Data una successione

$$\{a_n\}_{n\in\mathbb{N}}$$

la serie ad essa associata è la successione

$$\{S_k\}_{k\in\mathbb{N}}$$

$$S_k = \sum_{n=0}^{k} a_n$$

Serie "globale" è

$$\sum_{n=0}^{\infty} a_n$$

SERIE a TERMINI POSITIVI

SERIE di riferimento

- $$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$ serie ARMONICA generalizzata

SERIE converge $\iff \alpha > 1$
diverge $\iff \alpha \leq 1$

- $\sum_{n=0}^{\infty} b^n$ serie geometrica

SERIE converge $\Leftrightarrow$ $b < 1$
diverge $\Leftrightarrow$ $b \geq 1$

Studio di una serie:

1) confronto con serie armonica

$\Updownarrow$

criterio quoziente o ordine di INFINITESIMO

2) confronto con serie geometrica

$\Updownarrow$

criterio del RAPPORTO
criterio della RADICE

N.B. $\sum a_n$ converge

$\Downarrow$

$\lim_{n\to\infty} a_n = 0$

cond. necessaria

eq.nte :

$\lim_{n\to\infty} a_n = c > 0 \Rightarrow \sum a_n$ DIVERGE

Esercizi

N.B. serie a termini positivi

cioè $a_n \geq 0 \quad \forall n$

$\Downarrow$

$$S_k = \sum_{n=0}^{k} a_n \text{ è crescente}$$

$\Downarrow$

$$\exists \lim_{k \to \infty} S_k$$

$\Downarrow$

$$\sum_{n=0}^{\infty} a_n = \begin{cases} +\infty & \text{DIVERGE} \\ L \in \mathbb{R} & \text{CONVERGE} \end{cases}$$

1. $$\sum_{n=1}^{\infty} \frac{5^n}{n!}$$

$$n! = n \cdot (n-1)(n-2) \cdot \_\_\_ \cdot 3 \cdot 2 \cdot 1$$

$$(n+1)! = (n+1) \cdot (n!)$$

Quindi è utile considerare criterio del rapporto

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

$$a_{n+1} = \frac{5^{n+1}}{(n+1)!} = \frac{5 \cdot 5^n}{(n+1) \cdot n!}$$

$$a_n = \frac{5^n}{n!} \Rightarrow \frac{1}{a_n} = \frac{n!}{5^n}$$

$$\frac{a_{n+1}}{a_n} = \underbrace{\frac{5 \cdot 5^n}{(n+1) \cdot n!}}_{= a_{n+1}} \cdot \underbrace{\frac{n!}{5^n}}_{= \frac{1}{a_n}}$$

QUINDI

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{5}{n+1} = 0$$

$$\frac{5 \cdot \cancel{5^n}}{(n+1) \cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{5^n}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$$

$\Rightarrow$ SERIE CONVERGE

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2) $$\sum_{n=0}^{\infty} \frac{n^3 \cdot \left[\sqrt{2}+1\right]^n}{3^n}$$

ABBIAMO potenze n-ime

$\Rightarrow$ utilizziamo criterio della RADICE

$$\lim_{n\to\infty} \sqrt[n]{a_n} =$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^3 \cdot (\sqrt{2}+1)^n}{3^n}} =$$

$$\lim_{n\to\infty} \underbrace{\sqrt[n]{n^3}}_{\xrightarrow{n\to\infty} 1} \cdot \underbrace{\frac{(\sqrt{2}+1)}{3}}_{= \text{ numero } < 1}$$

perchè $\sqrt[n]{n} \xrightarrow{n\to\infty} 1$

quindi: $p(n)$ POLINOMIO di grado d

$$\sqrt[n]{p(n)} \xrightarrow{n\to\infty} 1$$

CONCLUSIONE:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{\sqrt{2}+1}{3} < 1$$

$\Rightarrow$ SERIE CONVERGE

3 $$\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$$

$$= 0$$

condizione necessaria è verificata

DOMANDA: come tende a 0?

