< M^{-1}V, M^{-1}W >= \frac{1}{2}\mathrm{trace}(M^{-1}VM^{-1}W)$$

for all $V, W \in M\mathsf{x}_d$. So we now have a Riemannian metric on X_d. Notice that $\mathsf{SL}_\pm(d, \mathbb{R})$ acts by isometries of this metric, since if $M, N \in X_d$, $A \in \mathsf{SL}_\pm(d, \mathbb{R})$ and $A(M) = N$, then $AMA^T = N$ and

$$
\begin{aligned}
< A(V), A(W) >_N \quad &= \quad < AVA^T, AWA^T >_N = \frac{1}{2}\mathrm{trace}(N^{-1}AVA^T N^{-1}AWA^T) \\
&= \quad \frac{1}{4}\mathrm{trace}((A^T)^{-1}M^{-1}A^{-1}AVA^T(A^T)^{-1}M^{-1}A^{-1}AWA^T) \\
&= \quad \frac{1}{4}\mathrm{trace}((A^T)^{-1}M^{-1}VM^{-1}WA^T) \\
&= \quad \frac{1}{4}\mathrm{trace}(M^{-1}VM^{-1}W) =< V, W >_M
\end{aligned}
$$

for all $V, W \in T_M X_d$.

The following is our key estimate for the distance function on X_d.

Proposition 26.1. *If $M \in X_d$, then*

$$d_{X_d}(I, M) = \frac{1}{2}\sqrt{(\log \sigma_1(M))^2 + \cdots + (\log \sigma_d(M))^2}.$$

Notice that if $A \in \mathsf{SL}(d, \mathbb{R})$, then $A(I) = AA^T$ and $\sigma_i(AA^T) = \sigma_i(A)^2$ for all i. Therefore, we see that the singular values of A record the translation distance of A at the origin.

Corollary 26.2. *If $A \in \mathsf{SL}(d, \mathbb{R})$, then*

$$d_{X_d}(I, A(I)) = \sqrt{(\log \sigma_1(A))^2 + \cdots + (\log \sigma_d(A))^2}.$$

Let's warm-up by showing that the "obvious shortest path" from I to M has the expected length. If you feel comfortable with the machinery here, you can skip this and move on to the proof. If $F \in X_d$ is a diagonal matrix, let $c : [0, 1] \to X_d$ be the smooth path joining I to F given by

$$c(t) = \begin{bmatrix} e^{t \log f_{11}} & 0 & 0 \cdots 0 & 0 \\ 0 & e^{t \log f_{22}} & 0 \cdots 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 \cdots 0 & e^{t \log f_{dd}} \end{bmatrix}$$

for all $t \in [0, 1]$. Then

$$c'(t) = \begin{bmatrix} \log f_{11} e^{t \log f_{11}} & 0 & 0 \cdots 0 & 0 \\ 0 & \log f_{22} e^{t \log f_{22}} & 0 \cdots 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 \cdots 0 & \log f_{dd} e^{t \log f_{dd}} \end{bmatrix}$$

so

$$4 < c'(t), c'(t) >_{c(t)} = (\log f_{11})^2 + \cdots + (\log f_{dd})^2 = (\log \sigma_1(F))^2 + \cdots + (\log \sigma_d(F))^2$$

for all t, so $c([0, 1])$ has length

$$\int_0^1 \sqrt{\langle c'(t), c'(t) \rangle} \, dt = \frac{1}{2} \sqrt{(\log \sigma_1(F))^2 + \cdots + (\log \sigma_d(F))^2}.$$

In general, since M is symmetric and positive definite, we can write $M = KDK^T$ where $K \in \mathsf{SO}(d)$ and $F \in X_d$ is diagonal. Then, since K act on X_d as an isometry and stabilizes I, the path $\hat{c}(t) = Kc(t)K^T$ joins I to M and has length

$$\frac{1}{2} \sqrt{\frac{(\log \sigma_1(F))^2 + \cdots + (\log \sigma_d(F))^2}{2}} = \frac{1}{2} \sqrt{(\log \sigma_1(M))^2 + \cdots + (\log \sigma_d(M))^2}.$$

However, this is not really a proof, since we don't yet know that these paths are length-minimizing.

Proof of Proposition 26.1: Let $F_d \subset X_d$ be the set of diagonal matrices with positive diagonal entries and determinant 1. Then $T_I F_d = \mathfrak{f}_d$ is the set of diagonal matrices with trace 0. The restriction of the Killing form to $T_I F_d$ is then just

$$4 < A, B >_I = a_{11}b_{11} + \cdots + a_{dd}b_{dd}.$$

and the restriction of the Killing form to $T_F F_d$ is simply

$$4 < A, B >_F = \text{trace}(F^{-1}AF^{-1}B) = \frac{a_{11}b_{11}}{f_{11}^2} + \cdots + \frac{a_{dd}b_{dd}}{f_{dd}^2}.$$

(Notice that $T_F F_d$ is also identified with the space of trace-free diagonal matrices so the F^{-1} term just introduces a scaling factor.) So the map $\tau : F_d \to \mathbb{E}^d$ given by

$$\tau(F) = 2(\log f_{11}, \ldots, \log f_{dd})$$

is an isometry from F_d, with its intrinsic metric, to the hyperplane $\{\vec{x} \in \mathbb{R}^d \mid x_1 + \cdots + x_d = 0\}$ with its usual Euclidean metric.

We claim that F_d is totally geodesic, so its intrinsic and extrinsic metrics agree. Since F_d asks transitively on itself, it suffices to check that if $\alpha : [0,1] \to X_d$ is a path joining I to F in X_d, then there is a path $\beta : [0,1] \to F_d$ joining I to F which is no longer than α. We may assume that $\alpha(s) = g(s)D(s)g(s)^T$ where $g(s) \in \mathsf{SO}(d)$ and $D(s) \in F_d$ both vary smoothly and $g(0) = g(1) = I$. So, we simply take $\beta(s) = D(s)$ and check that

$$||\alpha'(s)|| \geq ||\beta'(s)||$$

for all s. Notice that

$$\alpha'(s) = g(s)D'(s)g(s)^T + g'(s)D(s)g(s)^T + g(s)D(s)g'(s)^T \qquad \text{and} \qquad \beta'(s) = D'(s).$$

Since

$$
\begin{aligned}
4 < g(s)D'(s)g(s)^T, g(s)D'(s)g(s)^T >_{\alpha(s)} &= \text{trace}\left(g(s)D(s)^{-1}g(s)^T g(s)D'(s)g(s)^T g(s)D(s)^{-1}g(s)^T g(s)D'(s)g(s)^T\right) \\
&= \text{trace}\left((g(s)D(s)^{-1}D'(s)D(s)^{-1}D'(s)g(s)^T\right) \\
&= \text{trace}\left(D(s)^{-1}D'(s)D(s)^{-1}D'(s)\right) \\
&= 4 < D'(s), D'(s) >_{\beta(s)} = 4||\beta'(s)||^2
\end{aligned}
$$

it suffices to check that

$$C(s) = < g(s)D'(s)g(s)^T, g'(s)D(s)g(s)^T + g(s)D(s)g'(s)^T >_{\alpha(s)} = 0.$$

This is an unpleasant calculation which we include for completeness (but you should feel free to skip it). First notice that

$$\alpha(s)^{-1}g(s)D'(s)g(s)^T = g(s)D(s)^{-1}g(s)^T g(s)D'(s)g(s)^T = g(s)\Omega(s)g(s)^T$$

where $\Omega(s) = D(s)^{-1}D'(s)$ is diagonal, and

$$
\begin{aligned}
\alpha(s)^{-1}(g'(s)D(s)g(s)^T + g(s)D(s)g'(s)^T) &= g(s)D(s)^{-1}g(s)^T g'(s)D(s)g(s)^T + g(s)D(s)^{-1}g(s)^T g(s)D(s)g'(s)^T \\
&= g(s)D(s)^{-1}g(s)^T g'(s)D(s)g(s)^T + g(s)g'(s)^T
\end{aligned}
$$

So

$$
\begin{aligned}
4C(s) &= \text{trace}\left(g(s)\Omega(s)g(s)^T g(s)D(s)^{-1}g(s)^T g'(s)D(s)g(s)^T + g(s)\Omega(s)g(s)^T g(s)g'(s)^T\right) \\
&= \text{trace}\left(g(s)\Omega(s)D(s)^{-1}g(s)^T g'(s)D(s)g(s)^T + g(s)\Omega(s)g'(s)^T\right) \\
&= \text{trace}\left(D(s)g(s)^T g(s)\Omega(s)D(s)^{-1}g(s)^T g'(s) + \Omega(s)g'(s)^T g(s)\right) \\
&= \text{trace}\left(D(s)\Omega(s)D(s)^{-1}g(s)^T g'(s) + \Omega(s)g'(s)^T g(s)\right) \\
&= \text{trace}\left(\Omega(s)g(s)^T g'(s) - \Omega(s)g(s)^T g'(s)\right) = 0
\end{aligned}
$$

(In the transition from the second line to the third line we used the fact that $\text{trace}(AB+CD) = \text{trace}(BA + DC)$ and in the transition from the fourth line to the fifth line we used the facts that diagonal matrices commute and that $g'(s)^T g(s) + g(s)^T g'(s) = 0$, which one obtains by

differentiating the equation $g(s)^T g(s) = I$.) This completes our proof that F_d is totally geodesic, so if $F \in F_d$, then

$$d(I, F) = \frac{1}{2}\sqrt{(\log \sigma_1(F))^2 + \cdots + (\log \sigma_d(F))^2}.$$

In general, we can write $M \in X_d$ as $M = KFK^T$ where $K \in \mathsf{SO}(d)$ and $F \in F_d$. Then, since K acts on X_d as an isometry and stabilize I,

$$2d(I, A) = 2d(I, F) = \sqrt{(\log \sigma_1(F))^2 + \cdots + (\log \sigma_d(F))^2} = \sqrt{(\log \sigma_1(A))^2 + \cdots + (\log \sigma_d(A))^2}.$$

$\square$

We have proven that F_d is a totally geodesic $(d-1)$-submanifold of X_d isometric to Euclidean space. It is called a **maximal flat**, since it is known that every isometrically embedded, totally geodesic copy of Euclidean space has dimension at most $d-1$. The symmetric space X_d contains many maximal flats, since every translate of F_d by an element of $\mathsf{SL}(d, \mathbb{R})$ is a maximal flat. (In fact, all maximal flats are of this form, see [49, Proposition 10.45].) For example, any two points in X_d are contained in a single maximal flat.

The quotient $\mathsf{SO}_0(d-1, 1)/\mathsf{SO}(d-1)$ is a totally geodesic copy of $\mathbb{H}^{d-1}$ in X_d. If we embed $\mathsf{SL}(2, \mathbb{R})$ in the upper left-hand corner, then $\mathsf{SL}(2, \mathbb{R})/SO(2)$ gives a totally geodesic copy of $\frac{1}{2}\mathbb{H}^2$ within X_d. (Here $\frac{1}{2}\mathbb{H}^{d-1}$ is a simply connected Riemannian manifold with constant sectional curvature -4.) One can compute the intrinsic metric on these submanifolds directly and see that it has the claimed form. One can then use Proposition 26.1 to see that this agrees with the extrinsic metric, and hence that the submanifolds are totally geodesic (see also [49, Theorem 10.58]). One may further show that X_d has non-positive sectional curvature, see [49, Theorem 10.39].

Given an P_k-Anosov representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ we can define an **orbit map** $\tau_\rho : \Gamma \to X_d$ by letting $\tau_\rho(\gamma) = \rho(\gamma)(I)$. We can reprise the proof of Corollary 19.3 to show that τ_ρ is a quasi-isometric embedding.

Corollary 26.3. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a P_k-Anosov representation, then the orbit map $\tau_\rho : \Gamma \to X_d$ is a quasi-isometric embedding. In particular, $\rho(\Gamma)$ is discrete.*

Proof. Let S be a finite symmetric generating set for Γ and let $M = \max\{\log \sigma_1(\rho(s)) \mid s \in S\}$. Recall that if $A, B \in \mathsf{SL}(d, \mathbb{R})$, then

$$\sigma_1(A)\sigma_d(B)\sigma_1(AB) \le \sigma_1(A)\sigma_1(B) \qquad \text{and} \qquad \sigma_d(A^{-1}) = \frac{1}{\sigma_d(A)}.$$

Therefore, if $\gamma \in \Gamma$, then

$$\sigma_1(A) \le Md(1, \gamma)$$

where $d(1, \gamma)$ is the minimal word length of γ in the generating set S. Thus,

$$2Md(1, \gamma) \ge \log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))}\right) \ge \frac{1}{d}d_{X_d}(I, \rho(\gamma)(I)) = \frac{1}{d}d_{X_d}(\tau_\rho(id), \tau_\rho(\gamma)).$$

By Proposition 8.4, there exists $\eta \in \{1, \alpha, \beta\}$ so that $d(1, \gamma) \le 3\|\gamma\eta\| + K$. If J and B are the constants from Proposition 25.2, then

$$\log\left(\frac{|\lambda_1(\rho(\gamma\eta))|}{|\lambda_d(\rho(\gamma\eta))|}\right) \ge \log\left(\frac{|\lambda_k(\rho(\gamma\eta))|}{|\lambda_{k+1}(\rho(\gamma\eta))|}\right) \ge \frac{1}{J}\|\gamma\eta\| - B \ge \frac{1}{3J}d(1, \gamma) - \frac{K}{J} - B.$$

So,

$$\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))}\right) \geq \log\left(\frac{\sigma_1(\rho(\gamma\eta))}{\sigma_d(\rho(\gamma\eta))}\right) - 2Md(1,\eta) \geq \log\left(\frac{|\lambda_1(\rho(\gamma\eta))|}{|\lambda_d(\rho(\gamma\eta))|}\right) - 2Md(1,\eta)$$

so, if we set $G = \max\{d(1,\alpha), d(1,\beta)\}$, we may combine the last two inequalities to see that

$$d_{X_d}(\tau_\rho(id), \tau_\rho(\gamma)) \geq \frac{1}{2}\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))}\right) \geq \frac{1}{2}\left(\frac{1}{3J}d(1,\gamma) - \frac{K}{J} - B - 2MG\right).$$

Since τ_ρ is ρ-equivariant, this suffices to show that τ_ρ is a (K,C)-quasi-isometric embedding where $K = \max\{3J, 2dM\}$ and $C = \frac{K}{J} + B + 2MG$.

Notice that this implies that ρ is discrete, since if $\{\gamma_n\}$ is a sequence of distinct elements in Γ, then $d(id, \gamma_n) \to \infty$, so $d(I, \rho(\gamma_n)(I)) \to \infty$. $\qquad\square$

27. Singular values and Anosov representations

In this section we show that if ρ is a P_k-Anosov representation, the ratio of the kth and $(k+1)^{\text{st}}$ singular values grows uniformly exponentially in the word length. One may view this as a strengthening of the fact that the orbit map is a quasi-isometric embedding. Kapovich-Leeb-Porti [133] and Bochi-Potrie-Sambarino [32] showed that this property characterizes P_k-Anosov representations.

Proposition 27.1. (Guichard-Wienhard [109]) *If $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a P_k-Anosov representation, then there exists $D \geq 1$ and $L \geq 0$ so that*

$$Dd(1,\gamma) \geq \log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq \frac{1}{D}d(1,\gamma) - L$$

for all $\gamma \in \Gamma$.

Notice that one can use Proposition 27.1 to give a more direct proof of Corollary 26.3, which asserts that orbit maps of Anosov representations are quasi-isometric embeddings into X_d.

Proof. Let $\widehat{E}_\rho = \widehat{\Xi} \oplus \widehat{\Theta}$ be the splitting of the flat bundle associated to ρ over $\widehat{U}(\Gamma)$ and let $||\cdot||$ be a continuous family of norms on $\widehat{E}_\rho$ (which we may assume all come from bilinear forms on the fibers). We can then lift the norms $||\cdot||$ and the splitting $E(\Gamma) = \Xi \oplus \Theta$ equivariantly to the cover, which is a vector bundle over $U(\Gamma)$. We will assume, for simplicity, that Γ is torsion-free.

By Lemma 25.1, there exist $a, C > 0$ so that if $t > 0$, $Z \in U(\Gamma)$, $(Z, \vec{v}) \in \Xi|_Z$ and $(Z, \vec{w}) \in \Theta|_Z$, then

$$\frac{||\psi_t(Z, \vec{v})||}{||\psi_t(Z, \vec{w})||} \leq Ce^{-at}\frac{||(Z, \vec{v})||}{||(Z, \vec{w})||}.$$

Recall that $||(Z, \vec{v})||$ is just the norm of the vector $\vec{v}$ in the norm on $E(\Gamma)|_Z$, so we could also re-write $||(Z, \vec{v})|| = ||\vec{v}||_Z$, in which case we would write $||\psi_t(Z, \vec{v})|| = ||\vec{v}||_{\phi_t(Z)}$, since $\psi_t(Z, \vec{v}) = (\phi_t(Z), \vec{v})$. The above inequality can thus be rewritten as

$$\frac{||\vec{v}||_{\phi_t(Z)}}{||\vec{w}||_{\phi_t(Z)}} \leq Ce^{-at}\frac{||\vec{v}||_Z}{||\vec{w}||_Z}$$

which I hope will offer more intuition.

We now observe that there exists a compact subset R of $U(\Gamma)$ so that $\Gamma(R) = U(\Gamma)$ and if $\gamma \in \Gamma$, then there exists $Z \in R$ and $s_\gamma > 0$ so that $\phi_{s_\gamma}(Z) \in \gamma(R)$. If $\Gamma = \pi_1(M)$ and $M = \widetilde{M}/\Gamma$

is a closed negatively curved manifold, then we can simply let $S \subset \widetilde{M}$ be a compact submanifold (with boundary) so that $\Gamma(S) = \widetilde{M}$ and then let $R \subset T^1\widetilde{M} = U(\Gamma)$ be the set of all unit tangent vectors at points in S. The same argument works whenever Γ acts properly discontinuously and cocompactly on a CAT(-1)-space X, for example, if Γ is a convex cocompact subgroup of a rank one Lie group, where X is the convex core of the limit set of Γ. One may establish this fact more generally for the Gromov geodesic flow, but we will omit details here.

Let γ be a non-trivial element of Γ and suppose that $Z \in R$, $s_\gamma > 0$ and $\phi_{s_\gamma}(Z) \in \gamma(R)$. Let $W = \gamma^{-1}(\phi_{s_\gamma}(Z))$, so, by the equivariance of the norms,

$$||\rho(\gamma)^{-1}(\vec{u})||_W = ||\vec{u}||_{\phi_{s_\gamma}(Z)}$$

for all $\vec{u} \in \mathbb{R}^d$. So, in this case, the inequality above becomes

$$\frac{||\rho(\gamma)^{-1}(\vec{v})||_W}{||\rho(\gamma)^{-1}(\vec{w})||_W} = \frac{||\vec{v}||_{\phi_{s_\gamma}(Z)}}{||\vec{w}||_{\phi_{s_\gamma}(Z)}} \leq Ce^{-as_\gamma}\frac{||\vec{v}||_Z}{||\vec{w}||_Z}$$

if $\vec{v} \in \Xi|_Z$ and $\vec{w} \in \Theta|_Z$. However, since R is compact, there exists $B > 0$ so that if $||\cdot||_0$ is the standard norm on $\mathbb{R}^d$, then the identity map

$$id : (\mathbb{R}^d, ||\cdot||_0) \to (\mathbb{R}^d, ||\cdot||_Z)$$

is B-bilipschitz for all $Z \in R$. Therefore,

$$\frac{||\rho(\gamma)^{-1}(\vec{v})||_0}{||\rho(\gamma)^{-1}(\vec{w})||_0} \leq B^4 Ce^{-as_\gamma}\frac{||\vec{v}||_0}{||\vec{w}||_0}$$

if $\vec{v} \in \Xi|_Z$ and $\vec{w} \in \Theta|_Z$. Notice that, by equivariance, $\rho(\gamma^{-1})(\Xi_Z) = \Xi_W$ and $\rho(\gamma)^{-1}(\Theta|_Z) = \Theta_W$ So we may rewrite the inequality above to say that if $\vec{x} \in \rho(\gamma)^{-1}(\Xi|_Z) = \Xi_W$ and $\vec{y} \in \rho(\gamma)^{-1}(\Theta|_Z) = \Theta_W$, then

$$\frac{||\vec{x}||_0}{||\vec{y}||_0} \leq B^4 Ce^{-as_\gamma}\frac{||\rho(\gamma)(\vec{x})||_0}{||\rho(\gamma)(\vec{y})||_0}$$

so

$$\frac{||\rho(\gamma)(\vec{x})||_0}{||\rho(\gamma)(\vec{y})||_0} \geq \frac{1}{B^4 C}e^{as_\gamma}\frac{||\vec{x}||_0}{||\vec{y}||_0}.$$

Now consider the following alternative formulation of the definition of singular values

$$\sigma_k(A) = \sup\left\{\inf_{v \in P - \{0\}}\left(\frac{||A(v)||_0}{||v||_0}\right) \mid P \in \mathrm{Gr}_k(\mathbb{R}^d)\right\}.$$

It follows immediately from this definition, since $\Xi|_W \in \mathrm{Gr}_k(\mathbb{R}^d)$, that

$$\sigma_k(\rho(\gamma)) \geq \sigma_k(\rho(\gamma)|_{\Xi|_W})$$

On the other hand, if $P \in \mathrm{Gr}_{k+1}(\mathbb{R}^d)$, then $P \cap \Theta|_W$ is non-trivial, so

$$\sigma_{k+1}(\rho(\gamma)) \leq \sigma_1(\rho(\gamma)|_{\Theta|_W}).$$

By combining we see that

$$\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geq \frac{\sigma_k(\rho(\gamma)|_{\Xi|_W})}{\sigma_1(\rho(\gamma)|_{\Theta|_W})} \geq \frac{1}{BC^4}e^{as_\gamma}.$$

We now see that the Milnor-Svarc lemma implies that s_γ is coarsely comparable to $d(id, \gamma)$ (although we will only keep track of the lower bound we need). Choose $Z_0 \in R$ and $r > 0$ so

that $R \subset B(r, Z_0)$. The Milnor-Svarc Lemma implies that there exists (K, C) so that the orbit map $\tau : \Gamma \to U(\Gamma)$ given by $\tau(\gamma) = \gamma(Z_0)$ is a quasi-isometric embedding, so

$$d(Z_0, \gamma(Z_0)) \geq \frac{1}{K} d(1, \gamma) - C$$

which implies that

$$d(Z, \phi_{s_\gamma}(Z)) \geq \frac{1}{K} d(1, \gamma) - C - 2r.$$

On the other hand, by Theorem 24.1, there exists $(\hat{K}, \hat{C})$ so that all flow-lines are $(\hat{K}, \hat{C})$-quasi-isometric embeddings, so

$$s_\gamma \geq \frac{1}{\hat{K}} d(Z, \phi_{s_\gamma}(Z)) - \hat{C} \geq \frac{1}{K\hat{K}} d(id, \gamma) - \left(\frac{C + 2r}{\hat{K}} + \hat{C} \right)$$

which implies our desired lower bound

$$\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geq \left(\frac{1}{BC^4} e^{-\left(\frac{C+2r}{\hat{K}} + \hat{C} \right)} \right) e^{\frac{a}{K\hat{K}} d(id, \gamma)}.$$

The proof of the upper bound is standard at this point. Let $M = \sup \{ \sigma_1(\rho(s)) \mid s \in S \}$. Then

$$M^{2d(1, \gamma)} \geq \frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))} \geq \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}.$$

$\square$

Kapovich, Leeb and Porti [133, Theorem 1.5] proved that uniform growth of the kth singular value gap implies that a representation is P_k-Anosov. Moreover, they show that one doesn't need to assume that the domain group is Gromov hyperbolic. Bochi, Potrie and Sambarino [32] reproved this result using the theory of dominated splittings. Guéritaud, Guichard, Kassel and Wienhard [107] obtained closely related results.

Theorem 27.2. *If Γ is a finitely generated group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation, and there exists $L > 1$ and $D \geq 0$ so that*

$$Dd(1, \gamma) \geq \log \left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \right) \geq \frac{1}{D} d(1, \gamma) - L$$

for all $\gamma \in \Gamma$, then Γ is Gromov hyperbolic and ρ is a P_k-Anosov representation.

The key point of the approach of Bochi, Potrie and Sambarino [32] is that if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov, then the resulting splitting $\widehat{E}_\rho = \widehat{\Xi} \oplus \widehat{\Theta}$ is a dominated splitting of the flat bundle (which can be viewed as a strong form of the contraction property in Lemma 25.1). They use the singular value gap property to show that Γ is hyperbolic and to produce limit maps which give rise to a splitting. They then use work of Bochi and Gourmelon [31] to show that the resulting splitting is dominated and to recover that ρ is P_k-Anosov. Kapovich, Leeb and Porti [133] work more directly with the action of the group on the symmetric space and its boundary.

Remarks: One can derive the estimate on the eigenvalue gap in Theorem 25.2 from Proposition 27.1, by noticing that if $A \in \mathsf{SL}(d, \mathbb{R})$, then $\log \lambda_i(A) = \lim \log \frac{\sigma_i(A^n)}{n}$ and that $||\gamma||$ is uniformly comparable to $\lim \frac{d(1, \gamma^n)}{n}$ (see [72, Proposition 10.6.4]).

On the other hand if ρ is irreducible (or, more generally, a direct sum of irreducible representations), then one may use a result of Benoist [16], which is stated (and reproven) in the form needed here in [107, Theorem 4.12], to derive Proposition 27.1 from the estimate in Theorem 25.2.

Examples: Proposition 27.1 allows us to give a simple example of a discrete faithful representataion $\rho : F_2 \to \mathsf{SL}(4, \mathbb{R})$ whose orbit map is a quasi-isometric embedding, but which is not P_1-Anosov or P_2-Anosov. Since these are the two flavors of Anosov available for $\mathsf{SL}(4, \mathbb{R})$, we say that ρ is not Anosov. Our example is based on an example of Guichard [107, Appendix A].

Let $F_2 =< a, b >$ be the free group on two generators. Let $\rho_1 : F_2 \to \mathsf{SL}(2, \mathbb{R})$ be a representation which projects to a convex cocompact representation of F_2 into $\mathsf{PSL}(2, \mathbb{R})$. Therefore, $\log \sigma_1(\mu(\rho_1(\gamma))$ grows linearly in the word length of γ, i.e. there exist $J > 0$ and B so that

$$Jd(1, \gamma) \geq \log \sigma_1(\mu(\rho_1(\gamma)) \geq \frac{1}{J}d(1, \gamma) - B$$

for all $\gamma \in \Gamma$. Then let $\rho_2 : F_2 \to \mathsf{SL}(2, \mathbb{R})$ be defined so that $\rho_2(a) = \rho_1(a)$ and $\rho_2(b) = I$. Let $\rho : F_2 \to \mathsf{SL}(4, \mathbb{R})$ be defined by $\rho = \rho_1 \oplus \rho_2$ (where we consider the natural embedding of $\mathsf{SL}(2, \mathbb{R}) \oplus \mathsf{SL}(2, \mathbb{R})$ into $\mathsf{SL}(4, \mathbb{R})$ obtained by putting the first factor in the upper left hand corner and the second factor in the lower right hand corner.) Then, since

$$\sqrt{2} \log \sigma_1(\rho_1(\gamma)) \geq \sigma_1(\rho(\gamma) \geq \sigma_1(\rho_1(\gamma))$$

for all $\gamma \in F_2$. Proposition 26.1 then implies that the orbit map τ_ρ is a quasi-isometric embedding.

On the other hand,

$$\sigma_1(\rho(a^n)) = \sigma_1(\rho_1(a^n)) = \sigma_2(\rho(a^n))$$

for all n, so ρ is not P_1-Anosov. Moreover,

$$\sigma_2(\rho(b^n)) = 1 = \sigma_3(\rho(b^n))$$

for all n, so ρ is not P_2-Anosov. Therefore, ρ is not an Anosov representation into $\mathsf{SL}(4, \mathbb{R})$. However, if we view ρ as a representation into $\mathsf{SL}(2, \mathbb{R}) \times \mathsf{SL}(2, \mathbb{R})$ then it is Anosov with respect to the parabolic subgroup $\{I\} \times \mathsf{SL}(2, \mathbb{R})$. (See Section 49 for a discussion of what it means to be an Anosov representation into a semi-simple Lie group other than $\mathsf{SL}(d, \mathbb{R})$.)

In Guichard's example [107, Appendix A], he chooses ρ_1 to be a geometrically finite representation where a is taken to a parabolic (and every parabolic in $\rho_1(\Gamma)$ is conjugate, in $\rho_1(\Gamma)$, to a power of $\rho(a)$). One can then take $\rho_2 = \rho_1 \circ \iota$ where $\iota : F_2 \to F_2$ is the involution taking a to b, and let $\rho = \rho_1 \oplus \rho_2$. The orbit map of ρ is a quasi-isometric embedding, yet fails to be Anosov. Moreover, ρ is a limit of P_2-Anosov representations, but also a limit of indiscrete representations. Guichard's example also fails to be an Anosov representation into $\mathsf{SL}(2, \mathbb{R}) \times \mathsf{SL}(2, \mathbb{R})$.

Quite recently, Tsouvalas [207] has produced representations into $\mathsf{SL}(d, \mathbb{R})$, for $d \geq 4$, whose orbit maps are quasi-isometric embeddings into $\mathsf{SL}(d, \mathbb{R})$, but which are not P_k-Anosov for any k and are not limits of P_k-Anosov representations for any k.

28. **Stability**

The proof of stability makes use of Labourie's original (equivalent) definition of P_k-Anosov purely in terms of sections of $\widehat{E}^k_\rho$. If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation, let

$$G^j(\Gamma) = U(\Gamma) \times \mathrm{Gr}_j(\mathbb{R}^d) \qquad \text{and} \qquad \widehat{G}^j_\rho = G^j(\Gamma)/\Gamma$$

where, as usual, Γ acts on the first factor by Γ and on the second factor by $\rho(\Gamma)$. The flow $\tilde{\phi}_t$ extends to the trivial flow $\tilde{\psi}^j_t$ on $G^j(\Gamma)$ and descends to a flow ψ^j_t on $\widehat{G}^j_\rho$ extending the flow on $\widehat{U}(\Gamma)$.

We may endow $\widehat{G}^j_\rho$ with a continuously varying family of Riemannian metrics on the fibers. Then the flow is differentiable on the fibers in the sense that if $Z \in \widehat{U}(\Gamma)$, $(\widehat{G}^j_\rho)_Z$ is the fiber over Z and and $X \in (\widehat{G}^j_\rho)_Z$, then

$$D_X \psi^j_t : T_Z(\widehat{G}^j_\rho)_X \to T_{\psi_t(Z)}(\widehat{G}^j_\rho)_{\phi_t(Z)}$$

is well-defined. If $r : \widehat{U}(\Gamma) \to \widehat{G}^j_\rho$ is a flow-invariant section, we say ψ_t is *contracting along r* if there exists $a, A >$ so that if $t > 0$ and $Z \in \widehat{U}(\Gamma)$, then

$$||D_{r(Z)}\psi_t|| \le A e^{-at}.$$

Notice that a section $\sigma : \widehat{U}(\Gamma) \to \widehat{E}^k_\rho$ gives rise to sections $\sigma^k : \widehat{U}(\Gamma) \to \widehat{G}^k_\rho$ and $\sigma^{d-k} : \widehat{U}(\Gamma) \to \widehat{G}^{d-k}_\rho$ (and vice versa). We say that σ is *transverse* if $\sigma^k(Z)$ is transverse to $\sigma^{d-k}(Z)$ for all $Z \in \widehat{U}(\Gamma)$. Notice that σ lifts to a ρ-equivariant section $\sigma : \widehat{U}(\Gamma) \to \widehat{E}^k_\rho$, as do σ^k and σ^{d-k}. In this case, we obtain a splitting $E_\rho = \Xi_\sigma \oplus \Sigma_\sigma$ so that $\Xi_\sigma|_Z = \tilde{\sigma}^k(Z)$ and $\Theta_\sigma|_Z = \tilde{\sigma}^{d-k}(Z)$ which descends to a splitting $\widehat{E}_\rho = \widehat{\Xi}_\sigma \oplus \widehat{\Sigma}_\sigma$. At the beginning of the chapter we saw that $\sigma^*(\hat{V}^k_\rho)$ may be identified with $\mathrm{Hom}(\widehat{\Xi}_\sigma, \widehat{\Theta}_\sigma)$. Notice that the fiber of $\sigma^*(\hat{V}^k_\rho)$ above Z is exactly $T_{\sigma^k(Z)}(\widehat{G}^k_\rho)_Z$. So, if σ is flow-invariant, then the flow is contracting along σ^k if and only if the associated flow on $\mathrm{Hom}(\widehat{\Xi}_\sigma, \widehat{\Theta}_\sigma)$ is contracting.

We will make use of the following equivalence.

Lemma 28.1. ([45, Prop. 2.11]) *A representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov if and only if there exists a transverse flow invariant section $\sigma : \widehat{U}(\Gamma) \to \widehat{E}^k_\rho$ so that ψ^k_t is contracting along σ^k and ψ^{d-k}_{-t} is contracting along σ^{d-k}.*

Proof. Recall that if ρ is P_k-Anosov with P_k-limit maps ξ_ρ and θ_ρ, then one obtains a transverse flow-invariant section $\sigma : \widehat{U}(\Gamma) \to \widehat{E}^k_\rho$ so that the flow is contracting along $\sigma^*(\hat{V}^k_\rho)$ and the inverse flow is contracting along $\sigma^*(\hat{V}^{d-k}_\rho)$. Therefore, ψ^k_t is contracting along σ^k and ψ^{d-k}_{-t} is contracting along σ^{d-k}. This completes the forward direction of the proof.

Now suppose that there exists a transverse flow invariant section $\sigma : \widehat{U}(\Gamma) \to \widehat{E}^k_\rho$ so that ψ^k_t is contracting along σ^k and ψ^{d-k}_{-t} is contracting along σ^{d-k}. We may lift σ^k to a section $\tilde{\sigma}^k : U(\Gamma) \to U(\Gamma) \times \mathrm{Gr}_k(\mathbb{R}^d)$. Since σ is flow-invariant, there exists $\xi : \partial\Gamma \times \partial\Gamma - \Delta \to \mathrm{Gr}_k(\mathbb{R}^d)$ so that $\tilde{\sigma}^k(x, y, s) = ((x, y, s), \xi(x, y))$. Given $x \in \partial\Gamma$, consider $y, z \in \partial\Gamma \setminus \{x\}$. There exists $a \in \mathbb{R}$ so that $(x, y, 0)$ and (x, z, a) lie in the same leaf of the stable foliation of $U(\Gamma)$, so

$$\lim_{t\to\infty} d\Big((x, y, t), (x, y, a+t)\Big) = 0$$

where d is some Γ-invariant metric on $U(\Gamma)$. (We established the existence of an stable foliation for closed hyperbolic manifolds, and more generally closed strictly convex projective manifolds in Section 22. Mineyev [166, Theorem 60(h)] establishes the existence for all geodesic flows of hyperbolic groups.) But, since σ is continuous and $D\tilde{\psi}_t^k$ is contracting along σ^k, one see that

$$\lim_{t \to 0} d\Big(\tilde{\sigma}^k(x,y,t), \tilde{\sigma}^k(x,z,t+a)\Big) = 0$$

Therefore, $d\big(\xi(x,y), \xi(x,z)\big) = 0$, so $\xi(x,y) = \xi(x,z)$. So, ξ descends to a continuous, ρ-equivariant map $\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ so that $\tilde{\sigma}^k(x,y,s) = ((x,y,s), \xi_\rho(x))$. Similarly, there exists a continuous, ρ-equivariant map $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$ so that $\tilde{\sigma}^{d-k}(x,y,s) = ((x,y,s), \theta_\rho(y))$.

Since σ is transverse, ξ_ρ and θ_ρ are transverse. Since the flow is contracting along $\sigma^*(\hat{V}_\rho^k)$, we conclude that ρ is Anosov with limit maps ξ_ρ and θ_ρ.

$\square$

We are now ready to establish the stability of P_k-Anosov representations. Labourie [143, Proposition 2.1] first established stability by showing that $\sigma_\rho^k(U(\Gamma))$ and $\sigma_\rho^{d-k}(\widehat{U}(\Gamma))$ are isolated hyperbolic sets for ψ_t^k and ψ_{-t}^{d-k} and applying standard results from hyperbolic dynamics. We will give a more self-contained proof which follows the treatment of Guichard and Wienhard [109, Theorem 5.13]. This type of argument is fairly standard in hyperbolic dynamics, going back at least to the 1960's. However, it may be best to skip this argument if it feels alien to you, since we will not be using similar arguments elsewhere.

Theorem 28.2. *If $\rho_0 : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a P_k-Anosov representation, then there exists a neighborhood U of ρ_0 in $\mathrm{Hom}(\Gamma, \mathsf{SL}(d, \mathbb{R}))$ so that if $\rho \in U$, then ρ is P_k-Anosov.*

Moreover, the maps $X_k : U \to C^0(\partial\Gamma, \mathrm{Gr}_k(\mathbb{R}^d))$ and $X_{d-k} : U \to C^0(\partial\Gamma, \mathrm{Gr}_{d-k}(\mathbb{R}^d))$ given by $X_k(\rho) = \xi_\rho$ and $X_{d-k}(\rho) = \theta_\rho$ are continuous.

Proof. Suppose that W is a neighborhood of ρ_0 in $\mathrm{Hom}(\Gamma, \mathsf{SL}(d, \mathbb{R}))$. We can form the associated flat Grassmanian bundles $\widehat{G}_W^k$ and $\widehat{G}_W^{d-k}$ over $W \times \widehat{U}(\Gamma)$ by first considering

$$G_W^j = W \times U(\Gamma) \times \mathrm{Gr}_j(\mathbb{R}^d)$$

for all $j = 1, \dots, d-1$, and the projection map $\pi_j : G_W^j \to W \times U(\Gamma)$, which is a fibre bundle map. Let the flow $\{\psi_t^j\}_{t \in \mathbb{R}}$ on G_W^j be given by $\psi_t^j(\rho, Z, P) = (\rho, \phi_t(Z), P)$ for all t. We let the group Γ act on G_W^j by

$$\gamma(\rho, Z, P) = (\rho, \gamma(Z), \rho(\gamma)(P))$$

and set

$$\widehat{G}_W^j = G_W^j / \Gamma.$$

Then π_j descends to give a fiber bundle $\hat{\pi}_j : \widehat{G}_W^k \to W \times \widehat{U}(\Gamma)$ with fiber $\mathrm{Gr}_j(\mathbb{R}^d)$. Since all the ψ_t^W commute with the action of Γ, they descend to a flow $\{\hat{\psi}_t^j\}_{t \in \mathbb{R}}$ on $\widehat{G}_W^j$. (Notice that we are really just collecting all the $\widehat{G}_\rho^j$ for all $\rho \in W$.)

Let $\xi_0 : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta_0 : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$ be the limit maps for ρ_0. Let

$$\sigma_0^k : \{\rho_0\} \times \widehat{U}(\Gamma) \to \widehat{G}_W^k|_{\{\rho_0\} \times \widehat{U}(\Gamma)}$$

be the associated section. Let $||\cdot||$ be a continuous family of Riemannian metrics on the fibers of G_W^k. Recall that the flow ψ_t^j is contracting on $\sigma_0^k(\widehat{U}(\Gamma))$.

We can choose a closed neighborhood W_1 of σ_0, contained in W, a section

$$\sigma^k : W_1 \times \widehat{U}(\Gamma) \to \widehat{G}^k_W$$

$t_0 > 0$ and a closed neighborhood B of $\sigma_0(\{\rho_0\} \times \widehat{U}(\Gamma)$ in $G^k_{W_1}$ so that

(1) $\sigma^k|_{\{\rho_0\} \times \widehat{U}(\Gamma))} = \sigma^k_0$,
(2) $\sigma^k(W_1 \times \widehat{U}(\Gamma)) \subset B$,
(3) $||D\psi^k_t(\vec{v})|| \leq \frac{1}{2}||\vec{v}||$ for all $\vec{v} \in T_b(\widehat{G}^k_W)|_{\pi_k(b))}$ where $b \in B$ and $t \geq t_0$, and
(4) $\psi^k_t(B) \subset B$ for all $t \geq t_0$.

However, we cannot yet assume that σ^k is flow invariant.

Let $S(W_1, B)$ be the set of continuous sections $s : W_1 \times \widehat{U}(\Gamma) \to \widehat{G}^{d-k}_W$ so that $s(W_1 \times \widehat{U}(\Gamma)) \subset B$. For any $r \geq t_0$, we can define a uniform contraction F^r on $S(W_1, B)$, given by saying that $F^r(s)$ is the section so that

$$F^r(s)(\rho, Z) = \psi^k_r(s(\rho, \phi_{-r}(Z))).$$

The contraction mapping theorem, then assures that F^r has a unique fixed point

$$\nu^k_r = \lim(F^r)^n(\sigma^k).$$

(Here, the norm on the tangent spaces to the fibers gives rise to a continuous family of metrics on the fibers. We then get a metric on $S(W_1, B)$ by considering

$$d(s_1, s_2) = \max\{d(s_1(\rho, \widehat{Z}), s_2(\rho, \widehat{Z})) \mid (\rho, \widehat{Z}) \in W_1 \times \widehat{U}(\Gamma)\}$$

where $s_1, s_2 \in S(W_1, B)$ and the distance is measured in the fibers. Our assumptions imply that $d(F^r(s_1), F^r(s_2)) \leq \frac{1}{2}d(s_1, s_2)$ if $r \geq t_0$ and $s_1, s_2 \in S(W_1, B)$.)

Notice that since σ_0 is fixed by ψ^k, $\nu^k_r|_{\{\rho_0\} \times \widehat{U}(\Gamma)} = \sigma^k_0$. Since F^{r_1} and F^{r_2} commute for all $r_1, r_2 \geq t_0$ and the fixed points are unique, $\nu^k_{r_1} = \nu^k_{r_2}$. So we can simply define $\nu^k = \nu_r$ for some (any) $r \geq t_0$ and ν^k is fixed by all ψ^k_t with $t \geq t_0$ and hence for all $t \in \mathbb{R}$. Moreover, ψ^k_t is contracting along $\nu^k(W_1 \times \widehat{U}(\Gamma))$.

Similarly, there exists a section $\sigma^{d-k}_0 : \widehat{U}(\Gamma) \to \widehat{G}^{d-k}_{\rho_0}$ so that the inverse flow $\{\psi^{d-k}_{-t}\}$ is contracting along σ^{d-k}_0. We can then run the whole argument above to obtain a neighborhood W_2 of ρ_0 and a flow-invariant section $\nu^{d-k} : W_2 \times \widehat{U}(\Gamma) \to \widehat{G}^{d-k}_W$ so that the inverse flow is contracting along ν^{d-k}.

Since transversality is an open condition, and ξ_0 and ρ_0 are transverse, we can find an open neighborhood $U \subset W_1 \cap W_2$ of ρ_0, so that if $\rho \in U$, then ν^k_ρ and ν^{d-k}_ρ are transverse. One may then combine ν^k and ν^{d-k} to obtain a transverse flow invariant section $\nu : U \times \widehat{E}^k_\rho$ so that if $\rho \in U$, then then ψ^k_t is contracting along $\nu^k(\{\rho\} \times \widehat{U}(\gamma))$ and ψ^{d-k}_{-t} is contracting along $\sigma^{d-k}(\{\rho\} \times \widehat{U}(\Gamma))$. Therefore, if $\rho \in U$, then ρ is P_k-Anosov. Since ν is continuous, X_k and X_{d-k} are both continuous on U. □

Hirsch, Pugh and Shub [115, Theorem 3.8] proved that any section obtained in this manner is actually Hölder, which implies that our limit maps are Hölder.

