Esercizio $\int (x^2-2x)^2 e^{2x}\,dx$ $\qquad f' = e^{2x} \qquad f = \frac{e^{2x}}{2}$

parti $\qquad g = (x^2-2x)^2 \qquad g' = 2(x^2-2x)(2x-2)$

$$= \frac{e^{2x}}{2}(x^2-2x)^2 - \int \frac{e^{2x}}{2}(4x^3-12x^2+8x)\,dx$$

$g = x^4-4x^3+4x^2 \qquad g' = 4x^3-12x^2+8x$

$$= \frac{e^{2x}}{2}(x^2-2x)^2 - \int e^{2x}(2x^3-6x^2+4x)\,dx =$$

$f' = e^{2x} \qquad f = \frac{e^{2x}}{2}$

$g = 2x^3-6x^2+4x \qquad g' = 6x^2-12x+4$

$$= \frac{e^{2x}}{2}(x^2-2x)^2 - \left[\frac{e^{2x}}{2}(2x^3-6x^2+4x) - \int \frac{e^{2x}}{2}\cdot(6x^2-12x+4)\,dx\right] =$$

$$= \frac{e^{2x}}{2}(x^2-2x)^2 - e^{2x}(x^3-3x^2+2x) + \int e^{2x}(\underline{3x^2-6x+2})\,dx$$

Iterando si arriva a un'eq del tipo

$$e^{2x}\left[\text{Somma di polinomi}\right] \pm \underline{\int e^{2x}\cdot a\,dx}$$

Integrabile

Esercizio $\displaystyle\int_1^3 \frac{dx}{\sqrt[3]{x}+\sqrt[3]{x^2}}$

$y = \sqrt[3]{x}$ $\qquad x = y^3$

$dx = 3y^2\,dy$

$x = 3 \longrightarrow y = \sqrt[3]{3}$

$x = 1 \longrightarrow y = 1$

Sost

$$\int_1^{\sqrt[3]{3}} \frac{1}{y+y^2}\cdot 3y^2\,dy = \int_1^{\sqrt[3]{3}} \frac{3y}{1+y}\,dy =$$

$$\int_1^{\sqrt{3}} \frac{3y+3-3}{1+y}\,dy = \int_1^{\sqrt[3]{3}} 3 - \frac{3}{1+y}\,dy = 3y - 3\ln|1+y|\,\Big|_1^{\sqrt[3]{3}}$$

$$= 3\sqrt[3]{3} - 3\ln(1+\sqrt[3]{3}) - 3 + 3\ln(2).$$

In generale $\int f(x)\,dx$ e f funzione con radicali

$\sqrt[m_1]{x^{n_1}}\quad \sqrt[m_2]{x^{n_2}} \quad x^{n_1/m_1}, \; x^{n_2/m_2} \; \ldots$ Sost utile $y = x^{\frac{1}{mcm(m_i)}}$

Così ogni $x^{n_i/m_i} \longrightarrow$ potenza di $y \longrightarrow$ funzione razionale in y

Esercizio $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$ $x^3\sqrt{\text{qualcosa}}$

Sost 1 Sost $y=\sqrt{4-x^2}$ ... Try

$$\int \frac{2\cos(y)}{4\,\mathrm{sen}^2(y)}\cdot 2\cos(y)\,dy =$$

1 $x = 2\,\mathrm{sen}(y)$ $y = \mathrm{arcsen}\left(\frac{x}{2}\right)$

$$\sqrt{4-4\,\mathrm{sen}^2(y)} = \sqrt{4\cos^2(y)} = 2\cos(y)$$

$$= \int \frac{1}{tg^2(y)}\,dy$$

Sost 2

$dx = 2\cos(y)\,dy$

$$\int \frac{1}{z^2}\cdot\frac{1}{1+z^2}\,dz = \star$$

2 $z = tg(y)$ $y = \mathrm{arctg}(z)$

$$= \int \frac{A}{z^2} + \frac{B}{1+z^2}\,dz$$

$dy = \dfrac{1}{1+z^2}\,dz$ $k \in \mathbb{R}$

Sol $A=1$ $B=-1$

$$= \star \int \frac{1}{z^2} - \frac{1}{1+z^2}\,dz = -\frac{1}{z} - \mathrm{arctg}(z) + k$$

$$= -\frac{1}{z} - \operatorname{arctg}(z) + k \underset{(2)}{=} -\frac{1}{tg(y)} - y + k \underset{(1)}{=} -\frac{1}{tg\left(\operatorname{arcsen}\left(\frac{x}{2}\right)\right)} - \operatorname{arcsen}\left(\frac{x}{2}\right) + k$$

$z = tg(y)$

$y = \operatorname{arcsen}\left(\frac{x}{2}\right)$

$$\frac{1}{tg^2(y)} = \frac{\cos^2(y)}{sen^2(y)} = \frac{1 - sen^2(y)}{sen^2(y)} = \underbrace{\frac{1}{sen^2(y)}} - 1$$