ovvero: ordine di infinitesimo

STRUMENTI: → limite notevole

→ sviluppo di Taylor in $t=0$

limite notevole:

$$\lim_{n\to\infty} \frac{\left(1-\cos\left(\frac{1}{n}\right)\right)}{\frac{1}{n^2}} = \frac{1}{2}$$

$\frac{1}{2} \neq 0, +\infty$

Sviluppo di Taylor di $\cos(t)$ in $t=0$

$$\cos(t) = 1 - \frac{1}{2}t^2 + o(t^2)$$

$$\Downarrow$$

$$1-\cos(t) = \frac{1}{2}t^2 + o(t^2)$$

Pertanto

$$1-\cos\left(\frac{1}{n}\right) \sim \frac{1}{n^2}$$

eq.$^{\text{nte}}$: $1-\cos\left(\frac{1}{n}\right) = \frac{1}{2}\cdot\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$

Conclusione:

$\sum \frac{1}{n^2}$ converge $(2>1)$

QUINDI serie CONVERGE

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4 $\sum_{n=2}^{\infty} \frac{2+\mathrm{sen}(n^4)+e^{-n}}{2\cdot n^2-2}$

$a_n = \frac{2+\mathrm{sen}(n^4)+e^{-n}}{2\cdot n^2-2}$

numeratore: $1 \leq 2+\mathrm{sen}(n^4)+e^{-n} \leq 4$

denominatore: $\longrightarrow +\infty$

QUINDI

$$\frac{1}{2\cdot n^2-2} \leq a_n \leq \frac{4}{2\cdot n^2-2}$$

La serie converge

$$\Updownarrow$$

$$\sum \frac{1}{2n^2-2} \text{ converge}$$

$$\frac{1}{2n^2-2} \sim \frac{1}{n^2}$$

eq.$^{\text{nte}}$ $$\lim_{n\to\infty} \frac{\frac{1}{2n^2-2}}{\frac{1}{n^2}} = \frac{1}{2}$$

$\sum \frac{1}{n^2}$ converge

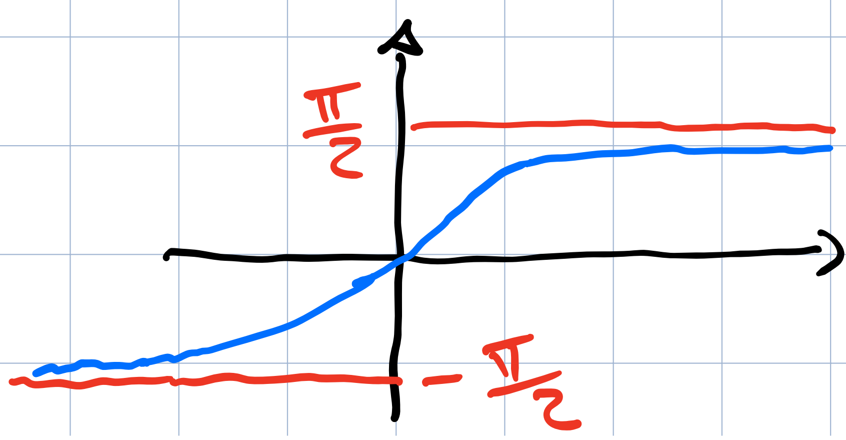
QUINDI SERIE converge

5 $$\sum_{n=1}^{\infty} \arctan\left(\frac{n^2+n+9}{2\sqrt{n+5}}\right) \cdot \arctan\left(\frac{2\sqrt{n+5}}{n^2+n+9}\right)$$

$$a_n = \arctan\left(\frac{n^2+n+9}{2\sqrt{n+5}}\right) \cdot \arctan\left(\frac{2\sqrt{n+5}}{n^2+n+9}\right)$$

Ricordo:

$\arctan(x)$



$$\lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}$$

In $x=0$ sviluppo di Taylor

$$\arctan(x) = x + o(x)$$

$$a_n = \underbrace{\arctan\left(\frac{n^2+n+9}{\underbrace{2\sqrt{n+5}}_{\to +\infty}}\right)}_{\to \frac{\pi}{2}} \cdot \underbrace{\arctan\left(\underbrace{\frac{2\sqrt{n+5}}{n^2+n+9}}_{\to 0}\right)}_{\to 0}$$

$\lim = \frac{\pi}{2} \Rightarrow$ definitivamente

preso $\varepsilon = 1$ $\exists\, n_0$ t.c.