Guichard and Wienhard [109, Theorem 5.14] note that since one can choose the contraction constants to be uniform over the neighborhood U in the proof above, one may also choose

uniform constants D and L so that if $\rho \in U$ and $\gamma \in \Gamma$, then

$$D||\gamma|| \geq \log\left(\frac{\lambda_k(\rho(\gamma))}{\lambda_{k+1}(\rho(\gamma))}\right) \geq \frac{1}{D}||\gamma|| - L.$$

Along with Proposition 8.3 this is all one needs to generalize Fricke's Theorem into the setting of P_k-Anosov representations.

Let

$$\mathrm{Anosov}_k(\Gamma, \mathsf{SL}(d,\mathbb{R})) \subset X(\Gamma, \mathsf{SL}(d,\mathbb{R}) = \mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{R}))/\mathsf{SL}(d,\mathbb{R})$$

be the set of conjugacy classes of P_k-Anosov representations.

Theorem 28.3. (Guichard-Wienhard [109, Corollary 5.4], see also [55, Theorem 6.2]) *If Γ is a torsion-free hyperbolic group, $d \geq 2$ and $1 \leq k \leq \frac{d}{2}$, then* $\mathrm{Out}(\Gamma)$ *acts properly discontinuously on* $\mathrm{Anosov}_k(\Gamma, \mathsf{SL}(d,\mathbb{R}))$.

Remarks: (1) Bridgeman, Canary, Labourie and Sambarino [45, Theorem 6.1] show that if ρ_0 is a smooth point of the (real algebraic) variety $\mathrm{Hom}(\Gamma, \mathsf{SL}(d,\mathbb{R}))$, then one can choose U so that X_k and X_{d-k} are real analytic on U. The proof is really just a more technically complicated version of the argument above.

(2) Kapovich, Leeb and Porti [133, Theorem 1.5] develop a notion of P_k-Morse quasigeodesic and show that a representation is P_k-Anosov if and only if the orbit map takes geodesic in C_Γ to (uniform) P_k-Morse-quasigeodesics. They also establish a local-to-global principle [134, Theorem 7.18] which allows them to establish stability of P_k-Anosov representations with a proof which is reminiscent of the proof in rank one, see [134, Theorem 7.33].

Part 6. **Anosov representations: Characterizations and Examples**

> *Where the eagle glides ascending*
> *There's an ancient river bending*
> *Down the timeless gorge of changes*
> *Where sleeplessness awaits*
> *I searched out my companions*
> *Who were lost in crystal canyons*
> *When the aimless blade of science*
> *Slashed the pearly gates*
> ———————————Neil Young [225]

In this chapter, we establish characterizations of Anosov representations due to Guéritaud-Guchard-Kassel-Wienhard [107], Guichard-Wienhard [109], Kapovich-Leeb-Porti [132, 134], Kassel-Potrie [136] and Tsouvalas [208]. Kapovich, Leeb and Porti develop other powerful and beautiful characterizations, but these will be out of the purview of these notes, as they involve a more sophisticated study of symmetric spaces.

The major early examples of Anosov representations were convex cocompact representations into rank one Lie groups, Benoist representations, Hitchin representations of surface groups into split real Lie groups (e.g. $\mathsf{SL}(d,\mathbb{R})$), and maximal representations of surface groups into Lie groups of Hermitian type (e.g. $\mathsf{SP}(2n,\mathbb{R})$). Our characterizations will allow us to show that convex cocompact representations in rank one Lie groups and Benoist representations are P_1-Anosov. We will show that d-Fuchsian representations into $\mathsf{SL}(d,\mathbb{R})$ are Borel Anosov (i.e. P_k-Anosov for all k), and hence that their small deformations are Borel Anosov. Labourie [143] showed that all deformations of d-Fuchsian representations into $\mathsf{SL}(d,\mathbb{R})$, i.e. all Hitchin representations, are Borel Anosov, but this will not be covered in these notes. The theory of maximal representations was pioneered by Burger-Iozzi-Wienhard [52] and we refer the reader to their original paper or to their survey article [53] for a discussion of maximal representations.

We again encourage the less experienced reader to focus exclusively on the P_1-Anosov case when first reading these notes. We will see that a representation is P_k-Anosov if and only if its kth exterior power is P_1-Anosov, see Theorem 34.3, so this restriction in viewpoint is not too limiting.

29. **More linear algebra in $\mathsf{SL}(d,\mathbb{R})$**

> *Mathematics and politics*
> *Three sixes and deadly tricks*
> ———————————Lucinda Williams [216]

In this section, we derive some algebraic consequences of the singular value decomposition which we will need later.

We recall that if $A \in \mathsf{SL}(d,\mathbb{R})$ then we may write $A = LDK$ where $K, L \in \mathsf{SO}(d)$ and D is the diagonal matrix with entries $d_{ii} = \sigma_i(A)$ for all i. In general, L and K can not be chosen canonically when some of the singular values agree. However, if $\sigma_k(A) > \sigma_{k+1}(A)$, we can define the subspaces $U_k(A) = L(< e_1, \ldots, e_k >)$ and $V_{d-k}(A) = L\big(< e_{k+1}, \ldots, e_d > \big)$. Geometrically,

$A(S^{d-1})$ is an ellipsoid and $U_k(A)$ is the k-plane spanned by the k longest axes of this ellipsoid and $V_{d-k}(A)$ is the subspace spanned by the $d - k$ shortest axes.

We begin with an elementary lemma about singular values of products.

Lemma 29.1. *If $A, B \in \mathsf{GL}(d, \mathbb{R})$, then*

$$\sigma_i(A)\sigma_1(B) \geq \sigma_i(AB) \geq \sigma_i(A)\sigma_d(B)$$

for all $i = 1, \ldots, d$.

Proof. Recall that if $T \in \mathsf{GL}(d, \mathbb{R})$ we can define $\sigma_i(T)$ iteratively, by letting

$$\sigma_1(T) = \max \left\{ \frac{||T(\vec{u})||}{||\vec{u}||} \,\middle|\, \vec{u} \in \mathbb{R}^d - \{\vec{0}\} \right\}$$

and choosing $\vec{v}_1(T)$ so that $||\vec{v}_1(T)|| = 1$ and $\sigma_1(T) = ||T(\vec{v}_1(T))||$ and then defining

$$\sigma_{j+1}(T) = \max \left\{ \frac{||T(\vec{u})||}{||\vec{u}||} \,\middle|\, \vec{u} \in \mathbb{R}^d - \{\vec{0}\}, \quad \vec{u} \perp \vec{v}_j(T) \text{ if } i \leq j \right\}$$

and $v_{j+1}(T)$ so that $\vec{v}_{j+1}(T) \perp \vec{v}_i(T)$ for all $i \leq j$, $||\vec{v}_{j+1}(T)|| = 1$ and $\sigma_{j+1}(T) = |||T(\vec{v}_{j+1})||$.

Let $\vec{u}_1$ be a unit vector in $< B^{-1}(v_1(A)) >$, then

$$\sigma_1(AB) \geq ||AB(\vec{u}_1)|| = \sigma_1(A)||\vec{B}(\vec{u}_1)|| \geq \sigma_1(A)\sigma_d(B).$$

For general i, choose a unit vector

$$\vec{u}_i \in \left(< v_1(AB), \ldots, v_{i-1}(AB) >^{\perp} \cap < B^{-1}(v_1(A)), \ldots, B^{-1}(v_i(A)) > \right).$$

Then, $B(\vec{u}_i) \in < v_1(A), \ldots, v_i(A) >$, so

$$\sigma_i(AB) \geq ||A(B\vec{u}_i)|| \geq \sigma_i(A)||B(\vec{u}_i)|| \geq \sigma_i(A)\sigma_d(B)$$

which completes the proof of the lower bound.

On the other hand, using the lower bound, we see that

$$\sigma_i(A) = \sigma_i((AB)B^{-1}) \geq \sigma_i(AB)\sigma_d(B^{-1}) = \frac{\sigma_i(AB)}{\sigma_1(B)}$$

which establishes the upper bound. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Recall that $d(< \vec{v} >, < \vec{w} >) = \sin\theta(\vec{v}, \vec{w})$ (where $\theta(\vec{v}, \vec{w})$ is the angle between $\vec{v}$ and $\vec{w}$) gives a metric on $\mathbb{RP}^{d-1}$. We will need the following well-known linear algebra lemma. The following proof is taken directly from Tsouvalas [208] (see also Bochi-Potrie-Sambarino [32, Lemma A.4]).

Lemma 29.2. *If $A, B \in \mathsf{SL}(d, \mathbb{R})$, $\sigma_1(A) > \sigma_2(A)$ and $\sigma_1(AB) > \sigma_2(AB)$, then*

$$d(U_1(AB), U_1(A)) \leq \sqrt{d-1}\frac{\sigma_1(B)}{\sigma_d(B)}\frac{\sigma_2(A)}{\sigma_1(A)}.$$

Proof. If $A = L_A D_A K_A$ is the singular value decomposition of A and $AB = L_{AB} D_{AB} K_{AB}$ is the singular values decomposition of AB, then $AB = L_{AB} D_{AB} K_{AB} = L_A D_A K_A B$. Thus, $L_A^{-1} L_{AB} D_{AB} = D_A K_A B K_{AB}^{-1}$, so

$$\langle L_A^{-1} L_{AB} D_{AB} e_1, e_i \rangle = \langle D_A K_A B K_{AB}^{-1} e_1, e_i \rangle$$

which implies that

$$\sigma_1(AB)\langle L_A^{-1} L_{AB} e_1, e_i \rangle = \sigma_i(A)\langle K_A B K_{AB}^{-1} e_1, e_i \rangle$$

for all i. We further note that

$$\sigma_1(AB) \geq \sigma_1(A)\sigma_d(B) \qquad \text{and} \qquad \left|\langle K_A B K_{AB}^{-1} e_1, e_i \rangle\right| \leq \sigma_1(B)$$

for all i, so

$$\left|\langle L_A^{-1} L_{AB} e_1, e_i \rangle\right| \leq \frac{\sigma_i(A)}{\sigma_1(A)} \frac{\sigma_1(B)}{\sigma_d(B)} \leq \frac{\sigma_2(A)}{\sigma_1(A)} \frac{\sigma_1(B)}{\sigma_d(B)}$$

for all $i \geq 2$. Now notice that

$$
\begin{aligned}
d(U_1(AB), U_1(A))^2 &= d(L_{AB}e_1, L_A e_1)^2 = \sin^2 \theta(L_{AB}e_1, L_A e_1) \\
&= 1 - |\langle L_{AB}e_1, L_A e_1 \rangle|^2 = 1 - |\langle L_A^{-1} L_{AB}e_1, e_1 \rangle|^2,
\end{aligned}
$$

so, since $\sum_{i=1}^{d} \left|\langle L_A^{-1} L_{AB}e_1, e_i \rangle\right|^2 = 1$,

$$d(U_1(AB), U_1(A))^2 = \sum_{i=2}^{d} \left|\langle L_A^{-1} L_{AB}e_1, e_i \rangle\right|^2 \leq (d-1)\left(\frac{\sigma_2(A)}{\sigma_1(A)} \frac{\sigma_1(B)}{\sigma_d(B)}\right)^2.$$

$\square$

We now explain how to get the corresponding result for general k. If this is your first time learning about Anosov representation and/or you're not very familiar with the algebra involved here, you can just ignore this and focus on the P_1 case.

If $2 \leq k \leq d-1$, we recall the **exterior power representation** $E_k^d : \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(\Lambda^k(\mathbb{R}^d))$ where $\Lambda^k(\mathbb{R}^d)$ is the kth exterior power of $\mathbb{R}^d$. Then if $A = LDK \in \mathsf{SL}(d, \mathbb{R})$ with $K, L \in \mathsf{SO}(d)$ and D diagonal, then $E_k^d(A) = E_k^d(L)E_k^d(D)E_k^d(K)$ where $E_k^d(L), E_k^d(K) \in \mathsf{SO}(\Lambda^k \mathbb{R}^d)$ and $E_k^d(D)$ is diagonal, is the singular value decomposition of $E_k^d(A)$. In particular, $\sigma_1(E_k^d(A)) = \sigma_1(A)\sigma_2(A)\cdots\sigma_k(A)$ and $\sigma_2(E_k^d(A)) = \sigma_1(A)\cdots\sigma_{k+1}(A)$. So,

$$\frac{\sigma_1(E_k^d(A))}{\sigma_2(E_k^d(A))} = \frac{\sigma_k(A)}{\sigma_{k+1}(A))}.$$

Moreover, there exists an embedding $G_k^d : \mathrm{Gr}_k(\mathbb{R}^d) \to \mathbb{P}(\Lambda^k \mathbb{R}^d)$, called the **Plücker embedding**, which takes $< \vec{v}_1, \ldots, \vec{v}_k >$ to $< \vec{v}_1 \wedge \cdots \wedge \vec{v}_k >$. We can then place a metric on $\mathrm{Gr}_k(\mathbb{R}^d)$ by letting $d(P, Q) = d(G_k^d(P), G_K^d(Q))$ for all $P, Q \in \mathrm{Gr}_k(\mathbb{R}^d)$. Finally, we notice that if $\sigma_k(A) < \sigma_{k+1}(A)$, then $U_1(E_k^d(A)) = G_k^d(U_k(A))$. If we put this all together with Lemma 29.2 we get the following corollary:

Corollary 29.3. *If $A, B \in \mathsf{SL}(d, \mathbb{R})$, $\sigma_k(A) > \sigma_{k+1}(A)$ and $\sigma_k(AB) > \sigma_{k+1}(AB)$, then*

$$d(U_k(AB), U_k(A)) \leq \sqrt{d-1}\left(\frac{\sigma_1(B)\cdots\sigma_k(B)}{\sigma_{d-k+1}(B)\cdots\sigma_d(B))}\right)\frac{\sigma_{k+1}(A)}{\sigma_k(A)}.$$

If A is proximal, we can bound the difference between $U_1(A)$ and the attracting eigenline A^+ of A. Our proof is again taken from Tsouvalas [208].

Lemma 29.4. *If $A \in \mathsf{SL}(d, \mathbb{R})$ and A is proximal, then*

$$d(U_1(A), A^+) \leq \frac{\sigma_2(A)}{|\lambda_1(A)|}.$$

Proof. Let $A = LDK$ be the singular value decomposition, so $K, L \in \mathsf{SO}d)$, D is diagonal, $d_{11} = \sigma_1(A)$ and $U_1(A) = <L(e_1)>$. Choose $M \in \mathsf{SO}(d)$ so that $<M(e_1)> = A^+$. Then

$$
\begin{aligned}
<M(e_1), L(e_i)> \;&=\; <L^{-1}M(e_1), e_i> = <D^{-1}K^{-1}A^{-1}M(e_1), e_i> = <K^{-1}(\lambda_1(A)M(e_1)), De_i> \\
&=\; \frac{\sigma_i(A)}{\lambda_1(A)} <K^{-1}M(e_1), e_i>
\end{aligned}
$$

for all i. Since $d(U_1(A), A^+) = \sin \angle (U_1(A), A^+) = \sin \angle (M(e_1), L(e_1))$, we see that

$$
\begin{aligned}
d(U_1(A), A^+)^2 \;&=\; 1 - |<M(e_1), L(e_1)>|^2 = \sum_{i=2}^{d} |<M(e_1), L(e_i)>|^2 \\
&=\; \sum_{i=2}^{d} \frac{\sigma_i(A)^2}{\lambda_1(A)^2} |<K^{-1}M(e_1), e_i>|^2 \\
&\leq\; \frac{\sigma_2(A)^2}{\lambda_1(A)^2}
\end{aligned}
$$

$\square$

We get the following consequence in the P_k setting.

Corollary 29.5. *If $A \in \mathsf{SL}(d, \mathbb{R})$, A is P_k-proximal, and A_k^+ is the attracting k-plane of A, then*

$$
d(U_k(A), A_k^+) \leq \frac{\sigma_1(A) \cdots \sigma_{k-1}(A)\sigma_{k+1}(A)}{|\lambda_1(A) \cdots \lambda_k(A)|}
$$

Proof. Notice that if A is P_k-proximal, then $E_k^d A$ is proximal,

$$
\sigma_2(E_k^d(A)) = \sigma_1(A) \cdots \sigma_{k-1}(A)\sigma_{k+1}(A) \qquad \text{and} \qquad `\lambda_1(E_k^d(A)) = \lambda_1(A) \cdots \lambda_k(A).
$$

Lemma 29.4 then immediately implies that

$$
d(U_k(A), A_k^+) \leq \frac{\sigma_2(E_k^d(A))}{|\lambda_1(E_k^d(A))|} = \frac{\sigma_1(A) \cdots \sigma_{k-1}(A)\sigma_{k+1}(A)}{|\lambda_1(A) \cdots \lambda_k(A)|}.
$$

$\square$

30. **The Cartan property**

Out in California
They're better than we are
They get no questions from their parents
They got no ceilings on their cars
And just as soon as they inspect their tan lines
They got a billion other things to do
————————*Pat McCurdy* [220]

We say that a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-**divergent** if whenever $\{\gamma_n\}$ is a sequence of distinct elements of Γ, then

$$\lim \log \left(\frac{\sigma_k(\rho(\gamma_n))}{\sigma_{k+1}(\rho(\gamma_n))} \right) = +\infty.$$

Proposition 27.1 implies that P_k-Anosov representations are P_k-divergent.

Suppose that Γ is Gromov hyperbolic, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-divergent and $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ is a continuous, ρ-equivariant map. We say that ξ has the **Cartan property** if whenever $\{\gamma_n\}$ is a sequence of distinct elements of Γ converging to $z \in \partial\Gamma$, then $\xi(z) = \lim U_k(\rho(\gamma_n))$. (Notice that since ρ is P_k-divergent, $U_k(\rho(\gamma_n))$ is defined for all sufficiently large n.)

We first observe that if a P_k-limit map has the Cartan property, then it is dynamics preserving at fixed points of P_k-proximal elements.

Lemma 30.1. *Suppose that Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation, and $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ is a continuous ρ-equivariant map with the Cartan property. If $\rho(\gamma)$ is P_k-proximal, then $\xi(\gamma^+)$ is the attracting k-plane of $\rho(\gamma)$.*

Proof. Recall that if γ has infinite order, then $\log |\lambda_i(\rho(\gamma))| = \lim \frac{\log \sigma_i(\rho(\gamma)^n)}{n}$ for all i.

Suppose $k = 1$ and $\rho(\gamma)$ is proximal, then

$$\lim \frac{1}{n} \log \left(\frac{\lambda_1(\rho(\gamma^n))}{\sigma_2(\rho(\gamma^n))} \right) = \log \left(\frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))} \right) > 0, \quad \text{so} \quad \lim \left(\frac{\sigma_2(\rho(\gamma^n))}{\lambda_1(\rho(\gamma^n))} \right) = 0.$$

Lemma 29.4 then implies that $\xi(\gamma^+) = \lim U_1(\rho(\gamma)^n)$ is the attracting eigenline of $\rho(\gamma)$.

If $k \neq 1$, we use Corollary 29.5 in place of Lemma 29.4. $\qquad\square$

As a consequence, we obtain a uniqueness property for limit maps with the Cartan property, when $\rho(\Gamma)$ contains a P_k-proximal element.

Corollary 30.2. *Suppose that Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation, and $\rho(\Gamma)$ contains a P_k-proximal element. If $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\hat{\xi} : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ are a continuous ρ-equivariant maps with the Cartan property, then $\xi = \hat{\xi}$.*

Proof. Suppose that $\alpha \in \Gamma$ and $\rho(\alpha)$ is P_k-proximal. Then, by Lemma 30.1, $\xi(\alpha^+) = \hat{\xi}(\alpha^+)$ is the attracting k-plane of $\rho(\alpha)$. By equivariance, ξ and $\hat{\xi}$ agree on the orbit $\Gamma(\alpha^+)$ of α^+. Since $\Gamma(\alpha^+)$ is dense in $\partial\Gamma$, by Proposition 5.6, and ξ and $\hat{\xi}$ are both continuous, this implies that $\xi = \hat{\xi}$. $\qquad\square$

We next show that if the ratio of singular values grow faster than linearly, then there is a limit map with the Cartan property.

Proposition 30.3. (Guéritaud-Guichard-Kassel-Wienhard [107, Theorem 1.1(1)]) *If Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation and there exist $c > 1$ and C so that*

$$\log \left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \right) \geq c \log(d(id, \gamma)) - C$$

then there exists a continuous ρ-equivariant map $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ which has the Cartan property.

One may calculate that if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\log \frac{\sigma_1(A)}{\sigma_2(A^n)} \sim 2 \log n$, so if $c \in (1,2]$, then the growth condition in Proposition 30.3 does not imply that every element is proximal. However, if $c > 2$, one can further prove that $\rho(\gamma)$ is proximal if γ has infinite order, see Guéritaud-Guichard-Kassel-Wienhard [107, Lemma 2.27], and so ξ is also dynamics-preserving, by Lemma 30.1, see [107, Theorem 1.1(2)]. If $\rho : F_n \to \mathsf{SL}(2, \mathbb{R})$ is a geometrically finite, but not convex cocompact, representation, then ρ satisfies the growth condition with $c = 2$, but not with respect to any $c > 2$.

Proof. We first give the proof when $k = 1$. Notice that, in this case, our assumption immediately implies that ρ is P_1-divergent.

Suppose $z \in \partial\Gamma$. Let $\gamma : [0, \infty) \to C_\Gamma$ be a geodesic in the Cayley graph C_Γ of Γ so that $\gamma(0) = id$ and $\gamma(\infty) = z$. Then $\gamma_n = \gamma(n) \in \Gamma$ for all n and Lemma 29.2 implies that

$$d(U_1(\rho(\gamma_n)), U_1(\rho(\gamma_{n+1}))) \leq \sqrt{d-1} \frac{\sigma_1(\gamma_n^{-1}\gamma_{n+1})}{\sigma_d(\gamma_n^{-1}\gamma_{n+1})} \frac{\sigma_2(\gamma_n)}{\sigma_1(\gamma_n)} \leq \sqrt{d-1} M^2 e^C n^{-c}$$

where $M = \sup\{\sigma_1(s) \mid s \in S\}$ and S is the symmetric generating set for Γ used to construct C_Γ (since $\gamma_n^{-1}\gamma_{n+1} \in S$). Since $c > 1$, it follows that $\{U_1(\rho(\gamma_n))\}$ is a Cauchy sequence and we define

$$\xi(z) = \lim U_1(\rho(\gamma_n)).$$

Now suppose that $\{\alpha_n\}$ is an arbitrary sequence converging to z. Given any $R \in \mathbb{N}$, there exists $N_R \geq R$ so that if $n > N_R$, then a geodesic $\{\beta_0 = \alpha_n, \beta_1, \ldots, \beta_s = \gamma_n\}$ in Γ joining α_n to γ_n does not pass within R of the origin. The above calculation, then implies that

$$\begin{aligned} d(U_1(\alpha_n), U_1(\gamma_n)) &\leq \sum_{i=1}^{s} d(U_1(\beta_{i-1}), U_1(\beta_i)) \leq \sum_{i=1}^{s} \sqrt{d-1} \frac{\sigma_1(\beta_i^{-1}\beta_{i+1})}{\sigma_d(\beta_i^{-1}\beta_{n+1})} \frac{\sigma_2(\beta_i)}{\sigma_1(\beta_i)} \\ &\leq \sqrt{d-1} M^2 C e^c \sum_{i=1}^{s} d(id, \beta_i)^{-c} \end{aligned}$$

Let α_m be the point on the geodesic closest to the identity. Choose $s = \lfloor 2\delta \rfloor + 1$. Then, by considering a triangle with vertice α_m, α_r and id, we see that α_{m+s} must lies a distance at most δ from a point x on a geodesic joining id to α_r and $d(x, id) \geq R - \delta$. It follows that $d(\alpha_{m+k}, id) \geq R - 3\delta + (k-s)$ for all k between 1 and $r - s$. One argues symmetrically to get $d(\alpha_{m-k}, id) \geq R - 3\delta + (k-s)$ for all k between 0 and m. Therefore,

$$\begin{aligned} d(U_1(\alpha_n), U_1(\gamma_n)) &\leq \sqrt{d-1} M^2 C e^c \sum_{i=1}^{s} d(id, \beta_i)^{-c} \\ &\leq 2\sqrt{d-1} M^2 C e^c \sum_{k=0}^{\infty} \left(R - 3\delta + (k-s) \right)^{-c} \end{aligned}$$

Since $\lim_{j\to\infty} \sum_{i=0}^{\infty} j^{-c} = 0$, we see that $\lim_{n\to\infty} d(U_1(\alpha_n), U_1(\gamma_n)) = 0$, so $\lim U_1(\alpha_n) = \lim U_1(\gamma_n)$. Therefore, ξ is well-defined and continuous.

In order to verify the result for general k, we simply use Corollary 29.3 in place of Lemma 29.2. □

Corollary 30.4. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a P_k-Anosov representation, then its limit maps have the Cartan property.*

Proof. Since, by Proposition 27.1, there exist $K > 0$ and C so that

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq Kd(id, \gamma) - C$$

for all $\gamma \in \Gamma$, Proposition 30.3 gives rise to a continuous, ρ-equivariant map $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ with the Cartan property. Moreover, by Proposition 25.2, ξ_ρ is P_k-dynamics-preserving. Lemma 30.2 then implies that $\xi = \xi_\rho$, so ξ_ρ has the Cartan property.

Similarly, since

$$\log\left(\frac{\sigma_{d-k}(\rho(\gamma))}{\sigma_{d-k+1}(\rho(\gamma))}\right) = \log\left(\frac{\sigma_k(\rho(\gamma^{-1}))}{\sigma_{k+1}(\rho(\gamma^{-1}))}\right) \geq c\log(d(id, \gamma)) - C,$$

Proposition 30.3 produces a continuous, ρ-equivariant map $\theta : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$ with the Cartan property. We again check that $\theta_\rho = \theta$, so θ_ρ has the Cartan property. $\square$

Remark: The Cartan property naturally arises in the background of the work of Guichard-Wienhard [109], Guéritaud-Guichard-Kassel-Wienhard [107] and Kapovich-Leeb-Porti [132], but Tsouvalas [208] identifies it as a central player in the theory of Anosov representations.

31. Characterization of P_k-Anosov representations

All the people we used to know
They're an illusion to me now
Some are mathematicians
Some are carpenters' wives
Don't know how it all got started
I don't know what they're doin' with their lives
But me, I'm still on the road
Headin' for another joint
We always did feel the same
We just saw it from a different point of view
Tangled up in blue
——————————Bob Dylan [93]

The following characterization of Anosov representations will allow us to construct our first examples of Anosov representation. In particular, it will easily follow that convex cocompact representations into $\mathsf{SO}_0(d-1, 1) \subset \mathsf{SL}(d, \mathbb{R})$ are projective Anosov.

Theorem 31.1. (Tsouvalas [208, Theorem 1.1]) *Suppose that Γ is word hyperbolic and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation. Then ρ is P_k-Anosov if and only if*

(1) *ρ is P_k-divergent,*
(2) *there exist continuous transverse ρ-equivariant maps $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$, and*
(3) *ξ has the Cartan property.*

Proof. We have already proven that P_k-Anosov representations satisfy (1), (2) and (3) (see Proposition 27.1, Proposition 25.2 and Corollary 30.4).

So suppose ρ satisfies (1), (2) and (3). Let $\widehat{E}_\rho = \widehat{\Xi} \oplus \widehat{\Theta}$ be the splitting of the flat bundle given by ξ and θ. Let $|| \cdot ||$ be a continuous family of norms on $\widehat{E}_\rho$. Our main claim is:

Claim: *If $\hat{Z} \in \widehat{U}(\Gamma)$ and $\vec{v} \in \widehat{\Xi}|_{\hat{Z}}$ and $\vec{w} \in \widehat{\Theta}|_{\hat{Z}}$ are non-zero, then*

$$\lim_{t \to \infty} \frac{||\hat{\psi}_t(\vec{v})||}{||\hat{\psi}_t(\vec{w})||} = 0.$$

We will then use the fact that $\widehat{U}(\Gamma)$ is compact, to show that $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$ is contracting and hence that ρ is P_k-Anosov.

Proof of Claim: Suppose that $\hat{Z} \in \widehat{U}(\Gamma)$ and that $\vec{v} \in \widehat{\Xi}|_{\hat{Z}}$ and $\vec{w} \in \widehat{\Theta}|_{\hat{Z}}$ are non-zero. Choose a compact subset R of $U(\Gamma)$ so that $\Gamma(R) = U(\Gamma)$ and then choose $Z \in R$ which covers $\hat{Z}$ and let $(Z, \vec{x}) \in \Xi|_Z$ and $(Z, \vec{y}) \in \Theta|_Z$ cover $\vec{v}$ and $\vec{w}$ respectively. Recall that, in the notation of Proposition 27.1,

$$\frac{||\hat{\psi}_t(\vec{v})||}{||\hat{\psi}_t(\vec{w})||} = \frac{||\vec{x}||_{\phi_t(Z)}}{||\vec{y}||_{\phi_t(Z)}}.$$

Suppose that $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ and $\lim t_n = +\infty$. For each n, choose γ_n so that

$$\gamma_n(\phi_{t_n}(Z)) = W_n \in R.$$

Recall that if $Z = (z^+, z^-, s)$, then $\phi_t(Z) = (z^+, z^-, s+t)$, so $\lim \phi_{t_n}(Z) = z^+$, since orbits are quasi-isometrically embedded. Therefore, $\gamma_n^{-1}(W_n) \to z^+$, so $\gamma_n^{-1} \to z^+$.

By the equivariance of the norms,

$$||\rho(\gamma_n)(\vec{u})||_{W_n} = ||\vec{u}||_{\phi_{t_n}(Z)}$$

for all $\vec{u} \in \mathbb{R}^d$. So,

$$\frac{||\rho(\gamma_n)(\vec{x})||_{W_n}}{||\rho(\gamma_n)(\vec{y})||_{W_n}} = \frac{||\vec{x}||_{\phi_t(Z)}}{||\vec{y}||_{\phi_t(Z)}}.$$

Since R is compact, there exists K so that $|| \cdot ||_W$ is K-bilipschitz to $|| \cdot ||_0$ for all $W \in R$. Therefore, it suffices to show, that if $\vec{x} \in \Xi_Z$ and $\vec{y} \in \Theta_Z$, then

$$\frac{||\rho(\gamma_n)(\vec{x})||_0}{||\rho(\gamma_n)(\vec{y})||_0} \to 0.$$

We may normalize so that $\Xi|_Z = \xi(z^+) = < e_1, \ldots, e_k >$ and $\theta|_Z = \theta(z^-) = < e_{k+1}, \ldots, e_d >$. Since ξ has the Cartan property, $U_k(\rho(\gamma_n)^{-1}) \to \xi(z^+) = < e_1, \ldots, e_k >$. Therefore, if $\vec{x} \in \Xi|_Z$ is non-zero, then

$$\limsup \left(\frac{||\rho(\gamma_n)(\vec{x})||}{\sigma_{d-k+1}(\rho(\gamma_n))||\vec{x}||} \right) \le 1$$

Since $U_k(\rho(\gamma_n)^{-1}) \perp \rho(\gamma_n)^{-1}\big(U_{d-k}(\rho(\gamma))\big)$ for all n, we see that

$$\rho(\gamma_n)^{-1}\Big(U_{d-k}(\rho(\gamma_n))\Big) \to < e_{k+1}, \ldots, e_d >,$$

so if $\vec{y} \in \Theta|_Z = <e_{k+1}, \ldots, e_d>$, then

$$\liminf \left(\frac{||\rho(\gamma_n)(\vec{y})||}{\sigma_{d-k}(\rho(\gamma_n))||\vec{y}||} \right) \geq 1.$$

So, if $\vec{x} \in \Xi_Z$ and $\vec{y} \in \Theta_Z$ are non-zero, then

$$\limsup \left(\frac{||\rho(\gamma_n)(\vec{x})||_0}{||\rho(\gamma_n)(\vec{y})||_0} \right) \leq \limsup \left(\frac{\sigma_{d-k+1}(\rho(\gamma_n))||\vec{x}||}{\sigma_{d-k}(\rho(\gamma_n))||\vec{y}||} \right) = \limsup \left(\frac{\sigma_{k+1}(\rho(\gamma_n^{-1}))}{\sigma_k(\rho(\gamma_n^{-1}))} \right) \frac{||\vec{x}||}{||\vec{y}||}$$

so, since ρ is P_k-divergent,

$$\frac{||\rho(\gamma_n)(\vec{x})||_0}{||\rho(\gamma_n)(\vec{y})||_0} \to 0$$

as desired. This completes the proof of the claim. $\qquad\square$

Since $\widehat{U}(\Gamma)$ is compact, there exists $t_0 > 0$ so that if $\hat{Z} \in \widehat{U}(\Gamma)$, $\vec{v} \in \widehat{\Xi}|_{\hat{Z}}$ and $\vec{w} \in \widehat{\Theta}|_{\hat{Z}}$ are non-zero, and $t \geq t_0$, then

$$\frac{||\hat{\psi}_t(\vec{v})||}{||\hat{\psi}_t(\vec{w})||} =\leq \frac{1}{2} \frac{||\vec{v}||}{||\vec{w}||}.$$

We choose $a = \frac{\log 2}{t_0}$ and

$$C = \sup \left\{ \frac{||\psi_t(\vec{v})||}{||\psi_t(|\vec{w})|||} \; \Big| \; \hat{Z} \in \widehat{U}(\Gamma), \vec{v} \in \widehat{\Xi}|_{\hat{Z}}, \vec{w} \in \widehat{\Theta}_{\hat{Z}}, ||\vec{v}|| = ||\vec{w}|| = 1, t \in [0, t_0] \right\}.$$

Therefore, if $\vec{v} \in \widehat{\Xi}|_{\hat{Z}}$ and $\vec{w} \in \widehat{\Theta}|_{\hat{Z}}$ are non-zero, and $t \geq 0$, then

$$\frac{||\hat{\psi}_t(\vec{v})||}{||\hat{\psi}_t(\vec{w})||} \leq C e^{-at} \frac{||\vec{v}||}{||\vec{w}||}.$$

Lemma 25.1 then implies that the flow $\{\hat{\psi}_t\}$ is contracting on $\mathrm{Hom}(\widehat{\Theta}, \widehat{\Xi})$. In Section 25, we observed that this implies that ρ is P_k-Anosov. $\qquad\square$

Remarks: 1) If ρ is irreducible (when $k = 1$) or Zariski dense (for general k), and you assume that that ξ_ρ and θ_ρ are compatible, then this result follows from work of Guichard and Wienhard [109, Proposition 4.10, Theorem 4.11]), see Theorem 33.1. We will see that if ρ is irreducible (when $k = 1$) or Zariski dense (for general k), then ξ must have the Cartan property, so condition (3) is un-necessary in these situations.

2) If you assume that both ξ and θ have the Cartan property, then Tsouvalas [208] observes that Theorem 31.1 follows from work of Kapovich-Leeb-Porti [132, Theorem 1.1]. Moreover, Theorem 31.1 implies the equivalence of conditions (i) and (iii) in [132, Theorem 1.1]. Theorem 31.1 also implies the equivalence of conditions (1), (2) and (3) in Guéritaud-Guichard-Kassel-Wienhard [107, Theorem 1.3]

3) The proof of Theorem 31.1 given in Tsouvalas [208] works more directly with the dynamical definition but our proof is simply a translation of his to the language used in this course.

32. Examples

In this section, we use Theorem 31.1 to exhibit examples of Anosov representations. Our first examples are convex cocompact representations into $\mathsf{SO}_0(d-1,1)$. (Similar results hold for convex cocompact representations into all rank one Lie groups, e.g. convex cocompact representations into $\mathsf{SU}(n,1)$ are P_2-Anosov.)

Corollary 32.1. *Suppose that $d \geq 3$, Γ is hyperbolic and $\rho : \Gamma \to \mathsf{SO}_0(d-1,1) \subset \mathsf{SL}(d,\mathbb{R})$ is a representation. Then ρ is convex cocompact if and only if ρ is P_1-Anosov.*

Proof. If ρ is P_1-Anosov, then the ratio of the first and second singular values of $\rho(\gamma)$ grows uniformly exponentially in $d(1,\gamma)$, by Proposition 27.1. Therefore, the ratio of the first and last singular values of $\rho(\gamma)$ also grows uniformly exponentially in $d(1,\gamma)$, so, by Lemma 11.2, ρ is convex cocompact.

Now suppose that ρ is convex cocompact. It follows from Lemma 11.2 that there exists K and C so

$$\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))}\right) \geq K d(id, \gamma) - C.$$

Since

$$\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))}\right) = \frac{1}{2}\log\left(\frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))}\right)$$

if $\rho(\gamma) \in \mathsf{SO}(d-1)$, ρ is P_1-divergent and Proposition 30.3 and Lemma 30.1 imply that there exists a ρ-equivariant continuous map $\xi : \partial\Gamma \to \mathbb{RP}^{d-1}$ with the Cartan property, such that if $\rho(\gamma)$ is proximal, then $\xi(\gamma^+)$ is the attracting eigenline. In particular, $\xi(\partial\Gamma) \subset \partial\mathbb{H}^{d-1} \subset \mathbb{RP}^{d-1}$ We then define, $\theta_\rho(z)$ to be the tangent plane to $\partial\mathbb{H}^n$ at $\xi_\rho(z)$ and notice that θ_ρ is continuous and ρ-equivariant and that ξ_ρ and θ_ρ are transverse. Theorem 31.1 then implies that ρ is P_1-Anosov. $\qquad\square$

It is also immediate from Theorem 31.1 that if you restrict an Anosov representation to a quasiconvex subgroup then it remains Anosov. However, the restriction to a non-quasiconvex subgroup need not be Anosov. For example, if $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold which fibers over the circle, then the inclusion map of Γ into $\mathsf{SO}_0(3,1)$ is convex cocompact, hence P_1-Anosov. However, it is well-known that the restriction to the fibre subgroup is not convex cocompact, see, for example, Cannon-Thurston [62].

Corollary 32.2. (Canary-Lee-Stover [57, Lemma 2.3]) *If $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is P_k-Anosov and Θ is a quasiconvex subgroup of Γ, then $\rho|_\Theta : \Theta \to \mathsf{SL}(d,\mathbb{R})$ is P_k-Anosov.*

Proof. Since ρ is P_k-Anosov, ρ is P_k-divergent and there exist transverse ρ-equivariant limit maps $\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$ with the Cartan property. Since Θ is quasiconvex, Θ is hyperbolic and there exists a Θ-equivariant embedding $\eta : \partial\Theta \to \partial\Gamma$ (by Corollary 4.2). Therefore, $\rho|_\Theta$ is P_k-divergent and $\xi_\rho \circ \eta$ and $\theta_\rho \circ \eta$ are transverse $\rho|_\Theta$-equivariant limit maps with the Cartan property. Theorem 31.1 then implies that $\rho|_\Theta$ is P_k-Anosov. $\quad\square$

We can also see that ρ is P_k-Anosov if and only if its restriction to a finite index subgroup is P_k-Anosov. This clearly fails for infinite index subgroups.

Corollary 32.3. (Guichard-Wienhard [109, Corollary 1.3]) *If Δ is a finite index subgroup of a hyperbolic group Γ and $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a representation, then ρ is P_k-Anosov if and only if $\rho|_\Delta : \Delta \to \mathsf{SL}(d,\mathbb{R})$ is P_k-Anosov.*

Proof. Since finite index subgroups are quasiconvex, Corollary 32.2 implies that $\rho|_\Delta$ is P_k-Anosov, if ρ is P_k-Anosov.

Suppose that $\rho|_\Delta$ is P_k-Anosov, then there exist transverse $\rho|_\Delta$-equivariant continuous maps $\xi : \partial\Delta \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta : \partial\Delta \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$ with the Cartan property with respect to $\rho|_\Delta$. We may assume that Δ is normal in Γ (by replacing Δ with the intersection of all its conjugates if necessary).

Since Δ has finite index, $\partial\Delta = \partial\Gamma$. We first check that ξ and θ are also ρ-equivariant. If $\delta \in \Delta$ has infinite order and $\gamma \in \Gamma$, then

$$\xi(\gamma(\delta^+)) = \xi\big((\gamma\delta\gamma^{-1})^+\big) = \rho(\gamma\delta\gamma^{-1})^+ = \rho(\gamma)(\rho(\delta)^+) = \rho(\gamma)(\xi(\delta^+)).$$

Since fixed points of infinite order elements of Δ are dense in $\partial\Delta = \partial\Gamma$, this implies that ξ is ρ-equivariant. The proof that θ is ρ-equivariant is similar.

We next check that ρ is P_k-divergent. Since Δ has finite index in Γ, there exists a finite set $\mathcal{B} \subset \Gamma$ so that if $\gamma \in \Gamma$ then there exists $\beta \in \mathcal{B}$ so that $\gamma\beta \in \Delta$. Let $M = \max\left\{\frac{\sigma_1(\rho(\beta))}{\sigma_d(\rho(\beta))} \mid \beta \in \mathcal{B}\right\}$. Let $\{\gamma_n\}$ be a sequence exiting every finite subset of Γ and, for all n, choose β_n so that $\gamma_n\beta_n \in \Delta$. Then $\{\gamma_n\beta_n\}$ leaves every finite subset of Δ, so, since $\rho|_\Delta$ is P_k-divergent, $\log\frac{\sigma_k(\rho(\gamma_n\beta_n))}{\sigma_{k+1}(\rho(\gamma_n\beta_n))} \to \infty$. Therefore, by Lemma 29.1,

$$\log\frac{\sigma_k(\rho(\gamma_n))}{\sigma_{k+1}(\rho(\gamma_n))} \geq \frac{1}{M^2}\log\frac{\sigma_k(\rho(\gamma_n\beta_n))}{\sigma_{k+1}(\rho(\gamma_n\beta_n))} \to \infty,$$

so ρ is also P_k-divergent.

Finally, we check that ξ has the Cartan property with respect to ρ. Let $\{\gamma_n\}$ be a sequence in Γ converging to $z \in \partial\Gamma$. For each n choose $\beta_n \in \mathcal{B}$ so that $\gamma_n\beta_n \in \Theta$. Notice that $\{\gamma_n\beta_n\}$ also converges to z. Since ξ has the Cartan property for $\rho|_\Theta$, $\{U_1(\rho(\gamma_n\beta_n))\}$ converges to $\xi(z)$. But, Corollary 29.3 implies that

$$d(U_1(\rho(\gamma_n\beta_n)), U_1(\rho(\gamma_n))) \leq \sqrt{d-1}M^k\left(\frac{\sigma_k(\rho(\gamma_n))}{\sigma_{k+1}(\rho(\gamma_n))}\right).$$

Since ρ is P_k-divergent, this implies that $U_1(\rho(\gamma_n)) \to \xi(z)$, so ξ has the Cartan property with respect to ρ.

Therefore, ρ is P_k-divergent and there exist continuous transverse ρ-equivariant maps $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$ and ξ has the Cartan property. Theorem 31.1 then implies that ρ is P_k-Anosov. $\qquad\square$

We may use Corollary **??** to show that a representation is P_k-Anosov if and only its dual is P_k-Anosov .

Corollary 32.4. *Let Γ be a Gromov hyperbolic group and $1 \leq k \leq \frac{d}{2}$. If $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a representation, then ρ is P_k-Anosov if and only if ρ^* is P_k-Anosov. Moreover, $\xi_{\rho^*} = \theta_\rho$ and $\theta_\rho^* = \xi_\rho$.*

Proof. Suppose that ρ is P_k-Anosov. Consider the map $\xi : \partial\Gamma \to \mathrm{Gr}_k\big((\mathbb{R}^d)^*\big)$ given by $\xi(z) = \theta_\rho(z)$ for all $z \in \partial\Gamma$. Here, we may either think of $\xi(z)$ as the k-plane of linear functionals having $\theta_\rho(z)$ in their kernel, or, more prosaically, identify it with the orthogonal subspace $\theta_\rho(z)^\perp$ of $\theta_\rho(z)$ in the standard identification of $(\mathbb{R}^d)^*$ with $\mathbb{R}^d$. Similarly, we define $\theta : \partial\Gamma \to \mathrm{Gr}_{d-k}\big((\mathbb{R}^d)^*\big)$

by $\theta(z) = \xi_\rho(z)$ (with similar identifications). Since ξ_ρ are θ_ρ are continuous, ρ-equivariant and transverse, ξ and θ are continuous, ρ^*-equivariant and transverse.