Esercizio

$$\int e^{\sqrt{x}}\, dx$$

Sost

$y = \sqrt{x}$ $\quad x = y^2$

$dx = 2y\, dy$

$$\int e^y \cdot 2y\, dy = \text{Parti}$$

$f' = e^y$ $\quad f = e^y$

$g = 2y$ $\quad g' = 2$

$$= 2ye^y - \int 2e^y\, dy =$$

$$= 2ye^y - 2e^y + k \qquad k \in \mathbb{R}$$

Integrazione di funzioni razionali $\frac{P(x)}{Q(x)}$ P e Q polinomi

Step 0 : Possiamo supporre che non abbiano radici / fattori in comune.

Step 1 : dividiamo P per Q

$P = Q \cdot S + R$ R resto $\deg R < \deg Q$

$$\frac{P}{Q} = \frac{Q \cdot S + R}{Q} = S + \frac{R}{Q}$$ S polinomio

Scopo $\int \frac{P}{Q}\,dx = \int S + R/Q \,dx$

integrabile

Posso considerare solo quozienti R/Q $\deg R < \deg Q$

Step 2 : Fattorizzazione di Q

$$x^2-3x+2 = (x-2)(x-1) \qquad x^2+2x+1=(x+1)^2$$

$$x^2+3x+5 \qquad \Delta = 9-20 \qquad \text{irriducibile}$$

Gli unici polinomi irriducibili a coefficienti in $\mathbb{R}$ sono

di grado 1 $\quad ax+b \quad a\neq 0$

di grado 2 $\quad ax^2+bx+c \quad a\neq 0$ e $b^2-4ac<0$

$$Q(x) = (a_1x+b_1)^{m_1}(a_2x+b_2)^{m_2}\cdots(a_tx+b_t)^{m_t}\cdot(\alpha_1x^2+\beta_1x+\gamma_1)^{r_1}\cdots$$
$$\cdots(\alpha_sx^2+\beta_sx+\gamma_s)^{r_s}$$

fattorizzazione più generale possibile.

Non è banale ottenerla in generale.

Step 3 Teo (scomposizione in fratti semplici)

$$\frac{R}{Q} = \frac{R}{\underbrace{(a_1x+b_1)}^{m_1} \cdots (\alpha_s x^2 + \beta_s x + \gamma_s)^{r_s}}$$ è sempre possibile

scriverlo come

$$\underbrace{\frac{\overset{\in\mathbb{R}}{A_{11}}}{(a_1x_1+b_1)} + \frac{\overset{\in\mathbb{R}}{A_{12}}}{(a_1x+b_1)^2} + \dots + \frac{\overset{\in\mathbb{R}}{A_{1m_1}}}{(a_1x+b_1)^{m_1}}}$$

mcm $\longrightarrow (a_1x+b_1)^{m_1}$

$$+ \frac{\overset{\in\mathbb{R}}{A_{t1}}}{(a_tx+b_t)} + \dots + \frac{\overset{\in\mathbb{R}}{A_{tm_t}}}{(a_tx+b_t)^{m_t}} + \frac{B_{11}x + C_{11}}{\alpha_1x^2+\beta_1x+\gamma_1} + \dots$$

$$+ \frac{B_{1r_1}x + C_{1r_1}}{(\alpha_1x^2+\beta_1x+\gamma_1)^{r_1}} + \dots + \frac{B_{sr_s}x + C_{sr_s}}{(\alpha_sx^2+\beta_sx+\gamma_s)^{r_s}}$$

mcm di tutti i denominatori è Q.

3 $\Longrightarrow$ Devo solo integrare funzioni del tipo

$$\frac{a}{(x+b)^m} \qquad \frac{ax+b}{(x^2+\beta x+\gamma)^n} \qquad \beta^2-4\gamma<0$$

Esempio $\displaystyle\int \frac{2x^5-3x^3+2}{\underline{(x+1)^2(x-2)}}\,dx$ Den $=$ $\underline{x^3-3x-2}$

0. -1 e 2 non sono radici di $2x^5-3x^3+2$

1. $\underset{P}{2x^5-3x^3+2} = \underset{Q}{(x^3-3x-2)}\underset{S}{(2x^2+3)}+\underset{R}{4x^2+9x+8}$

$$\int P/Q\,dx = \int \underline{2x^2+3} + R/Q\,dx$$

2. Fattorizzazione di Q $\longrightarrow$ già fatta

Teo $\quad \dfrac{4x^2+9x+8}{(x+1)^2(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x-2)}$

