Euler characteristic of convex projective manifolds

This note strengthens the compact Bochner estimate of [1, Proposition 3.4]. We retain its geometric setup and use the compact identity stated below.

Setup and input. Let M be a closed oriented properly convex projective n-manifold, put $N = n + 1$, and let $E \to M$ be its standard unimodular flat bundle. The Cheng–Yau level hypersurface defines a section σ and a splitting $E = \mathbb{R}\sigma \oplus D^E\sigma(TM)$. Identify a tangent vector Y with $D^E_Y\sigma$. The slice metric G satisfies

$$G(\sigma, \sigma) = N, \qquad G(\sigma, TM) = 0, \qquad G|_{TM} = k.$$

Let $\zeta(\sigma) = 1$, $\zeta(TM) = 0$, and let θ^a annihilate σ and be dual to a local frame e_a of TM. Set $F_p = \Lambda^p E^*$ and $\mathcal{B}^{p,q} = \Lambda^p E^* \otimes \Lambda^q E^*$. The inclusion of covectors annihilating σ gives an isometry

$$J_q : F_p \otimes \Lambda^q T^*M \longrightarrow \mathcal{B}^{p,q}.$$

Metrics on exterior powers are induced by G, and all L^2 inner products use $d\mathrm{vol}_k$. Let D^{tot} be the induced flat connection on both factors of $\mathcal{B}^{p,q}$ and define

$$P_{p,q}u = \sum_a \varepsilon(\theta^a) D^{\mathrm{tot}}_{e_a} u - (p+q)\varepsilon(\zeta)u, \qquad L_{p,q} = P^*_{p,q}P_{p,q} + P_{p-1,q}P^*_{p-1,q}.$$

Exterior multiplication and contraction below act in the first factor. Operators outside the exterior-degree range are zero. For the flat differential d_p of F_p, put $\Delta_{p,q} = d_p d_p^* + d_p^* d_p$. The input from [1, Proposition 3.4] is

$$\Delta_{p,q} = J_q^* L_{p,q} J_q + \left(p - q - \frac{p^2}{N}\right) \mathrm{Id}\,. \tag{1}$$

Proposition 1. *For every $\eta \in \Omega^q(M; F_p)$,*

$$\|d_p\eta\|^2 + \|d_p^*\eta\|^2 \geq \frac{(p-q)(N-p-q)}{N}\|\eta\|^2. \tag{2}$$

Consequently, $H^q(M; F_p) = 0$ if $q < \min\{p, N-p\}$.

Proof. Fix q and regard $P = \bigoplus_p P_{p,q}$ as a graded operator. Set

$$C = \iota(\sigma), \qquad W = \varepsilon(\zeta), \qquad R = W^* = \iota(\sigma/N) = C/N.$$

Since $\zeta^\sharp_G = \sigma/N$, these are pointwise adjoints, and

$$WR + RW = \frac{1}{N}\,\mathrm{Id}\,. \tag{3}$$

We claim that

$$PR + RP = -\frac{q}{N}\,\mathrm{Id}\,. \tag{4}$$

Indeed, on first exterior degree p,

$$CW + WC = \mathrm{Id}, \qquad \sum_a \varepsilon(\theta^a)\iota(e_a) = p\,\mathrm{Id} - WC.$$

Moreover $D^E_{e_a}\sigma = e_a$, so $D^{\mathrm{tot}}_{e_a}(Cu) = \iota(e_a)u + CD^{\mathrm{tot}}_{e_a}u$. Using $C\varepsilon(\theta^a) + \varepsilon(\theta^a)C = 0$, the first-order terms cancel and give

$$\begin{aligned} P_{p-1,q}Cu + CP_{p,q}u &= \sum_a \varepsilon(\theta^a)\iota(e_a)u - (p-1+q)WCu - (p+q)CWu \\ &= (p\,\mathrm{Id} - WC)u - (p-1+q)WCu - (p+q)(\mathrm{Id} - WC)u \\ &= -q\,u. \end{aligned}$$

Dividing by N proves (4).

Pair (4) with u and apply Cauchy–Schwarz:

$$\begin{aligned}\frac{q}{N}\|u\|^2 &= |(Ru, P^*u) + (Pu, Wu)| \\ &\le \big(\|Pu\|^2 + \|P^*u\|^2\big)^{1/2}\big(\|Ru\|^2 + \|Wu\|^2\big)^{1/2}.\end{aligned}$$

By (3), the last squared factor is $\|u\|^2/N$. Thus

$$(L_{p,q}u, u) = \|Pu\|^2 + \|P^*u\|^2 \ge \frac{q^2}{N}\|u\|^2.$$

Apply this to $u = J_q\eta$ in (1). Since J_q is isometric, the resulting coefficient is

$$p - q - \frac{p^2}{N} + \frac{q^2}{N} = \frac{(p-q)(N-p-q)}{N},$$

which proves (2). This coefficient is positive if $q < \min\{p, N-p\}$. Hodge theory for the flat de Rham complex on compact M identifies cohomology with $\ker \Delta_{p,q}$, so the vanishing follows. This use of Hodge theory requires a positive fiber metric, not a parallel one. □

Theorem 2. *Every closed properly convex projective manifold M^{2m} satisfies $(-1)^m\chi(M) \ge 0$. If M is oriented, then*

$$H^q(M; \Lambda^m E^*) = 0 \quad (q \ne m), \qquad \boxed{\binom{2m+1}{m}(-1)^m\chi(M) = \dim_{\mathbb{R}} H^m(M; \Lambda^m E^*).}$$

Proof. First suppose that M is oriented and set $N = 2m+1$. For both $p = m$ and $p = m+1$, the coefficient in (2) is

$$\frac{(m-q)(m+1-q)}{2m+1} > 0 \qquad (q < m).$$

Hence the cohomology of F_m and F_{m+1} vanishes below degree m. A parallel volume form on E identifies $F_m^* \cong F_{m+1}$. Poincaré duality with local coefficients therefore gives

$$H^q(M; F_m) \cong H^{2m-q}(M; F_{m+1})^* = 0 \qquad (q > m).$$

Only degree m remains. For every finite-rank flat bundle, the cellular cochain complex gives $\chi(M; F) = \operatorname{rank}(F)\chi(M)$. Thus

$$\binom{2m+1}{m}\chi(M) = \chi(M; F_m) = (-1)^m \dim_{\mathbb{R}} H^m(M; F_m).$$

This proves the formula and the sign. For nonorientable M, apply the oriented case to its orientation double cover and use $\chi(\widehat{M}) = 2\chi(M)$. □

In particular, $\chi(M) \ge 0$ in dimensions 4ℓ and $\chi(M) \le 0$ in dimensions $4\ell + 2$; the analytic argument is independent of the parity of m. In odd dimensions $\chi(M) = 0$ for every closed manifold.

References

[1] A. Tamburelli, *Euler characteristic and flat cohomology of convex projective 4-manifolds*, arXiv:2610.31085, 2026, Proposition 3.4.