Recall that, by Lemma 27.1, there exist $D > 0$ and $L \geq 0$ so that

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq D\, d(1, \gamma^{-1}) - L$$

for all $\gamma \in \Gamma$. So, for all $\gamma \in \Gamma$,

$$\sigma_k(\rho^*(\gamma)) = \sigma_k(\rho(\gamma^{-1})^T) = \sigma_k(\rho(\gamma^{-1})) \qquad \text{and} \qquad \sigma_{k+1}(\rho^*(\gamma)) = \sigma_{k+1}(\rho(\gamma^{-1}))$$

so

$$\log\left(\frac{\sigma_k(\rho^*(\gamma))}{\sigma_{k+1}(\rho^*(\gamma))}\right) \geq D\, d(1, \gamma^{-1}) - L = D\, d(1, \gamma) - L.$$

Therefore, ρ^* is P_k-divergent.

Finally, we show that ξ is dynamics-preserving. Suppose that $\gamma \in \Gamma$ has infinite order, then, since $\rho(\gamma^{-1})$ is P_k-biproximal, $\rho^*(\gamma) = \rho(\gamma^{-1})^T$ is also P_k-biproximal. Since θ_ρ is dynamics-preserving, $\theta_\rho(\gamma^+)$ is the attracting $(d - k)$-plane of $\rho(\gamma)$, which is identified by duality with the attracting k-plane of $\rho^*(\gamma)$. (To be more precise, if $\rho(\gamma) = KAL$ is the singular value decomposition of $\rho(\gamma)$, then $\theta_\rho(\gamma) = L(< e_1, \ldots, e_{d-k} >)$, so $\xi(\gamma^+)$ is identified with its dual $L(< e_{d-k+1}, \ldots, e_d >)$. Since $\rho^*(\gamma) = KA^{-1}L$ is the singular value decomposition of $\rho^*(\gamma)$, $L(< e_{d-k+1}, \ldots, e_d >)$ is the attracting k-plane of $\rho^*(\gamma)$.) Therefore, ξ is dynamics-preserving. Theorem **??** then implies that ρ^* is P_k-Anosov. Since, ξ and ξ_{ρ^*} are both continuous and dynamics-preserving, Lemma 30.2 implies that $\xi = \xi_{\rho^*}$. We similarly show that $\theta = \theta_{\rho^*}$.

After identifying $((\mathbb{R}^d)^*)^*$ with $\mathbb{R}^d$, one sees that $(\rho^*)^* = \rho$, so ρ is P_k-Anosov if and only if ρ^* is Anosov. $\qquad\square$

Another way to make new Anosov representations from existing Anosov representations, which was first used by Barbot [12], is to simply take the direct sum with the identity representation. Perhaps surprisingly, such representations and their deformations can exhibit interesting phenomena.

Corollary 32.5. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a P_1-Anosov representation and $T : \Gamma \to \mathsf{SL}(m, \mathbb{R})$ is the trivial representation, then $\rho \oplus T : \Gamma \to \mathsf{SL}(d + m, \mathbb{R})$ is P_1-Anosov.*

Proof. By definition, $\sigma_1(\rho \oplus T(\gamma)) = \sigma_1(\rho(\gamma))$ and $\sigma_2(\rho \oplus T(\gamma)) = \max\{1, \sigma_2(\rho)\}$. So

$$\frac{\sigma_1(\rho \oplus T(\gamma))}{\sigma_2(\rho \oplus T(\gamma))} = \min\left\{\frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))}, \sigma_1(\rho(\gamma))\right\} \geq \sqrt{\frac{\sigma_1(\rho)(\gamma))}{\sigma_2(\rho(\gamma))}}$$

since $\sigma_2(\rho(\gamma)) \geq \frac{1}{\sigma_1(\rho(\gamma))}$. Therefore, since ρ is P_1-divergent, $\rho \oplus T$ is also P_1-divergent.

Let $\xi = \partial\Gamma \to \mathbb{RP}^{d+m-1}$ be defined so that $\xi(x) = \xi_\rho(z) \oplus \{\vec{0}\}$. Notice that since $U_1((\rho \oplus T)(\gamma)) = U_1(\rho(\gamma)) \oplus \{\vec{0}\}$ and ξ_ρ has the Cartan property, ξ is a continuous $(\rho \oplus T)$-equivariant map with the Cartan property. Similarly, define $\theta = \partial\Gamma \to \mathrm{Gr}_{d+m-1}(\mathbb{R}^{d+m})$ by $\theta(z) = \theta_\rho(z) \oplus \mathbb{R}^m$ and notice that θ is a continuous $(\rho \oplus T)$-equivariant map transverse to ξ. Theorem 31.1 then implies that $\rho \oplus T$ is P_1-Anosov. $\qquad\square$

A favorite technique to obtain new Anosov representations is to take a convex cocompact representation into a rank one Lie group and then compose with a well-chosen representation of the rank one Lie group into another Lie group. If chosen correctly, the composition is Anosov

and hence, by stability, all small deformations of the composition are also Anosov. The most prominent example of this type are Hitchin representations, which are the most intensely studied class of Anosov representations.

We recall that for all d there exists an irreducible representation

$$\tau_d : \mathsf{SL}(2, \mathbb{R}) \to \mathsf{SL}(d, \mathbb{R})$$

given by regarding $\mathbb{R}^d$ as the vector space of degree $d-1$ homogeneous polynomials in 2 variables, i.e. $\mathbb{R}^d = \{a_1 x^{d-1} + a_2 x^{d-2} y + \cdots + a_d y^{d-1}\}$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\tau_d(A)$ acts on $\mathbb{R}^d$ by taking x to $ax + by$ and taking y to $cx + dy$. For example, if $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, then

$$\tau_d(A) \left(a_1 x^{d-1} + a_2 x^{d-2} y + \cdots + a_d y^{d-1} \right) = a_1 \lambda^{d-1} x^{n-1} + a_2 \lambda^{d-3} x^{n-2} y + \cdots + a_d \lambda^{1-d} y^{n-1}.$$

In other words,

$$\tau_d \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \right) = \begin{bmatrix} \lambda^{d-1} & 0 & \cdots & 0 \\ 0 & \lambda^{d-3} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{1-d} \end{bmatrix}$$

In particular, $\tau_d \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \right)$ is diagonalizable with distinct eigenvalues.

Since any hyperbolic element B of $\mathsf{SL}(2, \mathbb{R})$ is conjugate to one of the form $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, we see that $\tau_d(B)$ is conjugate to $\tau_d(A)$, so $\lambda_i(\tau_d(B)) = \lambda_1(B)^{d+1-2i}$ for all i. Similarly, if $B = LDK$ is the singular value decomposition of B, we see that $\tau_d(B) = \tau_d(L)\tau_d(D)\tau_d(K)$ is the singular value decomposition of $\tau_d(B)$, so $\sigma_i(\tau_d(B)) = \sigma_1(B)^{d+1-2i}$.

Recall that a (complete) **flag** F in $\mathbb{R}^d$ is a nested collection of vector subspaces

$$F_1 \subset F_2 \cdots \subset F_{d-1}$$

so that F_i has dimension i. The (complete) **flag variety** $\mathcal{F}_d$ is the space of all flags in $\mathbb{R}^d$. Notice that $\mathcal{F}_d = \mathsf{SL}(d, \mathbb{R})/B$ where B is the subgroup of upper triangular matrices. There is then an embedding $V_d : \mathbb{RP}^1 = \mathcal{F}_2 \to \mathcal{F}_d$, called the **Veronese embedding**, such that if $L \in \mathsf{SO}(2)$, then

$$V_d(< Le_1 >) = \Big(\tau_d(L)(< e_1 >), \tau_d(L)(< e_1, e_2 >), \ldots, \tau_d(L)(< e_1, \ldots, e_{d-1} >) \Big).$$

One may check that if $B \in \mathsf{SL}(2, \mathbb{R})$ has attracting eigenline B^+, then $V_d(B^+)$ is the attracting flag for $\tau_d(B)$.

If S is a closed orientable surface, the irreducible representation induces an embedding of $\mathrm{Hom}(\pi_1(S), \mathsf{SL}(2, \mathbb{R}))$ into $\mathrm{Hom}(\pi_1(S), \mathsf{SL}(d, \mathbb{R}))$ given by taking ρ to $\tau_d \circ \rho$. The **Hitchin component** $\mathcal{H}_d(S)$ is the component of

$$X(\pi_1(S), \mathsf{SL}(d, \mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathsf{SL}(d, \mathbb{R}))/\mathrm{GL}(d, \mathbb{R})$$

which contains the image of a Fuchsian representation. In particular, $\mathcal{H}_d(S)$ contains an image of Teichmüller space $\mathcal{T}(S)$ called the **Fuchsian locus.** All representations in the Fuchsian locus are d-**Fuchsian**, i.e. are of the form $\tau_d \circ \rho_0$ where ρ_0 is a discrete, faithful representation into

$\mathsf{SL}(2,\mathbb{R})$. Hitchin [114] showed that $\mathcal{H}_d(S)$ is an analytic manifold diffeomorphic to $\mathbb{R}^{(d^2-1)(2g-2)}$ and called it the Teichmüller component. Representations in the Hitchin component are called **Hitchin representations**. If $d = 3$, the Hitchin component agrees with the space of (conjugacy classes of lifts of) Benoist representations of $\pi_1(S)$.

We say that a representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is **Borel Anosov** if it is P_k-Anosov for all $1 \leq k \leq \frac{d}{2}$. (This name arises because in more Lie-theoretic language this means that ρ is Anosov with respect to the Borel subgroup of $\mathsf{SL}(d,\mathbb{R})$ which is simply the set of upper triangular matrices.) In a seminal paper in the field, Labourie [143] showed that Hitchin representations into $\mathsf{SL}(d,\mathbb{R})$ are all Borel Anosov. The proof is very demanding, so we will only show that d-Fuchsian representations, and hence small deformations of d-Fuchsian representations, are Borel Anosov. One can think of this as Step 0 of the actual proof.

Corollary 32.6. *If S is a closed orientable surface and $\rho : \pi_1(S) \to \mathsf{SL}(d,\mathbb{R})$ is a d-Fuchsian representation, then ρ is Borel Anosov.*

Proof. Let $\rho = \tau_d \circ \rho_0$. Recall that ρ_0 is convex cocompact (or, really, that its image under the projection to $\mathsf{PSL}(2,\mathbb{R})$ is convex cocompact). Therefore, ρ_0 is P_1-divergent and there exists a continuous ρ_0-equivariant map $\xi_{\rho_0} : \partial\Gamma \to \mathbb{RP}^1$ which has the Cartan property.

Notice that if $\gamma \in \pi_1(S)$ and the singular value decomposition of $\rho_0(\gamma)$ is given by

$$\rho_0(\gamma) = K \begin{bmatrix} \sigma_1(\rho_0(\gamma)) & 0 \\ 0 & \sigma_1(\rho_0(\gamma))^{-1} \end{bmatrix} L$$

where $K, L \in \mathsf{S})(d)$, then

$$\tau_d(\rho_0(\gamma)) = \tau_d(K) \begin{bmatrix} \sigma_1(\rho_0(\gamma))^{d-1} & 0 & \cdots & 0 \\ 0 & \sigma_1(\rho_0(\gamma))^{d-3} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \sigma_1(\rho_0(\gamma))^{1-d} \end{bmatrix} \tau_d(L).$$

Let $C = \sup\{\sigma_1(\tau_d(K)) \mid K \in \mathsf{SO}(2)\}$, which is finite since $\tau_d(\mathsf{SO}(2))$ is compact. Then, Lemma 29.1, implies that

$$\frac{1}{C^2} \geq \frac{\sigma_i(\rho(\gamma))}{\sigma_i(\rho_0(\gamma))^{d+1-2i}} \leq C^2,$$

so

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq 2\log\sigma_1(\rho_0(\gamma)) - 4C = \log\left(\frac{\sigma_1(\rho_0(\gamma))}{\sigma_2(\rho_0(\gamma))}\right) - 4C$$

for any k. Therefore, since ρ_0 is P_1-divergent, ρ is P_k-divergent for all k.

We then consider $\xi_\rho : \partial\pi_1(S) \to \mathcal{F}_d$ given by $\xi_\rho = V_d \circ \xi_{\rho_0}$ which is a continuous ρ-invariant map. We may decompose $\xi_\rho = (\xi_\rho^k)_{k=1}^{d-1}$ where $\xi_d^k : \partial\pi_1(S) \to \mathrm{Gr}_k(\mathbb{R}^d)$. We now claim that each ξ_ρ^k has the Cartan property. Since V_d takes the attracting fixed point of $\rho_0(\gamma)$ to the attracting eigenline of $\rho(\gamma)$, we see that $\xi_\rho^k(\gamma^+)$ is the attracting k-plane $\rho(\gamma)_k^+$ of $\rho(\gamma)$. On the other hand, Proposition 30.3 produces a continuous ρ-invariant map $\xi^k : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ with the Cartan property so that $\xi^k(\gamma^+) = \rho(\gamma)_k^+ = \xi_\rho^k$. So, $\xi_\rho^k = \xi^k$ has the Cartan property for all k.

Finally, notice that $\xi_\rho^k(z)$ is always transverse to $\xi^{d-k}(w)$ if $z \neq w$, since if we choose a hyperbolic element $A \in \mathsf{SL}(d,\mathbb{R})$ with attracting eigenline $\xi_{\rho_0}(z)$ and repelling eigenline $\xi_{\rho_0}(w)$,

then $\xi_\rho^k(z)$ is the attracting k-plane of $\tau_d(A)$ and $\xi^{d-k}(w)$ is the repelling $(d-k)$-plane of $\tau_d(A)$. Therefore, we may apply Theorem 31.1 to conclude that ρ is P_k-Anosov for all k. $\qquad\square$

33. **Irreducible representations and Benoist representations**

In the case that ρ is irreducible and $k = 1$, Guichard and Wienhard gave a simpler characterization.

Theorem 33.1. (Guichard-Wienhard [109, Proposition 4.10]) *Suppose that Γ is word hyperbolic and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is irreducible. Then ρ is P_1-Anosov if and only if there exist continuous ρ-equivariant transverse maps $\xi : \partial\Gamma \to \mathbb{RP}^{d-1}$ and $\theta : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$.*

We will derive a proof from Theorem 31.1. In their original statement, Guichard and Wienhard also require that ξ_ρ and θ_ρ are compatible, but our proof, taken from Tsouvalas [208], will not need this property.

We first observe that if a representation has limit map into projective space which spans $\mathbb{R}^d$, then it is P_1-divergent.

Lemma 33.2. (Canary-Tsouvalas [60, Lemma 9.2], Tsouvalas [208]) *If Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation, and $\xi : \partial\Gamma \to \mathbb{RP}^{d-1}$ is a continuous ρ-equivariant map such that $< \xi(\partial\Gamma) > = \mathbb{R}^d$, then ρ is P_1-divergent.*

In particular, if ρ is irreducible and $\xi : \partial\Gamma \to \mathbb{RP}^{d-1}$ is a continuous ρ-equivariant map, then ρ is P_1-divergent.

Proof. If ρ is not P_1-divergent, then there exists a sequence of distinct elements $\{\gamma_n\} \subset \Gamma$ so that

$$\lim \left(\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} \right) = R > 0.$$

Since Γ is a hyperbolic group there exists, perhaps up to passing to a subsequence z and $\hat{z}$ in $\partial\Gamma$ so that if $x \in \partial\Gamma - \{\hat{z}\}$, then $\gamma_n(x) \to z$, see Theorem 5.7. Let $\rho(\gamma_n) = L_n D_n K_n$ be the singular value decomposition of $\rho(\gamma_n)$. We may assume, perhaps after again passing to a subsequence, that $L_n \to L \in \mathsf{SO}(d)$ and $K_n \to K \in \mathsf{SO}(d)$.

Given $x \in \partial\Gamma - \{\hat{z}\}$, there exists $M_x \in \mathsf{SO}(d)$ such that $\xi(x) = < M_x e_1 >$. Since $\{\gamma_n(x)\}$ converges to z, we see that $\{\rho(\gamma_n)([M_x(e_1)])\}$ converges to $\xi(z)$ and we choose $M_z \in \mathsf{SO}(d)$ so that $\xi(z) = < M_z(e_1) >$. In other words, $< L_n D_n K_n M_x(e_1) >$ converges to $< M_z(e_1) >$, so $< D_n K_n M_x(e_1) >$ converges to $< L^{-1} M_z(e_1) >$. So, after perhaps again passing to a subsequence,

$$\lim \frac{D_n K_n M_x(e_1)}{||D_n K_n M_x(e_1)||} = \pm L^{-1} M_z(e_1).$$

Thus, since

$< D_n K_n M_x(e_1), e_1 > = \sigma_1(\rho(\gamma_n)) < K_n M_x(e_1), e_1 >$ and $< D_n K_n M_x(e_1), e_2 > = \sigma_2(\rho(\gamma_n)) < K_n M_x(e_1), e_2 >$

we see that

$$\frac{< KM_x(e_1), e_2 >}{< KM_x(e_1), e_1 >} = R \left(\frac{< L^{-1} M_z(e_1), e_2 >}{< L^{-1} M_z(e_1), e_1 >} \right) \tag{33.1}$$

whenever $< KM_x(e_1), e_1 > \neq 0$. Moreover, $< KM_x(e_1), e_1 > \neq 0$ if and only if $< L^{-1} M_z(e_1), e_1 > \neq 0$.

Notice that since $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$ and $\partial\Gamma$ is a perfect set, $\xi(\partial\Gamma - \{\hat{z}\})$ spans $\mathbb{R}^d$. Therefore, we may choose $x_0 \in \partial\Gamma - \{\hat{z}\}$, so that $< KM_{x_0}(e_1), e_1 > \neq 0$. Thus, $< L^{-1} M_z(e_1), e_1 > \neq 0$,

so, $< KM_x(e_1), e_1 > \neq 0$ for all $x \in \partial\Gamma - \{\hat{z}\}$. It follows that, for all $x \in \partial\Gamma - \{\hat{z}\}$, $M_x(e_1)$ lies in the hyperplane determined by equation (33.1). However, this contradicts the fact that $\xi(\partial\Gamma - \{\hat{z}\})$ spans $\mathbb{R}^d$, so ρ must be P_1-divergent.

If ρ is irreducible and there is a continuous ρ-equivariant limit map $\xi : \partial\Gamma \to \mathbb{R}^d$, then $\xi(\partial\Gamma)$ must span $\mathbb{R}^d$, since otherwise it spans a proper $\rho(\Gamma)$-invariant subspace of $\mathbb{R}^d$ (which is impossible). It follows from the above argument that ρ is P_1-divergent. $\square$

The key step in our proof of Theorem 33.1 will be Tsouvalas' observation that limit maps of irreducible representations have the Cartan property.

Proposition 33.3. (Tsouvalas [208]) *If Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is irreducible and there exists a continuous ρ-equivariant map $\xi : \partial\Gamma \to \mathbb{RP}^{d-1}$, then ξ has the Cartan property.*

Our proof will make use of a deep result of Benoist [16] which is based on work of Abels, Margulis and Soifer [1], see [107, Theorem 4.12] for a statement (and proof) in the form given here. We recall that a representation into $\mathsf{SL}(d, \mathbb{R})$ is **semi-simple** if it is a direct sum of irreducible representations.

Theorem 33.4. (Benoist [16]) *If Γ is a finitely generated group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a semi-simple representation, then there exists a finite subset B of Γ and K such that if $\gamma \in \Gamma$, then there exists $\beta \in B$ so that*

$$\left| \log \sigma_i(\rho(\gamma)) - \log\left(|\lambda_i(\rho(\gamma\beta))|\right) \right| \leq K$$

for all $i = 1, \ldots, d$.

Proof of Proposition 33.3: Lemma 33.2 implies that ρ is P_1-divergent. Recall also that $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$ if ρ is irreducible.

Fix $z \in \partial\Gamma$ and let $\{\gamma_n\} \subset \Gamma$ be a sequence converging to z. By Theorem 33.4, there exists K such that, for all n, there exists $\beta_n \in B$ such that

$$\left| \log \sigma_i(\rho(\gamma_n)) - \log|\lambda_i(\rho(\gamma_n\beta_n))| \right| \leq K$$

for all i. Let $C = \sup\left\{ \log\left(\frac{\sigma_1(\rho(\beta))}{\sigma_d(\rho(\beta))} \right) \mid \beta \in B \right\}$, so, by Lemma 29.1,

$$|\log \sigma_i(\rho(\gamma_n\beta_n)) - \log \sigma_i(\rho(\gamma_n))| \leq C,$$

hence

$$\left| \log \sigma_i(\rho(\gamma_n\beta_n)) - \log\left(|\lambda_i(\rho(\gamma_n\beta_n))|\right) \right| \leq K + C$$

for all n.

Since ρ is P_1-divergent, $\log\left(\frac{\sigma_1(\rho(\gamma_n\beta_n))}{\sigma_2(\rho(\gamma_n\beta_n))} \right) \to \infty$, so

$$\log\left(\frac{\lambda_1(\rho(\gamma_n\beta_n))}{\lambda_2(\rho(\gamma_n\beta_n))} \right) \to \infty.$$

In particular, $\rho(\gamma_n\beta_n)$ is proximal for all large enough n. Since $\partial\Gamma$ is perfect and $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$, we can find $x_n \in \partial\Gamma - \{(\gamma_n\beta_n)^-\}$, so that $\xi(x_n)$ does not lie in the repelling hyperplane

of $\rho(\gamma_n\beta_n)$, so $\lim \rho(\gamma_n\beta_n)^s(\xi_\rho(x_n)) = \rho(\gamma\beta_n)^+$. Then, since ξ is continuous and ρ-equivariant, and $\lim_{s\to\infty}(\gamma_n\beta_n)^s(x_n) \to (\gamma_n\beta_n)^+$, we see that

$$\xi((\gamma_n\beta_n)^+) = \lim_{s\to\infty} \xi((\gamma_n\beta_n)^s(x_n)) = \lim \rho(\gamma_n\beta_n)^s(\xi(x_n)) = \rho(\gamma_n\beta_n)^+.$$

Now notice that $z = \lim \gamma_n = \lim(\gamma_n\beta_n) = \lim(\gamma_n\beta_n)^+$ in $\Gamma \cup \partial\Gamma$, see Lemma 5.8, so

$$\xi(z) = \lim \xi((\gamma_n\beta_n)^+) = \lim \rho(\gamma_n\beta_n))^+.$$

We then apply Lemma 29.4 to see that

$$d(U_1(\rho(\gamma\beta_n)), \rho(\gamma_n\beta_n)^+) \leq \frac{\sigma_2(\rho(\gamma_n\beta_n))}{|\lambda_1(\rho(\gamma_n\beta_n))|} \leq e^{K+C}\frac{\sigma_2(\rho(\gamma_n\beta_n))}{\sigma_1(\rho(\gamma_n\beta_n))}$$

and, by Lemma 29.2,

$$d(U_1(\rho(\gamma_n\beta_n)), U_1(\rho(\gamma_n))) \leq \sqrt{d-1}\left(\frac{\sigma_1(\rho(\beta_n))}{\sigma_d(\rho(\beta_n))}\right)\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} \leq C\sqrt{d-1}\left(\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}\right).$$

Since ρ is P_1-divergent, we conclude that $\lim d(U_1(\rho(\gamma_n)), \rho(\gamma_n\beta_n)^+) = 0$. Then, since $\lim \rho(\gamma_n\beta_n)^+ = \xi(z)$, we see that $\lim U_1(\rho(\gamma_n)) = \xi(z)$, so ρ has the Cartan property. $\qquad\square$

It is now easy to derive Theorem 33.1 from Theorem 31.1.

Proof of Theorem 33.1: Suppose that Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is irreducible and $\xi : \partial\Gamma \to \mathbb{RP}^{d-1}$ and $\theta : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ are continuous, transverse ρ-equivariant maps. Lemma 33.2 implies that ρ is P_1-divergent and Proposition 33.3 implies that ξ has the Cartan property. Theorem 31.1 then implies that ρ is P_1-Anosov.

On the other hand, by definition, a P_1-Anosov representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ admits continuous, transverse ρ-equivariant maps $\xi_\rho : \partial\Gamma \to \mathbb{RP}^{d-1}$ and $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$.
$$\square$$

Since Benoist representations into $\mathsf{PGL}(d, \mathbb{R})$ have transverse limit maps into $\mathbb{RP}^{d-1}$ and $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$, see Proposition 18.1, and are irreducible, see Proposition 18.4, Theorem 33.1 implies that they are projective Anosov.

Corollary 33.5. *If $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is a Benoist representation, then ρ is P_1-Anosov.*

Remarks: (1) Corollary 33.5 was first established by Guichard and Wienhard [109, Proposition 6.1] as a consequence of Theorem 33.1. One can also prove that Benoist representations are P_1-Anosov by verifying directly that they are P_1-divergent (using either Proposition 33.3 or Lemma 42.1) and applying Corollary ??.

(2) It may worry you that the image of a Benoist representation is in $\mathsf{PGL}(d, \mathbb{R})$, rather than $\mathsf{SL}(d, \mathbb{R})$, but you can notice that all the definitions, statements and theorems work equally well if our image group is $\mathsf{PSL}(d, \mathbb{R})$, $\mathsf{PGL}(d, \mathbb{R})$, or even $\mathsf{GL}(d, \mathbb{R})$. Alternatively one can note that a Benoist represention into $\mathsf{PSL}(d, \mathbb{R})$ lifts to a representation into $\mathsf{SL}(d, \mathbb{R})$ (see Culler [79, Proposition 2.1]) and happily work there (passing to an index two subgroup if the image lies in $\mathsf{PGL}(d, \mathbb{R})$, but not $\mathsf{PSL}(d, \mathbb{R})$).

We can now use the stability property of Anosov representations to give a quick proof of Koszul's stability theorem for Benoist representations, which we earlier stated as Theorem 20.1.

Theorem 33.6. (Koszul [141]) *If Γ is a torsion-free hyperbolic group, then $\mathsf{Ben}(\Gamma, d-1)$ is an open subset of $\mathrm{Hom}(\Gamma, \mathsf{PGL}(d, \mathbb{R}))$.*

Proof. Suppose that $\rho_0 \in \mathsf{Ben}(\Gamma, n)$. Since ρ_0 is projective Anosov, Theorem 28.2 provides a neighborhood U of ρ_0 in $\mathrm{Hom}(\Gamma, \mathsf{PGL}(d, \mathbb{R}))$ so that if $\rho \in U$, then ρ is projective Anosov. The limit map ξ_{ρ_0} has image $\xi_{\rho_0}(\partial\Gamma)$ which is the boundary of a strictly convex domain Ω in an affine chart A. Since the limit maps vary continuously, we can pass to a subneighborhood $V \subset U$ of ρ_0 so that if $\rho \in V$, then $\xi_\rho(\partial\Gamma)$ is a compact subset of the affine chart A and spans $\mathbb{R}^d$. Let $\mathcal{C}_\rho$ be the convex hull of $\xi_\rho(\partial\Gamma)$ in A. Then $\rho(\Gamma)$ preserves $\mathcal{C}_\rho$ and hence it preserves its interior Ω_ρ. By definition, Ω_ρ is a properly convex domain, and since $\rho(\Gamma)$ is discrete, it acts properly discontinuously on Ω_ρ. Therefore, $\Omega_\rho/\rho(\Gamma)$ is a manifold which is homotopy equivalent to $\Omega/\rho_0(\Gamma)$. Since both manifolds are $(d-1)$-dimensional and one of them is closed, so is the other one. Since Γ is hyperbolic, Theorem 16.1 implies that Ω_ρ is strictly convex, so ρ is a Benoist representation. $\square$

34. The relationship between P_k-Anosov and P_1-Anosov representations

In this section, we prove a characterization of P_k-Anosov Zariski dense representations into $\mathsf{SL}(d, \mathbb{R})$ which is a generalization of Theorem 33.1. We further show that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov if and only if $E_d^k \circ \rho$ is P_1-Anosov where $E_d^k : \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(\Lambda^k \mathbb{R}^d)$ is the exterior power representation. This often allows one to reduce general questions about P_k-Anosov representations to the study of projective Anosov representations. I will assume basic facts about Zariski closures here. There is a nice treatment of the Zariski topology in our setting in Benoist-Quint [25, Chapter 6].

Recall from Section 29 that if $E_d^k : \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(\Lambda^k \mathbb{R}^d)$ is the exterior power representation, then

$$\frac{\sigma_1(E_d^k(A))}{\sigma_2(E_d^k(A))} = \frac{\sigma_k(A)}{\sigma_{k+1}(A)}$$

if $A \in \mathsf{SL}(d, \mathbb{R})$. Moreover,

$$U_1(E_d^k(A))) = G_d^k(U_k(A))$$

whenever both are defined, i.e. whenever $\sigma_k(A) > \sigma_{k+1}(A)$. One may also check that E_d^k is irreducible and that the Plücker embedding $G_d^k : \mathrm{Gr}_k(\mathbb{R}^d) \to \mathbb{P}(\Lambda^k \mathbb{R}^d)$ is E_d^k-equivariant.

We first observe the following consequence of Proposition 33.2:

Corollary 34.1. *If Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is Zariski dense and there exists a continuous ρ-equivariant map $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$, then ρ is P_k-divergent.*

Proof. Since E_d^k is irreducible and ρ is Zariski dense, we see that $E_d^k \circ \rho$ is irreducible. (The key point here is that the Zariski closure of $E_d^k(\rho(\Gamma))$ is simply the image under E_d^k of the Zariski closure of $\rho(\Gamma)$, which is all of $\mathsf{SL}(d, \mathbb{R})$. Then we observe that a subgroup of $\mathsf{SL}(\Lambda^k \mathbb{R}^d)$ is irreducible if and only if its Zariski closure is irreducible.)

The map $G_d^k \circ \xi : \partial\Gamma \to \mathbb{P}(\Lambda^k \mathbb{R}^d)$ is continuous and $E_d^k \circ \rho$-equivariant. Therefore, by Lemma 33.2, $E_d^k \circ \rho$ is P_1-divergent. But since,

$$\frac{\sigma_1(E_d^k(\rho(\gamma)))}{\sigma_2(E_d^k(\rho(\gamma)))} = \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}$$

for all $\gamma \in \Gamma$, this implies that ρ is P_k-divergent. $\square$

We are now ready to prove the analogue of Theorem 33.1.

Theorem 34.2. (Guichard-Wienhard [109, Theorem 4.11]) *Suppose that Γ is word hyperbolic and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is Zariski dense. Then ρ is P_k-Anosov if and only if there exist continuous ρ-equivariant transverse maps $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$.*

Proof. If ρ is P_k-Anosov, then, by definition, it admits continuous transverse ρ-equivariant maps $\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$. So the forward direction is obvious.

Now suppose that ρ is Zariski dense and there exist continuous ρ-equivariant transverse maps $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$. Corollary 34.1 implies that ρ is P_k-divergent. Since E_k^d is irreducible and ρ is Zariski dense, $E_k^d \circ \rho$ is also irreducible.

Then $\xi = G_d^k \circ \xi_\rho$ is a continuous $E_d^k \circ \rho$-equivariant map. Therefore, by Proposition 33.3, ξ has the Cartan property. However, since

$$U_1(E_d^k(\rho(\gamma))) = G_d^k(U_k(\rho(\gamma))$$

for all $\gamma \in \Gamma$ where $\sigma_k(\rho(\gamma)) > \sigma_{k+1}(\rho(\gamma))$, this implies that ξ_ρ has the Cartan property. Theorem 31.1 then implies that ρ is P_k-Anosov. $\qquad\square$

One may also define a map $F_d^k : \mathrm{Gr}_{d-k}(\mathbb{R}^d) \to \mathrm{Gr}_{\binom{d}{k}-1}(\Lambda^k(\mathbb{R}^d))$ with the property that

$$V_{\binom{d}{k}-1}(E_d^k(A)) = F_d^k(V_{d-k}(A))$$

whenever both are defined. (Recall that $\Lambda^k\mathbb{R}^d$ has dimension $\binom{d}{k}$.) Geometrically, if Q is a $(d-k)$-plane, let $\{\vec{u}_1, \ldots, \vec{u}_d\}$ be an orthonormal basis for $\mathbb{R}^d$, so that $\{\vec{u}_{k+1}, \ldots, \vec{u}_d\}$ is a basis for Q. Then $F_d^k(Q)$ is spanned by all elements of the form $\vec{u}_{n_1} \otimes \cdots \vec{u}_{n_k}$ where $n_1 > n_2 > \cdots > n_k$ and $n_k > k$. Notice that F_d^k is E_d^k-equivariant and that if $P \in \mathrm{Gr}_k(\mathbb{R}^d)$ and $Q \in \mathrm{Gr}_{d-k}(\mathbb{R}^d)$, then P and Q are transverse in $\mathbb{R}^d$ if and only if $G_d^k(P)$ and $F_d^k(Q)$ are transverse in $\Lambda^k(\mathbb{R}^d)$.

We are now ready to show that ρ is P_k-Anosov if and only if $E_k^d \circ \rho$ is P_1-Anosov.

Proposition 34.3. (Guichard-Wienhard [109, Theorem 4.11]) *Suppose that Γ is a hyperbolic group, $1 \le k \le \frac{d}{2}$ and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation. Then, ρ is P_k-Anosov if and only if $E_d^k \circ \rho$ is P_1-Anosov.*

One can derive this directly from the definitions, which is what Guichard and Wienhard do, but I feel like the following proof gives one a more visceral feeling for why the fact is true. You may disagree. Also, notice that one may derive Theorem 34.2 immediately from Theorem 33.1 and Proposition 34.3 (which is what Guichard and Wienhard do).

Proof. First suppose that ρ is P_k-Anosov. Since ρ is P_k-divergent and

$$\frac{\sigma_1(E_d^k(\rho(\gamma)))}{\sigma_2(E_d^k(\rho(\gamma)))} = \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \tag{34.1}$$

for all $\gamma \in \Gamma$, $E_d^k \circ \rho$ is P_1-divergent. If $\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ is the limit map of ρ, then $\xi = G_d^k \circ \xi_\rho$ is continuous and $E_d^k \circ \rho$-equivariant. Since ξ_ρ has the Cartan property and $U_1(E_d^k(\rho(\gamma))) = G_d^k(U_k(\rho(\gamma))$ for all $\gamma \in \Gamma$ where $\sigma_k(\rho(\gamma)) > \sigma_{k+1}(\rho(\gamma)$, ξ has the Cartan property. We then define $\theta = F_d^k \circ \theta_\rho$ and notice that ξ and θ are transverse, since ξ_ρ and θ_ρ are transverse. Theorem 31.1 then implies that $E_d^k \circ \rho$ is P_1-Anosov.

Now suppose that $E_d^k \circ \rho$ is P_1-Anosov, with limit maps ξ and θ. Since $E_k^d \circ \rho$ is P_1-divergent, Equation (34.1) implies that ρ is P_k-divergent. Moreover, by Proposition 27.1 applied to $E_d^k \circ \rho$, there exists K and C so that

$$\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} = \frac{\sigma_1(E_d^k(\rho(\gamma)))}{\sigma_2(E_d^k(\rho(\gamma)))} \geq Kd(1,\gamma) - C$$

for all $\gamma \in \Gamma$.

By examining the impact of E_d^k on the Jordan normal form, we conclude that

$$\frac{\lambda_1(E_d^k(\rho(\gamma)))}{\lambda_2(E_d^k(\rho(\gamma)))} = \frac{\lambda_k(\rho(\gamma))}{\lambda_{k+1}(\rho(\gamma))}.$$

If $\gamma \in \Gamma$ has infinite order, then $E_d^k(\rho(\gamma))$ is proximal, so $\rho(\gamma)$ is P_k-proximal. Moreover, the attracting eigenline of $E_d^k(\rho(\gamma))$ is $G_d^k(\rho(\gamma)_k^+)$ where $\rho(\gamma)_k^+$ is the attracting k-plane of $\rho(\gamma)$. It follows that

$$\xi(\gamma^+) = E_d^k(\rho(\gamma))^+ \in G_d^k(\mathrm{Gr}_k(\mathbb{R}^d))$$

for every attracting fixed point in $\partial\Gamma$, so $\xi(\partial\Gamma) \subset G_d^k(\mathrm{Gr}_k(\mathbb{R}^d))$ and we may define a continuous ρ-equivariant map $\xi_\rho = (G_d^k)^{-1} \circ \xi$. Notice that ξ_ρ has the Cartan property, since ξ does. (Or we could apply Proposition 30.3 and Corollary 30.2). We may similarly define $\theta_\rho = (F_d^k)^{-1} \circ \theta$ and notice that ξ_ρ and θ_ρ are transverse, since ξ and ρ are. Theorem 31.1 then implies that ρ is P_k-Anosov.

$\square$

Remark: Proposition 34.3 is a special case of a much more general principle established by Guichard-Wienhard [109, Proposition 4.3]. Given a semi-simple Lie group G and a parabolic subgroup P_θ of G, there is a notion of a P_θ-Anosov representation $\rho : \Gamma \to \mathsf{G}$. Guichard and Wienhard show that, at this level of generality, there is an irreducible representation $\tau : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ (for some d), so that ρ is P_θ-Anosov if and only if $\tau \circ \rho$ is P_1-Anosov. See Section 49 for more details.

35. A characterization in terms of singular values

We now show that uniform exponential growth of the kth singular value gap implies that ρ is P_k-Anosov. This characterization was first established by Kapovich-Leeb-Porti [133, Theorem 1.5], using their Morse Lemma for uniformly regular quasi-geodesics in the symmetric space. Bochi-Potrie-Sambarino [32] then gave a proof using the dynamical theory of dominated splittings. (Gueritaud-Guichard-Kassel-Wienhard [107, Theorem 1.3(iv)] prove a version of this theorem with somewhat stronger assumptions.)

We give a proof in the spirit of Feng Zhu's generalization [231] of the work on Bochi-Potrie-Sambarino [32] to the setting of relatively dominated representations (which are designed to be a higher rank analogue of geometrically finite representations into higher rank Lie groups, see also the earlier related work of Kapovich-Leeb [131]). The main difference is the use a concrete theorem from linear algebra, based on work of Quas-Thieullen-Zarrabi [180], in place of a result of Bochi-Gourmelon [31] on dominated splittings.

Theorem 35.1. (Kapovich-Leeb-Porti [133], Bochi-Potrie-Sambarino [32]) *If Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation and $1 \leq k \leq \frac{d}{2}$, then ρ is P_k-Anosov if and only if there exists $D > 0$ and $L \geq 0$ so that*

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq D\ d(1, \gamma) - L$$

for all $\gamma \in \Gamma$.

Proof. Theorem 27.1 implies that if ρ is P_k-Anosov, then the kth singular value gap grows exponentially, so we have already established the forward direction.

Now suppose that there exists $L \geq 0$ and $D > 0$ so that

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq D\ d(1, \gamma) - L$$

for all $\gamma \in \Gamma$. Proposition 30.3 implies that there exist limit maps $\xi : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$. Since ρ is P_k-divergent, Theorem 31.1 implies that it suffices to show that ξ and θ are transverse. Notice that by Theorem 34.3 it suffices to prove the result in the case where $k = 1$.

The work of Quas, Thieullen and Zarrabi is done in the context of linear transformations of Banach spaces. We make use of a generalization of their main result due to Zhu [231]. (If you would like to see the proof, I recommend looking at the Appendix to Zhu's paper [231] where he works in the simpler finite-dimensional setting.) We omit the explicit constants they obtain, since they will play no role in our work.

Theorem 35.2. (Quas-Thieullen-Zarrabi [180], Zhu [231, Theorem B.1]) *Suppose that $\{A_k\}_{k \in \mathbb{Z}}$ is a sequence in $\mathsf{GL}(d, \mathbb{R})$ and there exist constants $C \geq e^{1/3}$ and $\mu, \mu' \geq 1$ so that*

(1) *For all $r \in \mathbb{Z}$ and $n \geq 0$,*

$$\frac{\sigma_2(A_{r+n-1} \cdots A_r)}{\sigma_1(A_{r+n-1} \cdots A_r)} \leq Ce^{-n\mu}$$

(2) *For all $r \leq 0$ and $n \geq 0$,*

$$d\Big(U_1\big(A_{r-1} \cdots A_{r-n}\big), U_1\big(A_{r-1} \cdots A_{r-(n+1)}\big)\Big) \leq Ce^{-n\mu}$$

and

$$d\Big(U_{d-1}\big(A_r^{-1} \cdots A_{r+n-1}^{-1}\big), U_{d-1}\big(A_r^{-1} \cdots A_{r+n}^{-1}\big)\Big) \leq Ce^{-n\mu}.$$

(3) *For all $r \in \mathbb{Z}$ and $n, m \geq 0$,*

$$\frac{\sigma_1\big(A_{r+n-1} \cdots A_{r-m}\big)}{\sigma_1\big(A_{r+n-1} \cdots A_r\big)\sigma_1\big(A_{r-1} \cdots A_{r-m}\big)} \geq \frac{e^{-m\mu'}}{C}$$

Then, for each $r \in \mathbb{Z}$,

$$E^u(r) = lim_{n\to\infty}U_1\big(A_{r-1} \cdots A_{r-n}\big) \qquad and \qquad E^s(r) = lim_{n\to\infty}U_{d-1}\big(A_r^{-1} \cdots A_{r+n}^{-1}\big)$$

exist and are transverse.

Let x and y be distinct points in $\partial\Gamma$. It remains to show that $\xi(x)$ is transverse to $\theta(y)$. Let $(\gamma_k)_{k\in\mathbb{Z}}$ be a geodesic joining x to y. Since, both ξ and θ are ρ-equivariant, we may assume that $\gamma_0 = id$. Since ξ and θ have the Cartan property,

$$\xi(x) = \lim_{k\to-\infty} U_1\big(\rho(\gamma_k)\big) \qquad \text{and} \qquad \theta(y) = \lim_{k\to+\infty} U_{d-1}\big(\rho(\gamma_k)\big).$$

For all $k \in \mathbb{Z}$, let

$$A_k = \rho(\gamma_k^{-1}\gamma_{k-1}),$$

so

$$\rho(\gamma_k^{-1}) = A_k A_{k-1} \cdots A_1 \qquad \text{if} \quad k > 0$$

and

$$\rho(\gamma_k) = A_0 A_{-1} \cdots A_{k+1} \qquad \text{if} \quad k < 0.$$

Therefore,

$$\xi(x) = \lim_{k\to-\infty} U_1\big(A_0 A_{-1} \cdots A_{k+1}\big) \qquad \text{and} \qquad \theta(y) = \lim_{k\to+\infty} U_{d-1}\big(A_1^{-1} \cdots A_k^{-1}\big).$$

If we consider the sequence $\{A_k\}$, then $\xi(x) = E^u(1)$ and $\theta(y) = E^s(1)$. Theorem 35.2 will guarantee that $\xi(x)$ and $\theta(y)$ are transverse if we can verify conditions (1), (2) and (3).

First notice that

$$\frac{\sigma_2(A_{r+n-1}\cdots A_r)}{\sigma_1(A_{r+n-1}\cdots A_r)} = \frac{\sigma_2(\rho(\gamma_{r+n-1}^{-1}\gamma_{r-1}))}{\sigma_1(\rho(\gamma_{r+n-1}^{-1}\gamma_{r-1}))} \le (e^L)e^{-Dn}$$

since $d(\gamma_{r+n-1}^{-1}\gamma_{r-1}, id) = n$. Now we check that

$$
\begin{aligned}
d\Big(U_1\big(A_{r-1}\cdots A_{r-n}\big), U_1\big(A_{r-1}\cdots A_{r-(n+1)}\big)\Big) &= d\Big(U_1\big(\rho(\gamma_{r-1}\gamma_{r-n-1}^{-1})\big), U_1\big(\rho(\gamma_{r-1}\gamma_{r-n-2}^{-1})\big)\Big) \\
&\le \sqrt{d-1}\frac{\sigma_1(\rho(\gamma_{r-n-1}\gamma_{r-n-2}^{-1}))\sigma_2(\rho(\gamma_{r-1}\gamma_{r-n-1}^{-1}))}{\sigma_d(\rho(\gamma_{r-n-1}\gamma_{r-n-2}^{-1}))\sigma_1(\rho(\gamma_{r-1}\gamma_{r-n-1}^{-1}))} \\
&\le \sqrt{d-1}e^L M^2 e^{-Dn}
\end{aligned}
$$

where $M = \max\{\sigma_1(\rho(s)) \mid s \in S\}$. In the second line we applied Lemma 29.2 and in the third line we used our main assumption and the fact that $d(\gamma_{r-1}\gamma_{r-n-1}, id) = n$. Similarly,

$$d\Big(U_{d-1}\big(A_r^{-1}\cdots A_{r+n-1}^{-1}\big), U_{d-1}\big(A_r^{-1}\cdots A_{r+n}^{-1}\big)\Big) \le \sqrt{d-1}e^L M^2 e^{-Dn}.$$

.