Calcoliamo A, B, C $\qquad \dfrac{A(x+1)(x-2)+B(x-2)+C(x+1)^2}{(x+1)^2(x-2)}$

Q

$$x^2(A+C) + x(-A+B+2C) - 2A - 2B + C = 4x^2 + 9x + 8$$

$$\begin{cases} A + C = 4 \\ -A + B + 2C = 9 \\ -2A - 2B + C = 8 \end{cases} \quad \rightsquigarrow \text{Sol} \qquad A = -\frac{2}{3} \qquad B = -1 \qquad C = \frac{14}{3}$$

$$\int P/Q \, dx = \int 2x^2 + 3 - \frac{2}{3}\left(\frac{1}{x+1}\right) - \left(\frac{1}{(x+1)^2}\right) + \frac{14}{3}\left(\frac{1}{x-2}\right) dx$$

$$= \frac{2}{3}x^3 + 3x - \frac{2}{3}\ln|x+1| + \frac{1}{(x+1)} + \frac{14}{3}\ln|x-2| + k \qquad k \in \mathbb{R}$$

Per ora dobbiamo integrare

- $\dfrac{a}{(bx+c)^m} \rightsquigarrow \dfrac{a}{(x+d)^m}$, mettendo in evidenza b^m al den

$$\int \frac{a}{(bx+c)^m}\,dx = \frac{a}{b^m}\int \frac{1}{\left(x+\frac{c}{b}\right)^m}\,dx = \begin{cases} \dfrac{a}{b^m} \ln\left|x+\dfrac{c}{b}\right| & m=1 \\ \dfrac{a}{b^m}\dfrac{(x+c)^{1-m}}{1-m} & m>1 \end{cases}$$

Tutti i termini lineari sono integrabili

- $\dfrac{Ax+B}{(\underbrace{x^2+\beta x+\gamma}))^m}$ $\qquad \boxed{\dfrac{B}{(x^2+\beta x+\gamma)^m}}$ $\quad A=0$

NO

$\underset{\sim\sim\sim}{A\neq 0}$ $\quad Ax+B = \dfrac{A}{2}\left(2x+\dfrac{2B}{A}\right) = \dfrac{A}{2}\left(\underbrace{2x+\beta}_{X}\ \overbrace{-\beta+\dfrac{2B}{A}}\right)$

Der di $x^2+\beta x+\gamma$

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^m}\,dx$$

Sost $x^2+\beta x+\gamma = y$

$(2x+\beta)\,dx = dy$

$\downarrow$ Sost

$$\int \frac{dy}{y^m} = \text{integrabile}$$

Rimaniamo con $\dfrac{\beta}{(x^2+\beta x+\gamma)^m}$ // $\boxed{\beta^2-4\gamma<0}$ //

Esempio con $m=1$

$$\int \frac{1}{2x^2-x+2}\,dx$$ // Non ha radici

Trasf lineari $x \to ay+b$

Radici $\longrightarrow$ Radici

$$\int \frac{1}{z^2+1}\,dx = \operatorname{arctg}(z) + k \qquad k>0$$ // Non può avere radici

$$\int \frac{1}{2x^2 - x + 2}\,dx$$

$$\Vert$$

$$\int \frac{1}{\left(\sqrt{2}\,x - \frac{1}{2\sqrt{2}}\right)^2 + \frac{15}{8}}\,dx$$

Sost $\downarrow$ y

$$\int \frac{1}{y^2 + \frac{15}{8}} \frac{1}{\sqrt{2}}\,dy$$

sost $\downarrow$ $z^2 + 1$

$$\int \frac{1}{\frac{15}{8} z^2 + \frac{15}{8}} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{15}{8}}\,dz$$

Completo il quadrato

$$2x^2 - x + 2 = (ax + b)^2 + c$$

$$= a^2x^2 + 2abx + b^2 + c$$

$$2 = a^2 \qquad 2ab = -1 \qquad b^2 + c = 2$$

$$a = \sqrt{2} \qquad b = -\frac{1}{2\sqrt{2}} \qquad c = 2 - \frac{1}{8} = \frac{15}{8}$$

Sost $y = \sqrt{2}\,x - \frac{1}{2\sqrt{2}}$ $\quad dy = \sqrt{2}\,dx$

Sost $y = \sqrt{\frac{15}{8}} \cdot z$ $\quad dy = \sqrt{\frac{15}{8}}\,dz$

$$= \frac{\sqrt{15}}{4} \cdot \frac{8}{15} \int \frac{1}{z^2 + 1}\,dz = \frac{4}{\sqrt{15}} \operatorname{arctg}(z)$$

$$\frac{4}{\sqrt{15}} \operatorname{arctg}(z) \longrightarrow \frac{4}{\sqrt{15}} \operatorname{arctg}\left(\sqrt{\frac{8}{15}}\, y\right) \longrightarrow$$

$$\longrightarrow \frac{4}{\sqrt{15}} \operatorname{arctg}\left(\sqrt{\frac{8}{15}} \cdot \left(x\sqrt{2} - \frac{1}{2\sqrt{2}}\right)\right) + k \qquad k \in \mathbb{R}$$