$\forall n \geq n_0$ fissato

$$\frac{\pi}{2} - 1 \leq \arctan\left(\frac{n^2+n+9}{2\sqrt{n+5}}\right)$$

$$\leq \frac{\pi}{2} + 1$$

CIOE':

$$\left(\frac{\pi}{2} - 1\right) \cdot \arctan\left(\frac{2\sqrt{n+5}}{n^2+n+9}\right) \leq a_n$$

$$a_n \leq \left(\frac{\pi}{2} + 1\right) \cdot \arctan\left(\frac{2\sqrt{n+5}}{n^2+n+9}\right)$$

Serie converge $\Longleftrightarrow$

$$\sum \arctan\left(\frac{2\sqrt{n+5}}{n^2+n+9}\right) \text{ converge}$$

Taylor o limite notevole:

$$\arctan\left(\frac{2\sqrt{n+5}}{n^2+n+9}\right) \sim \frac{2\sqrt{n+5}}{n^2+n+9}$$

POICHE' $\arctan(x) = x + o(x)$

in $x=0$

$$\frac{2\sqrt{n+5}}{n^2+n+9} \qquad \sim n^{\frac{1}{2}} \quad \sim n^2$$

$$\Rightarrow \frac{2\sqrt{n+5}}{n^2+n+9} \sim \frac{n^{\frac{1}{2}}}{n^2} = \frac{1}{n^{3/2}}$$

Eq.^{nte}

$$\lim_{n\to\infty} \frac{\left(\frac{2\sqrt{n+5}}{n^2+n+9}\right)}{\left(\frac{1}{n^{3/2}}\right)} = 2 \neq 0, +\infty$$

QUINDI la serie ha lo stesso comportamento di

$$\sum \frac{1}{n^{3/2}}$$

$\frac{3}{2} > 1 \implies$ SERIE CONVERGE

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6) $$\sum_{n=2}^{\infty} \frac{1}{\sqrt[10]{n} \cdot \left(e^{\frac{1}{\ln(n)}} - 1\right)}$$

$$a_n = \frac{1}{\sqrt[10]{n} \cdot \left(e^{\frac{1}{\ln(n)}} - 1\right)}$$

$$\frac{1}{\ln(n)} \longrightarrow 0 \quad \text{per} \quad n \to +\infty$$

Sviluppo di Taylor di e^x in $x=0$

$$e^x = 1 + x + o(x)$$

$$\Rightarrow \quad e^{\frac{1}{\ln(n)}} = 1 + \frac{1}{\ln(n)} + o\left(\frac{1}{\ln(n)}\right)$$

$$\Rightarrow \quad e^{\frac{1}{\ln(n)}} - 1 \sim \frac{1}{\ln(n)}$$

$$a_n \sim \frac{1}{\sqrt[10]{n}} \cdot \frac{1}{\frac{1}{\ln(n)}}$$

$$= \frac{\ln(n)}{\sqrt[10]{n}}$$

$$e_n \longrightarrow 0$$

$$e_n \geq \frac{1}{\sqrt[10]{n}} \qquad \forall\, n \geq 3$$

$$\sum e_n \geq \sum \frac{1}{\sqrt[10]{n}}$$

$$\forall\, n \geq 3$$

$$\frac{1}{\sqrt[10]{n}} = \frac{1}{n^{\frac{1}{10}}}$$

$$\frac{1}{10} < 1 \Rightarrow \text{serie } \sum \frac{1}{n^{\frac{1}{10}}}$$

DIVERGE

$$\Rightarrow \sum e_n \geq \sum \frac{1}{n^{\frac{1}{10}}}$$

DIVERGE

$\Rightarrow$ SERIE DIVERGE

7 $$\sum_{n=1}^{\infty} \frac{n!\, n^2}{n^n}$$

criterio rapporto:

$$\frac{a_{n+1}}{a_n} = \overbrace{\left(\frac{(n+1)!\,(n+1)^2}{(n+1)^{n+1}}\right)}^{a_{n+1}} \cdot \underbrace{\left(\frac{n^n}{n!\, n^2}\right)}_{=\frac{1}{a_n}}$$