Finally, applying Lemma 29.1, we see that

$$
\begin{aligned}
\frac{\sigma_1\big(A_{r+n-1}\cdots A_{r-m}\big)}{\sigma_1\big(A_{r+n-1}\cdots A_r\big)\sigma_1\big(A_{r-1}\cdots A_{r-m}\big)} &\ge \frac{\sigma_1\big(A_{r+n-1}\cdots A_r\big)\sigma_d\big(A_{r-1}\cdots A_{r-m}\big)}{\sigma_1\big(A_{r+n-1}\cdots A_r\big)\sigma_1\big(A_{r-1}\cdots A_{r-m}\big)} \\
&\ge M^{-2m} = e^{-(2\log M)m}
\end{aligned}
$$

So, (1), (2) and (3) hold with $C = \sqrt{d-1}e^L M^2$, $\mu = \frac{1}{4D}$ and $\mu' = 2\log M$. We then apply Theorem 35.2 to see that $\xi(x)$ and $\theta(y)$ are transverse, which completes our proof. $\qquad\square$

Remark: Both Kapovich-Leeb-Porti [133] and Bochi-Potrie-Sambarino [32] also show that if $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation of a finitely generated group and the kth singular value gap grows uniformly exponentially in word length, then the group is hyperbolic. In both cases, the main idea is to show that the action of $\rho(\Gamma)$ on its Benoist limit set is a convergence group action so that every point is conical. It then follows from a deep result of Bowditch [36], see Theorem 5.9, that $\rho(\Gamma)$, and hence Γ, is hyperbolic.

36. A characterization in terms of eigenvalues

Kassel and Potrie [136] used Theorem 35.1 to prove a characterization of P_k-Anosov representations in terms of eigenvalues:

Theorem 36.1. (Kassel-Potrie [136, Corollary 4.6]) *If Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation and $1 \leq k \leq \frac{d}{2}$, then ρ is P_k-Anosov if and only if there exists $J, B > 0$, so that*

$$\log \left(\frac{|\lambda_k(\rho(\gamma))|}{|\lambda_{k+1}(\rho(\gamma))|} \right) \geq J||\gamma|| - B$$

for any element $\gamma \in \Gamma$.

We will give the proof under the additional assumption that ρ is semi-simple, i.e. a direct sum of irreducible representation. This will allow us to use an important result of Abels, Margulis and Soifer on semi-simple representations. (Semi-simple representations are sometimes called completely reducible representations, which results in the absurdist fact that irreducible representations are completely reducible.). In general, one completes the proof by consider the semi-simplification ρ^{ss} of ρ. This suffices since Guichard-Gueritaud-Kassel-Wienhard [107, Lemma 2.40 and Proposition 1.8] show that $\lambda_i(\rho^{ss}(\gamma)) = \lambda_i(\rho(\gamma))$ for all i and all $\gamma \in \Gamma$ and that ρ is P_k-Anosov if and only if ρ^{ss} is P_k-Anosov.

Proof. We use the following (special case of a) lemma of Delzant-Guichard-Labourie-Mozes.

Lemma 36.2. (Delzant-Guichard-Labourie-Mozes [87, Lemma 2.0.1]) *If Γ is a hyperbolic group, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation and there exists J and B so that*

$$\log |\lambda_1(\rho(\gamma))| \geq J||\gamma|| - B$$

for all $\gamma \in \Gamma$, then there exists R and D so that

$$\log \sigma_1(\rho(\gamma)) \geq Rd(1, \gamma) - D$$

for all $\gamma \in \Gamma$. In particular, the orbit map $\tau_\rho : \Gamma \to X_d$ is a quasi-isometric embedding.

Proof. Recall that since Γ is hyperbolic, see Proposition 8.4, there exists $\alpha, \beta \in \Gamma$ and $K > 0$ so that

$$d(1, \gamma) \leq 3 \max\{||\gamma||, ||\gamma\alpha||, ||\gamma\beta||\} + K$$

for all $\gamma \in \Gamma$. Let $\mathcal{C} = \{id, \alpha, \beta\}$, $x_0 = [\mathsf{SO}(d)] \in X_d = \mathsf{SL}(d, \mathbb{R})/\mathsf{SO}(d)$ and $C = \max_{\beta \in \mathcal{C}}\{d(x_0, \beta(x_0))\}$. So, given $\gamma \in \Gamma$, we can choose $\eta \in C$ so that

$$d(id, \gamma) \leq 3||\gamma\eta|| + K.$$

Therefore, since $d(x_0, \rho(\gamma\eta)(x_0)) \geq \log \sigma_1(\rho(\gamma\eta))$ and $\sigma_1(\rho(\gamma\eta) \geq |\lambda_1(\rho(\gamma\eta))|$, we see that

$$d(x_0, \gamma(x_0)) \geq d(x_0, \gamma\eta(x_0)) - 2C \geq \log \sigma_1(\rho(\gamma\eta)) - 2C \geq J||\gamma\eta|| - B - 2C \geq \frac{J}{3}d(1, \gamma) - (K + B + 2C).$$

But notice that

$$d \log \sigma_1(\rho(\gamma)) > d(x_0, \gamma(x_0)) \geq \log \sigma_1(\rho(\gamma))$$

so this implies that

$$\log \sigma_1(\rho(\gamma)) \geq \frac{J}{3d} d(1, \gamma) - (K + B + 2C)$$

which establishes our main claim. Moreover, if $M = \max\{\log \sigma_1(\rho(s)) \mid s \in S\}$, then

$$\log \sigma_1(\rho(\gamma)) \leq M^2 d(1, \gamma), \quad \text{so} \quad dM^2 d(1, \gamma) \geq d(x_0, \gamma(x_0))$$

which implies that τ_ρ is a quasi-isometric embedding. □

Notice that since

$$|\lambda_1(\rho(\gamma))| \geq \left(\frac{|\lambda_k(\rho(\gamma))|}{|\lambda_{k+1}(\rho(\gamma))|} \right)^{\frac{1}{d-1}},$$

ρ satisfies the assumptions of Lemma 36.2, so there exists R and D so that

$$\log \sigma_1(\rho(\gamma)) \geq Rd(1, \gamma) - D$$

for all $\gamma \in \Gamma$.

Then, since we are assuming that ρ is semi-simple Theorem 33.4 implies that there exists a finite subset $\mathcal{B}$ of Γ and $A \geq 0$ such that if $\gamma \in \Gamma$, then there exists $\beta \in \mathcal{B}$ so that

$$\left| \log \sigma_i(\rho(\gamma)) - \log \left(|\lambda_i(\rho(\gamma\beta))| \right) \right| \leq A$$

for all $i = 1, \ldots, d$. In particular,

$$\log |\lambda_1(\rho(\gamma\beta))| \geq \log \sigma_1(\rho(\gamma)) - A \geq Rd(1, \gamma) - (A + B),$$

but if $M = \max\{\log \sigma_1(\rho(s)) \mid s \in S\}$, then

$$M||\gamma\beta|| \geq \log |\lambda_1(\rho(\gamma\beta))|$$

so

$$||\gamma\beta|| \geq \frac{R}{M} d(1, \gamma) - \frac{A + B}{M}.$$

Finally,

$$
\begin{aligned}
\log \sigma_k(\rho(\gamma)) - \log \sigma_{k+1}(\rho(\gamma) &\geq& \log |\lambda_k(\rho(\gamma))| - \log |\lambda_{k+1}(\rho(\gamma))| - 2A \\
&\geq& J||\gamma\beta|| - B - 2A \\
&\geq& \frac{JR}{M} d(1, \gamma) - (B + 2A) - \frac{AB}{M}
\end{aligned}
$$

Theorem 35.1 then implies that ρ is P_k-Anosov.

□

Part 7. Convex cocompactness revisited

> *It was then I knew I'd had enough*
> *Burned my credit card for fuel*
> *Headed out to where the pavement*
> *Turns to sand*
> *With a one-way ticket*
> *To the land of truth*
> *And my suitcase in my hand*
> —————————Neil Young [225]

Although Kleiner-Leeb [140] and Quint [181] have shown that one cannot construct a robust theory by studying convex cocompact actions on symmetric spaces, Danciger-Guéritaud-Kassel [84] and Andrew Zimmer [232] have recently shown that many Anosov representations can profitably be viewed as convex cocompact actions on properly convex domains in projective space. The ideas in these papers are very beautiful and seem likely to have many future applications.

Their work was preceded, and partially inspired by, by Benoist's work on Benoist representations. Crampon-Marquis [77, 78], Cooper-Long-Tillman [70, 71], Ballas-Danciger-Lee [10], and others, have also worked on more general actions of discrete groups on properly convex domains, although their work does not focus on the relationship with Anosov representations.

37. Definitions and goals

We begin with some basic definitions. If Ω is a properly convex subset of $\mathbb{RP}^{d-1}$, we recall that $\mathrm{Aut}(\Omega) \subset \mathsf{SL}(d, \mathbb{R})$ is the group of projective automorphisms preserving Ω. If Γ is a discrete subgroup of $\mathrm{Aut}(\Omega)$, then its **full orbital limit set** $\Lambda^{orb}(\Gamma) \subset \partial\Omega$ is the set of accumulation points of any Γ-orbit $\Gamma(x)$ where $x \in \Omega$, i.e.

$$\Lambda^{orb}(\Gamma) = \{z \in \partial\Omega \mid \text{there exists } x \in \Omega \text{ and } \{\gamma_n\} \subset \Gamma \text{ so that } \lim \gamma_n(x) = z\}.$$

Let $\mathcal{C}(\Lambda^{orb}(\Gamma))$ be the convex hull of the full orbital limit set and let $\mathcal{C}_0(\Lambda^{orb}(\Gamma)) = \mathcal{C}(\Lambda^{orb}(\Gamma)) \cap \Omega$. The quotient

$$\widehat{\mathcal{C}_\Gamma} = \mathcal{C}_0(\Lambda^{orb}(\Gamma))/\Gamma$$

is called the **convex core** of Ω/Γ. (In the study of hyperbolic manifolds, it is common to refer to the convex hull of the limit set as its intersection with $\mathbb{H}^n$ and a similar convention is often used in this theory. However, we will try to be careful in our treatment, since we will have occasion to use both the full convex hull and its intersection with Ω.)

Suppose that Γ is a discrete subgroup of $\mathrm{Aut}(\Omega)$, where $\Omega \subset \mathbb{RP}^{d-1}$ is a properly convex domain. We say that the action of Γ on Ω is **naively convex cocompact** if Γ acts cocompactly on some non-empty convex subset of Ω. We say that the action of Γ on Ω is **convex cocompact** if its convex core is compact, i.e. if Γ acts cocompactly on $\mathcal{C}_0(\Lambda^{orb}(\Gamma))$. We say that Γ acts **regularly convex cocompactly** if it acts convex cocompactly on a convex subset of $\mathcal{C}$ of Ω, and every point in $\overline{\mathcal{C}} \cap \partial\Omega$ is a C^1 extreme point of $\partial\Omega$. (Recall that $z \in \partial\Omega$ is an extreme point if it is not contained in the interior of any line segment in $\partial\Omega$.) We say that Γ acts **strongly convex cocompactly** on Ω if it acts convex cocompactly and Ω is strictly convex and has

a C^1 boundary. These definitions give increasingly restrictive generalizations of the classical definition in the setting of discrete subgroups of $\mathsf{SO}_0(d-1,1) \subset \mathsf{SL}(d,\mathbb{R})$.

We will see that there is an intimate connection between convex cocompact groups and Anosov representations. We first show that the inclusion map of any regular convex cocompact subgroup of $\mathsf{SL}(d,\mathbb{R})$ is projective Anosov. Conversely, we will see that if a projective Anosov representations preserves a properly convex domain, then its image is regularly convex cocompact. We will also see that if the domain group of a projective Anosov representation is freely indecomposable, but not a surface group, then its image is strongly convex cocompact. All these results are due to Danciger-Guérituad-Kassel [84] and/or Zimmer [232].

Danciger, Guéritaud and Kassel [84, Theorem 1.15] show that if a group acts regularly convex cocompactly on some properly convex domain, then it acts strongly convex cocompactly on some, perhaps different, strictly convex domain. Zimmer [232, Corollary 1.30] also showed that there exists a finite-dimensional vector space W and a representation $\tau : \mathsf{SL}(d,\mathbb{R}) \to \mathsf{SL}(W)$ so that if $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is projective Anosov, then $\tau \circ \rho$ is regularly convex cocompact. So one can always find a regularly convex cocompact group naturally associated to a projective Anosov representation.

However, not all images of projective Anosov representations are convex cocompact. We will soon see (Lemma 38.1) that if $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is projective Anosov and $\rho(\Gamma)$ preserves Ω, then $\xi(\partial\Gamma) \subset \partial\Omega$. So, if $\rho(\Gamma)$ is convex cocompact, then $\xi_\rho(\Gamma)$ must be contained in an affine chart. For example, if $\rho : \pi_1(S) \to \mathsf{SL}(3,\mathbb{R})$ is the direct sum of a Fuchsian representation into $\mathsf{SL}(2,\mathbb{R})$ and the trivial representation, then $\xi_\rho(\partial\Gamma)$ is a projective line, so cannot be contained in any affine chart. More interesting examples are provided by even-dimensional Hitchin representations. It is not hard to check that if $\rho : \pi_1(S) \to \mathsf{SL}(2n,\mathbb{R})$ is $2n$-Fuchsian, then $\xi_\rho(\partial\pi_1(S)) = V_{2n}(\mathbb{RP}^1)$ does not lie in any affine chart, so ρ is not convex cocompact. In general, Danciger-Gueritaud-Kassel [84, Proposition 1.7] show that a Hitchin representation $\rho : \pi_1(S) \to \mathsf{SL}(d,\mathbb{R})$ is convex cocompact if and only if d is odd and $d \geq 3$.

In Sections 38 through 41, we restrict to representations or discrete groups which are irreducible and discuss regular convex cocompactness. When first encountering this material it is probably best to focus on this simpler setting. (In fact, Zimmer [232] works almost entirely in this setting). In Section 42 we extend these results to the general setting, while in Section 43 we prove that regularly convex cocompact representations are strongly convex cocompact. In Section 44 we discuss, largely without proof, some further topics in the subject.

38. **First principles**

We now make some relatively simple observations about actions on properly convex domains. We first see that if ρ is projective Anosov and its image preserves a properly convex domain Ω, then the image of its limit map ξ_ρ is the full orbital limit set and ξ_ρ is a continuous extension of the orbit map (for any choice of basepoint).

Lemma 38.1. *Suppose that $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is projective Anosov, $\Omega \subset \mathbb{RP}^{d-1}$ is properly convex and $\rho(\Gamma) \subset \mathrm{Aut}(\Gamma)$. Then*

(1) $\xi_\rho(\partial\Gamma) \subset \partial\Omega$.
(2) *If $z \in \partial\Gamma$, then $\theta_\rho(z)$ is a support plane for $\partial\Omega$.*
(3) *If $x \in \Omega$ and a sequence $\{\gamma_n\}$ converges to $z \in \partial\Gamma$, then $\{\rho(\gamma_n)(x)\}$ converges to $\xi_\rho(z)$.*
(4) *The full orbital limit set $\Lambda^{orb}(\rho(\Gamma)) = \xi_\rho(\partial\Gamma)$.*

Proof. We first observe that if γ has infinite order then there exists some point $x \in \Omega$ which does not lie in the repelling hyperplane for $\rho(\gamma)$, so $\{\rho(\gamma^n)(x)\}$ converges to the attracting eigenline $\xi_\rho(\gamma^+)$ for $\rho(\gamma)$. But $\rho(\Gamma)$ acts properly discontinuously on Ω, so $\xi_\rho(\gamma^+) \in \partial\Omega$. Since ξ_ρ is continuous and attracting fixed points of infinite order elements are dense in $\partial\Gamma$ (Proposition 5.6), $\xi_\rho(\partial\Gamma) \subset \partial\Omega$. So (1) holds.

Corollary 32.4 implies that $\rho^* : \Gamma \to \mathsf{SL}((\mathbb{R}^d)^*)$ is also projective Anosov and that $\xi_{\rho^*} = \theta_\rho$. Lemma 15.4 then implies that Ω^* is properly convex and that $\rho^*(\Gamma)$ preserves Ω^*. Part (1) implies that $\xi_{\rho^*}(\partial\Gamma)$ lies in $\partial\Omega^*$. Therefore, if $z \in \partial\Gamma$, then $\theta_\rho(z) = \xi_{\rho^*}(z)$ is a support plane to Ω. Since $\xi_\rho(z) \in \theta_\rho(z)$, $\theta_\rho(z)$ is a support plane to Ω at $\xi_\rho(z)$, so (2) holds.

Since the closure of Ω is compact, to establish (3) it suffices to prove that every convergent subsequence of $\{\rho(\gamma_n)(x)\}$ converges to $\xi_\rho(w)$. Since every convergent subsequence of $\{U_{d-1}(\rho(\gamma_n)^{-1})\}$ converges to a hyperplane in $\theta_\rho(\partial\Omega)$ (since θ_ρ has the Cartan property), there exists $\delta > 0$ so that

$$d_{\mathbb{RP}^{d-1}}\Big(x, U_{d-1}(\rho(\gamma_n)^{-1})\Big) = \sin\angle(x, U_{d-1}(\rho(\gamma_n)^{-1}) \geq \delta$$

for all large enough n. But then, by Lemma 38.2 below,

$$d_{\mathbb{RP}^{d-1}}(\rho(\gamma_n)(x), U_1(\rho(\gamma_n))) = \sin\angle(\rho(\gamma_n(x)), U_1(\rho(\gamma_n))) \leq \frac{\sigma_2(\rho(\gamma_n))}{\delta\sigma_1(\rho(\gamma_n))}$$

for all large enough n. So, since ρ is P_1-divergent,

$$d_{\mathbb{RP}^{d-1}}(\rho(\gamma_n)(x), U_1(\rho(\gamma_n))) \to 0.$$

Since ξ_ρ has the Cartan property, $d(U_1(\rho(\gamma_n)), \xi_\rho(z)) \to 0$, so

$$\lim \rho(\gamma_n)(x) = \xi_\rho(z)$$

as required, so (3) holds.

Then, (3) implies that the full orbital limit set $\Lambda^{orb}(\rho(\Gamma))$ is exactly $\xi_\rho(\partial\Gamma)$, so (4) holds. $\square$

In the proof of (3) we made use of the following elementary lemma from linear algebra (see, for example, Bochi-Potrie-Sambarino [32, Lemma A.6].)

Lemma 38.2. *If $A \in \mathsf{SL}(d, \mathbb{R})$ is biproximal, and $x \in \mathbb{RP}^{d-1}$, then*

$$d_{\mathbb{RP}^{d-1}}(\rho(\gamma_n)(x), U_1(\rho(\gamma_n)) \leq \frac{\sigma_2(\rho(\gamma_n))}{\Big(\sin\angle(x, U_{d-1}(\rho(\gamma_n)^{-1})\Big)\sigma_1(\rho(\gamma_n))}.$$

It will sometimes be useful to work in the unit sphere rather than projective space. If V is a vector space, usually $\mathbb{R}^d$ or $(\mathbb{R}^d)^*$, let $S(V)$ be the unit sphere in the vector space V and let $p : S(V) \to \mathbb{P}(V)$ be the quotient map. Often $\mathsf{Gr}_{d-1}(\mathbb{R}^d)$ will be identified with $\mathbb{P}((\mathbb{R}^d)^*)$.

If ρ is projective Anosov, it will be useful to be able to lift the limit maps to maps having image in the unit spheres. We see that we can do so whenever $\rho(\Gamma)$ preserves a properly convex domain.

Lemma 38.3. (Danciger-Guéritaud-Kassel [84, Proposition 4.5], Zimmer [232, Theorem 3.1])
Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation and $\rho(\Gamma)$ preserves a properly convex domain Ω_0. Then, there exist ρ-equivariant lifts

$$\bar{\xi}_\rho : \partial\Gamma \to S(\mathbb{R}^d) \qquad \text{and} \qquad \bar{\theta}_\rho : \partial\Gamma \to S((\mathbb{R}^d)^*)$$

of the Anosov limit maps $\xi_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$ so that

$$\bar{\theta}_\rho(z)(\bar{\xi}_\rho(w)) > 0$$

if $w \neq z \in \partial\Gamma$.

Proof. Notice that since Ω_0 is properly convex, $p^{-1}(\Omega_0)$ has two disjoint components Ω_0^+ and Ω_0^- which are separated by a hyperplane H_0. Since $\rho(\gamma)$ is discrete, it acts properly discontinuously on Ω_0. We have seen that $\xi_\rho(\partial\Gamma) \subset \partial\Omega_0$.

So, for all $z \in \partial\Gamma$ choose $\bar{\xi}_\rho(z) \in p^{-1}(\xi_\rho(z)) \cap \Omega_0^+$ and choose $\bar{\theta}_\rho(z) \in p^{-1}(\theta_\rho(z))$ so that $\bar{\theta}_\rho(\bar{\xi}_\rho(x)) > 0$ for all $x \in \partial\Gamma - \{z\}$. $\square$

We now show that one can use the lifted limit maps from Lemma 38.3 to find a "maximal" convex domain preserved by $\rho(\Gamma)$. (It is maximal in the sense that every properly convex $\rho(\Gamma)$-invariant domain containing Ω is contained in Ω_{max}, see Danciger-Gueritaud-Kassel [84, Proposition 4.5].)

Lemma 38.4. (Danciger-Guéritaud-Kassel [84, Proposition 4.5], Zimmer [232, Lemma 3.3])
Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation and there exist ρ-equivariant lifts

$$\tilde{\xi}_\rho : \partial\Gamma \to S(\mathbb{R}^d) \qquad \text{and} \qquad \tilde{\theta}_\rho : \partial\Gamma \to S((\mathbb{R}^d)^*)$$

of the Anosov limit maps $\xi_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$, and

$$\bar{\theta}_\rho(z)(\bar{\xi}_\rho(w)) > 0$$

for all $w \neq z \in \partial\Gamma$. Let

$$\Omega_{max} = \mathbb{P}\Big(\big\{\vec{v} \mid \bar{\theta}_\rho(z)(\vec{v}) > 0 \ \text{ if } \ z \in \partial\Gamma\big\}\Big).$$

Then

(1) $\rho(\Gamma) \subset \operatorname{Aut}(\Omega_{max})$.
(2) $\mathcal{C}_0(\xi_\rho(\partial\Gamma)) \subset \Omega_{max}$.
(3) *If $z \in \partial\Gamma$, then $\ker(\theta_\rho(z))$ is a support plane to $\partial\Omega_{max}$ at $\xi_\rho(z)$.*
(4) *If $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$, then Ω_{max} is properly convex.*
(5) *The closure of the dual Ω_{max}^* of Ω_{max} is the convex hull of $\theta_\rho(\partial\Gamma)$ in $(\mathbb{RP}^{d-1})^*$. More precisely,*

$$\overline{\Omega_{max}^*} = \mathbb{P}\left(\left\{\sum_{i=1}^d a_i \bar{\theta}_\rho(z_i) \mid a_i \geq 0 \text{ and } z_i \in \partial\Gamma \text{ for all } i\right\}\right).$$

Proof. Let

$$\widetilde{\Omega}_{max} = \left\{\vec{v} \in S(\mathbb{R}^d) \mid \bar{\theta}_\rho(z)(\vec{v}) > 0 \ \text{ if } \ z \in \partial\Gamma\right\}$$

so $\Omega_{\max} = p(\widetilde{\Omega}_{max})$. Since $\bar{\theta}_\rho$ is ρ-equivariant, $\widetilde{\Omega}_{max}$, and thus Ω_{max}, is invariant under $\rho(\Gamma)$, which establishes (1).

The convex hull $\mathcal{C}(\bar{\xi}_\rho(\partial\Gamma))$ of $\bar{\xi}_\rho(\partial\Gamma)$ may be written as:

$$\mathcal{C}(\bar{\xi}_\rho(\partial\Gamma)) = \left\{ \sum_{i=1}^d a_i \bar{\xi}_\rho(z_i) \;\Big|\; \sum_{i=1}^d a_i = 1, \;\; a_1, \ldots, a_d \geq 0, \;\; z_1, \ldots, z_d \in \partial\Gamma \right\}.$$

If $x = \sum_{i=1}^d a_i \xi_\rho(z_i) \in \mathcal{C}_0(\bar{\xi}_\rho(\partial\Gamma))$, then we may assume that $z_1 \neq z_2$ and $a_1, a_2 > 0$. Thus, if $z \in \partial\Gamma$, then $\bar{\theta}_\rho(z)(x) > 0$. Therefore, $\mathcal{C}_0(\bar{\xi}_\rho(\partial\Gamma)) \subset \widetilde{\Omega}_{max}$, so $\mathcal{C}_0(\xi_\rho(\partial\Gamma)) \subset \Omega_{max}$. Notice that, by construction, if $z \in \partial\Gamma$, then $\ker(\theta_\rho(z))$ is a support plane to Ω_{max} at $\xi_\rho(z)$ since it is disjoint from Ω_{max} and intersects $\partial\Omega_{max}$ at $\xi_\rho(z)$. So, we have established (2) and (3).

If $\bar{\theta}_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$, there exists $z_1, \ldots, z_d$ so that $\{\bar{\theta}_\rho(z_1), \ldots, \bar{\theta}_\rho(z_d)\}$ span $(\mathbb{R}^d)^*$. Then

$$\widetilde{\Omega}_{max} = \left\{ \vec{v} \mid \bar{\theta}_\rho(z)(\vec{v}) > 0 \;\; \text{if} \;\; z \in \partial\Gamma \right\} \subset \left\{ \vec{v} \mid \bar{\theta}_\rho(z_i)(\vec{v}) > 0, \;\; \text{for all} \;\; i = 1, \ldots, d \right\}$$

which is a simplex in an affine chart for $\mathbb{RP}^{d-1}$. Since $\widetilde{\Omega}_{max}$ is convex by construction, it is thus properly convex. Therefore, $\Omega_{max} = p(\widetilde{\Omega}_{max})$ is properly convex, so (4) holds.

Finally, we show that Ω^*_{max} is the convex hull of $\theta_\rho(\partial\Gamma)$. Let

$$\widehat{\mathcal{C}} = \left\{ \sum_{i=1}^d a_i \bar{\theta}_\rho(z_i) \;\Big|\; \sum_{i=1}^d a_i = 1, \; a_i \geq 0 \text{ and } z_i \in \partial\Gamma, \text{ for all } i \right\}.$$

First notice that if $\phi = \sum_{i=1}^d a_i \eta_i(z_i) \in \widehat{\mathcal{C}}$, then $\phi(\vec{v}) \geq 0$ for all $\vec{v} \in \widetilde{\Omega}_{max}$, so ϕ lies in $\overline{\Omega^*_{max}}$. If $[\phi]$ does not lie in the convex hull of $\theta_\rho(\partial\Gamma)$, then it is contained in a plane through the origin in $(\mathbb{R}^d)^*$ which intersects the cone on $\widehat{\mathcal{C}}$ only at the origin. Therefore, there exists $\vec{w} \in \mathbb{R}^d$ so that $\phi(\vec{w}) = 0$, but $\eta(\vec{w}) \neq 0$ for any $\eta \in \widehat{\mathcal{C}}$. Since $\widehat{\mathcal{C}}$ is connected, either $\eta(\vec{w}) > 0$ for all $\eta \in \mathcal{C}$ or $\eta(\vec{w}) < 0$ for all $\eta \in \mathcal{C}$. So, perhaps after replacing $\vec{w}$ with $-\vec{w}$, we see that $\bar{\theta}_\rho(z)(\vec{w}) > 0$ for all $z \in \partial\Gamma$. Therefore, $\vec{w} \in \Omega_{max}$, which implies that $p(\phi)$ does not lie in Ω^*_{max}. This completes the proof of (5). $\qquad\square$

39. **Convex cocompact groups which are Anosov**

It is clear that not every convex cocompact subgroup of $\mathsf{SL}(d, \mathbb{R})$ is projective Anosov. For example, Benoist [23] exhibited many groups acting cocompactly on properly convex domains which are not strictly convex. These groups will be convex cocompact, but not even Gromov hyperbolic (by Theorem 16.1). However, if we require that the action is regularly convex cocompact, then the inclusion map will be projective Anosov.

Theorem 39.1. (Danciger-Guéritaud-Kassel [84, Theorem 1.15], Zimmer [232, Theorem 5.1]) *If $\Omega \subset \mathbb{RP}^{d-1}$ is properly convex and $\Gamma \subset \mathrm{Aut}(\Omega)$ acts regularly convex cocompactly on Ω, then the inclusion map of Γ into $\mathsf{SL}(d, \mathbb{R})$ is projective Anosov.*

In this section we will restrict to the setting of irreducible subgroups of $\mathsf{SL}(d, \mathbb{R})$ in the final portion of the proof, but in Section 42 we will give a complete proof.

Proof. (*When Γ is irreducible.*) We first show, assuming only that there are no line segments in Λ^{orb}_Γ, that Γ is hyperbolic.

Proposition 39.2. (Danciger-Guéritaud-Kassel [84, Lemma 6.3], Zimmer [232, Lemma 5.4]) *Suppose that $\Omega \subset \mathbb{RP}^{d-1}$ is properly convex and $\Gamma \subset \mathrm{Aut}(\Gamma)$ acts convex cocompactly on Ω. If $\Lambda^{orb}(\Gamma)$ contains no line segments, then $\mathcal{C}_0(\Lambda^{orb}_\Gamma)$ and Γ are Gromov hyperbolic.*

This result and its proof are a natural generalization of Proposition 16.2 from the convex divisible situation.

Proof. We first show that there exists R so that any geodesic in $\mathcal{C}_0(\Lambda_\Gamma^{orb})$ lies in a neighborhood of radius at most R from a projective line segment. If not, there exist a sequence α_n of geodesic line segments in $\mathcal{C}_0$ joining points $a_n, b_n \in \mathcal{C}_0$ and a point $y_n \in \alpha_n$ so that $d(y_n, [a_n, b_n]) \geq n$ for all n, where $[a_n, b_n]$ is the projective line segment in $\mathcal{C}_0$ joining a_n to b_n. Since the action of Γ on $\mathcal{C}_0$ is cocompact, there exists a compact set K and sequence $\{\gamma_n\} \subset \Gamma$ so that $\gamma_n(y_n) \in K$. After replacing, α_n with $\gamma_n(\alpha_n)$, we may pass to a subsequence so that $\{y_n\}$ converges to $y \in K$, $\{[a_n, b_n]\}$ converges to a line segment $[a, b]$ in $\overline{\mathcal{C}_0}$, and α_n converges to a geodesic joining a to b. Since $d(y, [a_n, b_n]) \to \infty$, we see that $[a, b]$ is contained in Λ_Γ^{orb}. Since Λ_Γ^{orb} contains no line segments, $a = b$. But Proposition 15.5 implies that no geodesic in a properly convex domain Ω can join a point in $\partial\Omega$ to itself. We have achieved a contradiction, so there does exist R so any geodesic in $\mathcal{C}_0(\Lambda_\Gamma^{orb})$ lies in a neighborhood of radius at most R from a projective line segment.

We next show that there exists δ so that any geodesic triangle in $\mathcal{C}_0$ whose edges are projective line segments is δ-thin, i.e. if $[x, z]$ and $[y, z]$ are projective line segments in $\mathcal{C}_0$ and $u \in [x, y]$, then $d(u, [x, z] \cup [y, z]) \leq \delta$. If not, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ of points in $\mathcal{C}_0$ such that $u_n \in \overline{x_n y_n}$ and $d(u_n, \overline{x_n z_n} \cup \overline{y_n z_n}) \geq n$ for all $n \in \mathbb{N}$. Since Γ acts cocompactly on $\mathcal{C}_0$, there is a compact set K in $\mathcal{C}_0$ such that we can always choose $\gamma_n \in \Gamma_n$ so that $\gamma_n(u_n) \in K$. So, after replacing x_n, y_n, z_n, u_n with $\gamma(x_n), \gamma(y_n), \gamma(z_n), \gamma(u_n)$, we may assume that $u_n \in K$ for all n. We may then pass to a subsequence so that $u_n \to u$, $x_n \to x$, $y_n \to y$ and $z_n \to z$. Since $d(u_n, \overline{x_n z_n} \cup \overline{y_n z_n}) \to \infty$, we must have $x, y, z \in \partial \Lambda^{orb}(\Gamma)$. Since $u \in \overline{xy}$, $x \neq y$. If $x = z$, then $\overline{y_n z_n} \to \overline{yz}$, so $d(u_n, \overline{y_n z_n}) \to 0$, which is a contradiction, so $x \neq z$. Similarly $y \neq z$. But then, since $\Lambda^{orb}(\Gamma)$ contains no line segments, the open line segment (xz) is contained in Ω, so $d(u, \overline{xz}) < \infty$. We have achieved a contradiction, so there must exist δ so that any geodesic triangle in $\mathcal{C}_0$ whose edges are projective line segments is δ-thin.

Combining these two observations, we see that any geodesic triangle is $\mathcal{C}_0$ is $(2R + \delta)$-thin, so $\mathcal{C}_0$ is Gromov hyperbolic. Therefore, Γ is also Gromov hyperbolic (by Proposition 4.1). $\square$

We now establish a more general lemma that guarantees that there exists a well-defined limit map from $\partial\Gamma$ to Λ_Γ^{orb}. The proof is a generalization of the proof of Proposition 18.1.

Lemma 39.3. *If $\Omega \subset \mathbb{RP}^{d-1}$ is properly convex, $\Gamma \subset \text{Aut}(\Omega)$ is hyperbolic and acts convex cocompactly on Ω and Λ_Γ^{orb} contains no line segments, then one may identify the Gromov boundary of $\mathcal{C}_0(\Lambda_\Gamma^{orb})$ with Λ_Γ^{orb}. Moreover, there exists a Γ-equivariant homeomorphism*

$$\xi : \partial\Gamma \to \Lambda_\Gamma^{orb} \subset \partial\Omega \subset \mathbb{RP}^{d-1}.$$

Proof. We observe that one may identify the Gromov boundary $\partial_\infty \mathcal{C}_0$ of $\mathcal{C}_0$ with $\Lambda^{orb}(\Gamma)$. Pick $x_0 \in \mathcal{C}_0$. Let $r : [0, \infty) \to \mathcal{C}_0$ be a geodesic ray. For all n let L_n be the projective line segment joining x_0 to $r(n)$ in $\mathcal{C}_0$. Since $\mathcal{C}_0$ is Gromov hyperbolic, the Fellow Traveller Property implies that there exists K so that the Hausdorff distance between $r([0, n])$ and L_n is at most K, for all n. We may pass to a subsequence so that $\{L_n\}$ converges to a projective line segment L joining x_0 to a point $z \in \Lambda^{orb}(\Gamma)$. Therefore, $r([0, \infty))$ lies a Hausdorff distance at most K from L. It follows that every geodesic ray in $\mathcal{C}_0$ lies a bounded Hausdorff distance from a projective line segment ending in Λ_Γ^{orb}. On the other hand, two projective line segments in a properly convex domain Ω lie a bounded Hausdorff distance apart if and only if their endpoints lie in the interior

of a line segment in $\partial\Omega$, see Lemma 15.3. Therefore, since every point of Λ_Γ^{orb} is an extreme point, no two geodesic rays with distinct endpoints in Λ_Γ^{orb} lie a finite Hausdorff distance apart. Finally, we conclude that $\partial_\infty \mathcal{C}_0$ may be identified with Λ_Γ^{orb}.

The Milnor-Svarc Lemma implies that the orbit map $\tau : \Gamma \to \mathcal{C}_0$ given by $\gamma \to \gamma(x_0)$ is a quasi-isometry, if $x_0 \in \mathcal{C}_0$, so Proposition 3.5 and Corollary 3.6 imply that there exists a homeomorphism $\xi : \partial\Gamma \to \Lambda_\Gamma^{orb}$ so that if $\{\gamma_n\} \subset \Gamma$ is a sequence so that $\lim \gamma_n = z \in \partial\Gamma$, then $\lim \gamma_n(x_0) = \xi(z)$. $\qquad\square$

Since every point in Λ_Γ^{orb} is a C^1 point of $\partial\Omega$, we can define $\theta : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ by letting $\theta(z)$ be the tangent space to $\partial\Omega$ at the point $\xi(z)$. Since $\partial\Omega$ is properly convex, ξ and θ are transverse.

If Γ is irreducible, then Theorem 33.1 immediately implies that the inclusion map of Γ into $\mathsf{SL}(d, \mathbb{R})$ is projective Anosov. In the reducible case, we will have to prove that the inclusion map is P_1-divergent and that ξ is dynamics preserving, so that we can apply Corollary **??**, see Section 42. $\qquad\square$

Danciger, Guéritaud and Kassel further show that you can weaken the assumptions of Theorem 39.1.

Theorem 39.4. (Danciger-Guéritaud-Kassel [84, Theorem 1.15]) *If $\Omega \subset \mathbb{RP}^{d-1}$ is a properly convex domain, $\Gamma \subset \mathrm{Aut}(\Omega)$ acts convex cocompactly on Ω and* **either**

(1) *the full orbital limit set $\Lambda^{orb}(\Gamma)$ contains no line segments,* **or**
(2) *Γ is Gromov hyperbolic,*

then the inclusion map of Γ into $\mathsf{SL}(d, \mathbb{R})$ is projective Anosov.

Tsouvalas [208] weakens the assumptions, by requiring that there is a convex Gromov hyperbolic subset of Ω preserved by Γ, not necessarily the convex hull of the full orbital limit set, and only requiring that the orbit map into this set is a quasi-isometric embedding (and not requiring that the action is cocompact).

Theorem 39.5. (Tsouvalas [208]) *If $\Omega \subset \mathbb{RP}^{d-1}$ is strictly convex, $\partial\Omega$ is C^1, $\Gamma \subset \mathrm{Aut}(\Omega)$, and*

(1) *the inclusion of Γ into $\mathsf{SL}(d, \mathbb{R})$ is a semisimple, quasi-isometric embedding, and*
(2) *Ω contains a Γ-invariant convex subset $\mathcal{C}$ so that $\mathcal{C}$ is Gromov hyperbolic with respect to the Hilbert metric on Ω,*

then the inclusion of Γ into $\mathsf{SL}(d, \mathbb{R})$ is projective Anosov.

40. **Anosov groups which are convex cocompact**

We are now ready to show that if the image of a projective Anosov representation preserves a properly convex domain, then it is regularly convex cocompact.

Theorem 40.1. (Danciger-Guéritaud-Kassel [84, Theorem 1.4], Zimmer [232, Theorem 1.27]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation and $\rho(\Gamma)$ preserves a properly convex domain in $\mathbb{RP}^{d-1}$, then $\rho(\Gamma)$ is regularly convex cocompact. If ρ is irreducible, then $\rho(\Gamma)$ acts regularly convex cocompactly on Ω_{max}.*

For the moment, we will only fully establish Theorem 40.1 in the case when $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$. Notice that if ρ is irreducible, then $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$. (Its dual ρ^* is also irreducible,

so if $\theta_\rho(\partial\Gamma) = \xi_{\rho^*}(\partial\Gamma)$ does not span $(\mathbb{R}^d)^*$ then $\rho^*(\Gamma)$ would preserve the subspace spanned by $\xi_{\rho^*}(\partial\Gamma)$ which would contradict irreducibility.) Our proof only uses the extra assumption that $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$ to show that $\rho(\Gamma)$ preserves a properly convex domain which contains $\mathcal{C}_0(\xi_\rho(\Gamma))$ and to show that every point of $\xi_\rho(\partial\Gamma)$ is a C^1 point of $\partial\Omega$. The argument in the general case, which we will give in Section 42, is more complicated and is due to Danciger-Guéritaud-Kassel [84].

Proof. (*When* $\theta_\rho(\partial\Gamma)$ *spans* $(\mathbb{R}^d)^*$) Recall that, by Lemmas 38.1, 38.3 and 38.4, $\rho(\Gamma) \subset \mathrm{Aut}(\Omega_{max})$, $\Lambda^{orb}(\rho(\Gamma)) = \xi_\rho(\partial\Gamma)$, $\xi_\rho(\partial\Gamma) \subset \partial\Omega_{max}$, $\mathcal{C}_0(\xi_\rho(\partial\Gamma)) \subset \Omega_{max}$, Ω_{max} is properly convex, and if $z \in \partial\Gamma$, then $\theta_\rho(z)$ is a support plane for Ω_{max} at the point $\xi_\rho(z)$.

The following more general lemma guarantees that $\rho(\Gamma)$ acts convex cocompactly on Ω_{max}.

Lemma 40.2. (Danciger-Guéritaud-Kassel [84, Proposition 8.1], Zimmer [232, Lemma 3.7]) *If* $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ *is projective Anosov,* Ω *is properly convex,* $\rho(\Gamma) \subset \mathrm{Aut}(\Omega)$ *and* $\mathcal{C}_0(\xi_\rho(\partial\Gamma)) \subset \Omega$, *then* $\rho(\Gamma)$ *acts convex cocompactly on* Ω.

Proof. If the convex core $\widehat{\mathcal{C}_\Gamma}$ is not compact, then there exists a sequence $\{x_n\}$ in $\mathcal{C}_0$ and $x_0 \in \mathcal{C}_0$ so that
$$d(x_n, x_0) = d(x_n, \Gamma(x_0)) \to \infty$$
where distance is measured in the Hilbert metric on Ω. We may pass to a subsequence, still called $\{x_n\}$, so that $\{x_n\}$ converges to some point $z \in \partial\Omega$. Since $x_n \in \mathcal{C}_0$ for all n, $z \in \Lambda^{orb}(\Gamma)$, so $z = \xi_\rho(w)$ for some $w \in \partial\Gamma$. Let $B_n = B(x_n, d(x_0, x_n))$ and notice that $\Gamma(x_0) \cap B_n$ is empty for all n and B_n converges to
$$H(x_0, z) = \bigcup_{x \in \overrightarrow{x_0 z}} B(x, d(x, x_0))$$
since $\overline{x_0 x_n}$ converges to $\overrightarrow{x_0 z}$. So $H(x_0, z)$ is disjoint from $\Gamma(x_0)$.

Since every point of $\partial\Gamma$ is a conical limit point for the action of Γ on $\partial\Gamma$, see Theorem 5.7, there exist distinct points $b \neq c \in \partial\Gamma$ and a sequence $\{\gamma_n\} \subset \Gamma$, so that $\gamma_n(w) \to b$ and $\gamma_n(v)$ converges to c uniformly on compact subsets of $\partial\Gamma - \{b\}$. Therefore, since ξ_ρ is ρ-equivariant, $\rho(\gamma_n)(z) \to \xi_\rho(b)$ and $\rho(\gamma_n)(x_0) \to \xi_\rho(c)$. We may assume that we have chosen x_0 so that there exists $u \neq v \in \xi_\rho(\Gamma) - \{z\}$, so that x_0 lies on the open line segment (uv). Since $\{\rho(\gamma_n)(u)\}$ and $\{\rho(\gamma_n)(v)\}$ both converge to z, we see that $\{\rho(\gamma_n)(x_0)\}$ converges to z. Since $\{d(\overrightarrow{\gamma_n(x_0)\gamma_n(z)}, x_0))\}$ converges to $d(\overleftrightarrow{\xi(b)\xi(c)}, x_0) < \infty$ and $d(x_0, \gamma_n(x_0)) \to \infty$, we see that for all large values of n,
$$x_0 \in H(\gamma_n(x_0), \gamma_n(z)) = \gamma_n(H(x_0, z))$$
which contradicts the fact that $H(x_0, z)$ is disjoint from $\Gamma(x_0)$. Therefore, Γ is convex cocompact. $\square$

We next establish that every point of $\xi_\rho(\partial\Gamma)$ is an extreme point.

Lemma 40.3. (Danciger-Guéritaud-Kassel [84, Theorem 1.15], Zimmer [232, Lemma 3.9]) *If* $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ *is projective Anosov and* $\rho(\Gamma)$ *acts convex cocompactly on a properly convex domain* $\Omega \subset \mathbb{RP}^{d-1}$, *then every point of* $\xi_\rho(\partial\Gamma)$ *is an extreme point of* $\partial\Omega$. *In particular,* $\xi_\rho(\partial\Gamma)$ *contains no line segments.*

Proof. Suppose that $z = \xi_\rho(w)$ lies in the interior of a line segment $[a, b]$ contained in $\partial\Omega$. Let y be another point in the interior of $[a, b]$ and choose $x_0 \in \mathcal{C}_0$. Lemma 15.3 implies that the geodesic rays $\overrightarrow{x_0 y}$ and $\overrightarrow{x_0 z}$ lie a bounded Hausdorff distance apart. So choose sequences $\{p_n\}$ in $\overrightarrow{x_0 z}$ and $\{q_n\}$ in $\overrightarrow{x_0 y}$, so that $p_n \to z$, $q_n \to y$ and $d(p_n, q_n) \leq R$ for some R. Since $\overrightarrow{x_0 z} \subset \mathcal{C}_0(\xi_\rho(\Gamma))$ and Γ acts cocompactly on $\mathcal{C}_0(\xi_\rho(\Gamma))$, there exists a sequence $\{\gamma_n\}$ in Γ and $S > 0$ so that $d(\rho(\gamma_n^{-1})(p_n), x_0) \leq S$ for all n. Therefore, $d(\rho(\gamma_n^{-1})(q_n), x_0) \leq R + S$ for all n. We pass to a subsequence so that $\{\gamma_n\}$ converges to $v \in \partial\Gamma$, $\{\rho(\gamma_n^{-1}(p_n))\}$ converges to $p_0 \in \Omega$, and $\{\rho(\gamma_n^{-1}(q_n)\}$ converges to $q_0 \in \Omega$. Then $\rho_n(\gamma_n)(p_0) \to z$ and $\rho(\gamma_n)(q_0) \to y$, so, by Lemma 38.1, $\xi_\rho(v) = z$ and $\xi_\rho(v) = y$. However, this impossible, so every point in $\xi_\rho(\partial\Gamma)$ is an extreme point of $\partial\Omega$. $\square$

It only remains to prove that every point in $\xi_\rho(\partial\Gamma)$ is a C^1 point of $\partial\Omega$. Suppose that $x \in \xi_\rho(\partial\Gamma) = \Lambda^{orb}(\rho(\Gamma))$ and $x = \xi_\rho(w)$. Then $H_x = \theta_\rho(w)$ is a support plane to $\partial\Omega_{max}$ at z. In order to show that $\partial\Omega_{max}$ is C^1 at $\theta_\rho(w)$ it suffices to show that H_x is the unique support plane to $\partial\Omega_{max}$ at z.

We observed in Lemma 38.4 that the closure of Ω^*_{max} is the convex hull $\mathcal{C}(\theta_\rho(\partial\Gamma)))$ of $\theta_\rho(\partial\Gamma)$ in $\mathbb{P}((\mathbb{R}^d)^*)$. Recall that any support plane H for Ω_{max} is the kernel of a linear functional in $\partial\Omega^*_{max}$, so $H = \ker\phi$ where $\phi = \sum_{i=1}^d a_i\phi_i$, $a_i \geq 0$ for all i and $\phi_i = \bar\theta_\rho(z_i)$ for some $z_i \in \partial\Gamma$. However, $\bar\theta_\rho(z_i)(\bar\xi_\rho(\tilde w)) > 0$ if $z_i \neq w$, where $\tilde w$ lies in $p^{-1}(w) \cap \partial\Omega^+$, so we must have $\phi = \bar\theta_\rho(w)$ and $H = H_x$. We conclude that every point in $\xi_\rho(\partial\Gamma)$ is a C^1 point of $\partial\Omega_{max}$. Therefore, $\rho(\Gamma)$ acts regularly convex cocompactly on Ω_{max}. $\square$

Combining Theorems 40.1 and 39.2 we obtain a characterization of which discrete groups preserving a properly convex domain are projective Anosov. (Of course, the proof we have given is incomplete in the setting of reducible subgroups.)

Corollary 40.4. (Danciger-Guéritaud-Kassel [84, Theorem 1.4], Zimmer [232, Theorems 1.22/1.27]) *Suppose that $\Gamma \subset \mathsf{SL}(d, \mathbb{R})$ is discrete and preserves a properly convex domain Ω. Then Γ is regularly convex cocompact if and only if the inclusion map of Γ into $\mathsf{SL}(d, \mathbb{R})$ is projective Anosov.*

41. Zimmer's criterion

It can be difficult to verify that the image of a projective Anosov representation preserves a properly convex domain, but Andrew Zimmer [232] provided a simple group-theoretic criterion. He shows that the image of an irreducible projective Anosov representation is regularly convex cocompact if the domain group is freely indecomposable, but not a surface group.

Zimmer's proof makes use of fundamental results of Bowditch [37] who analyzed the topology of the boundaries of hyperbolic groups. The theorem below records what we will need in the proof of Zimmer's result. For convenience, we state his results in the torsion-free setting. Recall that $z \in \partial\Gamma$ is a **cut point** if $\partial\Gamma - \{z\}$.

Theorem 41.1. (Bowditch [37, Theorem 0.1 and Proposition 5.29]) *Suppose that Γ is a torsion-free hyperbolic group, then*

(1) *$\partial\Gamma$ is connected if and only if Γ is freely indecomposable.*
(2) *If $\partial\Gamma$ is connected, then $\partial\Gamma$ has no cut points.*

(3) *If $\partial\Gamma$ is connected and Γ is not a surface group, then there exists $u, v \in \partial\Gamma$ so that $\partial\Gamma - \{u, v\}$ is connected.*

We are now ready to give the simple and elegant proof of Zimmer's result.

Theorem 41.2. (Zimmer [232, Theorem 1.25]) *Suppose Γ is a torsion-free hyperbolic group which is not a surface group or a cyclic group and does not split non-trivially as a free product. If $\rho: \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation, then $\rho(\Gamma)$ is regularly convex cocompact.*

Notice that Zimmer's result applies to many of our favorite groups. For example, if Γ is freely indecomposable, but not a surface group, and $\rho_0: \Gamma \to \mathsf{SO}(d, 1)$ is convex cocompact, then any small deformation of ρ_0 within $\mathsf{SL}(d+1, \mathbb{R})$ is regularly convex cocompact. For the moment, we will only give the proof when $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$, but we will later return to the proof and remove this assumption.

Proof. (When $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$) Choose $u, v \in \partial\Gamma$, so that $\partial\Gamma - \{u, v\}$ is connected (see Theorem 41.1). We may normalize so that $\xi_\rho(u) = [e_1]$, $\xi_\rho(v) = [e_2]$, $\ker(\theta_\rho(u)) = e_2^\perp$ and $\ker(\theta_\rho)(v) = e_1^\perp$. Then $\mathbb{RP}^{d-1} - (\theta_\rho(u) \cup \theta_\rho(v))$ has two components. Since $\xi_\rho(\partial\Gamma) - \{u, v\}$ is connected and disjoint from $\theta_\rho(u) \cup \theta_\rho(v)$, it must be in one of these two components. Without loss of generality, we may assume that

$$\xi_\rho(\partial\Gamma - \{u, v\}) \subset B = \{[x_1, \ldots, x_d] \mid x_1 > 0, x_2 > 0\}$$

so $\xi_\rho(\partial\Gamma)$ is a compact subset of the affine chart A determined by the hyperplane $x_1 + x_2 = 1$.

Let $[\bar{\phi}] \in S((\mathbb{R}^d)^*)$ be given by $\bar{\phi}(\vec{x}) = x_1 + x_2$. Then for all $z \in \partial\Gamma$, choose $\bar{\xi}_\rho(z) \in p^{-1}(\xi_\rho(z))$ so that $\bar{\phi}(\bar{\xi}_\rho(\vec{x})) > 0$.

Similarly, choose $\tilde{\theta}_\rho(z) \in p^{-1}(\theta_\rho(z))$ so that $\tilde{\theta}_\rho(z)(\tilde{\xi}_\rho(x)) > 0$ if $x \in \partial\Gamma - \{z\}$ (which is possible since $\bar{\xi}_\rho(\partial\Gamma - \{z\})$ is connected and disjoint from $\ker(\theta_\rho(z))$, by transversality.)

Since $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$, Lemma 38.4 implies that $\rho(\Gamma)$ preserves a properly convex domain. Theorem 40.1 then implies that $\rho(\Gamma)$ is regularly convex cocompact. $\qquad\square$

Zimmer also gives a simple way to regard any projective Anosov representation as regularly convex cocompact, perhaps after postcomposing with another representation.

Let $\mathrm{Sym}(d, \mathbb{R})$ be the vector space of real symmetric $d \times d$ matrices. We can define

$$S_d: \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(\mathrm{Sym}(d, \mathbb{R})) \qquad \text{by} \qquad S_d(A)(X) = AXA^T.$$

Notice that $S_d(\mathsf{SL}(d, \mathbb{R}))$ preserves the properly convex domain

$$\mathcal{P}_d = \mathbb{P}\Big(\{X \in \mathrm{Sym}(d, \mathbb{R}) \mid X > 0\}\Big)$$

of (projective classes of) positive matrices in $\mathrm{Sym}(d, \mathbb{R})$.

Theorem 41.3. (Zimmer [232, Corollary 1.30]) *If $\rho: \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov, then $S_d(\rho(\Gamma))$ is regularly convex cocompact.*

Proof. We first notice that $\sigma_1(S_d(A)) = \sigma_1(A)^2$ and $\sigma_2(S_d(A)) = \sigma_1(A)\sigma_2(A)$. Since ρ is projective Anosov, Proposition 27.1 implies that there exists K and C so that if $\gamma \in \Gamma$, then

$$\log \frac{\sigma_1(S_d(\rho(\gamma)))}{\sigma_2(S_d(\rho(\gamma)))} = \log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \geq Kd(1, \gamma) - C.$$

So, $S_d \circ \rho$ is projective Anosov and preserves the properly convex domain $\mathcal{P}_d$ in $\mathbb{P}(\mathrm{Sym}(d, \mathbb{R}))$. Theorem 40.2 then implies that $S_d(\rho(\Gamma))$ is regularly convex cocompact. $\square$

Remarks: (1) Zimmer actually restricts to irreducible representations in both of these results, but this is un-necessary when we apply Danciger, Guéritaud and Kassel's version of Theorem 40.1 (which we will prove in the next section.)

(2) One may remove the assumption that Γ is torsion-free in Theorem 41.2, at the cost of requiring that no finite index subgroup of Γ splits as a non-trivial free product or is a surface group.

42. **Reducible convex cocompact representations**

In this section we explain how to extend the proofs of the main theorems above to the case where the representation (or discrete group) is reducible. The majority of the new arguments in this section are due to Danciger-Guéritaud-Kassel [84], since Zimmer [232] worked almost entirely in the irreducible setting.

We first complete the proof of Theorem 39.1, which we restate below.

Theorem 39.1. (Danciger-Guéritaud-Kassel [84, Theorem 1.15], Zimmer [232, Theorem 5.1]) *If $\Omega \subset \mathbb{RP}^{d-1}$ is properly convex and $\Gamma \subset \mathrm{Aut}(\Omega)$ acts regularly convex cocompactly on Ω, then the inclusion map of Γ into $\mathsf{SL}(d, \mathbb{R})$ is projective Anosov.*

Proof of Theorem 39.1. Proposition 39.2 implies that Γ is hyperbolic. Lemma 39.3 implies that the Gromov boundary of $\mathcal{C}_0(\Lambda_\Gamma^{orb})$ may be identified with Λ_Γ^{orb} and that there exists a continuous, transverse Γ-equivariant homeomorphism

$$\xi : \partial\Gamma \to \Lambda_\Gamma^{orb} \subset \mathbb{RP}^{d-1}.$$

One may again define $\theta : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ by defining for all $z \in \partial\Gamma$, $\theta(z)$ to be the tangent plane (unique support plane) to $\partial\Omega$ at $\xi_\rho(z)$. One again checks that ξ and θ are compatible, by definition, and transverse, since there are no line segments in Λ_Γ^{orb}.

Theorem 39.1 will follow from Corollary **??** once we establish that the inclusion map is P_1-divergent and that ξ is dynamics preserving.

Lemma 42.1. (Danciger-Guéritaud-Kassel [84, Lemma 7.5]) *If $\Omega \subset \mathbb{RP}^{d-1}$ is properly convex, $\Gamma \subset \mathrm{Aut}(\Omega)$ and Λ_Γ^{orb} contains no line segments, then the inclusion map of Γ into $\mathsf{SL}(d, \mathbb{R})$ is P_1-divergent.*

Proof. If the inclusion map of Γ into $\mathsf{SL}(d, \mathbb{R})$ is not P_1-divergent, then there exists a sequence $\{\gamma_n\}$ of distinct elements in Γ so that $\lim \frac{\sigma_2(\gamma_n)}{\sigma_1(\gamma_n)} = C_2 > 0$.

We may pass to a subsequence so that $\lim \gamma_n = w \in \partial\Gamma$ and $\lim \frac{\sigma_k(\gamma_n)}{\sigma_{k+1}(\gamma_n)} = C_{k+1}$ exists for all $k = 2, \ldots, d-1$. We write $\gamma_n = K_n A_n L_n$ where $K_n, L_n \in \mathsf{SO}(d)$ and A_n is a diagonal matrix with diagonal entries $(\sigma_1(\gamma_n), \ldots, \sigma_d(\gamma_n))$. We pass to a subsequence so that $K_n \to K$ and $L_n \to L$. Since Ω is open, there exist non-trivial vectors $\vec{u}, \vec{v} \in \mathbb{R}^d$ so that $[\vec{u}], [\hat{\vec{v}}] \in \Omega$, where $L(\vec{u}) = (a_1, a_2, \ldots, a_d)$, $L(\vec{v}) = (b_1, b_2, \ldots, b_d)$, and $a_1 = b_1$ but $a_2 \neq b_2$.

Then

$$\lim \gamma_n([\vec{u}]) = \left[K\left(a_1 e_1 + C_2 a_2 e_2 + \sum_{i=3}^{d} C_i a_i e_i\right) \right] \neq \left[K\left(b_1 e_1 + C_2 b_2 e_2 + \sum_{i=3}^{d} C_i b_i e_i\right) \right] = \lim \gamma_n([\hat{\vec{v}}]).$$

However, since $d(\gamma_n([\vec{u}]), \gamma_n([\vec{v}])) = d([\vec{u}], [\vec{v}])$ for all n, this would imply that $\lim \gamma_n([\vec{u}])$ and $\lim \gamma_n([\vec{v}])$ span a line segment in Λ_Γ^{orb}, see Lemma 15.3, which contradicts our assumption that there are no line segments in Λ_Γ^{orb}. Therefore, the inclusion map is P_1-divergent. $\qquad\square$

Since the action of the hyperbolic group Γ on $\mathcal{C}_0(\Lambda_\Gamma^{orb})$ is cocompact, Proposition 3.5 implies that if γ is an infinite order element of Γ, then if $x \in \mathcal{C}_0$, then $\lim \gamma^n(x) = \xi_\rho(\gamma^+)$. Since Γ is P_1-divergent, and ξ and θ are continuous, transverse, and Γ-equivariant Lemma 18.3 implies that ξ is P_1-dynamics preserving.

Since Γ is P_1-divergent and there exist continuous, transverse, Γ-equivariant limit maps ξ and θ so that ξ is dynamics preserving, Corollary ?? implies that the inclusion map is projective Anosov. $\qquad\square$

We now prepare to complete the proof of Theorem 40.1 which asserts that the image of a projective Anosov representation which preserves a properly convex domain, acts regularly convex cocompactly on some (probably different) properly convex domain.

The first obstacle in extending the proof is that we can't invoke Lemma 38.4 to guarantee the existence of a properly convex domain which is preserved by $\rho(\Gamma)$ and contains $\mathcal{C}_0 = \mathcal{C}_0(\xi_\rho(\partial\Gamma))$. The following result fills that gap.

Lemma 42.2. (Danciger-Guéritaud-Kassel [84, Lemma 8.5]) *Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov and $\rho(\Gamma)$ preserves a properly convex domain in $\mathbb{R}^d$. Then there exists a properly convex domain Ω so that $\rho(\Gamma) \subset \mathrm{Aut}(\Omega)$, $\mathcal{C}_0(\xi_\rho(\partial\Gamma)) \subset \Omega$. and if $z \in \partial\Gamma$, then $\theta_\rho(\partial\Gamma)$ is a support plane for Ω at $\xi_\rho(z)$. Moreover, if $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$, then we can take $\Omega = \Omega_{max}$.*

Proof. First notice that since $\rho(\Gamma)$ preserves a properly convex domain Ω_0, the limit maps ξ_ρ and θ_ρ lift to maps $\bar{\xi}_\rho$ and $\bar{\theta}_\rho$, so Lemma 38.4 gives rise to a convex region Ω_{max} containing $\mathcal{C}_0(\xi_\rho(\partial\Gamma))$ and Ω_0. If $\theta_\rho(\partial\Gamma)$ spans $(\mathbb{R}^d)^*$, then Lemma 38.4 tells us that Ω_{max} is properly convex and so we may take $\Omega = \Omega_{max}$.

If not, let $\mathcal{H} = \{H_0, H_1, \ldots, H_d\}$ be a collection of projective hyperplanes which determine an open simplex Δ containing $\mathcal{C}(\xi_\rho(\partial\Gamma))$. Consider

$$\Omega = \bigcap_{\gamma \in \Gamma} \gamma(\Delta \cap \Omega_{max}).$$

Then, by definition, Ω is $\rho(\Gamma)$-invariant, convex, bounded in the affine chart containing the closure of Δ and contains $\mathcal{C}_0(\xi_\rho(\partial\Gamma))$.

It only remains to check that Ω is open. If not, there exists $x \in \Omega$, so that Ω contains no open neighborhood of z. Therefore, there exists $H \in \mathcal{H}$ and a sequence $\{\gamma_n\} \subset \Omega$ so that $\gamma_n(H)$ converges to a hyperplane H_∞ containing x. We may pass to a subsequence so that $\gamma_n \to z \in \partial\Gamma$. We then consider the dual representation $\rho^* : \Gamma \to \mathsf{SL}((\mathbb{R}^d)^*)$ and recall that $\rho^*(\Gamma)$ preserves Ω^* (Lemma 15.4), ρ^* is projective Anosov (Corollary 32.4) and $\xi_\rho^* = \theta_\rho$ (Corollary 32.4). Lemma 38.1 then implies that $\{H_n = \gamma_n(H)\}$ converges to $\xi_{\rho^*}(z) = \theta_\rho(z)$. But since $x \in \Omega_{max}$, x cannot lie in $\theta_\rho(z)$ for any $z \in \partial\Gamma$. Therefore, Ω is open and the proof is complete. $\qquad\square$

We will also need the following improvement of Lemma 40.3.

Proposition 42.3. (Danciger-Guéritaud-Kassel [84, Lemma 4.1]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov and $\rho(\Gamma)$ acts convex cocompactly on a properly convex domain Ω, then Ω contains*

a properly convex domain Ω_0 so that $\rho(\Gamma)$ acts convex cocompactly on Ω_0 and no point in $\xi_\rho(\partial\Gamma)$ lies on a non-trivial line segment in $\partial\Omega_0$.

Proof. Let $\mathcal{C}_0 = \mathcal{C}_0(\xi_\rho(\partial\Gamma)) \subset \Omega$. We claim that we may choose

$$\Omega_0 = \{x \in \Omega \mid d_\Omega(x, \mathcal{C}_0) < 1\}.$$

The following lemma implies that Ω_0 is properly convex.

Lemma 42.4. *Suppose that Ω is properly convex, $X \subset \Omega$ is convex, and $d > 0$, then*

$$\mathcal{N}_d(X) = \{x \in \Omega \mid d_\Omega(x, X) < d\}$$

is properly convex.

Proof. If $x, y \in \mathcal{N}_d(X)$, let $[x, y]$ be the line segment joining x to y in Ω, let x_0 and y_0 be points in X so that $d(x, x_0) < d$ and $d(y, y_0) < d$. Let $u, v, w, z \in \partial\Omega$ so that $x, x_0 \subset [u, v]$ and $y, y_0 \subset [w, z]$. Let Q be the quadrilateral with vertices u, v, w and z. Then if $p \in [x, y]$, one may check that

$$d_\Omega(p, [x_0, y_0]) \leq d_Q(p, [x_0, y_0]) \leq \max\{d_Q(x, x_0), d_Q(y, y_0)\} = \max\{d_\Omega(x, x_0), d_\Omega(y, y_1)\} < 1,$$

so, since $[x_0, y_0] \subset X$, $d_\Omega(p, X) < d$.. Therefore, $[x, y] \subset \mathcal{N})d(X)$, so $\mathcal{N}_d(X)$ is properly convex. $\square$

We next check that $\rho(\Gamma)$ acts convex cocompactly on Ω_0. By construction, $\rho(\Gamma) \subset \mathrm{Aut}(\Omega_0)$ and $\mathcal{C}_0 \subset \Omega_0$. Lemma 40.2 then implies that $\rho(\Gamma)$ acts convex cocompactly on Ω.

It only remains to check that no point in $\xi_\rho(\partial\Gamma)$ lies on a line segment in $\partial\Omega_0$. Since every point in $\xi_\rho(\partial\Gamma)$ is an extreme point for $\partial\Omega_0$, by Lemma 40.3, it suffices to rule out line segments in $\partial\Omega_0$ ending at points in $\xi_\rho(\partial\Gamma)$. Since, $\xi_\rho(\partial\Gamma) = \Lambda^{orb}_{\rho(\Gamma)}$ (by Lemma 38.1), the following more general lemma, which we record for future use, completes the proof of Proposition 42.3.

say more about calculation?

Lemma 42.5. *Suppose that a discrete subgroup Γ of $\mathsf{SL}(d, \mathbb{R})$ acts convex cocompactly on a properly convex domain Ω. If Ω_0 is properly convex and Γ-invariant, and there exist $a > b > 0$ so that*

$$\mathcal{N}_b(\mathcal{C}_0(\Lambda^{orb}_\Gamma)) \subset \Omega_0 \subset \mathcal{N}_a(\mathcal{C}_0(\Lambda^{orb}_\Gamma)),$$

then there does not exist a line segment $[z, y] \in \partial\Omega_0$ so that $z \in \Lambda^{orb}_\Gamma$ and $(z, y] \subset \partial\Omega_0 - \Lambda^{orb}_\Gamma$.

Proof. Suppose that there is a line segment $[z, y] \in \partial\Omega_0$ such that $z \in \Lambda^{orb}_\Gamma$ and $(z, y] \subset \partial\Omega_0 - \Lambda^{orb}_\Gamma$. Fix $x_0 \in \mathcal{C}_0 = \mathcal{C}_0(\Lambda^{orb}_\Gamma)$ and let $\{w_n\}$ be a sequence in $(z, y]$ converging to z. Since $\rho(\Gamma)$ acts cocompactly on $\mathcal{C}_0$ and $b \leq d_\Omega(w_n, \mathcal{C}_0) \leq a$ for all n, there exists a compact subset K of $\mathcal{C}_0$ and a sequences $\{\gamma_n\} \subset \Gamma$ and $\{x_n\} \subset K$ so that

$$b \leq d(\gamma_n(w_n), x_n) \leq a.$$

We may pass to a subsequence so that $\{\gamma_n(w_n)\}$ converges to $w_\infty \in \overline{\Omega}_0 \cap \Omega$, $\{x_n\}$ converges to $x_\infty \in K$, $\{\gamma_n(y)\}$ converges to y_∞ and $\{\gamma_n(z)\}$ converges to $z_\infty \in \Lambda^{orb}_\Gamma$. Lemma 38.1 implies that $y_\infty \in \Lambda^{orb}_\Gamma$. So, $w_\infty \subset [z_\infty, y_\infty] \subset \mathcal{C}_0$ and $d(w_\infty, \mathcal{C}_0) \geq b > 0$, and we have achieved a contradiction. Therefore, there no such line segment exists, which completes the proof. $\square$

$\square$

We are now ready to complete the proof of Theorem 40.1, which we recall below.

Theorem 40.1. (Danciger-Guéritaud-Kassel [84, Theorem 1.4], Zimmer [232, Theorem 1.27]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation and $\rho(\Gamma)$ preserves a properly convex domain in $\mathbb{RP}^{d-1}$, then $\rho(\Gamma)$ is regularly convex cocompact. If ρ is irreducible, then $\rho(\Gamma)$ acts regularly convex cocompactly on Ω_{max}.*

Proof of Theorem 40.1. Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation and $\rho(\Gamma)$ preserves a properly convex domain in $\mathbb{RP}^{d-1}$. We have already seen that if $\theta_\rho(\partial\Gamma)$ does not span $(\mathbb{R}^d)^*$, then $\rho(\Gamma)$ acts regularly convex cocompactly on Ω_{max}.

Lemma 42.2 implies that there exists a properly convex domain Ω which is preserved by $\rho(\Gamma)$ and contains $\mathcal{C}_0 = \mathcal{C}_0(\xi_\rho(\partial\Gamma))$. Lemma 15.4, that the dual Ω^* of Ω is properly convex and that $\rho^*(\Gamma)$ preserves Ω^* and that $\xi_{\rho^*} = \theta_\rho$. Corollary 32.4 implies that ρ^* is projective Anosov. Lemmas 40.2 and 42.3 together imply that there exists a properly convex domain $\Delta \subset \Omega^*$ so that $\rho(\gamma)$ acts convex cocompactly on Δ and that no point in $\xi_{\rho^*}(\partial\Gamma)$ lies in a line segment in $\partial\Delta$. Lemma 15.4, that the dual Δ^* of Δ is properly convex and that $(\rho^*)^*(\Gamma) = \rho(\Gamma)$ preserves Δ^*. If $z \in \xi_\rho(\Gamma)$, then, since $\mathcal{C}_0(\theta_\rho(\partial\Gamma) \subset \Delta$, $\Delta^* \subset \Omega_{max}$. Then, since $\theta_\rho(z)$ is a support plane to Ω_{max} at $\xi_\rho(z)$, it is also a support plane to Δ^* at $\xi_\rho(z)$. Since $\theta_\rho(z)$ does not lie on a non-trivial line segment in Δ. Lemma 15.4 then implies that $\xi_\rho(z)$ is a C^1 point of $\partial\Omega$. Since $\mathcal{C}_0 \subset \Omega$ and $\Omega \subset \Delta^*$, one has that $\mathcal{C}_0 \subset \Delta^*$. Lemmas 40.2 and 40.3 imply that $\rho(\gamma)$ acts convex cocompactly Δ^* and that every point in $\xi_\rho(\partial\Gamma)$ is an extreme point of Δ^*. Therefore, $\rho(\Gamma)$ acts regularly convex cocompactly on Δ^*. $\square$

Finally, we establish Zimmer's Theorem 41.2 in the general case.

Theorem 41.2. (Zimmer [232, Theorem 1.25]) *Suppose Γ is a torsion-free hyperbolic group which is not a surface group or a cyclic group and does not split non-trivially as a free product. If $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is a projective Anosov representation, then $\rho(\Gamma)$ is regularly convex cocompact.*

Proof of Theorem 41.2. Just as in the original proof, choose $u, v \in \partial\Gamma$, so that $\partial\Gamma - \{u, v\}$ is connected and normalize so that $\xi_\rho(u) = [e_1]$, $\xi_\rho(v) = [e_2]$, $\ker(\theta_\rho(u)) = e_2^\perp$ and $\ker(\theta_\rho)(v) = e_1^\perp$. Then $\mathbb{RP}^{d-1} - (\theta_\rho(u) \cup \theta_\rho(v))$ has two components. Since $\xi_\rho(\partial\Gamma) - \{u, v\}$ is connected, it must be in one of these two components. Without loss of generality, we may assume that

$$\xi_\rho(\partial\Gamma - \{u, v\}) \subset B = \{[x_1, \ldots, x_d] \mid x_1 > 0, x_2 > 0\}$$

so $\xi_\rho(\partial\Gamma)$ is a compact subset of the affine chart A determined by the hyperplane $x_1 + x_2 = 1$. Let $[\bar{\phi}] \in S((\mathbb{R}^d)^*)$ be given by $\bar{\phi}(\vec{x}) = x_1 + x_2$. Then for all $z \in \partial\Gamma$, choose $\bar{\xi}_\rho(z) \in p^{-1}(\xi_\rho(z))$ so that $\bar{\phi}(\bar{\xi}_\rho(\vec{x})) > 0$. Similarly, choose $\tilde{\theta}_\rho(z) \in p^{-1}(\theta_\rho(z))$ so that $\tilde{\theta}_\rho(z)(\tilde{\xi}_\rho(x)) > 0$ if $x \in \partial\Gamma - \{z\}$.

We now notice that the proof of Lemma 42.2 also establishes establishes the following statement.

Lemma 42.6. *Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov, $\xi_\rho(\partial\Gamma)$ is contained in an affine chart A and there exist ρ-equivariant lifts*

$$\tilde{\xi}_\rho : \partial\Gamma \to S(\mathbb{R}^d) \qquad \text{and} \qquad \tilde{\theta}_\rho : \partial\Gamma \to S((\mathbb{R}^d)^*)$$

of the Anosov limit maps $\xi_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$, and

$$\bar{\theta}_\rho(z)(\bar{\xi}_\rho(w)) > 0$$

for all $w \neq z \in \partial\Gamma$. Then, there exists a properly convex domain Ω containing $\mathcal{C}_0(\xi_\rho(\Gamma))$ so that $\rho(\Gamma) \subset \mathrm{Aut}(\Omega)$.

Therefore, $\rho(\Gamma)$ preserves a properly convex domain containing $\mathcal{C}_0(\xi_\rho(\Gamma))$, so Theorem 40.1 implies that $\rho(\Gamma)$ is regularly convex cocompact. $\square$

43. **Strongly convex cocompact actions**

> *Horses wept jewels the size of fists*
> *swept by scholars with a mind*
> *to twist and level facets*
> *of each plane to be raffled*
> *when the bombing ceased*
> ————Patti Smith [191]

In this section, we will see that if Γ acts regularly convex cocompactly on a properly convex domain, then one can construct a strictly convex C^1 domain which Γ acts on convex cocompactly.

Theorem 43.1. (Danciger-Guéritaud-Kassel [84, Theorem 1.15]) *If a discrete, torsion-free subgroup Γ of $\mathsf{SL}(d, \mathbb{R})$ is regularly convex cocompact, then Γ is strongly convex cocompact.*

We may combine Theorem 40.1 and 43.1 to obtain the following immediate corollary:

Corollary 43.2. (Danciger-Guéritaud-Kassel [84, Theorem 1.4]) *If Γ is torsion-free, $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation and $\rho(\Gamma)$ preserves a properly convex domain in $\mathbb{RP}^{d-1}$, then $\rho(\Gamma)$ is strongly convex cocompact.*

Theorem 43.1 and Corollary 43.2 remain true without the assumption that Γ is torsion-free, but the proof is simplified somewhat by this assumption. The crucial ingredient in the proof is the following lemma:

Lemma 43.3. (Danciger-Guéritaud-Kassel [84, Lemma 9.2]) *Suppose that a torsion-free discrete subgroup Γ of $\mathsf{SL}(d, \mathbb{R})$ acts convex cocompactly on a properly convex domain Ω. If $b > 0$, there exists a properly convex domain Ω_0 and $a > 0$ so that*

$$\mathcal{N}_a(\mathcal{C}_0(\Lambda_\Gamma^{orb})) \subset \Omega_0 \subset \mathcal{N}_b(\mathcal{C}_0(\Lambda_\Gamma^{orb})),$$

and every point in $\partial\Omega_0 - \Lambda_\Gamma^{orb}$ is a C^1 extreme point.

In hyperbolic geometry, one knows that the metric neighborhood of convex set is strictly convex and C^1. However, in Hilbert geometry in general, metric neighborhoods of convex sets will only be convex and need not be C^1. All techniques, that we are aware of, for proving results of this form in Hilbert geometry use somewhat ad hoc methods. We note that the method of Cooper-Long-Tillman [71, Corollary 8.2] prove an analogous result by constructing a smooth convex function to use in place of the the distance function.

Proof. In the first step of the proof we produce a Γ-invariant convex domain which is C^1 off the full orbital limit set, while in the second, more complicated, step we make it both C^1 and strictly convex away from the limit set.

Choose $c = \frac{b}{2}$. Since Γ acts cocompactly on $\mathcal{C}_0 = \mathcal{C}_0(\Lambda^{orb}(\Gamma))$ and distance to $\mathcal{C}_0$ is Γ-invariant, we see that Γ acts cocompactly on $\overline{\mathcal{N}_c(\partial C_0)}$. Let K be a compact subset of $\overline{\mathcal{N}_c(\partial C_0)}$ so that $\Gamma(K) = \overline{\mathcal{N}_c(\partial C_0)}$. Let $\mathcal{D}$ be a closed neighborhood of K with smooth boundary which lies inside inside of $\mathcal{N}_b(\mathcal{C}_0)$ and let Ω_1 be the convex hull of $\Gamma(\mathcal{D})$. Notice that $\Gamma(\mathcal{D}) \subset \mathcal{N}_b(\mathcal{C}_0)$ since $\mathcal{N}_b(\mathcal{C}_0)$ is convex, by Lemma 42.4, and that $\mathcal{N}_c(\mathcal{C}_0) \subset \Omega_1$ by construction.

Notice that Ω_1 is properly convex by definition. We now claim that $\partial_0 \Omega_1 = \partial \Omega_1 - \Lambda_\Gamma^{orb}$ is C^1. Suppose that $y \in \partial_0 \Omega_1$. If H_y is a supporting plane of Ω_1 at y, then H_y cannot intersect Λ_Γ^{orb}, by Lemma 42.5. It follows that there exists a neighborhood U of y in $\partial \Omega_1$ and V of Λ_Γ^{orb} in $\mathbb{RP}^{d-1}$ so that if $x \in U$, then any support plane to Ω through x is disjoint from V. On the other hand, if $\gamma_n \to \infty$ and H is a support plane to $H(\mathcal{D})$, then $\gamma_n(H)$ intersects V for all large enough n. Therefore, there exists a finite subset $\{\gamma_1, \ldots, \gamma_m\}$ of Γ so that U lies in the boundary of the convex hull of $\bigcup_{i=1}^m \gamma_i(\mathcal{D})$. However, the dual of the convex hull of $\{\gamma_1, \ldots, \gamma_m\}$ is exactly $\bigcap_{i=1}^m (\gamma_i(\mathcal{D}))^*$. Now, since $\gamma_i(\mathcal{D})$ is smooth and properly convex, its dual $(\gamma_i(\mathcal{D}))^*$ is strictly convex, by Lemma 15.4. Therefore, $\bigcap_{i=1}^m (\gamma_i(\mathcal{D}))^*$ is strictly convex, so the boundary of its dual is C^1, again by Lemma 15.4. Therefore, U is C^1, and since y was arbitrary, $\partial_0 \Omega_1$ is C^1.

We now explain how to alter Ω_1 to obtain a new properly convex domain Ω_0 so that $\mathcal{C}_0 \subset \Omega_0$ and $\partial_0 \Omega_0 = \partial \Omega - \Lambda_\Gamma^{orb}$ is strictly convex and C^1.

Given $y \in \partial_0 \Omega_1$, let F_y be the stratum of $\partial \Omega_1$ containing y (i.e. the intersection of $\partial \Omega_1$ with the unique supporting hyperplane H_y to $\partial \Omega_1$ through y). Notice that F_y is compact and cannot intersect Λ_Γ^{orb}, by Lemma 42.5. We claim that if $\gamma \in \Gamma$ is non-trivial, then F_y is disjoint from $\gamma(F_y) = F_{\gamma(y)}$. If not, there exist $\hat{y} \in F_y \cap F_{\gamma(y)}$, which would imply, since support planes at points in $\partial_0 \Omega_1$ are unique, that

$$H_y = H_{\hat{y}} = H_{\gamma(y)} = \gamma(H_y).$$

But then,

$$H_{\gamma^n(y)} = \gamma^n(H_y) = H_y$$

for all n, so $\gamma^n(y) \in F_y$ for all y. Since there is a subsequence of $\{\gamma^n(y)\}_{n \in \mathbb{N}}$ which converges to $z \in \Lambda_\Gamma^{orb}$ and F_y is compact, $z \in F_y$, which contradicts the fact that F_y is disjoint from Λ_Γ^{orb}. Therefore, $\gamma(F_y)$ and $F_{\gamma(y)}$ are disjoint if $\gamma \in \Gamma$ is non-trivial.

We first explain the proof in the simple case that there exists $y \in \partial_0 \Omega_1$ so that any line segment in $\partial_0 \Omega_1$ is contained in $\Gamma(F_y)$, which is a disjoint union of copies of F_y. This case contains all the crucial ideas of the construction. Here it will suffice to alter Ω_1 on a neighborhood of F_y and then extend this alteration equivariantly. In general, since the action of Γ on $\partial_0 \Omega_1$ is properly discontinuous and cocompact, the line segments all lie in the iterates of "nice" open neighborhoods of finitely many strata and we will work iteratively.

Suppose that any line segment in $\partial_0 \Omega_1$ is contained in $\Gamma(F_y)$ for some stratum $y \in \partial_0 \Omega_1$. We will assume that Ω_1 is contained in an affine chart A and make use of its associated Euclidean metric. Choose $d - 1$ hyperplanes $H_1, \ldots, H_{d-1}$ which are in general position in A so that

(1) Each H_i separates F_y from Λ_Γ^{orb}.
(2) If R_i is the component of $\Omega_1 - H_i$ containing y, then R_i is disjoint from $\gamma(R_i)$ if $\gamma \in \Gamma$ is non-trivial.
(3) If p_i is Euclidean orthogonal projection onto H_i, then $p_i(R_i) = \Omega_1 \cap H_i$

Let $R = \bigcup R_i$, and if $Q_i = \partial R_i \cap \partial \Omega_1$, let $Q = \bigcup Q_i$. Our assumptions imply that $F_y \subset Q$, Q is disjoint from $\gamma(Q)$ and R is disjoint from $\gamma(R)$ if $\gamma \in \Gamma$ is non-trivial.

We further assume that in the affine chart $A \cong \mathbb{R}^{d-1}$, each Q_i is the hyperplane $\{x_i = 0\}$ and y lies in the positive orthant. Then R is the intersection of Ω with the complement of the negative orthant, Q is the intersection of $\partial\Omega_1$ with the complement of the negative orthant and $\mathcal{C}_0$ is contained in the negative orthant. Consider a C^1 function $h : \mathbb{R} \to \mathbb{R}$ which is strictly concave on $(0, \infty)$, $h(x) = x$ if $x \leq 0$ $h'(0) = 1$ and $0 < h(x) < x$ if $x > 0$. One example is provided by letting $h(x) = \tanh x$ if $x \geq 0$. We then define

$$\phi : A \to A \qquad \text{by} \qquad \phi(x_1, \ldots, x_{d-1}) = (h(x_1), \ldots, h(x_{d-1})).$$

By construction $\phi(\partial_0\Omega_0)$ is strictly convex and $\phi(Q)$ is C^1. Let $\hat{R}$ be the portion of the complement of the negative orthant bounded by $\hat{Q} = \phi(Q)$, so $\phi(\Omega_1) = (\Omega_1 - R) \cup \hat{R}$. Since $h'(0) = 1$, $\phi(\Omega_1)$ remains strictly convex and C^1 on $\partial Q = \partial \hat{Q}$. Since $\Gamma(R \cup Q)$ is a disjoint union of translates of $R \cup Q$, we can define

$$\Omega_0 = \left(\Omega_1 - \Gamma(R)\right) \cup \Gamma(\hat{R}) = \bigcap_{\gamma \in \Gamma}(\gamma(\phi(\Omega_1))), \qquad \text{so} \qquad \partial\Omega_0 = \left(\partial\Omega_1 - \Gamma(Q)\right) \cup \Gamma(\hat{Q}).$$

Since $\partial_0\Omega_1 - \Gamma(Q)$ is C^1 and strictly convex, we easily see that $\partial_0\Omega_0 = \partial\Omega_0 - \Lambda_\Gamma^{orb}$ is C^1 and strictly convex. Since $\Gamma(R)$ is disjoint from $\mathcal{C}_0$ and $\Gamma(\hat{R}) \subset \Omega_1$, we see that

$$\mathcal{C}_0 \subset \Omega_0 \subset \Omega_1 \subset \mathcal{N}_a(\mathcal{C}_0)$$

as required.

The extension to the general case uses the same ideas but requires careful book-keeping. If $y \in \partial\Omega_1 - \Lambda_\Gamma^{orb}$, then one may choose, just as above, $d-1$ hyperplanes $H_1^y, \ldots, H_{d-1}^y$ which are in general position in A so that

(1) Each H_i^y separates F_y from Λ_Γ^{orb}.
(2) If R_i^y is the component of $\Omega_1 - H_i$ containing y, then R_i^y is disjoint from $\gamma(R_i^y)$ if $\gamma \in \Gamma$ is non-trivial.
(3) If p_i^y is Euclidean orthogonal projection onto H_i^y, then $p_i(R_i^y) = \Omega_1 \cap H_i^y$

Let $R_y = \bigcup R_i^y$, and if $Q_i^y = \partial R_i^y \cap \partial\Omega_1$, let $Q_y = \bigcup Q_i^y$. Then $F_y \subset Q_y$ and Q_y is disjoint from $\gamma(Q_y)$ and R_y is disjoint from $\gamma(R_y)$ if $\gamma \in \Gamma$ is non-trivial.

Since the action of Γ on $\partial\Omega_1 - \Lambda_\Gamma^{orb}$ is cocompact, there exist a finite number of points $y_1, \ldots, y_m$, so that

$$\Gamma(Q_{y_1} \cup \cdots Q_{y_n}) = \partial_0\Omega_1.$$

We first construct a properly convex domain Ω_2 which has the form

$$\Omega_2 = \left(\Omega_1 - \Gamma(R_{y_1})\right) \cup \Gamma(\hat{R}_{y_1})$$

(for some $\hat{R}_{y_1}$ to be defined later), is C^1 on $\partial_0\Omega_2 = \partial\Omega_2 - \Lambda_\Gamma^{orb}$ and is strictly convex at any point in $\Gamma(\hat{Q}_y)$ and at any point in $\partial_0\Omega_2 \cap \partial_0\Omega_1$ which is strictly convex in $\partial\Omega_1$.

There exists $T_1 \in \mathsf{SL}(A, \mathbb{R})$ so that $T_1(y_1)$ is in the positive orthant and, for all i, $T(H_i^{y_1})$ is the hyperplane $\{y_i = 0\}$ in the affine chart A. Notice that $\phi(T_1(\partial\Omega_1)) \subset \Omega_1$ is C^1 on $\phi(T_1(\partial_0\Omega_1))$ (since $\partial_0\Omega_1$ is C^1), and is C^1 on $\phi(Q_{y_1})$ and at any point $\phi(z)$ where $z \in \partial_0\Omega_1$ is C^1. Let $\hat{R}_{y_1} = T_1^{-1}(\phi(T_1(R_{y_1})) \subset R_{y_1}$ and $\hat{Q}_{y_1} = T_1^{-1}(\phi(T_1(Q_{y_1})))$. Then $\hat{\Omega}_2 = (\Omega - R_{y_1}) \cup \hat{R}_{y_1}$ contain

$\mathcal{C}_0$, is C^1 on $\partial_0 \Omega_2$, and strictly convex on $\hat{Q}_{y_1}$ and at any point in $\partial \Omega_1 - \hat{Q}_{y_1}$ where $\partial \Omega_1$ is strictly convex. Then let

$$\Omega_2 = (\Omega_1 - \Gamma(R_{y_1})) \cup \Gamma(\hat{R}_{y_1}) = \bigcap_{\gamma \in \Gamma} \gamma(\hat{\Omega}_2)$$

so if $\partial_0 \Omega_2 = \partial \Omega_2 - \Lambda_\Gamma^{orb}$, then

$$\partial_0 \Omega_2 = \left(\partial_0 \Omega_1 \right) - \Gamma(Q_{y_1}) \bigcup \Gamma(\hat{Q}_{y_1}).$$

One easily checks that Ω_2 has all the claimed properties.