Poiché $(n+1)! = (n+1)\cdot(n!)$

$$(n+1)^{n+1} = (n+1)^n \cdot (n+1)$$

$\frac{a_{n+1}}{a_n}$ DIVENTA

$$\frac{(n+1)^3}{(n+1)\cdot n^2} \cdot \left(\frac{n}{n+1}\right)^n =$$

$$= \underbrace{\frac{(n+1)^3}{(n+1)\,n^2}}_{\substack{\downarrow\, n\to\infty \\ 1}} \cdot \underbrace{\left(1 - \frac{1}{n+1}\right)^n}_{\substack{\downarrow\, n\to\infty \\ \frac{1}{e}}}$$

Poiché $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$

Infatti:

$$\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n} =$$

$$\lim_{n\to\infty}\underbrace{\left(1-\frac{1}{n+1}\right)^{n+1}}_{\downarrow \; 1/e}\cdot\underbrace{\left(1-\frac{1}{n+1}\right)^{-1}}_{\downarrow \; 1}$$

Formule generale:

$$\lim_{n\to\infty}\left(1+\frac{\beta}{n}\right)^{n} = e^{\beta}$$

CONCLUSIONE:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$

$\Rightarrow$ SERIE converge

ESERCIZIO x CASA:

Studiare al variare di $\alpha \in \mathbb{R}$ la convergenza della serie

$$\sum_{n=1}^{\infty} \frac{n! \cdot n^2}{n^{\alpha n}}$$

suggerimento: criterio del rapporto

Esercizio compito 9-6-2021

Studiare

$$\sum_{n=1}^{\infty} \frac{n^{n+1} + \ln(n^{200})}{5^n \cdot n!}$$

sol

$$a_n = \underbrace{\frac{n^{n+1}}{5^n \cdot n!}}_{b_n} + \underbrace{\frac{\ln(n^{200})}{5^n \cdot n!}}_{c_n}$$

Studio c_n :

$$\frac{1}{5^n} \cdot \frac{\ln(n^{200})}{n!} \leq \frac{1}{5^n}$$

per $n \geq n_0$

Poiché $\frac{\ln(n^{200})}{n!} \longrightarrow 0$

$\sum \frac{1}{5^n}$ converge $\Rightarrow \sum c_n$ converge

Studio b_n :

$$b_n = \frac{n^{n+1}}{5^n \cdot n!}$$

criterio rapporto:

$$b_{n+1} = \frac{(n+1)^{n+2}}{5^{n+1} \cdot (n+1)!}$$

$$(n+1)^{n+2} = (n+1)^{n+1} \cdot (n+1)$$

$$(n+1)! = (n+1) \cdot (n!)$$

$$5^{n+1} = 5^n \cdot 5$$

$$\frac{b_{n+1}}{b_n} = \left(b_{n+1}\right) \cdot \left(\frac{1}{b_n}\right)$$

$$= \frac{(n+1)^{n+1} \cdot \cancel{(n+1)}}{\cancel{5^n} \cdot 5 \cdot \cancel{(n+1)} \cdot \cancel{(n!)}} \cdot \frac{\cancel{5^n} \cdot \cancel{n!}}{n^{n+1}}$$

$$= \frac{1}{5} \cdot \underbrace{\left(\frac{n+1}{n}\right)^{n+1}}_{e}$$

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \frac{1}{5} \cdot e < 1$$

conclusione: $\sum b_n$ converge

Pertanto

$$\sum a_n = \sum b_n + \sum c_n$$

converge   converge

$$\Rightarrow \sum a_n \text{ converge}$$