One then iteratively defines $\Omega_{i+1} = \left(\Omega_1 - \Gamma(Q_{y_i}) \right) \cup \Gamma(\hat{Q}_{y_i})$ for all $i = 1, \ldots, n$ so that $\partial_0 \Omega_{i+1} = \partial \Omega_{i+1} - \Lambda_\Gamma^{orb}$ is C^1, is strictly convex on $\partial \Omega_{i+1} - \bigcup_{j=i+2}^n Q_{y_j}$ and

$$\mathcal{C}_0 \subset \Omega_{i+1} \subset \Omega_i \subset \mathcal{N}_b(\mathcal{C}_0).$$

If we set $\Omega_0 = \Omega_{n+1}$, then $\partial_0 \Omega_0$ is C^1 and strictly convex.

Finally, we claim that there exists $a > 0$, so that $\mathcal{N}_a(\mathcal{C}_0) \subset \Omega_0$. Let K be a compact subset of $\mathcal{C}_0$ so that $\Gamma(K) = \mathcal{C}_0$. Since Ω_0 is an open neighborhood of K, there exists $a > 0$ so that $\mathcal{N}_a(K) \subset \Omega_0$. Since Ω_0 is Γ-invariant, we see that $\mathcal{N}_a(\mathcal{C}_0) \subset \Omega_0$, and we have completed the proof. $\qquad \square$

Armed with Lemma 43.3 we are ready to complete the proof of Theorem 43.1.

Proof of Theorem 43.1. Suppose that Γ is a discrete, torsion-free subgroup of $\mathsf{SL}(d, \mathbb{R})$ which acts regularly convex cocompactly on a properly convex domain $\Omega \subset \mathbb{RP}^{d-1}$. Lemma 43.3 provides a properly convex domain $\Omega_0 \subset \Omega$ so that, if $\mathcal{C}_0 = \mathcal{C}_0(\Lambda^{orb}(\Gamma))$, then there exists $a > 0$ so that

$$\mathcal{N}_a(\mathcal{C}_0) \subset \Omega_0 \subset \mathcal{N}_1(\mathcal{C}_0)$$

and $\partial_0 \Omega_0 = \partial \Omega - \Lambda_\Gamma^{orb}$ is C^1 and strictly convex. Since Γ acts regularly convex cocompactly on $\partial \Omega$, there are no line segments in Λ_Γ^{orb}. Since there are no line segments in $\partial_0 \Omega_0$, Ω_0 is strictly convex.

It remains to show that $\partial \Omega_0$ is C^1. We already know that $\partial_0 \Omega_0$ is C^1, so consider a point $z \in \Lambda_\Gamma^{orb}$. Since Γ acts regularly convex cocompactly on Ω, z is a C^1 point of $\partial \Omega$. Let H_z be the unique support plane to Ω at z. Notice that H_z is also a support plane to Ω_0 at z, since $\Omega_0 \subset \Omega$. If z is not a C^1 point of $\partial \Omega_0$, then there is a support plane H to $\partial \Omega_0$ which does not agree with H_z. Therefore, H intersects Ω, and that intersection contains an open line segment (w, z) joining $w \in \partial \Omega_0$ to z.

Let (v, z) be a line segment in $\mathcal{C}_0$ joining some point $v \in \mathcal{C}_0$ to z. Let $\{v_n\}$ be a sequence of points in (v, z) converging to z and choose $\{w_n\} \subset (w, z)$, so that $[v_n, w_n]$ is parallel to H for all n. We claim that $\lim d_\Omega(v_n, w_n) = 0$. Suppose that $[w_n, z_n] \subset [u_n, y_n]$ and $u_n, y_n \in \partial \Omega$, so $d_\Omega(v_n, w_n) = \frac{1}{2} \log[u_n, v_n, w_n, y_n]$ (where we assume that the points appear in the order u_n, v_n, w_n, y_n on the line segment $[u_n, y_n]$). Let U_n be the line through z and u_n and Y_n be the line through y_n and z. Since Ω is C^1 at z, U_n and Y_n converge to the same line in H_z. This implies that $\lim[u_n, v_n, w_n, y_n] = 1$, so $\lim d_\Omega(v_n, w_n) = 0$. Since $\mathcal{N}_a(\mathcal{C}_0) \subset \Omega_0$ and $v_n \in \mathcal{C}_0$ for all n, we see that $v_n \in \Omega_0$ for all large enough n, which contradicts the fact that H is a support plane to $\partial \Omega_0$ (since $v_n \in H$ for all n). Therefore, $\partial \Omega_0$ is C^1 at z. Since z was an arbitrary point in Λ_Γ^{orb}, this completes the proof that $\partial \Omega_0$ is C^1. $\qquad \square$

44. Convex cocompactness: Further topics

I see nothing to be gained by any explanation
There's no words that need to be said
——————Bob Dylan [94]

New examples. One of the first applications of Danciger, Guéritaud and Kassel's work was a proof that any hyperbolic right-angled Coxeter group admits a projective Anosov representation (in fact, a $P_1^{p,q}$-Anosov representation into $\mathsf{O}(p,q)$ for some p and q). Many such Coxeter groups are not isomorphic to lattices in rank one Lie groups.

Theorem 44.1. (Danciger-Guéritaud-Kassel [83, Theorem 1.20]) *If Γ is an infinite, Gromov hyperbolic, right-angled Coxeter group, then there exists $d \geq 2$ and a projective Anosov representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ so that $\rho(\Gamma)$ is strongly convex cocompact.*

Lee and Marquis [146] subsequently exhibited Coxeter groups which are not isomorphic to lattices in rank one Lie groups, but are isomorphic to $\mathrm{AdS}^{d,1}$ strictly GHC-regular groups with d equal to 4, 5, 6, 7 and 8. In particular, such groups admit projective Anosov representations into $\mathsf{SL}(d+2,\mathbb{R})$ with image in $\mathsf{O}(d,2)$. Danciger, Guéritaud, Kassel, Lee and Marquis are now preparing a manuscript which will handle all hyperbolic Coxeter groups.

This work addresses the relative paucity of examples of "new" Anosov representations. The majority of currently known examples are either representations of surface groups or free groups or arise by considering a convex cocompact subgroup Γ of a rank one Lie group G and a "well-behaved representation $\tau : \mathsf{G} \to \mathsf{SL}(d,\mathbb{R})$ and then deforming $\tau|_\Gamma : \Gamma \to \mathsf{SL}(d,\mathbb{R})$.

Benoist's criterion and Hitchin representations. Benoist [18] provides a characterization of strongly irreducible subgroups of $\mathsf{SL}(d,\mathbb{R})$ which preserve a properly convex domain. We recall that a subgroup Γ of $\mathsf{SL}(d,\mathbb{R})$ is said to be *positively proximal* if and only if Γ contains a proximal element and if $A \in \Gamma$ is proximal, then its first eigenvalue is positive, i.e. $\lambda_1(A) > 1$.

Theorem 44.2. (Benoist [18, Proposition 1.1]) *If Γ is a strongly irreducible subgroup of $\mathsf{SL}(d,\mathbb{R})$, then Γ preserves a properly convex domain $\Omega \subset \mathbb{RP}^{d-1}$ if and only if it contains a finite index subgroup which is positively proximal.*

One consequence of Benoist's criterion is that the image of a Hitchin representation into $\mathsf{SL}(d,\mathbb{R})$ preserves a properly convex domain if and only if d is odd. In particular, its image is strongly convex cocompact if and only if d is odd.

Corollary 44.3. (Danciger-Guéritaud-Kassel [84, Proposition 1.7], Zimmer [232, Corollary 1.33]) *If S is a closed, orientable surface and $\rho : \pi_1(S) \to \mathsf{SL}(d,\mathbb{R})$ is a Hitchin representation, then $\rho(\Gamma)$ is strongly convex cocompact if and only if d is odd. Moreover, if d is even, then $\rho(\pi_1(S))$ does not preserve a properly convex domain.*

We provide a brief sketch of the proof of Corollary 44.3. Let $\tau_d : \mathsf{SL}(2,\mathbb{R}) \to \mathsf{SL}(d,\mathbb{R})$ be an irreducible representation (which is well-defined up to conjugacy). If $\eta : \pi_1(S) \to \mathsf{SL}(2,\mathbb{R})$ is a Fuchsian representation and $g \in \pi_1(S)$, then $\lambda_1(\tau_d(\eta(g)) = \lambda_1(\tau(g))^{d-1}$.

First suppose that d is odd. Then, $\lambda_1(\tau_d(\eta)) > 1$ if $g \in \pi_1(S)$ is non-trivial, so the d-Fuchsian representation $\tau_d \circ \eta$ is positively proximal. If $g \in \pi_1(S)$, then $\lambda_1(\rho(g))$ varies continuously as ρ varies over the Hitchin component $\mathcal{H}_d(\pi_1(S))$. Labourie [143] showed that any Hitchin

representation ρ is Borel Anosov, so if $g \in \pi_1(S)$ is non-trivial, then $|\lambda_1(\rho(g))| > 1$. Therefore, since every Hitchin component contains a d-Fuchsian representation, by definition, it follows that $\lambda_1(\rho(g)) > 1$ for all $\rho \in \mathcal{H}_d(S)$, so every $\rho \in \mathcal{H}_d(S)$ is positvely proximal. Since all Hitchin representations are strongly irreducible, see Labourie [143, Lemma 10.1], Theorem 44.2 then implies that every $\rho \in \mathcal{H}_d(S)$ preserves a properly convex domain. Corollary 43.2 then implies that $\rho(\Gamma)$ is strongly convex cocompact.

Now suppose that d is even. Recall that there exists a non-trivial element $g \in \pi_1(S)$ so that $\lambda_1(\eta(g)) < 0$ (see Choi-Goldman [66, Lemma 2]). Therefore, $\lambda_1(\tau_d(\eta(g)) < 0$, so the d-Fuchsian representation $\tau_d \circ \eta$ is not positively proximal. Since $\lambda_1(\rho(g))$ varies continuously over $\mathcal{H}_d(S)$ and always has modulus greater than 1, we see that $\lambda_1(\rho(g)) < -1$ for all $\rho \in \mathcal{H}_d(S)$. Since $\rho(g)$ is proximal, since ρ is Borel Anosov, it follows that if $\rho \in \mathcal{H}_d(S)$, then ρ is not positively proximal. Theorem 44.2 then implies that $\rho(\Gamma)$ does not preserve a properly convex domain. In particular, $\rho(\Gamma)$ is not strongly convex cocompact.

Entropy rigidity for convex cocompact groups. Zimmer [232] obtains a generalization of Crampon's [74] rigidity result, see Theorem 23.1, in the setting of strongly convex cocompact subgroups of $\mathsf{SL}(d, \mathbb{R})$. We recall that the **Hilbert entropy** of strongly convex cocompact subgroup Γ of $\mathsf{SL}(d, \mathbb{R})$ is defined to be

$$h_H(\rho) = \lim_{T \to \infty} \frac{\# \left\{ [\gamma] \in [\Gamma] \mid \log\left(\frac{\log |\lambda_1(\rho(\gamma))|}{\log |\lambda_d(\rho(\gamma))|}\right) \leq T \right\}}{T}.$$

Theorem 44.4. (Zimmer [232, Theorem 1.35]) *If $d \geq 3$ and Γ is a strongly convex cocompact subgroup of $\mathsf{SL}(d, \mathbb{R})$, then*

$$h_H(\rho) \leq d - 2,$$

and $h_H(\rho) = d - 2$ if and only if Γ is conjugate to a lattice in $\mathsf{SO}(d - 1, 1)$.

Tholozan [201, Theorem 2] earlier proved the closely related fact that the volume growth entropy of the Hilbert metric on a properly convex domain in $\mathbb{RP}^{d-1}$ is at most $d - 2$.

Convex cocompact groups which are not strongly convex cocompact. Although we have focussed almost exclusively on the case of Anosov convex cocompact groups the analysis of Danciger-Guéritaud-Kassel [84] extends to give signification information about groups which are only convex cocompact.

They establish the following general properties of convex cocompact groups, all of which are reminiscent of standard properties of Anosov groups.

Theorem 44.5. (Danciger-Guéritaud-Kassel [84, Theorem 1.17]) *If $\Gamma \subset \mathsf{SL}(d, \mathbb{R})$ acts convex cocompactly on the properly convex domain Ω, then*

(1) *The group Γ is finitely generated and any orbit map $\tau : \Gamma \to \Omega$ is a quasi-isometric embedding.*
(2) Γ *contains no unipotent elements.*
(3) *The dual Γ^* of Γ acts convex cocompactly on Ω^*.*
(4) *There exists a neighborhood U of the inclusion map in $\mathrm{Hom}(\Gamma, \mathsf{SL}(d, \mathbb{R}))$, so that if $\rho \in U$, then $\rho(\Gamma)$ is convex cocompact.*

They also demonstrate that there are interesting examples of convex cocompact groups which are not Anosov, by exhibiting convex cocompact groups that are isomorphic to non-hyperbolic Coxeter groups, hence not Anosov, see [83].

Geometrically finite actions on projective spaces. Crampon and Marquis [77, 78] studied geometrically finite actions of projective automorphism groups on strictly convex C^1 domains in $\mathbb{RP}^{d-1}$. This theory is designed to extend the well-developed theory of geometrically finite subgroups of $\mathsf{SO}(n, 1)$.

The simplest definition is to say that a finitely generatated, discrete subgroup Γ of $\mathsf{SL}(d, \mathbb{R})$ is **geometrically finite** it it preserves a strictly convex C^1 domain Ω, $\mathcal{C}_0(\Lambda_\Gamma^{orb})$ is contained in Ω and its convex core $\mathcal{C}_0(\Lambda_\Gamma^{orb})/\Gamma$ has finite volume. They prove [77, Theorem 8.1] that many of the standard alternative definitions from Kleinian groups generalized to this setting and remain equivalent. They show [77, Theorem 9.1] that Γ is relatively hyperbolic with respect to its maximal parabolic subgroups.

In a second paper [78] they study the geodesic flow of geometrically finite actions on projective space. As a cautionary tale, they point out [78, Proposition 8.1] that the geodesic flow need not be uniformly hyperbolic, unlike in the Kleinian setting. However, they do show that the non-wandering portion of the geodesic flow is uniformly hyperbolic if the cusps are "asymptotically hyperbolic," see [78, Theorem 5.2]. As a consequence [78, Corollary 7.3] they see that $\partial\Omega$ is $C^{1+\alpha}$ at all points in Λ_Γ^{orb} in this case (which places one more firmly in the setting of classical dynamics).

Part 8. **Anosov representations: Extra for Experts**

> *But Lather still finds it a nice thing to do,*
> *To lie about nude in the sand,*
> *Drawing pictures of mountains that look like bumps,*
> *And thrashing the air with his hands*
> ————Grace Slick [122]

In this chapter, we will briefly discuss a handful of topics. Each of these sections should be regarded as a brief introduction to a research area and we will not make any attempt to make the discussion complete. The choice of topics reflects our own idiosyncratic tastes. I intended to include more topics here and I may come back and do so someday.......

45. **Sambarino's geodesic flow**

Andres Sambarino [185] originated the idea of constructing a flow associated to a projective Anosov representation whose periods record the spectral radii of elements of $\rho(\Gamma)$. This idea was further developed by Sambarino and his co-authors in [45]. His flow turns out to be a reparametrization of the geodesic flow $\widehat{U}(\Gamma)$, so allows one to bring to bear the powerful tools of the Thermodynamic Formalism.

Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov and let

$$F_\rho = \big\{ ((x,y), \vec{v}) \mid (x,y) \in \partial\Gamma \times \partial\Gamma - \Delta, \ \vec{v} \in \xi_\rho(x) - \{\vec{0}\} \big\} / \vec{v} \sim -\vec{v}.$$

If $\pi : F_\rho \to \partial\Gamma \times \partial\Gamma - \Delta$ is the projection map, then π is a fibre bundle with fibre homeomorphic to $\mathbb{R}$. The only reasonable flow $\{\eta_t\}$ on F_ρ is given by

$$\eta_t(x, y, \vec{u}) = (x, y, e^t \vec{u})$$

for all $t \in \mathbb{R}$.

The group Γ acts on F_ρ by

$$\gamma((x,y), \vec{v}) = \big((\gamma(x)\gamma(y)), \rho(\gamma)(\vec{v}) \big)$$

for all $\gamma \in \Gamma$. We will show that the action of Γ on F_ρ is properly discontinuous, so, since the action commutes with the action of $\{\eta_t\}_{t \in \mathbb{R}}$, $\{\eta_t\}_{t \in \mathbb{R}}$ descends to a flow $\{\hat{\eta}_t\}_{t \in \mathbb{R}}$ on the quotient

$$\widehat{U}_\rho(\Gamma) = F_\rho / \Gamma.$$

We call the resulting flow the **spectral radius geodesic flow** of the representation ρ, since as we will see, the periods of this flow are exactly the spectral radii of elements of $\rho(\Gamma)$. (Recall that the spectral radius of an element $A \in \mathsf{SL}(d, \mathbb{R})$ is simply $|\lambda_1(A)|$.)

Proposition 45.1. ([45, Propositions 4.1 and 4.2]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation, there exists a Γ-equivariant orbit equivalence $g_\rho : U(\Gamma) \to F_\rho$. In particular, Γ acts properly discontinuously on F_ρ and g_ρ descends to an orbit equivalence $\hat{g}_\rho : \widehat{U}(\Gamma) \to \widehat{U}_\rho(\Gamma)$.*

We recall that a homeomorphism $g : (X, a_t) \to (Y, b_t)$ between flow spaces is an **orbit equivalence** if it takes flow lines to flow lines, i.e. $g(\{a_t(x) \mid t \in \mathbb{R}\}) = \{b_t(g(x)) \mid t \in \mathbb{R}\}$ for all $x \in X$. A map $f : (X, a_t) \to (Y, b_t)$ between flow spaces is said to **conjugate** (X, a_t) to (Y, b_t) if it is an orbit equivalence which preserves time, i.e. $f(a_t(x)) = b_t(f(x))$ for all $t \in \mathbb{R}$ and $x \in X$.

Proof. We first show that $\widehat{\Xi}$ is a contracting line bundle. There is a simple proof using tensor analysis (see [45, Lemma 2.4]), but we will give a more complicated proof based on the techniques developed in our earlier work (since tensor analysis always confuses me).

Lemma 45.2. ([45, Lemma 2.4]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation, then $\widehat{\Xi}$ is a contracting line bundle.*

Proof. Since $\widehat{U}(\Gamma)$ is compact,it suffices to prove that if $(\hat{Z}, \vec{v}) \in \widehat{\Xi}$, then $\lim_{t\to\infty} ||\hat{\psi}_t(\hat{Z}, \vec{v})|| = 0$

As usual, we choose a compact subset R of $U(\Gamma)$ so that $\Gamma(R) = U(\Gamma)$ and then choose $(Z, \vec{x}) \in \Xi|_Z$ so that $Z \in R$ and $(Z, \vec{x})$ covers $(\hat{Z}, \vec{v})$. Suppose that $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\lim t_n = +\infty$. For each n, choose γ_n so that

$$\gamma_n(\phi_{t_n}(Z)) = W_n \in R.$$

As we have previously seen, in the proof of Theorem 31.1,

$$||\hat{\psi}_{t_n}(\hat{z}, \vec{v})|| = ||(\phi_{t_n}(Z), \vec{x})|| = ||\vec{x}||_{\phi_{t_n}(Z)} = ||\rho(\gamma_n(\vec{x}))||_{W_n}$$

and if $Z = (z^-, z^+, s)$, then $\gamma_n^{-1} \to z^+$. We also note that $|| \cdot ||_{W_n}$ is uniformly bilipschitz to $|| \cdot ||_0$, so it suffices to prove that $||\rho(\gamma_n)(\vec{x})||_0 \to 0$.

Since ξ_ρ has the Cartan property, by Corollary 30.4,

$$U_1(\rho(\gamma_n)^{-1}) \to \xi_\rho(z^+) = \Xi|_Z = \rho(z)^+.$$

Therefore, since $\vec{x} \in \xi_\rho(z^+)$,

$$\lim \left(\frac{||\rho(\gamma_n)(\vec{x})||_0}{\sigma_d(\rho(\gamma_n))||\vec{x}||_0} \right) = 1.$$

Since

$$\sigma_d(\rho(\gamma_n)) = \frac{1}{\sigma_1(\rho(\gamma_n^{-1}))} \leq \sqrt{\frac{\sigma_2(\rho(\gamma_n^{-1}))}{\sigma_1(\rho(\gamma_n^{-1}))}},$$

and ρ is P_1-divergent, we see that

$$\lim ||\rho(\gamma_n)(\vec{x})||_0 = 0.$$

which completes the proof. (In the displayed equation above the final inequality follows from the fact that $\sigma_2(A) \geq \left(\frac{1}{\sigma_1(A)} \right)^{1/(d-1)}$ for any $A \in \mathsf{SL}(d, \mathbb{R})$.) $\qquad\square$

It will also be convenient to choose a particularly well-behaved norm on $\widehat{\Xi}$. We do so by averaging the norms over a sufficiently large time frame. (I recommend skipping the proof, since it is just a technical calculation which is here for completeness.)

Lemma 45.3. ([45, Lemma 4.3]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation, there exists a norm $|| \cdot ||$ on $\widehat{\Xi}$ and $\beta > 0$, such that*

$$||\hat{\psi}_t(\vec{v})|| \leq e^{-\beta t}||\vec{v}||$$

for all $\vec{v} \in \widehat{\Xi}$ and $t \geq 0$.

Proof. We start with a continuous family $||\cdot||^E$ of norms on $\widehat{\Xi}$. Since the flow $\hat{\psi}_t$ is contracting on $\widehat{\Xi}$, there exists $t_0 > 0$ such that

$$||\hat{\psi}_{t_0}(\vec{v})||^E \leq \frac{1}{4}||\vec{v}||^E$$

for all $\vec{v} \in \widehat{\Xi}$. Choose $\beta > 0$ so that $2 < e^{\beta t_0} < 4$ and, for all s, let

$$||\vec{v}||^{E,s} = ||\hat{\psi}_s(\vec{v})||^E,$$

so, by definition,

$$e^{\beta t_0}||\vec{v}||^{E,s+t_0} < 4||\vec{v}||^{E,s+t_0} \leq ||\vec{v}||^{E,s} \tag{45.1}$$

for all $s \geq 0$ and $\vec{v} \in \widehat{\Xi}$. Then let

$$||\vec{v}|| = \int_0^{t_0} e^{\beta s}||\vec{v}||^{E,s}\, ds$$

for all $\vec{v} \in \widehat{\Xi}$. If $t > 0$ and $\vec{v} \in \widehat{\Xi}$, then

$$
\begin{aligned}
||\hat{\psi}_t(\vec{v})|| &= \int_0^{t_0} e^{\beta s}||\hat{\psi}_t(\vec{v})||^{E,s}\, ds \\
&= e^{-\beta t}\int_t^{t+t_0} e^{\beta u}||\vec{v}||^{E,u}\, du \\
&= e^{-\beta t}\left(||\vec{v}|| + \int_{t_0}^{t+t_0} e^{\beta u}||\vec{v}||^{E,u}\, du - \int_0^{t} e^{\beta u}||\vec{v}||^{E,u}\, du\right) \\
&= e^{-\beta t}\left(||\vec{v}|| + \int_0^{t} e^{\beta u}\left(e^{\beta t_0}||\vec{v}||^{E,u+t_0} - ||\vec{v}||^{E,u}\right)\, du\right) \\
&< e^{-\beta t}||\vec{v}||
\end{aligned}
$$

where the last inequality follows from equation (45.1). Therefore our new metric has the desired property. $\qquad\square$

We lift the norm $||\cdot||$ from Lemma 45.3 to an equivariant norm, still denoted $||\cdot||$, on Ξ and we define $g_\rho : U(\Gamma) \to F_\rho$ so that

$$g_\rho(x, y, s) = (x, y, \vec{u}(x, y, s)) \qquad \text{where} \qquad ||\vec{u}(x, y, s)||_{(x,y,s)} = 1.$$

By our chosen contraction property, $||\vec{u}(x, y, s)||_{(x,y,s+t)} < 1$ if $t > 0$, so g_ρ is injective. Moreover, $\lim_{s\to\infty} ||\vec{u}(x, y, s)||_{(x,y,0)} = +\infty$ and $\lim_{s\to\infty} ||\vec{u}(x, y, s)||_{(x,y,0)} = 0$, so g_ρ is proper on each fibre, and hence a homeomorphism. By definition, g_ρ is equivariant and takes flow lines to flow lines. $\qquad\square$

Proposition 45.1 implies that the closed orbits of $\widehat{U}_\rho(\Gamma)$ are in one-to-one correspondence with the closed orbits of $\widehat{U}(\Gamma)$ and hence with the conjugacy classes of infinite order elements of Γ.

We now check that the period of the orbit associated to the conjugacy class $[\gamma]$ in Γ is the spectral radius of $\rho(\gamma)$.

Lemma 45.4. ([45, Proposition 4.1]) *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a projective Anosov representation, then the closed orbit associated to the conjugacy class $[\gamma]$ in Γ has period $|\lambda_1(\rho(\gamma))|$.*

Proof. The orbit of $\widehat{U}_\rho(\Gamma)$ associated to $[\gamma]$ is the quotient of the fiber of F_ρ over $(\gamma^+, \gamma^-) \in \partial \Gamma \times \partial \Gamma - \Delta$. If $\vec{v} \in \xi_\rho(\gamma^+)$, then

$$\gamma(\gamma^+, \gamma^-, \vec{v}) = (\gamma^+, \gamma^-, \rho(\gamma)(\vec{v})) = (\gamma^+, \gamma^-, \lambda_1(\rho(\gamma))\vec{v}) = \phi_{\log|\lambda_1(\rho(\gamma))|}(\gamma^+, \gamma^-, \vec{v})$$

since $\xi_\rho(\gamma^+)$ is the attracting eigenline of $\rho(\gamma)$. Therefore, the orbit has period $\log|\lambda_1(\rho(\gamma))|$. $\square$

Remarks: If $\rho : \pi_1(S) \to \mathsf{SL}(d, \mathbb{R})$ is a Hitchin representation and $1 \le k \le \frac{d}{2}$, then one may construct a geodesic flow which is a Hölder reparametrization of $T^1 S$ so that the orbit associated to $[\gamma]$ in $\pi_1(S)$ has period $\log\left(\frac{|\lambda_k(\rho(\gamma))|}{|\lambda_{k+1}(\rho(\gamma))|}\right)$, see Martone-Zhang [160] and [46].

46. Thermodynamic Formalism and the entropy of the geodesic flow

> *Where the vulture glides descending*
> *On an ancient highway bending*
> *Through libraries and museums*
> *Galaxies and stars*
> *Down the windy halls of friendship*
> *To the rose clipped by the bullwhip*
> *The motel of lost companions*
> *Waits with heated pool and bar*
> ——————————Neil Young [225]

It turns out that the natural regularity class for our constructions is Hölder regularity. The geodesic flow $\widehat{U}(\Gamma)$ is only well-defined up to Hölder orbit equivalence (even in the case of a closed surface group $\pi_1(S)$). If one examines the proof of stability more closely one can conclude that the limit map is Hölder. Then, one may observe that the orbit equivalence constructed above is Hölder, and hence that $\widehat{U}_\rho(\Gamma)$ is a Hölder conjugate to a Hölder **reparameterization** of $\widehat{U}(\Gamma)$. Roughly, this means that there exists a Hölder function $f_\rho : \widehat{U}(\Gamma) \to (0, \infty)$, so that $\widehat{U}_\rho(\Gamma)$ is Hölder conjugate to the unit speed flow associated to the new element of arc length $f_\rho ds$ along the flow lines of $\widehat{U}(\Gamma)$, where ds is the element of arc length for the flow lines in $\widehat{U}(\Gamma)$. We call the resulting flow $\widehat{U}(\Gamma)^{f_\rho}$ and notice that the identity map is a Hölder orbit equivalence between $\widehat{U}(\Gamma)$ and $\widehat{U}(\Gamma)^{f_\rho}$. (See [45, Sections 3 and 6] for more details and references.)

Recall that $\widehat{U}(\Gamma)$ is Anosov if $\Gamma = \pi_1(M)$ where M is a closed manifold and is a compact flow-invariant subset of an Anosov flow if Γ admits a convex cocompact representation into a rank one Lie group. In all cases it is topologically transitive. In either case, we are immediately in the setting of the Thermodynamic Formalism in which there are powerful tools and invariants. This theory was developed by Bowen [39], Ruelle [183] and Parry-Pollicott [175]. In general, one can prove that $\widehat{U}_\rho(\Gamma)$ is always "metric Anosov" (see [45, Proposition 5.1] and [69, Theorem C]), so by work of Pollicott [176] one can still apply many of the tools of the Thermodynamic Formalism in this setting. In this section we will discuss, without proof, some of these applications.

One of the most studied invariants is the topological entropy. If $T > 0$ and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov, let

$$R_\rho(T) = \{[\gamma] \in [\Gamma] \mid \log(|\lambda_1(\rho(\gamma))|) \leq T\}.$$

The **topological entropy** of $\widehat{U}_\rho(\Gamma)$ is then defined to be

$$h(\rho) = \lim_{T \to \infty} \frac{\log(\#(R_\rho(T)))}{T}.$$

(In general, the topological entropy of a topologically transitive Anosov flow may be defined to be the exponential growth rate of the number of close orbits with period at most T.) The Thermodynamic Formalism guarantees that this limit always exists for a Hölder reparamterization of an Anosov flow.

Let $\{\rho_u : \Gamma \to \mathsf{SL}(d, \mathbb{R})\}_{u \in M}$ be a family of projective Anosov representations parameterized by an real analytic manifold M. We say that $\{\rho_u\}_{u \in M}$ is an **analytic family** of projective Anosov representations if $\rho_u(\gamma)$ varies analytically for all $\gamma \in \Gamma$. For example, every Hitchin component is an analytic family of projective Anosov representations (see [114]). Johnson and Millson [124] observed that Benoist components can have singularities. However, they contain many analytic submanifolds.

The choice of reparameterization function f_ρ is not canonical (although any two choices are Livsic gologous). However, Bridgeman, Canary, Labourie and Sambarino [45, Theorem 6.1] show that the limit map ξ_ρ does vary analytically. They further show that:

Proposition 46.1. ([45, Proposition 6.2]) *If $\{\rho_u\}_{u \in M}$ is an analytic family of projective Anosov representations, then one may choose $\{f_{\rho_u}\}$ to vary locally analytically over M. More explicitly, at every point $u_0 \in M$ there exists $\alpha > 0$, a neighborhood U of u_0, and an analytic map F from U into the space of α-Hölder maps of $\widehat{U}(\Gamma)$ into $(0, \infty)$ so that if $u \in U$, then $\widehat{U}_{\rho_u}(\Gamma)$ is Hölder conjugate to $\widehat{U}(\Gamma)^{F(u)}$.*

The Thermodynamic Formalism then gives that entropy varies analytically.

Corollary 46.2. ([45, Theorem 1.3]) *If $\{\rho_u\}_{u \in M}$ is an analytic family of projective Anosov representations, then*

$$u \to h(\rho_u)$$

is an analytic function on M.

If $\rho \in CC(\Gamma, \mathsf{SO}_0(n, 1))$, then Sullivan [196] showed that $h(\rho)$ is (a scalar multiple of) the Hausdorff dimension of the limit set $\Lambda(\rho)$ of $\rho(\Gamma)$. If n is 2 or 3, then $CC(\Gamma, \mathsf{SO}_0(n, 1))$ is an analytic manifold, see Bers [28]. In this special case we obtain:

Corollary 46.3. ([45, Corollary 1.8]) *The Hausdorff dimension of the limit set varies analytically over analytic submanifolds of $CC(\Gamma, \mathsf{SO}_0(n, 1))$. In particular, it varies analytically over $CC(\Gamma, \mathsf{SO}_0(2, 1))$ and $CC(\Gamma, \mathsf{SO}_0(3, 1))$.*

We encourage the interested reader to consult the work of Sambarino [46, 185, 186], Potrie-Sambarino [179] and the very recent work of Edwards, Lee, and Oh [95, 149] for further results on the dynamical properties of Anosov representations.

Remarks: (1) Corollary 46.3 was established when Γ is a surface group or free group by Ruelle [184] and for certain other groups by Anderson and Rocha [7].

(2) If G is any rank one Lie group, Corlette-Iozzi [73] and Yue [226] showed that the Hausdorff dimension is a scalar multiple of the topological entropy and one may similarly show that the Hausdorff dimension of the limit set varies analytically over analytic families of convex cocompact representations into G.

(3) Pollicott and Sharp [177] earlier showed that entropy varies analytically over the Hitchin component. Tapie [200] previously established that the Hausdorff dimension varies continuously differentiably over (smooth points of) $CC(\Gamma, \mathsf{SO}_0(n, 1))$.

Pressure metrics.

> I got a couple of opinions that I hold dear
> Got a whole lot of debt and a whole lot of fear
> I got an itch that needs scratching but it feels alright
> I got the need to blow it out on Saturday night
> I got a grill in the backyard and a case of beers
> Got a boat that ain't seen the water in years
> More bills than money, I can do the math
> I'm trying to keep focused on the righteous path
> —————————-Patterson Hood [89]

If $\{\rho_u\}_{u \in M}$ is an analytic family of projective Anosov representations, one can use Proposition 46.1 to construct a locally analytic map of M into the space of (Livsic cohomology classes) of positive Hölder functions on $\widehat{U}(\Gamma)$. If all the representations are irreducible and ρ_u is not conjugate to ρ_v, in $\mathsf{GL}(d, \mathbb{R})$, if $u \neq v$, this map will be injective, see [45, Theorem 1.2].

Notice that if $\{\rho_u\}_{u \in M}$ is an analytic family of P_k-Anosov representations, then $\{E_d^k \circ \rho_u\}_{u \in M}$ is an analytic family of projective Anosov representations, by Theorem 34.3. Thus, one can map M into the the space of (Livsic cohomology classes) of positive Hölder functions on $\widehat{U}(\Gamma)$ by taking u to $f_{E_d^k \circ \rho}$.

McMullen [163] used the Thermodynamical Formalism to construct a pressure form on the space of positive Hölder functions which is non-negative. One may pull this form back to obtain an analytic form on M, which we again call the **pressure form**. If this form is positive definite, it gives rise to an analytic Riemannian metric on M. McMullen implemented this strategy in the case of the classical Teichmüller space of a closed orientable surface S and in this case it gives rise to Thurston's reformulation of the Weil-Petersson metric as "the Hessian of the length of a random geodesic," see also Bonahon [33] and Wolpert [218]. Bridgeman [44] showed that on the space $QF(S) = \widehat{CC}(\pi_1(S), \mathsf{SO}(2, 1))$ the pressure form is only non-degenerate at the Fuchsian locus and only in pure bending directions, so one gets a **pressure metric** which is a path metric that is Riemannian off the submanifold of Fuchsian representations. Moreover, Bridgeman's pressure metric restricts to (a scalar multiple of) the Weil-Petersson metric on the Fuchsian locus (which is naturally a copy of $\mathcal{T}(S)$).

These are both examples of a much more general phenomenon. We recall that a representation into $\mathsf{SL}(d, \mathbb{R})$ is **generic** if some element of the image is diagonalizable with distinct eigenvalues

Theorem 46.4. ([45, Theorem 1.4]) *If $\{\rho_u\}_{u \in M}$ is an analytic family of generic irreducible P_1-Anosov representations, then the pressure metric on M is an analytic Riemannian metric.*

More generally, if $\{\rho_u\}_{u \in M}$ is an analytic family of Zariski dense P_k-Anosov representations, then the pressure metric on M is an analytic Riemannian metric.

Notice that Fuchsian representation are not Zariski dense in $\mathsf{SO}(3,1)$ so Bridgeman's work tells us that the Zariski density assumption is essential in this case. (We note that all convex cocompact representations of a group Γ into $\mathsf{SO}(d,1)$ are generic, viewed as representations into $\mathsf{SO}(d,1)$, rather than $\mathsf{SL}(d+1,\mathbb{R})$.)

We cite two examples of special interest (to me).

Corollary 46.5. ([45, Corollary 1.7]) *If Γ does not have a finite index subgroup which is either free or a surface group, then the pressure metric is an analytic Riemannian metric on $\widehat{CC}(\Gamma, \mathsf{SO}(3,1))$ which is invariant under the action of $\mathrm{Out}(\Gamma)$.*

Corollary 46.6. ([45, Corollary 1.6]) *If S is a closed orientable surface and $d \geq 3$, then the pressure metric is an analytic Riemannian metric on the Hitchin component $\mathcal{H}_d(S)$ which is invariant under the action of the mapping class group $\mathrm{Mod}(S)$. Moreover, the restriction of the pressure metric to the Fuchsian locus is a scalar multiple of the Weil-Petersson metric.*

We give a brief definition of the pressure metric here, but we refer the reader to the survey paper [48] for a more complete expository treatment. Given two projective Anosov representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ and $\eta : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ we define there **pressure intersection**

$$I(\rho, \eta) = \lim_{T \to \infty} \frac{1}{|R_T(\rho)|} \sum_{[\gamma] \in R_T(\rho)} \frac{\log(|\lambda_1(\eta(\gamma))|}{\log|\lambda_1(\rho(\gamma))|}$$

which one may think of as the (spectral radius) length (in η) of a random geodesic (with respect to length in ρ). One then considers the **renormalized pressure intersection** given by

$$J(\rho, \eta) = \frac{h(\eta)}{h(\rho)} \lim_{T \to \infty} \frac{1}{|R_T(\rho)|} \sum_{[\gamma] \in R_T(\rho)} \frac{\log(|\lambda_1(\eta(\gamma))|}{\log|\lambda_1(\rho(\gamma))|}.$$

(Thurston and McMullen did not need to renormalize the pressure intersection since entropy is constant on the Teichmüller space of a closed surface.) If $\{\rho_u\}_{u \in M}$ is an analytic family of generic irreducible P_1-Anosov representations and $u_0 \in M$, we consider $J(\rho_{u_0}, \cdot)$ as a map from M to $\mathbb{R}$. The Thermodynamic formalism gives that J has a minimum at u_0, so we define the pressure metric by

$$\mathbb{P}|_{T_{u_0}M} = \mathrm{Hess}J(\rho_{u_0}, \cdot)$$

and see immediately that $\mathbb{P}$ is non-negative at every point. The most difficult part of the proof of Theorem 46.4 involves showing that P is actually positive definite at every point.

Remarks: (1) Li [150] produced another mapping class group invariant Riemannian metric on $\mathcal{H}_3(S)$, which she calls the Loftin metric. The Loftin metric also restricts to a scalar multiple of the Weil-Petersson metric on the Fuchsian locus.

(2) Marc Burger [51] was the first one to consider the pressure intersection, in the context of convex cocompact rank one representations.

(2) Bridgeman, Canary, Labourie and Sambarino [46] construct another pressure metric on the Hitchin component, called the simple root pressure metric, by replacing the spectral radius geodesic flow with the (first) simple root geodesic flow whose periods have the form $\log\left(\frac{|\lambda_1(\rho(\gamma))|}{|\lambda_2(\rho(\gamma))|}\right)$ for $[\gamma]$ in $\pi_1(S)$.

47. **Marked length rigidity**

It's a pure perfect world that tells no lies
Burn you down you try to improvise
If kerosene works why not gasoline....
—————Brian Henneman [34]

We begin with Dal'bo and Kim's simple and elegant proof that any two faithful Zariski dense representations into $\mathsf{PSL}(d, \mathbb{R})$ with the same marked length spectrum differ by precomposition by an automorphism of $\mathsf{PSL}(d, \mathbb{R})$.

In the statement

$$\ell(\rho_i(\gamma)) = \sqrt{\big(\log \lambda_1(\rho_i(\gamma))\big)^2 + \cdots + \big(\log \lambda_1(\rho_i(\gamma))\big)^2}$$

is the translation length of $\rho_i(\gamma)$ on the symmetric space X_d associated to $\mathsf{PSL}(d, \mathbb{R})$.

Theorem 47.1. (Dal'bo-Kim[81]) *If Γ is a group and $\rho_1 : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ are faithful, Zariski dense representations and $\ell(\rho_1(\gamma)) = \ell(\rho_2(\gamma))$ for all $\gamma \in \Gamma$, then there exists an automorphism τ of $\mathsf{SL}(d, \mathbb{R})$ so that $\rho_2 = \tau \circ \rho_1$.*

We note that the subgroup of automorphisms given by conjugation by an element of $\mathsf{PGL}(d, \mathbb{R})$ has index two in the group of automorphisms of $\mathsf{PSL}(d, \mathbb{R})$ and that the contragredient involution given by $A \to (A^{-1})^T$ is a representative of the non-trivial coset. So, one may strengthen the conclusion by saying that ρ_1 is conjugate to either ρ_2 or ρ_2^*.

Proof. Consider the **Jordan projection**

$$\lambda : \mathsf{PSL}(d, \mathbb{R}) \to \mathfrak{a}^+$$

where

$$\mathfrak{a}^+ = \{\vec{x} \in \mathbb{R}^d \mid x_1 \geq x_2 \geq \cdots \geq x_d, \ x_1 + \cdots x_d = 0\}$$

given by

$$\lambda(g) = \big(\log(|\lambda_1(g)|), \ldots, \log(|\lambda_d(g)|)\big).$$

If Δ is a subgroup of $\mathsf{PSL}(d, \mathbb{R}) \times \mathsf{PSL}(d, \mathbb{R})$, then its **Benoist limit cone** $\Lambda(\Delta)$ is the closure in $\mathfrak{a}^+ \times \mathfrak{a}^+$ of the cone on $(\lambda \times \lambda)(\Delta)$.

The crucial tool in the proof is a powerful result of Benoist, which we state in our setting, although it is a special case of a result for limit cones of Zariski dense subgroups of connected, algebraic, semisimple Lie groups.

Theorem 47.2. (Benoist [16]) *If Δ is a Zariski dense subgroup of $\mathsf{PSL}(d, \mathbb{R}) \times \mathsf{PSL}(d, \mathbb{R})$, then its limit cone $\Lambda(\Delta)$ is convex and has non-empty interior.*

We consider the representation $\rho_1 \times \rho_2 : \Gamma \to \mathsf{PSL}(d, \mathbb{R}) \times \mathsf{PSL}(d, \mathbb{R})$. If $\ell(\rho_1(\gamma)) = \ell(\rho_2(\gamma))$ for all $\gamma \in \Gamma$, then $||\lambda(\rho_1(\gamma))|| = ||\lambda(\rho_2(\gamma))||$ for all $\gamma \in \Gamma$, so $\{(\lambda(\rho(\gamma)), \lambda(\sigma(\gamma)))\}$ lies in the codimension one submanifold $E = \{(\vec{x}, \vec{y}) \mid ||\vec{x}|| = ||\vec{y}||\}$ of $\mathfrak{a}^+ \times \mathfrak{a}^+$. Moreover, its limit cone $\Lambda((\rho_1 \times \rho_2)(\Gamma))$ lies in E. Theorem 47.2 then implies that $(\rho_1 \times \rho_2)(\Gamma)$ is not Zariski dense in $\mathsf{PSL}(d, \mathbb{R}) \times \mathsf{PSL}(d, \mathbb{R})$.

Let Z be the Zariski closure of $(\rho_1 \times \rho_2)(\Gamma)$ in $\mathsf{PSL}(d, \mathbb{R}) \times \mathsf{PSL}(d, \mathbb{R})$. Let $p_i : Z \to \mathsf{SL}(d, \mathbb{R})$ be the projection of Z onto the ith factor. Since $\rho_i(\Gamma)$ is Zariski dense, p_i is surjective. Consider the kernel K_i of p_1. Then $K_1 \subset \{I\} \times \mathsf{SL}(2, \mathbb{R})$, so may be identified with a normal subgroup

of $\mathsf{PSL}(d,\mathbb{R})$. Since $\mathsf{PSL}(d,\mathbb{R})$ is simple, K_1 is either trivial or all of $\mathsf{PSL}(d,\mathbb{R})$. Since Z is not all of $\mathsf{PSL}(d,\mathbb{R}) \times \mathsf{PSL}(d,\mathbb{R})$, K_1 must be trivial. Similarly, K_2 is trivial. Thus, p_1 and p_2 are isomorphisms. Therefore, the isomorphism $\rho_2 \circ \rho_1^{-1} : \rho_1(\Gamma) \to \rho_2(\Gamma)$ extends to an automorphism $\tau = p_2|_Z \circ (p_1|_Z)^{-1}$ of $\mathsf{PSL}(d,\mathbb{R})$. $\qquad\square$

Using the fact, due to Benoist [18], that the Zariski closure of the image of a Benoist representation into $\mathsf{PSL}(d,\mathbb{R})$ is either $\mathsf{PSO}(d-1,1)$ or $\mathsf{PSL}(d,\mathbb{R})$ one may use essentially the same argument to prove the following theorem.

Theorem 47.3. (Cooper-Delp [68], Kim [139]) *If Γ is a hyperbolic group and $\rho_1 : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ are faithful, Benoist representations so that*

$$\ell^H(\rho_1(\gamma)) = \frac{1}{2}\log\left(\frac{\lambda_1(\rho_1(\gamma))}{\lambda_d(\rho_1(\gamma))}\right) = \frac{1}{2}\log\left(\frac{\lambda_1(\rho_2(\gamma))}{\lambda_d(\rho_2(\gamma))}\right) = \ell^H(\rho_2(\gamma))$$

for all $\gamma \in \Gamma$, then either ρ_1 is conjugate, in $\mathsf{PGL}(d,\mathbb{R})$, to either ρ_2 or ρ_2^*.

One can show that the eigenvalues of maximal modulus determine an irreducible projective Anosov representations.

Theorem 47.4. (Bridgeman-Canary-Labourie-Sambarino [45, Theorem 1.2]) *Suppose that $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ and $\eta : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ are projective Anosov representations and*

$$\lambda_1(\rho(\gamma)) = \lambda_1(\eta(\gamma))$$

for all $\gamma \in \Gamma$ then

   (1) *If ρ and η are irreducible, then ρ and η are conjugate in $\mathsf{GL}(d,\mathbb{R})$.*
   (2) *If $< \xi_\rho(\partial\Gamma) >$ and $< \xi_\rho(\partial\Gamma) >$ have the same dimension, then there exists $g \in \mathsf{GL}(d,\mathbb{R})$ so that $g \circ \xi_\rho = \xi_\eta$.*

We will sketch a proof of part (1) of this result. A key tool in the proof is a cross ratio introduced and studied by Labourie [144, 145]. Let

$$\mathbb{T}_d = \{(\varphi,\psi,u,v) \in \mathbb{P}(\mathbb{R}^d))^{*2} \times \mathbb{P}(\mathbb{R}^d)^2 : (\varphi,v) \text{ and } (\psi,u)\text{are transverse pairs}\}.$$

We then define the **cross ratio b** $: \mathbb{T}_d \to \mathbb{R}$ by

$$\mathbf{b}(\varphi,\psi,u,v) = \frac{\vec{\varphi}(\vec{u})\vec{\psi}(\vec{v})}{\vec{\varphi}(\vec{v})\vec{\psi}(\vec{u})}$$

where we choose non-zero representatitves $\vec{u} \in u$, $\vec{v} \in v$, $\vec{\varphi} \in \varphi$ and $\vec{\psi} \in \psi$ and notice that our choice of representatives doesn't matter.

Let $\partial\Gamma^{(4)} \subset \partial\Gamma^4$ denote the space of ordered 4-tuples of distinct points in $\partial\Gamma$. If $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is projective Anosov, Labourie defines an associated cross-ratio $\mathbf{b}_\rho : \partial\Gamma^{(4)} \to \mathbb{R}$ given by

$$\mathbf{b}_\rho(x,y,z,w) = \mathbf{b}(\theta(x),\theta(y),\xi(z),\xi(w)).$$

We first observe that the cross-ratio determines an irreducible projective Anosov representation into $\mathsf{PSL}(d,\mathbb{R})$ up to conjugacy in $\mathsf{PGL}(d,\mathbb{R})$. Let $\pi : \mathsf{SL}(d,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ be the obvious projection map.

Proposition 47.5. *Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ and $\eta : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ are irreducible, projective Anosov representations, and $\mathbf{b}_\rho = \mathbf{b}_\eta$, then there exists $g \in \mathsf{PGL}(d, \mathbb{R})$ such that*

$$g \circ (\pi \circ \rho) \circ g^{-1} = \pi \circ \eta.$$

Recall that a collection of $(d + 1)$ lines in $\mathbb{R}^d$ is said to be a **projective frame** if every d lines in the collection span $\mathbb{R}^d$.

Proof. Choose $\{x_0, \ldots, x_d\} \subset \Lambda(\Gamma)$ so that

$$\{\xi_\rho(x_0), \ldots, \xi_\rho(x_d)\} \text{ and } \{\xi_\eta(x_0), \ldots, \xi_\eta(x_d)\}$$

are projective frames for $\mathbb{R}^d$ and

$$\{\theta_\rho(x_0), \ldots, \theta_\rho(x_d)\} \text{ and } \{\theta_\eta(x_0), \ldots, \theta_\eta(x_d)\}$$

are projective frames for $(\mathbb{R}^d)^*$. (One can do since ρ is irreducible, so ξ_ρ must span $\mathbb{R}^d$ and ρ^* is irreducible, so $\theta_\rho = \xi_{\rho^*}$ must span $(\mathbb{R}^d)^*$.)

Choose $\vec{u}_0 \in \xi_\rho(x_0)$ and $\{\varphi_1, \ldots, \varphi_d\} \subset (\mathbb{R}^d)^*$ such that $\varphi_i \in \theta_\rho(x_i)$ and $\varphi_i(\vec{u}_0) = 1$. For $y \in \Lambda(\Gamma) - \{x_0, \ldots, x_d\}$, the projective coordinates of $\xi_\rho(y)$ with respect to the basis for $\mathbb{R}^d$ dual to $\{\varphi_1, \ldots, \varphi_d\}$ are given by

$$[\ldots : \varphi_i(\vec{y}) : \ldots] = [\ldots : \frac{\varphi_i(\vec{y})}{\varphi_1(\vec{y})} \frac{\varphi_1(\vec{u}_0)}{\varphi_i(\vec{u}_0)} : \ldots]$$

(where $\vec{y}$ is a non-zero vector in $\xi_\rho(y)$) which reduces to

$$[\mathbf{b}_\rho(x_1, x_1, y, x_0), \ldots, \mathbf{b}_\rho(x_d, x_1, y, x_0),].$$

Now choose $\vec{v}_0 \in \xi_\eta(x_0)$ and $\{\psi_1, \ldots, \psi_d\}$ such that $\psi_i \in \theta_\eta(x_i)$ and $\psi_i(\vec{v}_0) = 1$. Again $\{\psi_1, \ldots, \psi_d\}$ is a basis for $(\mathbb{R}^d)^*$ and in the dual basis, $\xi_\eta(y)$ has projective coordinates

$$[\mathbf{b}_\eta(x_1, x_1, y, x_0), \ldots, \mathbf{b}_\eta(x_p, x_1, y, x_0), 0, \ldots, 0].$$

We now choose $g \in \mathsf{GL}(d, \mathbb{R})$ so that $g\varphi_i = \psi_i$ for all i. Since $\mathbf{b}_\rho(x_i, x_1, y, x_0) = \mathbf{b}_\eta(x_i, x_1, y, x_0)$ for all i, we see that $g \circ \xi_\rho = \xi_\eta$. Since an element $A \in \mathsf{PGL}(d, \mathbb{R})$ is determined entirely by its action on a projective frame and ξ_ρ and ξ_θ are both ρ-equivariant, this implies that $g \circ (\pi \circ \rho) \circ g^{-1} = \pi \circ \eta$. $\qquad\square$

One can compute the cross-ratio on pairs of fixed points of co-prime infinite order elements in Γ. We say that α and β are **co-prime** if the group they generate $< \alpha, \beta >$ is not virtually cyclic. If A is proximal with attracting eigenline A^+ and repelling hyperplane A^-, we define $\mathbf{p}(A)$ to be the projection onto A^+ parallel to A^-. Let $\mathbf{r}(A) = A - \lambda_1(A)p(A)$.

Proposition 47.6. *If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is projective Anosov and $\alpha, \beta \in \Gamma$ are infinite order and co-prime, then*

$$\mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+). = \mathrm{Tr}\big(\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))\big) = \lim_{n \to \infty} \frac{\lambda_1(\rho(\alpha^n\beta))}{\lambda_1(\rho(\alpha))^n} \neq 0$$

Proof. Choose $\vec{a} \in \xi(\alpha^+)$, $\vec{A} \in \theta(\alpha^-)$, $\vec{b} \in \xi(\beta^+)$ and $\vec{B} \in \theta(\beta^-)$. We may write

$$\mathbf{p}(\rho(\alpha))(\vec{u}) = \left(\frac{\vec{A}(\vec{u})}{\vec{A}(\vec{a})}\right)\vec{a} \qquad \text{and} \qquad \mathbf{p}(\rho(\beta))(\vec{u}) = \left(\frac{\vec{B}(\vec{u})}{\vec{B}(\vec{b})}\right)\vec{b}$$

for all $\vec{u} \in \mathbb{R}^d$, so

$$\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))(\vec{u}) = \frac{\vec{B}(\vec{u})\vec{A}(\vec{b})}{\vec{A}(\vec{a})\vec{B}(\vec{u})} \ \vec{a}.$$

Therefore,

$$\mathrm{Tr}\big(\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))\big) = \frac{\vec{B}(\vec{a})\vec{A}(\vec{b})}{\vec{A}(\vec{a})\vec{B}(\vec{b})} = \mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+).$$

We now prove the second equality. First notice that

$$\mathrm{Tr}\big(\rho(\alpha^n)\rho(\beta^n)\big) = \lambda_1(\alpha^n\beta^n)(1 + g_n)$$

where

$$g_n = \frac{\mathrm{Tr}\big(\mathbf{r}(\rho(\alpha^n\beta^n))\big)}{\lambda_1(\rho(\alpha^n\beta^n))}.$$

Since ρ is P_1-Anosov, $\lim_{n\to\infty} g_n = 0$.

On the other hand,

$$\rho(\alpha^n)\rho(\beta^n) = \lambda_1\rho(\alpha))^n\lambda_1(\rho(\beta))^n\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))+\lambda_1(\rho(\alpha))^n\mathbf{p}(\rho(\alpha))\mathbf{r}(\rho(\beta)^n)+\lambda_1(\rho(\beta))^n\mathbf{r}(\rho(\alpha^n))\mathbf{p}(\rho(\beta))+\mathbf{r}(\rho(\alpha^n))\mathbf{r}(\rho(\beta^n))),$$

so

$$\mathrm{Tr}\big(\rho(\alpha^n\beta^n)\big) = \lambda_1(\rho(\alpha))^n\lambda_1(\rho(\beta))^n\mathrm{Tr}\big(\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))\big)(1 + \hat{g}_n)$$

where

$$\hat{g}_n = \frac{\lambda_1(\rho(\alpha))^n\mathrm{Tr}\big(\mathbf{p}(\rho(\alpha))\mathbf{r}(\rho(\beta^n))\big) + \lambda_1(\rho(\beta)^n)\mathrm{Tr}\big(\mathbf{r}(\rho(\alpha^n))\mathbf{p}(\rho((\beta)))\big) + \mathrm{Tr}\big(\mathbf{r}(\rho(\alpha^n))\mathbf{r}(\rho(\beta^n))\big)}{\mathrm{Tr}\big(\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))\big)\lambda_1(\rho(\alpha))^n\lambda_1(\rho(\beta))^n}.$$

Since ρ is P_1-Anosov, $\lim \hat{g}_n(\rho) = 0$.

Combining, we see that

$$\mathrm{Tr}\big(\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))\big) = \frac{\lambda_1(\rho(\alpha^n\beta^n))(1 + g_n)}{\lambda_1(\rho(\alpha)^n)\lambda_1(\rho(\beta))^n(1 + \hat{g}_n)},$$

which implies, since $\lim g_n = 0$ and $\lim \hat{g}_n = 0$, that

$$\mathrm{Tr}\big(\mathbf{p}(\rho(\alpha))\mathbf{p}(\rho(\beta))\big) = \lim_{n\to\infty} \frac{\lambda_1(\rho(\alpha^n\beta^n))}{\lambda_1(\rho(\alpha))^n\lambda_1(\rho(\beta))^n}$$

which completes the proof.   □

Proof of Theorem 47.4(1): Suppose that $\lambda_1(\rho(\gamma)) = \lambda_1(\eta(\gamma))$ for all $\gamma \in \Gamma$. Proposition 47.6 implies that if α and β are co-prime infinite order elements, then

$$\mathbf{b}_\rho(\alpha^-, \beta^-, \beta^+, \alpha^+) = \mathbf{b}_\eta(\alpha^-, \beta^-, \beta^+, \alpha^+).$$

Since pairs of the form (α^+, α^-) (for some $\alpha \in \Gamma$) are dense in $\partial\Gamma \times \partial\Gamma$ and the cross-ratio is continuous, this implies that $\beta_\rho = \beta_\eta$.

Proposition 47.5 then implies that there exists $g \in \mathsf{PGL}(d, \mathbb{R})$ so that $g \circ (\pi \circ \rho) \circ g^{-1} = \pi \circ \eta$. Let h be a representative of g in $\mathsf{GL}(d, \mathbb{R})$. Then $h \circ \rho \circ h^{-1}(\gamma) = \pm\eta(\gamma)$ for any $\gamma \in \Gamma$. However, since $\lambda_1(\rho(\gamma)) = \lambda_1(\eta(\gamma))$, we must have $h \circ \rho \circ h^{-1}(\gamma) = \eta(\gamma)$ for all $\gamma \in \Gamma$. Therefore, ρ and η are conjugate in $\mathsf{GL}(d, \mathbb{R})$.   □

For Hitchin representations, it suffices to consider the spectral radii of simple closed curves on the surface S if the genus of S is at least 3.

Theorem 47.7. (Bridgeman-Canary-Labourie [47, Theorem 1.1]) *If S is a closed orientable surface of genus $g \geq 3$, $\rho_1 : \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ and $\rho_2 : \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ are Hitchin representations, and $|\lambda_1(\rho_1(\gamma))| = |\lambda_1(\rho_2(\gamma))|$ for every $\gamma \in \Gamma$ which is represented by a simple closed curve in S, then ρ_1 and ρ_2 are conjugate in $\mathsf{PGL}(d,\mathbb{R})$.*

In the special case of Hitchin representations into $\mathsf{PSL}(3,\mathbb{R})$, which are also Benoist representations, we see that it suffices to consider Hilbert lengths of simple closed curves.

Theorem 47.8. (Bridgeman-Canary-Labourie [47, Theorem 9.1]) *If S is a closed orientable surface of genus $g \geq 3$, $\rho_1 : \pi_1(S) \to \mathsf{PSL}(3,\mathbb{R})$ and $\rho_2 : \pi_1(S) \to \mathsf{PSL}(3,\mathbb{R})$ are Benoist representations, and*

$$\ell^H(\rho_1(\gamma)) = \frac{1}{2}\log\left(\frac{\lambda_1(\rho_1(\gamma))}{\lambda_d(\rho_1(\gamma))}\right) = \frac{1}{2}\log\left(\frac{\lambda_1(\rho_2(\gamma))}{\lambda_d(\rho_2(\gamma))}\right) = \ell^H(\rho_2(\gamma))$$

for any $\gamma \in \Gamma$ which is represented by a simple closed curve on S, then either ρ_1 is conjugate to ρ_2 or ρ_1 is conjugate to ρ_2^*.

Remarks: Dal'bo and Kim actually establish Theorem 47.1 in the more general the setting of semi-simple Lie groups (see [82] for the relevant definitions).

Theorem 47.9. (Dal'bo-Kim [82]) *Suppose that G_1 and G_2 are connected, semi-simple Lie groups with trivial center and without compact factors. If $\rho_1 : \Gamma \to \mathsf{G}_1$ and $\rho_2 : \Gamma \to \mathsf{G}_2$ are faithful representations so that $\rho_i(\Gamma)$ is Zariski dense in G_i for $i = 1,2$ and $\ell(\rho_1(\gamma)) = \ell(\rho_2(\gamma))$ for all $\gamma \in \Gamma$, then there exists an isomorphism $\tau : \mathsf{G}_1 \to \mathsf{G}_2$ so that $\tau \circ \rho_1 = \rho_2$.*

(2) Notice that the assumption that $< \xi_{\rho_1}(\partial\Gamma) >$ and $< \xi_{\rho_2}(\partial\Gamma) >$ have the same dimension in part (2) of Theorem 47.4 is not given in [45, Theorem 11.1], but is used implicitly in the proof, so the statement there is incorrect in this case.

48. Topological restrictions

It is natural to wonder which groups admit Anosov representations into $\mathsf{SL}(d,\mathbb{R})$ for some d. The only obvious restriction is that the groups must be linear hyperbolic groups, or perhaps almost linear if the group is allowed to have torsion. Another potential obstruction is that their geodesic flows must be "metric Anosov." However, it is unknown whether there are linear hyperbolic groups whose geodesic flows fail to be metric Anosov. In fact, we know of no example of a linear hyperbolic group which does not admit an Anosov representation.

On the other hand, if you restrict your Lie group one can sometimes characterize, or at least place restrictions, on which groups admit Anosov representations. For example, every Anosov representation into $\mathsf{SL}(2,\mathbb{R})$ is Fuchsian, so the only (torsion-free) groups admitting Anosov representations into $\mathsf{SL}(2,\mathbb{R})$ are free groups and surface groups. Similarly, every Anosov representation with image in $\mathsf{SO}_0(3,1) \subset \mathsf{SL}(4,\mathbb{R})$ is convex cocompact, so the Geometrization Theorem gives a topological classification of which torsion-free groups arise.

The only examples we have seen of Anosov representations into $\mathsf{SL}(3,\mathbb{R}) = \mathsf{PSL}(3,\mathbb{R})$ are Benoist representations of surface groups. One can show that surface groups and free groups are the only (torsion-free) groups which arise. Notice that not all representations of surface groups are Benoist groups, since one also has the representations constructed by Barbot [12] by considering the direct sum of a Fuchsian representation into $\mathsf{SL}(2,\mathbb{R})$ and the trivial representation, and its deformations, see Corollary 32.5.

Theorem 48.1. (Canary-Tsouvalas [60, Theorem 1.1]) *If $\rho : \Gamma \to \mathsf{SL}(3, \mathbb{R})$ is P_1-Anosov and Γ is torsion-free, then Γ is either a free group or a surface group.*

We will bring in a few outside results in the proof, but we may try to fill these in later.

Proof. Let $\Gamma = \Gamma_1 * \cdots * \Gamma_r * F_s$ be the maximal free product decomposition of Γ where each Γ_i is freely indecomposable (i.e. does not split as a non-trivial free product) and not cyclic. If Γ is not a free group, then $r \geq 1$. It is known that Γ_1 is quasiconvex in Γ (see [37, Prop. 1.2]), so $\rho|_{\Gamma_1}$ is projective Anosov (see [57, Lemma 2.3]).

First suppose that $\rho|_{\Gamma_1}$ is irreducible. If Γ_1 is not a surface group, then a result of Zimmer [232], see Theorem 41.2, implies that $\rho(\Gamma_1)$ acts convex cocompactly on a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^3)$. Then Γ_1 is isomorphic to the fundamental group of the surface $\Omega/\rho(\Gamma_1)$, so Γ_1 is either a surface group or a free group. Since Γ_1 is freely indecomposable and not cyclic, it must be a surface group, so $\rho(\Gamma_1)$ acts cocompactly on Ω and $\rho|_{\Gamma_1}$ is a Benoist representation.

Now suppose that Γ_1 has infinite index in Γ (i.e. suppose that there is more than one factor in the free decomposition above). Then $\partial \Gamma_1$ is a proper subset of $\partial \Gamma$. Let $z \in \partial \Gamma - \partial \Gamma_1$. Notice that if $\xi_\rho(z) \in \Omega$, then $\theta_\rho(z)$ must intersect $\partial\Omega = \xi_\rho(\partial\Gamma_1)$, which would contradict the transversality of ξ_ρ and θ_ρ. On the other hand, $\partial\Omega$ is C^1 and Ω is strictly convex, so every point in $\mathbb{RP}^2 - \Omega$ lies in some tangent line to $\partial\Omega$. Therefore,

$$\mathbb{RP}^2 - \Omega = \bigcup_{x \in \partial\Gamma_1} \theta_\rho(x)$$

so, again by transversality, $\xi_\rho(z)$ cannot lie in $\mathbb{RP}^2 - \Omega$. Therefore, $\Gamma = \Gamma_1$ is a surface group.

If $\rho|_{\Gamma_1}$ is reducible, then one may check that there exists a projective Anosov representation of Γ_1 into $\mathsf{SL}^\pm(W)$ where W is a proper subspace of $\mathbb{R}^3$, see [60, Corollary 2.5]. However, every torsion-free discrete subgroup of $\mathsf{SL}^\pm(W)$ is either a free group or a surface group if W is one or two dimensional. Since Γ_1 is freely indecomposable, it is a surface group. If $\Gamma \neq \Gamma_1$, we again choose $z \in \partial\Gamma - \partial\Gamma_1$. Then $\xi_\rho(\partial\Gamma)$ lies in the affine chart $A = \mathbb{RP}^2 - \theta_\rho(z)$ and $\rho(\Gamma_1)$ preserves the convex hull of $\xi_\rho(\partial\Gamma_1)$ in A, so is a Benoist representation. Since Benoist representations are irreducible, see Proposition 18.4, we must actually be in the previous case. This completes the proof. $\square$

One may also characterize groups which admit projective Anosov representations into $\mathsf{SL}(4, \mathbb{R})$. All these groups also admit P_2-Anosov representations into $\mathsf{SL}(4, \mathbb{R})$, and it is conjectured that these are the only such groups. However, only surface groups and free groups admit representations into $\mathsf{SL}(4, \mathbb{R})$ which are Borel Anosov, i.e. both P_1-Anosov and P_2-Anosov, see [60, Theoremv1.6].

Theorem 48.2. (Canary-Tsouvalas [60]) *If Γ is torsion-free and $\rho : \Gamma \to \mathsf{SL}(4, \mathbb{R})$ is projective Anosov, then Γ is isomorphic to a convex cocompact subgroup of $\mathsf{SO}_0(3, 1)$. In particular, Γ is the fundamental group of a compact hyperbolizable 3-manifold.*

Moreover, only surface groups and free groups admit representations into $\mathsf{SL}(4, \mathbb{R})$ which are Borel Anosov, i.e. both P_1-Anosov and P_2-Anosov.

Theorem 48.3. (Canary-Tsouvalas [60]) *If Γ is torsion-free and $\rho : \Gamma \to \mathsf{SL}(4, \mathbb{R})$ is Borel Anosov, then Γ is either a free group or a surface group.*

In general, we can place restrictions on groups admitting P_k-Anosov representations into $\mathsf{SL}(d, \mathbb{R})$ which depend on k and d, in terms of the cohomological dimension of the group.

Recall that Γ has **cohomological dimension** m, if m is the minimal dimension so that if R is any $\mathbb{Z}\Gamma$-module, then $H^r(\Gamma, R) = 0$ if $r > m$. For example, if X is a closed n-manifold with contractible universal cover, then $\pi_1(M)$ has cohomological dimension n. More generally, if X is a finite CW-complex of dimension n, with contractible universal cover so that $H_n(X, R) \neq 0$ for some R (usually $\mathbb{Z}$ or $\mathbb{Z}_2$), then $\pi_1(X)$ has cohomological dimension n. Notice that if Γ is torsion-free and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is Benoist, then Γ has cohomological dimension $d - 1$. (A group with non-trivial torsion has infinite cohomological dimension, so in this case one considers virtual cohomological dimension which registers the cohomological dimension of finite index torsion-free subgroups, if they exist.)

We will use Bestvina and Mess' characterization of the cohomological dimension of a hyperbolic group, in terms of the topological dimension of its Gromov boundary. Their result generalizes the fact that the Gromov boundary of the fundamental group of a closed negatively curved n-manifold is S^{n-1}.

Theorem 48.4. (Bestvina-Mess [29]) *If a torsion-free hyperbolic group Γ has cohomological dimension n, then its Gromov boundary $\partial\Gamma$ has topological dimension $n - 1$.*

Canary and Tsouvalas obtain the following restrictions:

Theorem 48.5. (Canary-Tsouvalas [60]) *Suppose that Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is P_k-Anosov.*

(1) *If (d, k) is not $(2, 1)$, $(4, 2)$, $(8, 4)$ or $(16, 8)$, then Γ has cohomological dimension at most $d - k$.*

(2) *If (d, k) is $(2, 1)$, $(4, 2)$, $(8, 4)$ or $(16, 8)$, then Γ has cohomological dimension at most $k + 1 = d - k + 1$. Moreover, if Γ has cohomological dimension $k + 1$, then $\partial\Gamma$ is homeomorphic to S^k and, if $(d, k) \neq (2, 1)$, ρ is not projective Anosov.*

Remarks: The exceptional values of (d, k) are associated to the four Hopf fibrations and one may produce examples from lattices in $\mathsf{SL}(2, \mathbb{R})$, $\mathsf{SL}(2, \mathbb{C}) \subset \mathsf{SL}(4, \mathbb{R})$, $\mathsf{SL}(2, \mathcal{Q}) \subset \mathsf{SL}(8, \mathbb{R})$. and $\mathsf{SL}(2, \mathcal{O}) \subset \mathsf{SL}(16, \mathbb{R})$ where $\mathcal{H}$ is the quaternions and $\mathcal{O}$ is the octonions (see [60, Section 10]).

Proof. We may assume that $d \geq 4$, since we have already classified Anosov representations into $\mathsf{SL}(2, \mathbb{R})$ and $\mathsf{SL}(3, \mathbb{R})$.

Suppose that ρ is P_k-Anosov and has limit maps $\xi_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\theta_\rho : \partial\Gamma \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$. Let m be the topological dimension of $\partial\Gamma$. Fix $x_0 \in \partial\Gamma$ and a $(d - k + 1)$-plane V in $\mathbb{R}^d$ which contains $\theta_\rho(x_0)$. We define a map

$$F : \partial\Gamma - \{x_0\} \to \mathbb{P}(V - \xi^k(x_0))$$

by letting $F(y)$ be the line which is the intersection of $\xi_\rho(y)$ with V, i.e.

$$F(y) = \xi_\rho(y) \cap V.$$

(Transversality implies that the intersection of $\xi_\rho(y)$ and $\theta_\rho(x_0)$ is trivial if $y \neq x_0$, so the intersection of $\xi_\rho(y)$ with V must be a line.)

If $x \neq y \in \partial\Gamma$, then $\xi_\rho(x)$ and $\xi_\rho(y) \subset \theta_\rho(y)$ have trivial intersection (by transversality). Therefore, F is injective. Moreover, F is proper, since if $\{y_n\}$ is a sequence in $\partial\Gamma - \{x_0\}$

converging to x_0, then, by continuity of limit maps, $\{\xi^k_\rho(y_n)\}$ is converging to $\xi^k_\rho(x_0)$, so $\{F(y_n)\}$ leaves every compact subset of $\mathbb{P}(V - \xi^k(x_0))$. Therefore, F is an embedding. Since $\partial\Gamma - \{x_0\}$ embeds in a $(d-k)$-manifold, $\partial\Gamma$ has topological dimension at most $d-k$. (See Hurewicz-Wallman [119, Theorem III.1] for details on topological dimension, but I think it is reasonable just to take this plausible fact on faith.)

Now suppose that $\partial\Gamma$ has topological dimension exactly $d-k$. Then, $F(\partial\Gamma)$ contains an open subset of $\mathbb{P}(V)$ (see [119, Thm. IV.3/Cor. 1]), so $\partial\Gamma$ has a manifold point. Thus, by a result of Benakli and Kapovich [126, Theorem 4.4], $\partial\Gamma$ is homeomorphic to S^{d-k}. Let

$$E = \bigcup_{x \in \partial\Gamma} S(\xi^k_\rho(x)) \subset S(\mathbb{R}^d)$$

where $S(V)$ is the unit sphere in V. One obtains a fibre bundle $p : E \to \partial\Gamma \cong S^{d-k}$ by letting $p(S(\xi^k_\rho(x))) = x$ and E is a closed manifold of dimension $(d-k) + k - 1 = d-1$. (Notice that, since $\xi_\rho(x)$ and $\xi_\rho(y)$ intersect trivially, $S(\xi_\rho(x))$ and $S(\xi_\rho(y))$ are disjoint if $x \neq y$.) Hence, E is a closed submanifold of $S(\mathbb{R}^d) \cong S^{d-1}$ of dimension $d-1$, which implies that $E = S(\mathbb{R}^d) = S^{d-1}$. However, by the classification of sphere fibrations (Adams [3]), this is only possible if $(d-1, k-1)$ is $(3,1)$, $(7,3)$ or $(15,7)$. Moreover, in these cases, ρ cannot be projective Anosov, since if ρ is projective Anosov, then it admits a P_1-limit map $\xi^1_\rho : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ which lifts to a section $s : \partial\Gamma \to E$ of p, which is impossible (since $p_* \circ s_*$ is the identity map on $H_{d-k}(S^{d-k}) \cong \mathbb{Z}$, while, p_* is the zero map on $H_{d-k}(E)$.)

If $\partial\Gamma$ has topological dimension at most $d-k-1$, then, by Bestvina-Mess Γ has cohomological dimension at most $d-k$. If $\partial\Gamma$ has topological dimension $d-k$, then Γ has cohomological dimension $d-k+1$ and, by the previous paragraph, (d,k) is $(2,1)$, $(4,2)$, $(8,4)$ or $(16,8)$, $\partial\Gamma \cong S^{d-k}$ and ρ is not projective Anosov if (d,k) is $(4,2)$, $(8,4)$ or $(16,8)$.   □

One nearly immediate consequence is that Benoist representations are only P_1-Anosov.

Corollary 48.6. ([60, Corollary 1.4]) *If Γ is torsion-free, $\rho : \Gamma \to \mathsf{SL}(d+1, \mathbb{R})$ is a Benoist representation, and $2 \le k \le \frac{d}{2}$, then ρ is not P_k-Anosov.*

Proof. Notice that Γ has cohomological dimension d. If ρ is P_k-Anosov for $k \ge 2$, then Theorem 48.5 implies that Γ has cohomological dimension at most $d+1-k < d$, which is a contradiction.   □

49. Other Lie groups

I will sit here until dawn tripping the spine
of the stars, a Pythagorean traveller marveling
another numerical scheme, adding to his shoulder
a music not heard but attained
—————————Patti Smith [192]

If G is a semi-simple Lie group (with no compact factors and finite center) and P is a parabolic subgroup of G then there is a notion of a P-**Anosov representation** $\rho : \Gamma \to \mathsf{G}$. In the case that G is $\mathsf{SL}^\pm(d, \mathbb{R})$, $\mathsf{PSL}(d, \mathbb{R})$, or $\mathsf{PGL}(d, \mathbb{R})$, and $P = P_k$, we may use exactly the same definitions as we did for $\mathsf{SL}(d, \mathbb{R})$, and all the characterizations we earlier established go

through nearly word for word. (One can even extend the definition trivially for the reductive Lie group $\mathsf{GL}(d,\mathbb{R})$, but we will limit our discussion to semi-simple Lie groups.)

We will briefly recall some basic definitions from Lie theory which will allow us to define P-Anosov representations, but it would be a mistake to learn Lie theory from me and I encourage you to consult the standard references for a more complete treatment if you haven't seen this material before. I will be pretending that the reader (and the author) are familiar with basic Lie theory and that I am just reminding us of the definitions.

If G is a (real) Lie group, its Lie algebra $\mathfrak{g}$ is its tangent space at the identity $T_{id}\mathsf{G}$. Recall that a (real) **Lie algebra** $\mathfrak{h}$ is a real vector space with an bilinear, anti-commutative Lie bracket $[\cdot,\cdot] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ satisfying the Jacobi identity $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$ for all $x,y,z \in \mathfrak{h}$. The **Adjoint representation** $Ad : \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$ is given by $Ad(g) = D_{id}\Psi_g$ where $\Psi_g : \mathsf{G} \to \mathsf{G}$ is the inner automorphism $\Psi_g(h) = ghg^{-1}$. The derivative of the Adjoint representation is the **adjoint representation** $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ where $\mathfrak{gl}(\mathfrak{g})$ is the Lie Algebra of $\mathsf{GL}(\mathfrak{g})$. One may compute that $ad_x(y) = [x,y]$ if $x,y \in \mathfrak{g}$. One then considers the **Killing form** $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B(x,y) = \mathrm{Tr}(ad(x)ad(y))$ where Tr denotes the trace. We say that G is **semi-simple** if its Killing form is non-degenerate. (If $\mathsf{G} \subset \mathsf{SL}(d,\mathbb{R})$, then $Ad(g)(X) = gXg^{-1}$, $[X,Y] = XY - YX = ad_X(Y)$ and $B(X,Y) = 2d\,\mathrm{Tr}(XY)$.)

Any semisimple Lie algebra $\mathfrak{g}$ decomposes as a direct product of simple Lie algebras

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m.$$

We recall that a Lie algebra is **simple** if it is non-abelian and contains no non-zero proper ideals. (In fact, the existence of a decomposition into simple Lie algebras is equivalent to being semi-simple.) We say that $\mathfrak{g}_i$ is a **compact factor** if the Killing form restricts to a negative definite form on $\mathfrak{g}_i$. (These simple Lie algebras are called compact factors, since they are lie algebras of compact semisimple Lie groups.) We say that G has **no compact factors** if $\mathfrak{g}$ has no compact factors. If G has finite center and no compact factors, we may consider a maximal compact subgroup K and the Killing form descends to a metric of non-positive sectional curvature on the symmetric space $Z = \mathsf{G}/K$. In section 26, we saw how this theory worked out explicitly when $\mathsf{G} = \mathsf{SL}(d,\mathbb{R})$ and $K = \mathsf{SO}(n)$.

If $g_p \in \mathsf{G}$ and $Ad(g_p)$ is diagonalizable (over $\mathbb{R}$), then we let $\mathfrak{p}$ be the sum of the eigenspaces of $Ad(g_p)$ with eigenvalues of modulus at least 1 and let $\mathfrak{p}^{opp}$ be the sum of the eigenspaces of $Ad(g_p)$ with eigenvalues of modulus at most 1. If $\mathfrak{n}$ is the sum of the eigenspaces of $Ad(g_p)$ with eigenvalue of modulus strictly greater than 1 and $\mathfrak{m}$ is the sum of the eigenspaces of $Ad(g_p)$ with eigenvalue of modulus exactly 1, then $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{m}$. Then the normalizers P and P^{opp} of $\mathfrak{p}$ and $\mathfrak{p}^{opp}$ are a pair of **opposite parabolic subgroups**. The quotients G/P and G/P^{opp} are (generalized) flag varieties. Notice that G acts naturally on G/P and G/P^{opp} by $g([h]) = [gh]$. If $[g] \in \mathsf{G}/P$ and $[h] \in \mathsf{G}/P^{opp}$ we say that $[g]$ and $[h]$ are **transverse** if $gPg^{-1} \cap hP^{opp}h^{-1}$ is conjugate to $P \cap P^{opp}$. (In $\mathsf{SL}(d,\mathbb{R})$, the stabilizer P_k of a k-plane is a parabolic subgroup, and P_k^{opp} is the stabilizer of a complementary $(d-k)$-plane.)

Given a representation $\rho : \Gamma \to \mathsf{G}$ of a hyperbolic group and a pair (P, P^{opp}) of opposite parabolic subgroups, we let $E_\rho^P = U(\Gamma) \times \mathsf{G}/P \times \mathsf{G}/P^{opp}$ and form the bundle

$$\widehat{E}_\rho^P = E_\rho^P/\Gamma \to \widehat{U}(\Gamma) = U(\Gamma)/\Gamma$$

where $\gamma(Z,[g],[h]) = (\gamma(Z),[\rho(\gamma)g],[\rho(\gamma)h])$ for all $\gamma \in \Gamma$. The flow

$$\tilde{\psi}_t(z,w,s,[g],[h]) = (z,w,s+t,[g],[h])$$

on E_ρ^P descends to a flow ψ_t on E_ρ^P which lifts the geodesic flow ϕ_t on $\widehat{U}(\Gamma)$.

We define vector bundles V and V^{opp} over E_ρ^P whose fibers over the point $(Z, [g], [h])$ are $T_{[g]}\mathsf{G}/P$ and $T_{[h]}\mathsf{G}/P^{opp}$. The flow $\tilde{\psi}_t$ on E_ρ^P lifts to flows $\tilde{\eta}_t$ and $\tilde{\eta}_t^{opp}$ on V and V^{opp}. The flows $\tilde{\eta}_t$ and $\tilde{\eta}_t^{opp}$ then descend to flows η_t and η_t^{opp} on the quotient vector bundles $\widehat{V} = V/\Gamma$ and $\widehat{V}^{opp} = V^{opp}/\Gamma$ over $\hat{E}_\rho^P$.

Two continuous ρ-equivariant maps $\xi : \partial\Gamma \to \mathsf{G}/P$ and $\theta : \Gamma \to \mathsf{G}/P^{opp}$ are **transverse limit maps** for a representation $\rho : \Gamma \to \mathsf{G}$ if $\xi(z)$ and $\theta(w)$ are transverse whenever $z \neq w \in \partial\Gamma$. A pair of transverse limit maps induce a section $\sigma : U(\Gamma) \to E_\rho^P$ given by $\sigma(z, w, s) = (z, w, s, \xi(z), \theta(w))$. Then σ descends to a flow-invariant section $\hat{\sigma} : \widehat{U}(\Gamma) \to \widehat{E}_\rho^P$. The bundles $\hat{\sigma}^*(\widehat{V})$ and $\hat{\sigma}^*(\widehat{V}^{opp})$ over $\widehat{U}(\Gamma)$ admit flows which lift the geodesic flow on $\widehat{U}(\Gamma)$.

We say that a representation $\rho : \Gamma \to \mathsf{G}$ of a hyperbolic group Γ is P-**Anosov** if it admits transverse limit maps giving rise to a section $\hat{\sigma}$ so that the flow on $\hat{\sigma}^*(\hat{V})$ is expanding and the flow on $\hat{\sigma}^*(\hat{V}^{opp})$ is contracting.

Many of our favorite properties of Anosov representations have direct analogues in the setting of P-Anosov representations, see Labourie [143] and Guichard-Wienhard [109] for details.

Theorem 49.1. *Suppose that G is a semi-simple Lie group with finite center and no compact factors, P is a parabolic subgroup and $\rho : \Gamma \to \mathsf{G}$ is P-Anosov. Then,*

  (1) *ρ has finite kernel.*
  (2) *$\rho(\Gamma)$ is a discrete subgroup of G.*
  (3) *Any orbit map $\tau_\rho : \Gamma \to \mathsf{G}/K$ is a quasi-isometric embedding.*
  (4) *If $\gamma \in \Gamma$ has infinite order, then $\rho(\gamma)$ is P-proximal, i.e. $\rho(\gamma)$ has an attracting fixed point on G/P.*
  (5) *There exists an open neighborhood U of ρ in $\mathrm{Hom}(\Gamma, \mathsf{G})$, so that if $\sigma \in U$, then σ is P-Anosov.*

We will now develop the background needed to clearly state Guichard and Wienhard's result that given G and P there exists an irreducible representation $\tau : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ (for some d), so that $\rho : \Gamma \to \mathsf{G}$ is P-Anosov if and only if $\tau \circ \rho$ is projective Anosov. We have already established this when $\mathsf{G} = \mathsf{SL}(d, \mathbb{R})$ and $P = P_k$, see Theorem 34.3, where τ is the kth exterior power representation. In practice, this allows one to reduce many questions about general Anosov representations to questions about projective Anosov representations.

We first construct the representation τ that we will need.

Lemma 49.2. (Guichard-Wienhard [109, Remark 4.12]) *Suppose that G is a a semi-simple Lie group with finite center and no compact factors, and (P, P^{opp}) is a pair of opposite parabolic subgroups. There exists an irreducible representation $\tau : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ (for some d) so that $\tau(P)$ is the stabilizer, in $\tau(\mathsf{G})$, of a line in $\mathbb{R}^d$ and $\tau(P^{opp})$ is the stabilizer, in $\tau(\mathsf{G})$ of a complementary hyperplane in $\mathbb{R}^d$.*

Proof. Let k be the dimension of $\mathfrak{p}$ and consider the composition of the Adjoint representation with the kth exterior power representation, i.e.

$$\alpha : \mathsf{G} \to \mathsf{sl}(\Lambda^k \mathfrak{g}) \qquad \text{where} \qquad \alpha = \Lambda^k \circ Ad.$$

Since G is semi-simple, the representation α splits into a product $\alpha = \oplus\alpha_i$ of irreducible representations $\alpha_i : \mathsf{G} \to \mathfrak{h}_i$ and $\oplus\mathfrak{h}_i = \Lambda^k \mathfrak{g}$. Since $\mathfrak{p}$ is the attracting k-plane of $Ad(g_p)$,

$\Lambda^k Ad(g_p)$ is proximal and has attracting eigenline $\Lambda^k\mathfrak{p}$. Therefore, $\Lambda^k\mathfrak{p}$ must lie in one of the factors, say $\mathfrak{h}_1$, of our decomposition. We then take $\tau = \alpha_1$. It is immediate that τ is irreducible and that P is the stabilizer, within $\tau(\mathsf{G})$ of the line $\Lambda^k\mathfrak{p}$ in $V = \mathfrak{h}_1$.

Let $\{e_1, \ldots, e_r\}$ be a basis for $\mathfrak{g}$ so that $\{e_1, \ldots, e_k\}$ is a basis for $\mathfrak{p}$, $\{e_{q+1}, \ldots, e_r\}$ is a basis for $\mathfrak{p}^{opp}$. Let Z be the subspace of $\Lambda^k\mathfrak{g}$ spanned by all k-fold wedge products of the form $e_{i_1} \wedge \cdots \wedge e_{i_k}$ which do not lie in $\Lambda^k\mathfrak{p}$. If we consider the dual representation α^*, then Z is the kernel of any non-trivial linear functional in the attracting eigenline of $\alpha^*(g_p)$ and $\tau(P^{opp})$ is its stabilizer. If we let $W = Z \cap V$, then W is a hyperplane in V, is complementary to the line $\Lambda^k\mathfrak{p}$, and $\tau(P^{opp})$ is the stabilizer, in $\tau(\mathsf{G})$, of W. $\qquad\square$

We say that a representation $\rho : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ is a **Plücker representation** for (P, P^{opp}) if it satisfies the conclusions of Lemma 49.2. Notice that given a Plücker representation τ one obtains τ-equivariant maps $\beta_\tau : \mathsf{G}/P \to \mathbb{RP}^{d-1}$ and $\beta_\tau^{opp} : \mathsf{G}/P^{opp} \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$.

We are now ready to formally state Guichard and Wienhard's result.

Theorem 49.3. (Guichard-Wienhard [109, Proposition 4.3], see also [107, Proposition 3.5]) *Suppose that* G *is a a semi-simple Lie group with finite center and no compact factors,* (P, P^{opp}) *is a pair of opposite parabolic subgroups, and* $\tau : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ *is a Plücker representation for* (P, P^{opp}). *Then a representation* $\rho : \Gamma \to \mathsf{G}$ *is P-Anosov if and only if* $\tau \circ \rho$ *is projective Anosov.*

Furthermore, if $\xi : \partial\Gamma \to \mathsf{G}/P$ *and* $\theta : \partial\Gamma \to \mathsf{G}/P^{opp}$ *are the limit maps of* ρ, *then* $\beta_\tau \circ \xi$ *and* $\beta_\tau^{opp} \circ \theta$ *are the limit maps of* $\tau \circ \rho$.

Once one has made this observation, all the characterizations of P_k-Anosov representations into $\mathsf{SL}(d, \mathbb{R})$ we obtained in Chapter 6 have analogees for more general P-Anosov representations. The easiest one to state is the analogue of Theorem 33.1. One may view this as a vast generalization of Proposition 34.3.

Theorem 49.4. (Guichard-Wienhard [109, Theorem 4.11]) *Suppose that* G *is a a semi-simple Lie group with finite center and no compact factors and* (P, P^{opp}) *is a pair of opposite parabolic subgroup. Then a Zariski dense representation* $\rho : \Gamma \to \mathsf{G}$ *is P-Anosov if and only if there exist continuous ρ-equivariant transverse maps* $\xi : \partial\Gamma \to \mathsf{G}/P$ *and* $\theta : \partial\Gamma \to \mathsf{G}/P^{opp}$.

The proof of Theorem 49.4 is simple given Theorem 49.3. We consider a Plücker representation $\tau : \mathsf{G} \to \mathsf{SL}(d, \mathbb{R})$ for (P, P^{opp}) and check that $\beta_\tau \circ \xi$ and $\beta_\tau^{opp} \circ \theta$ are continuous, transverse, $\tau \circ \rho$-equivariant maps. Theorem 33.1 then implies that $\tau \circ \rho$ is projective Anosov and Theorem 49.3 then implies that ρ is P-Anosov.

All the other characterizations have similarly immediate analogues, but we would need to develop more Lie theory to state them. We encourage the energetic reader to consult the original sources for the statements.

50. **Open problems**

> *Though my problems are meaningless,*
> *That don't make them go away*
> ————Neil Young [222]

I hope to eventually produce a slightly fuller list of open problems, but for the moment, here is a sampling of problems that occurred to me while writing this section. So, they reflect mostly

things that I have thought about myself over the last few years. Please suggest other problems which belong here.

Which hyperbolic groups are Anosov.
We currently know very little about the class of linear hyperbolic groups which admit Anosov representations. We begin with elementary existential questions.

Problem 50.1. *Exhibit explicit examples of linear hyperbolic groups which do not admit Anosov representations.*

Question 50.2. *Are there hyperbolic groups which admit discrete faithful linear representations, but do not admit Anosov representations?*

Problem 50.3. *Exhibit explicit examples of linear hyperbolic groups which admit Anosov representations, but do not admit convex cocompact representations into any rank one Lie groups.*

Kapovich [128] asked whether or not there are Gromov hyperbolic right-angled Coxeter groups which do not admit convex cocompact representations into $\mathsf{PO}(n, 1)$ for any n. Since Danciger, Guéritaud and Kassel [83] showed that every Gromov hyperbolic right-angled Coxeter representation admits an Anosov representation, so Kapovich's question suggest that right-angled Coxeter group could provide examples for Problem 50.3.

Question 50.4. *Are there Gromov hyperbolic right-angled Coxeter groups which do not admit convex cocompact representations into any rank one Lie groups?*

Which groups are Borel Anosov.
Sambarino asked whether all Borel Anosov groups are (virtually) either free groups or surface groups.

Question 50.5. (Sambarino) *If Γ is torsion-free and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is Borel Anosov, must Γ be either free or a surface group?*

I was initially very skeptical, but there is now ample evidence for Sambarino's question. In particular, Canary and Tsouvalas [60] answered this question in the affirmative when $d = 3, 4$, and Tsouvalas [206] answered it in the affirmative when d has the form $4n + 2$ for some $n \in \mathbb{N}$.

One might more ambitiously ask whether Borel Anosov representations in even dimensions are Hitchin. (In odd dimensions one may use Barbot's construction [12] to construct counterexamples.)

Question 50.6. *Is every Borel Anosov representation of a surface group into $\mathsf{SL}(4, \mathbb{R})$ Hitchin? What about in $\mathsf{SL}(2n, \mathbb{R})$ for $n \geq 2$?*

Canary-Tsouvalas [60] proved that every Borel Anosov representation of a surface group into $\mathsf{SL}(4, \mathbb{R})$ is irreducible. As an example of how little we know, I have asked several experts on maximal representations the following innocent question, but still don't know the answer.

Question 50.7. *Is it true that a maximal representation $\rho : \pi_1(S) \to \mathsf{Sp}(4, \mathbb{R})$ is Borel Anosov if and only if it is Hitchin?*

Pressure metrics. Very little is known about the geometry of pressure metrics, beyond their existence. We propose a number of natural questions one might start with.

One would like to know that that there are points arbitrarily far away from the Fuchsian locus, so that the Hitchin component isn't just a thin neighborhood of the Fuchsian locus in the pressure metric. One may interpret this question as an indication of how little we know about the Hitchin component, since the answer is not known for even a single example of a Hitchin component. This is the question I find most personally embarassing not to be able to answer.

Question 50.8. *Do there exist points arbitrarily far away the Fuchsian locus in the pressure metric on a Hitchin component?*

Explicit lower bounds for the translation distance of pseudo-Anosov mapping on Teichmüller space, with either the Teichmüller or Weil-Petersson metric, are a subject of intense current interest. In the case of the Weil-Petersson metric on Teichmüller space, Daskalopoulos and Wentworth [86] showed that the translation length of a pseudo-Anosov mapping class is realized exactly along a geodesic axis. For the pressure metric on the Hitchin component it is not even known whether or not there is a lower bound.

Question 50.9. *Is there a lower bound for the translation distance, in the pressure metric, for the action of a pseudo-Anosov mapping class on the Hitchin component? If there is a lower bound, is it achieved? Is there anything resembling an axis for the action of a pseudo-Anosov mapping class?*

It is known that the isometry of Teichmüller space with the Teichmüller (see Royden [182]) or Weil-Petersson metric (see Masur-Wolf [162]) is the extended mapping class group (i.e. $\mathrm{Out}(\pi(S))$).

Problem 50.10. *Is the isometry group of a Hitchin component $\mathcal{H}_d(S)$ generated by the (extended) mapping class group of S and the contragredient involution? More generally, explore whether the relevant outer automorphism group is a finite index subgroup of the isometry group of a higher Teichmüller space with the pressure metric.*

Bridgeman, Canary, and Labourie [47] have showed that any diffeomorphism of $\mathcal{H}_3(S)$ which preserves the presure intersection is an element of the extended mapping class group or the composition of an element in the extended mapping class group with the contragredient involution. A solution of the following problem would thus solve Problem 50.10 when $d = 3$.

Problem 50.11. *Prove that if $g : \mathcal{H}_d(S) \to \mathcal{H}_d(S)$ is an isometry with respect to the pressure metric then g preserves the pressure intersection, i.e. $I(g(\rho), g(\sigma)) = I(\rho, \sigma)$ for all $\rho, \sigma \in \mathcal{H}_3(S)$.*

If S is a closed surface, the Hitchin component $\mathcal{H}_3(S)$ may be interpreted as the space of (marked) projective structures on S. In analogy, with the classical augmented Teichmüller space, one may form an augmented Hitchin component $\widehat{\mathcal{H}}_3(S)$ by appending noded projective structures on S. The augmented Teichmüller space is the metric completion of Teichmüller space with the Weil-Petersson metric (see Masur [161]). By analogy, one might make the following conjecture.

Conjecture 50.12. *The augmented Hitchin component $\widehat{\mathcal{H}}_3(S)$ is homeomorphic to the metric completion of the Hitchin component $\mathcal{H}_3(S)$, with respect to the pressure metric.*

The Weil-Petersson metric is known to be negatively curved (see Tromba [205] or Wolpert [217]), but not bounded away from 0 or $-\infty$ (see Huang [116, 117]). Virtually nothing else is known about curvature properties of pressure metrics on Hitchin components.

Problem 50.13. *Investigate the curvature of pressure metrics on Hitchin components.*

One warning sign here is the work of Pollicott-Sharp [178] (see also Kao [125]) showing that an analogously defined pressure metric on spaces of metric graphs has both positive and negative curvature.

Simple root length. Potrie and Sambarino's proof [179] that the entropy of a Hitchin representation with respect to simple root length is constant, suggest that the simple root length may have properties more reminiscent of classical Teichmüller theory than either the translation length on the symmetric space or the length given by considering the logarithm of the spectral radius (as studied in [45]). Here is one question which makes this suggestion precise.

Question 50.14. *Given a closed surface S and d, does there exist $L > 0$ so that if $\rho \in \mathcal{H}_d(S)$, there exists $\alpha \in \pi_1(S)$ so that*

$$\log\left(\frac{\sigma_1(\rho(\alpha))}{\sigma_2(\rho(\alpha))}\right) \leq L?$$

Bridgeman and Canary have answered this question in the affirmative for Hitchin components of representations of (certain) triangle groups into $\mathsf{SL}(3, \mathbb{R})$. (These deformation spaces are one-dimensional so it is easy to check this computationally.)

Bridgeman, Canary, Labourie and Sambarino [45] constructed a pressure metric on the Hitchin component associated to the first simple root length. Deroin and Tholozan [88] showed that one can embed any Hitchin component $\mathcal{H}_d(S)$ into the Teichmüller space of foliated complex structures on the unit tangent bundle T^1S of S. Sullivan [197] develops the theory of this Teichmüller space and there is a natural associated Weil-Petersson metric. Deroin and Tholozan show that the pull-back of this Weil-Petersson metric is a scalar multiple of the simple root pressure metric constructed in [46].

Problem 50.15. *Use the Teichmüller theory developed by Sullivan [197] to study the simple root pressure metric on a Hitchin component.*

Leftover questions. Here are some leftover questions which don't fit one of the themes above. I wonder whether there are "exotic" Benoist representations of lattices in $\mathsf{SO}_0(n, 1)$, i.e.

Question 50.16. *Does there exists a cocompact lattice $\Gamma \to \mathsf{SO}(n, 1)$ and a Benoist representation $\rho : \Gamma \to \mathsf{SL}(n+1, \mathbb{R})$ which cannot be continuously deformed to the identity representation?*

Note: Sam Ballas, Jeff Danciger, Gye-Seon Lee and Ludo Marquis [11] have recently produced examples when $n = 3$.

In Canary-Tsouvalas [60] we prove that if a torsion-free group admits a projective Anosov representation into $\mathsf{SL}(4, \mathbb{R})$, then it is isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3, 1)$. One expects that a similar result holds for P_2-Anosov representations.

Question 50.17. *If Γ is a torsion-free hyperbolic group and $\rho : \Gamma \to \mathsf{SL}(4, \mathbb{R})$ is P_2-Anosov, must Γ be isomorphic to a convex cocompact subgroup of $\mathsf{PO}(3, 1)$?*

We men pour out our problems
Like we think that they're unique
They cheer when a baby starts to speak
Ought to give 'em a prize for stopping
————————Robbie Fulks [99]

I'm singing this borrowed tune
I took from the Rolling Stones
—————Neil Young [223]

References

[1] H. Abels, G. Margulis and G. Soifer, "Semigroups containing proximal linear maps," *Israel J. Math.* **9**(1995), 1–30.

[2] W. Abikoff, *The Real Analytic Theory of Teichmüller Space*, Lecture Notes in Math. vol. 820, Springer-Verlag, 1980.

[3] J.F. Adams, "On the non-existence of elements of Hopf invariant one," *Ann. of Math.* **72**(1960), 20–104.

[4] I. Adeboye, H. Bray and D. Constantine, "Entropy rigidity and Hilbert volume," *Disc. Contin. Dyn. Syst.* **39**(2019), 1731–1744.

[5] L. Ahlfors, "Some remarks on Teichmüller's space of Riemann surfaces," *Ann. Math.* **74**(1961), 171–191.

[6] R. Alperin, "An elementary account of Selberg's lemma," *L'Enseign. Math.* **33**(1987), 269–273.

[7] J.W. Anderson and A. Rocha, "Analyticity of Hausdorff dimension of limit sets of Kleinian groups," *Ann. Acad. Sci. Fenn.* **22**(1997), 349–364.

[8] D. Anosov, "Geodesic flows on closed Riemannian manifolds with negative curvature," *Proc. Steklov Inst. Math.* **90**(1969), 1–235.

[9] B. Apanasov, "Nontriviality of Teichmüller space for Kleinian groups in space," in *Riemann surfaces and related topics*, Princeton University Press, 1981, 21–31.

[10] S. Ballas, J. Danciger and G.-S. Lee, "Convex projective structures on non-hyperbolic three-manifolds," *Geom. Top.* **22**(2018), 1593–1646.

[11] S. Ballas, J. Danciger, G.-S. Lee, and L. Marquis, "Exotic real projective Dehn surgery space," in preparation.

[12] T. Barbot, "Three-dimensional Anosov flag manifolds, " *Geom. Top.* **14**(2010), 153–191.

[13] T. Barthelmé, L. Marquis and A. Zimmer, "Entropy rigidity of Hilbert and Riemannian metrics," *I.M.R.N.* **22**(2017), 6841–6866.

[14] M. Batty and P. Papasoglu, *Notes on Hyperbolic and Automatic Groups*, 2003, available at: https://www.math.ucdavis.edu/~kapovich/280-2009/hyplectures_papasoglu.pdf

[15] A. Beardon and B. Maskit, "Limit points of Kleinian groups and finite sided fundamental polyhedra," *Acta Math.* **132**(1974), 1–12.

[16] Y. Benoist, "Propriétés asymptotiques des groupes linéaires," *Geom. Funct. Anal.* **7**(1997), 1–47.

[17] Y. Benoist, "Propriétés asymptotiques des groupes linéaires. II," in *Analysis on homogeneous spaces and representation theory of Lie groups*, Math. Soc. Japan, 2000, 33—48.

[18] Y. Benoist, "Automorphismes des cônes convexes," *Invent. Math.* **141**(2000), 149–193.

[19] Y. Benoist, "Convexes hyperboliques et fonctions quasisymétriques," *Publ. Math. I.H.E.S.* **97**(2003), 181–237.

[20] Y. Benoist, "Convexes divisibles I," in *Algebraic groups and arithmetic*, Tata Inst. Fund. Res. Stud. Math. **17**(2004), 339–374.

[21] Y. Benoist, "Convexes divisible II," *Duke Math. J.* **120**(2003), 97–120.

[22] Y. Benoist, "Convexes divisible III," *Ann. Sci. de l'E.N.S.* **38**(2005), 793–832.

[23] Y. Benoist, "Convexes divisible IV: structure du bord en dimension 3," *Invent. Math.* **164**(2006), 249–278.

[24] Y. Benoist, "Convexes hyperboliques et quasiisométries," *Geom. Ded.* **122**(2006), 109–134.

[25] Y. Benoist and J.F. Quint, *Random Walks on Reductive Groups*, Springer-Verlag, 2016.

[26] J.-P. Benzecri, "Sur les variétés localement affines et localement projectives," *Bull. de la S.M.F.* **88**(1960), 229–332.

[27] L. Bers, "Quasiconformal mappings and Teichmüller's theorem," in *Analytic functions*, Princeton University Press, 1960, 89–119.

[28] L. Bers, "Spaces of Kleinian groups," in *Maryland conference in Several Complex Variables I*, Springer-Verlag Lecture Notes in Math, No. 155(1970), 9–34.

[29] M. Bestvina and G. Mess, "The boundary of negatively curved groups," *J. Amer. Math. Soc.* **4**(1991), 469–481.

[30] M. Bobb, "Codimension-1 simplices in divisible convex domains," preprint, https://arxiv.org/abs/2001.11096

[31] J. Bochi and N. Gourmelon, "Some characterizations of domination," *Math. Z.* **263**(2009), 221–231.

[32] J. Bochi, R. Potrie and A. Sambarino, "Anosov representations and dominated splittings," *J.E.M.S.* **21**(2019), 3343–3414.

[33] F. Bonahon, "The geometry of Teichmüller space via geodesic currents," *Invent. Math.* **92**(1988), 139–162.

[34] The Bottle Rockets, "Kerosene," *The Bottle Rockets*, East Side Digital, 1992.

[35] B. Bowditch, "A short proof that a subquadratic isoperimetric inequality implies a linear one," *Mich. Math. J.* **42** (1995), 103–107.

[36] B. Bowditch, "A topological characterization of hyperbolic groups," *J.A.M.S.* **11**(1998), 643–667.

[37] B. Bowditch, "Cut points and canonical splittings of hyperbolic groups," *Acta Math.* **180**(1998), 145–186.

[38] B. Bowditch, "Spaces of geometrically finite representations," *Ann. Acad. Sci. Fenn.* **23**(1998), 389–414.

[39] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, Springer-Verlag, Vol. 470(1975).

[40] N. Brady, "Branched coverings of cubical complexes and subgroups of hyperbolic groups," *Jour. L.M.S.* **60**(1999), 461–480.

[41] H. Bray, "Geodesic flow of nonstrictly convex Hilbert geometries," *Ann. Inst. Four.*, to appear, available at: https://arxiv.org/abs/1710.06938

[42] H. Bray, "Ergodicity of Bowen-Margulis measure for the Benoist 3-manifolds," preprint, https://arxiv.org/abs/1705.08519

[43] H. Bray and D. Constantine, "Entropy rigidity for finite volume strictly convex projective manifolds," preprint, https://arxiv.org/abs/2006.13619

[44] M. Bridgeman, "Hausdorff dimension and the Weil-Petersson extension to quasifuchsian space," *Geom. and Top.* **14**(2010), 799–831.

[45] M. Bridgeman, R. Canary, F. Labourie and A. Sambarino, "The pressure metric for Anosov representations," *G.A.F.A.* **25**(2015), 1089–1179.

[46] M. Bridgeman, R. Canary, F. Labourie and A. Sambarino, "Simple root flows for Hitchin representations," *Geom. Ded.* **38**(2018), 57–86.

[47] M. Bridgeman, R. Canary, and F. Labourie "Simple length rigidity for Hitchin representations," *Advances in Mathematics* **360**(2020).

[48] M. Bridgeman, R. Canary and A. Sambarino, "An introduction to pressure metrics on higher Teichmüller spaces," *Erg. Thy. Dyn. Sys.* **38**(2018), 2001–20035.

[49] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, 1999.

[50] M. Burger, A. Iozzi, A. Parreau, and M. Pozzetti, "The real spectrum compactification of character varieties: characterizations and applications," preprint, available at: https://arxiv.org/abs/2005.00120

[51] M. Burger, "Intersection, the Manhattan curve and Patterson-Sullivan theory in rank 2," *Internat. Math. Res. Notices* **7**(1993), 217–225.

[52] M. Burger, A. Iozzi, and A Wienhard, "Surface group representations with maximal Toledo invariant," *Annals of Math.* **172**(2010), 517–566.

[53] M. Burger, A. Iozzi, and A Wienhard, "Higher Teichmüller spaces: From $\mathsf{SL}(2,\mathbb{R})$to other Lie groups," in *Handbook of Teichmüller Theory IV*, European Mathematical Society Publishing House, 2014, 539–618.

[54] M. Burger and M. Pozzetti, "Maximal representations, non Archimedean Siegel spaces, and buildings," *Geom. Top.* **21**(2017), 3539–3599.

[55] R. Canary, "Dynamics on character varieties: a survey," in *Handbook of Group Actions, Vol. II*, International Press, 2015, 175–200.

[56] R. Canary, D. Epstein, and P. Green, "Notes on notes of Thurston," in *Analytical and Geometric Aspects of Hyperbolic Space*, Cambridge University Press, 1987, 3–92.

[57] R. Canary, M. Lee, A. Sambarino and M. Stover, "Amalgam Anosov representations," *Geom. Top.* **21**(2017), 215–251.

[58] R.D. Canary and P. Storm, "Moduli spaces of hyperbolic 3-manifolds and dynamics on character varieties," *Comm. Math. Helv.*, **88**(2013), 221–251.

[59] R. Canary, M. Stover, and K. Tsouvalas, "New non-linear hyperbolic groups," *Bull. L.M.S.* **51**(2019), 547–553.

[60] R. Canary and K. Tsouvalas, "Topological restrictions on Anosov representations," available at: https://arxiv.org/abs/1904.02002

[61] J. Cannon, "The combinatorial structure of cocompact discrete hyperbolic groups," *Geom. Ded.* **16**(1984), 123–148.

[62] J. Cannon and W. Thurston, "Group invariant Peano curves," *Geom. Top.* **11**(2007), 1315–1355.

[63] Captain Beefheart and the Magic Band, "Ink Mathematics," *Ice Cream for Crow*, Epic Records, 1982.

[64] C. Champetier, "Petite simplification dans les groupes hyperboliques," *Ann. Fac. Sci. Toulouse Math.* **3**(1994), 161–221.

[65] S. Choi, "The Margulis lemm and the thick and thin decomposition for convex real projective surfaces," *Adv. Math.* **122**(1996), 150–191.

[66] S. Choi and W. Goldman, "Convex real projective structures on closed surfaces are closed," *Proc. A.M.S.* **118**(1993), 657–661.

[67] B. Colbois, C. Vernicos and P. Verovic, "Area of ideal triangles and Gromov hyperbolicity," *Illinois J. Math.* **52**(2008), 319–343.

[68] D. Cooper and K. Delp, "The marked length spectrum of a projective manifold or orbifold," *Proc. A.M.S.* **138**(2010), 3361–3376.

[69] D. Constantin, J.-F. Lafont and D. Thompson, "Strong symbolic dynamics for geodesic flow on CAT(−1) spaces and other metric Anosov flows," *J. École Poly. Math.* **7**(2020), 201–231.

[70] D. Cooper, D. Long and S. Tillman, "On convex projective manifolds and cusps," *Adv. Math.* **277**(2015), 181–251.

[71] D. Cooper, D. Long and S. Tillman, "Deforming convex projective manifolds," *Geom. Top.* **22**(2018), 1349–1404.

[72] M. Coornaert, T. Delzant and A. Papadopoulos, *Géométrie et théorie des groupes*, Springer-Verlag Lecture Notes in Mathematics vol. 1441, 1990.

[73] K.Corlette and A. Iozzi, "Limit sets of discrete groups of isometries of exotic hyperbolic spaces," *Trans. A.M.S.* **351**(1999), 1507–1530.

[74] M. Crampon, "Entropies of strictly convex projective manifolds." *J. Mod. Dyn.* **3**(2009), 511–547.

[75] M. Crampon, "The geodesic flow of Finsler and Hilbert Geometries," *Handbook of Hilbert Geometry*, European Mathematical Society Publishing House, 2014, 161–206.

[76] M. Crampon and L. Marquis, "Un lemme de Kazhdan-Margulis-Zassenhaus pour les géométries de Hilbert," *Ann. Blaise Pascal* **20**(2013), 363–376.

[77] M. Crampon and L. Marquis, "Finitude géométrique en géométrie de Hilbert," *Ann. Inst. Four.* **64**(2014), 2299–2377.

[78] M. Crampon and L. Marquis, "Le flot géodésique des quotients géométriquement finis des géométries de Hilbert," *Pac. J. Math.* **268**(2014), 313–369.

[79] M. Culler, "Lifting representations to covering groups," *Adv. Math.* **59**(1986), 64–70.

[80] F. Dahmani and V. Guirardel, "The isomorphism problem for all hyperbolic groups," *G.A.F.A.* **21**(2011), 223–300.

[81] F. Dal'Bo and I. Kim, "A criterion of conjugacy for Zariski dense subgroups," *Comptes Rendus Math.* **330** (2000), 647–650.

[82] F. Dal'Bo and I. Kim, "Marked length rigidity for symmetric spaces," *Comm. Math. Helv.* **77**(2002), 399–407.

[83] J. Danciger, F. Guéritaud and F. Kassel, "Convex cocompactness in pseudo-Riemannian symmetric spaces," *Geom. Ded.* **192**(2018), 87–126.

[84] J. Danciger, F. Guéritaud and F. Kassel, "Convex cocompact actions in real projective geometry," preprint, available at: https://arxiv.org/abs/1704.08711

[85] J. Danciger, F. Guéritaud, F. Kassel, G.-S. Lee and L. Marquis, "Convex cocompactness for Coxeter groups," in preparation.

[86] G. Daskalapoulos and R. Wentworth, "Classification of Weil-Petersson isometries," *Amer. J. Math.* **125**(2003), 941–975.

[87] T. Delzant, O. Guichard, F. Labourie, and S. Mozes, "Well displacing representations and orbit maps," in *Geometry, Rigidity, and Group Actions*, University of Chicago Press, 2011, 494–514.

[88] B. Deroin and N. Tholozan, in preparation.

[89] Drive By Truckers, "Righteous path," *Brighter than Creation's Dark*, New West Records, 2008.

[90] Drive By Truckers, "Perfect Timing," *Brighter than Creation's Dark*, New West Records, 2008.

[91] C. Drutu and M. Kapovich, *Geometric Group Theory*, American Mathematical Society, 2018.

[92] B. Dylan, "With God on our side," *The times they are a-changin'*, Columbia Records, 1964.

[93] B. Dylan, "Tangled up in blue," *Blood on the tracks*, Columbia Records, 1975.

[94] B. Dylan, "Standing in the doorway," *Time out of mind*, Columbia Records, 1997.

[95] S. Edwards, M. Lee and H. Oh, "Anosov groups: local mixing, counting, and equidistribution," preprint, available at: https://arxiv.org/abs/2003.14277

[96] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton University Press, 2011.

[97] V. Fock and A. Goncharov, "Moduli spaces of local systems and higher Teichmüller theory," *Publ. Math. I.H.E.S.* **103**(2006), 1–211.

[98] T. Foertsch and A. Karlsson, "Hilbert metrics and Minkowski norms," *J. Geom.* **83**(2005), 22–31.

[99] R. Fulks, "Fare thee well, Carolina gals," *Upland Stories*, Bloodshot Records, 2016.

[100] F. Gehring and G. Martin, "Discrete quasiconformal groups I," *Proc. L.M.S.* **55**(1987), 331–358.

[101] E. Ghys and P. de la Harpe, *Sur les Groupes Hyperboliques d'apres Mikhael Gromov*, Birkhäuser, 1990.

[102] W. Goldman, "Topological components of spaces of representations," *Invent. Math.* **93**(1988), 557–607.

[103] W. Goldman, "The symplectic nature of fundamental groups of surfaces," *Adv. Math.* **54**(1984), 200-225.

[104] W. Goldman, "Convex projective structures on compact surfaces," *J. Diff. Geom.* **31**(1990), 791–845.

[105] W. Goldman, "Mapping class group dynamics on surface group representations," in *Problems on mapping class groups and related topics*, *A.M.S. Proc. Symp.. Pure Math.* **74**(2006), 189–214.

[106] M. Gromov, "Hyperbolic groups," in *Essays in Group Theory*, MSRI Publ. 8 (1987), 75–263.

[107] F. Guéritaud, O. Guichard, F. Kassel and A. Wienhard, "Anosov representations and proper actions," *Geom. Top.* **21**(2017), 485–584.

[108] F. Guéritaud, O. Guichard, F. Kassel and A. Wienhard, "Tameness of Riemannian locally symmetric spaces arising from Anosov representations," preprint, available at: https://arxiv.org/abs/1508.04759

[109] O. Guichard and A. Wienhard, "Anosov representations: Domains of discontinuity and applications," *Invent. Math.* **190**(2012), 357–438.

[110] M. Haggard, "The roots of my raising," *The Roots of My Raising*, Capitol Records, 1976.

[111] P. de la Harpe, "On Hilbert's metric for simplices," in *Geometric Group Theory, vol 1 (Sussex 1991)*, Cambridge University Press, 1993, 97–119.

[112] B. Hasselblatt and A. Katok, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.

[113] J. Hendrix, "If 6 was 9," *Axis: Bold as Love*, Reprise Records, 1967.

[114] N. Hitchin, "Lie groups and Teichmüller space," *Topology* **31**(1992), 449–473.

[115] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant manifolds*, Springer-Verlag Lecture Notes in Mathematics, Vol. 583, 1977.

[116] Z. Huang, "Asymptotic flatness of the Weil-Petersson metric on Teichmüller space," *Geom. Ded.* **110**(2005), 81–102.

[117] Z. Huang, "On Asymptotic Weil-Petersson geometry of Teichmüller space of Riemann surfaces," *Asian J. Math.* **11**(2007), 459–485.

[118] I. Hunter, "Shades Off," *Ian Hunter*, Columbia Records, 1975.

[119] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1941.

[120] M. Islam and A. Zimmer, "A flat torus theorem for convex co-compact actions of projective linear groups," preprint, https://arxiv.org/abs/1907.03277

[121] M. Islam and A. Zimmer, "Convex co-compact actions of relatively hyperbolic groups," preprint, https://arxiv.org/abs/1910.08885

[122] Jefferson Airplane, "Lather," *Crown of Creation*, RCA Victor, 1968.

[123] W. Jenning and W. Nelson, "Mamas don't let your babies grow up to be cowboys," *Waylon and Willie*, RCA Victor, 1978.

[124] D. Johnson and J. Millson, "Deformation spaces associated to compact hyperbolic manifolds," in *Discrete groups in geometry and analysis (New Haven, Conn., 1984*, Progr. Math. vol. 67 (1987), Birkhäuser 48–106.

[125] L. Kao, "Pressure type metrics on spaces of metric graphs," *Geom. Ded.* **187**(2017), 151–177.

[126] I. Kapovich and N. Benakli "Boundaries of hyperbolic groups," in *Combinatorial and geometric group theory*, American Mathematical Society, 2002, 39–93.

[127] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Birkhäuser, 2000.

[128] M. Kapovich, "Representations of polygons of finite groups," *Geom. Top.* **9**(2005), 1915–1951.

[129] M. Kapovich, "Convex projective structures on Gromov-Thurston manifolds," *Geom. Top.* **11**(2007), 1777–1830.

[130] M. Kapovich and B. Leeb, "Finsler bordifications of symmetric and certain locally symmetric spaces," *Geom. Top.* **22**(2018), 2533–2646.

[131] M. Kapovich and B. Leeb, "Relativizing characterizations of Anosov subgroups, I," preprint, available at: https://arxiv.org/abs/1807.00160

[132] M. Kapovich, B. Leeb and J. Porti, "Anosov subgroups: Dynamical and geometric characterizations," *Eur. Math. J.* **3**(2017), 808–898.

[133] M. Kapovich, B. Leeb and J. Porti, "A Morse Lemma for quasigeodesics in symmetric spaces and Euclidean buildings," *Geom. Top* **22**(2018), 2533–2646.

[134] M. Kapovich, B. Leeb and J. Porti, "Morse actions of discrete groups on symmetric space," preprint, available at: https://arxiv.org/abs/1403.7671

[135] A. Karlsson and G. Noskov, "The Hilbert metric and Gromov hyperbolicity," *L'Enseignement Mathématique* **48**(2002), 73–89.

[136] F. Kassel and R. Potrie, "Eigenvalue gaps for hyperbolic groups and semigroups," preprint, available at: https://arxiv.org/abs/2002.07015

[137] D. Kay, "The Ptolemaic inequality in Hilbert geometries," *Pac. J. Math.* **21**(1967), 293–301.

[138] D. Kazhdan and G. Margulis, "A proof of Selberg's conjecture," *Mat. Sb.* **75**(1968), 163–168.

[139] I. Kim, "Rigidity and deformation spaces of strictly convex real projective structures on compact manifolds," *J. Diff. Geom.* **58**(2001), 189–218.

[140] B. Kleiner and B. Leeb, "Rigidity of invariant convex sets in symmetric spaces," *Invent. Math.* **163**(2006), 657–676.

[141] J.-L. Koszul, "Déformation des connexions localement plates," *Ann. Inst. Fourier* **18**(1968), 103–114.

[142] C. Kourouniotis, "Deformations of hyperbolic manifolds," *Math. Proc. Camb. Phil. Soc.* **98**(1985), 247–261.

[143] F. Labourie, "Anosov flows, surface groups and curves in projective space," *Invent. Math.* **165**(2006), 51–114.

[144] F. Labourie. "Cross Ratios, Surface Groups, $\mathsf{SL}_n(\mathbb{R})$ and Diffeomorphisms of the Circle," *Publ. Math. de l'I.H.E.S.* **106**(2007), 139–213.

[145] F. Labourie, "Cross ratios, Anosov representations and the energy functional on Teichmüller space," *Ann. E.N.S.* **41**(2008), 439–471.

[146] G,-S. Lee and L. Marquis, "Anti-de Sitter strictly GHC-regular groups which are not lattices," *Trans. AMS* **372**(2019), 153–186.

[147] G.-S. Lee and T. Zhang, "Collar lemma for Hitchin representations," *Geom. Top.* **21**(2017), 2243–2280.

[148] M. Lee, "Dynamics on the PSL(2, $\mathbb{C}$)-character variety of a compression body," *Alg. Geom. Top.*, **14** (2014), 2149–2179.

[149] M. Lee and H.Oh, "Invariant measures for horospherical actions and Anosov groups," preprint, available at: https://arxiv.org/abs/2003.14277

[150] Q. Li, "Teichmüller space is totally geodesic in Goldman space," *Asian J. Math.* **20**(2016), 21–46.

[151] A. Manning, "Topological hboxentropy for geodesic flows," *Ann. of Math.* **110**(1979), 567–573.

[152] J. Marché and M. Wolff, "The modular action on PSL(2, $\mathbb{R}$)-characters in genus 2," *Duke Math. J.* **165**(2016), 371–412.

[153] J. Marché and M. Wolff, " Six-point configurations in the hyperbolic plane and ergodicity of the mapping class group," *Groups, Geom. Dyn.* **13**(2019), 731–766.

[154] A. Marden, "The geometry of finitely generated Kleinian groups," *Ann. of Math.* **99**(1974), 383–462.

[155] L. Marquis, "Espaces des modules des surfaces projectives proprement convexes de volume fini," *Geom. Top.* **14**(2010), 2103–2149.

[156] L. Marquis, "Surface projective proprement convexe de volume fini," *Ann. Inst. Four.* **62**(2012), 325–392.

[157] L. Marquis, "Exemples de variétés projectives strictement convexes de volume fini en dimension quelconque," *L'Enseignement Mathématique* **58**(2012), 3–47.

[158] L. Marquis, "Around groups in Hilbert geometry," in *Handbook of Hilbert Geometry*, European Mathematical Society Publishing House, 2014, 207–261.

[159] B. Martelli, *An Introduction to Geometric Topology*, CreateSpace, 2016, also available at http://people.dm.unipi.it/martelli/Geometric_topology.pdf

[160] G. Martone and T. Zhang, "Positively ratioed representations," *Comm. Math. Helv.* **94**(2019), 273–345.

[161] H. Masur, "Extension of the Weil-Petersson metric to the boundary of Teichmüller space," *Duke Math. J.* **43**(1976), 623–635.

[162] H. Masur and M. Wolf, "The Weil-Petersson isometry group," *Geom. Ded.* **93**(2002), 177–190.

[163] C. McMullen, "Thermodynamics, dimension and the Weil-Petersson metric," *Invent. Math.* **173**(2008), 365–425.

[164] J. McMurtry, "Just us kids," *Just Us Kids*, Lightning Rod Records, 2008.

[165] H. Meschkowski, *Noneuclidean geometry*, Academic Press, 1964.

[166] I. Mineyev, "Flows and joins of metric spaces," *Geom. Top.* **9**(2005), 403–482.

[167] Y. Minsky, "Note on dynamics of $\mathrm{Out}(F_n)$ on $\mathrm{PSL}(2,\mathbb{C})$-characters," *Israel J. Math.*, **193**(2013), 47–70.

[168] M. Morse, "A fundamental class of geodesics on any closed surface of genus greater than one," *Trans. A.M.S.* **26**(1924), 25–60.

[169] G. Mostow, "Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms," *Publ. Math. I.H.E.S.* **34**(1968), 53–104.

[170] X. Nie, "Entropy degeneration of convex projective surfaces," *Conform. Geom. Dyn.* **19**(2015), 318–322.

[171] Old 97s, "Longer than you've been alive," *Most Messed Up*, 2014.

[172] P. Papasoglu, "On the sub-quadratic isoperimetric inequality," in *Geometric Group Theory*, De Gruyter, 1995, 149–158.

[173] J. Parker, *Hyperbolic spaces*, Jyväskylä Lectures in Mathematics **2**, 2008.

[174] A. Parreau, "Compactification déspaces de représentations de groupes de type fini," *Math. Z.* **272**(2012), 51–86.

[175] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamic*, Astérisque, Vol. 187-188 (1990).

[176] M. Pollicott, "Symbolic dynamics for Smale flows," *Am. J. of Math.* **109**(1987), 183–200.

[177] M. Pollicott and R. Sharp, "Length asymptotics in higher Teichmüller theory," *Proc. A.M.S.* **142**(2014), 101–112.

[178] M. Pollicott and R.Sharp, "A Weil-Petersson type metric on spaces of metric graphs," *Geom. Ded.* **172**(2014), 229–244.

[179] R. Potrie and A. Sambarino,"Eigenvalues and entropy of a Hitchin representation", *Invent. Math.* **209**(2017), 885–925.

[180] A. Quas, P. Thieullen, and M. Zarrabi, "Explicit bounds for separation between Oseledets subspaces," *Dyn. Syst.* **34**(2019), 517—560.

[181] J.-F. Quint, "Groupes convexes cocompacts en rang supérieur," *Geom. Ded.* **113**(2005), 1–19.

[182] H. Royden, "Automorphisms and isometries of Teichmüller spaces," in *Advances in the theory of Riemann surfaces*, Annals of Math. Studies, Vol. 66(1970), 369–383.

[183] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, 1978.

[184] D. Ruelle, "Repellers for real analytic maps," *Ergodic Theory Dynamical Systems* **2**(1982), 99–107.

[185] A. Sambarino, "Quantitative properties of convex representations," *Comm. Math. Helv.* **89**(2014), 443–488.

[186] A. Sambarino, "The orbital counting problem for hyperconvex representations," *Ann. Inst. Fourier.* **65**(2015), 1755–1797.

[187] R. Schwartz, "The symmetric space for $\mathsf{SL}(n,\mathbb{R})$," lecture notes, available at https://www.math.brown.edu/~res/M2710M/symm.pdf

[188] Z. Sela, "The isomorphism problem for hyperbolic groups I," *Ann. Math.* **141**(1995), 217–283.

[189] A. Selberg, "On discontinuous groups in higher-dimensional symmetric spaces," in *Contributions to Function Theory*, Tata Institute for Fundamental Research, 1960, 147–164.

[190] A. Sisto, *Lecture notes on Geometric Group Theory*, 2014, available at: https://people.math.ethz.ch/~alsisto/LectureNotesGGT.pdf

[191] P. Smith, "The geometry blinked ruin unimaginable," *Auguries of Innocence*, Ecco books, 2005.

[192] P. Smith, "A Pythagorean traveller," *Auguries of Innocence*, Ecco books, 2005.

[193] P. Smith, "Constantine's Dream," *Banga*, Columbia Records, 2012.

[194] P. Smith, *Year of the Monkey*, Alfred A. Knopf, 2019.

[195] D. Seuss, *One Fish, Two Fish, Red Fish, Blue Fish*, Random House, 1960.

[196] D. Sullivan, "The density at infinity of a discrete group of hyperbolic motions," *Inst. Hautes Études Sci. Publ. Math.* **50**(1979), 171–202.

[197] D. Sullivan, "Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers," in *Topological methods in modern mathematics*, Publish or Perish, 1993, 543–564.

[198] Z. Sun, A. Wienhard and T. Zhang, "Flows on the $\mathsf{PSL}(V)$-Hitchin component," *Geom. Funct. Anal.*, to appear, available at: https://arxiv.org/abs/1709.03580

[199] Z. Sun and T. Zhang, " The Goldman symplectic form on the $\mathsf{PSL}(V)$-Hitchin component," preprint, available at: https://arxiv.org/abs/1709.03589

[200] S. Tapie, "A variation formula for the topological entropy of convex-cocompact manifolds," *Erg. Thy. Dynam. Sys.* **31**(2011), 1849–1864.

[201] N. Tholozan, "Entropy of Hilbert metrics and length spectrum of Hitchin representations in $\mathsf{PSL}(3, \mathbb{R})$," *Duke Math. J.* **166**(2017), 1377–1403.

[202] W. Thurston, *The geometry and topology of 3-manifolds*, lecture notes, Princeton University, 1980, available at: http://library.msri.org/nonmsri/gt3m/

[203] W. Thurston, *Three-dimensional geometry and topology*, Princeton University Press, 1997.

[204] J. Tits, "Free subgroups in linear groups," *J. Algebra* **20**(1972), 250–270.

[205] A. Tromba, "On a natural algebraic affine connection on the space of almost complex structures and the curvature of Teichmüller space with respect to its Weil-Petersson metric," *Manu. Math.* **56**(1986), 475—497.

[206] K. Tsouvalas, "On Borel Anosov representations in even dimensions," available at: https://arxiv.org/abs/1909.13034.

[207] K. Tsouvalas, "Quasi-isometric embeddings inapproximable by Anosov representations," preprint, available at: https://arxiv.org/abs/2002.11782

[208] K. Tsouvalas, "Anosov representations and strongly convex cocompact groups," in preparation.

[209] P. Tukia, "Convergence groups and Gromov's metric hyperbolic spaces," *New Zealand J. Math.* **23**(1994),157–187.

[210] Velvet Underground, "Rock and Roll," *Loaded*, Cotillion Records, 1970.

[211] J. Vey, "Sur les automorphismes affines des ouverts convexes saillants," *Ann. Sc. Norm. Sup. Pisa* **24**(1970), 641–665.

[212] K. Vogtmann, *Geometric Group Theory Course Spring 2014*, available at: https://warwick.ac.uk/fac/sci/maths/people/staff/karen_vogtmann/lectures/

[213] B. Warren, "Birmingham," *The Sick Season*, 2020.

[214] W. Willie, "Country side of life," *Keep on Smilin'*, Capricorn Records, 1974.

[215] A. Wienhard and T. Zhang, "Deforming convex real projective structures," *Geom. Ded.* **192**(2018), 327–360.

[216] L. WIlliams, "Bone of contention," *Good souls, better angels*, Thirty Tigers, 2020.

[217] S. Wolpert, "Noncompleteness of the Weil-Petersson metric for Teichmüller space," *Pac. J. Math.* **61**(1975), 573–577.

[218] S. Wolpert, "Thurston's Riemannian metric for Teichmüller space," *J. Diff. Geom.* **23**(1986), 143–174.

[219] S. Wolpert, *Families or Riemann Surfaces and Weil-Petersson Geometry*, CMBS Regional Conference series, vol. 113, American Mathematical Society, 2010.

[220] Yipes!, "Out in California," *Yipes!*, Millenium Records, 1979.

[221] N. Young, "Ambulance Blues," *On the Beach*, Reprise Records, 1974.

[222] N. Young, "On the Beach," *On the Beach*, Reprise Records, 1974.

[223] N. Young, "Borrowed Tune," *Tonight's the Night*, Reprise Records, 1975.

[224] N. Young, "Tired Eyes," *Tonight's the Night*, Reprise Records, 1975.

[225] N. Young, "Thrasher," *Rust Never Sleeps*, Reprise Records, 1979.

[226] C. Yue, "The ergodic theory of discrete isometry groups on manifolds of variable negative curvature," *Trans. A.M.S.* **348**(1996), 4965–5005.

[227] H. Zassenhaus, "Beweis eines satzes über diskrete gruppen," *Abh. Math. Hamburg* **12**(1937), 289–312.

[228] W. Zevon, "Run Straight Down," *Transverse City*, Virgin Recods, 1989.

[229] W. Zevon, "Transverse City," *Transverse City*, Virgin Recods, 1989.

[230] T. Zhang, "The degeneration of convex $\mathbb{RP}^2$ structures on surfaces," *Proc. L.M.S.* **111**(2015), 967–1012.

[231] F. Zhu, "Relatively dominated representations," preprint, available at: https://arxiv.org/abs/1912.13152.

[232] A. Zimmer, "Projective Anosov representations, convex cocompact actions, and rigidity," *J. Diff. Geom.*, to appear, available at: arXiv:1704.08582.