Teo fond + Calcolo delle aree

$f : [a,b] \to \mathbb{R}$ limitata fissato $c \in [a,b]$

$F_c(x) = \int_c^x f(t)\,dt$ è una primitiva di f

sotto opportune condizioni ×

Primitiva $F_c'(x) = f(x)$ •

$\exists$ infinite primitive di $f(x)$ tutte del tipo $F_c(x) + k$ $k \in \mathbb{R}$

$\forall$ primitiva F di f

$\int_a^b f(x)\,dx = \underline{F(b) - F(a)}$

$\int f(x)\,dx$

$\int_a^b f(x)\,dx$

Calcolo delle primitive

- $f(x) = x^n \qquad n \in \mathbb{N}$ $\qquad (\alpha \in \mathbb{R} \quad x^\alpha)$

Primitiva: $\quad F' = \underbrace{x^n}$ $\qquad F = \dfrac{x^{n+1}}{n+1}$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + k \qquad n \in \mathbb{N}$$

$$F' = \underbrace{\cancel{(n+1)}}\,\frac{x^n}{\cancel{n+1}}$$

- $\forall \alpha \in \mathbb{R} - \{-1\} \qquad \displaystyle\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + k \qquad k \in \mathbb{R}$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + k \qquad k \in \mathbb{R}$$

- $\displaystyle\int f + g \, dx = \int f \, dx + \int g \, dx$ $\qquad$ Potenze $\longrightarrow$ Polinomi

$$\int a f \, dx = a \int f \, dx \qquad a \in \mathbb{R}$$

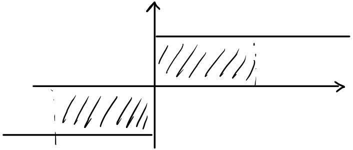
• $f(x) = \mathrm{sgn}(x)$ $\qquad f(x) = \begin{cases} 1 & x \in (0, +\infty) \\ -1 & x \in (-\infty, 0) \end{cases}$

$F(x) = |x|$ è una primitiva di $f(x)$

Dominio $\mathbb{R}$ $\qquad$ f limitata $\Longrightarrow$ F continua

$\cdot 50$ $\qquad$ f non def in 0 $\qquad$ F non è derivabile in 0

$g = \begin{cases} \mathrm{sgn}(x) & x \neq 0 \\ 50 & x = 0 \end{cases}$ $\qquad$ Funzione def su $\mathbb{R}$
Limitata
discontinua in 0

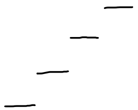


$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ $\qquad$ $|x|$ è primitiva
F non deriv in 0

di g $\forall x \in \mathbb{R}$ ma g non continua in 0

Esempio $f(x) = \lfloor x \rfloor$ parte intera di x

Discontinua in ogni $n \in \mathbb{Z}$



Cerchiamo una primitiva

$f(x) = n$ nell'intervallo $[n, n+1)$

$F(x) = nx$ nell'intervallo $[n, n+1)$

Integrabile a tratti e discontinua in ogni $n \in \mathbb{Z}$

Esempi $\int e^x \, dx = e^x + k$ $\quad \int \operatorname{sen}(x) \, dx = -\cos(x) + k$

$\int \cos(x) \, dx = \operatorname{sen}(x) + k$ ... $\quad \int e^{ax} \, dx = \frac{e^{ax}}{a} + k$

Esempio $\int \operatorname{sen}(x)\cos(x)\,dx = \int \operatorname{sen}(x)\operatorname{sen}'(x)\,dx$ ✶

$f(x) = x^2 \qquad g(x) = \operatorname{sen}(x) \qquad (f \circ g)(x) = f(\operatorname{sen}(x)) = \operatorname{sen}^2(x)$

$(f \circ g)' = f'(g(x)) \cdot g'(x) \qquad (f \circ g)'(x) = 2\operatorname{sen}(x)\cos(x)$

✶ $\int \frac{1}{2} \underbrace{\frac{d \operatorname{sen}^2(x)}{dx}}_{= (\operatorname{sen}^2(x))'} \cdot dx = \frac{1}{2}\operatorname{sen}^2(x) + k \qquad k \in \mathbb{R}$

$\int f'(x)\,dx = f(x) + k$

$\int \operatorname{sen}(x)\cos(x)\,dx = \int \cos(x)(-\cos(x))'\,dx = -\frac{1}{2}\cos^2(x) + k \qquad k \in \mathbb{R}$

$\frac{1}{2}\operatorname{sen}^2(x) - \frac{1}{2} = \cancel{\frac{1}{2}\operatorname{sen}^2(x)} - \frac{1}{2}(\cancel{\operatorname{sen}^2(x)} + \cos^2(x)) = -\frac{1}{2}\cos^2(x)$

Esempio $\int \frac{\ln(x)}{x}\,dx = \int \ln(x)(\ln(x))'\,dx$

$= \frac{(\ln(x))^2}{2} + k \qquad k \in \mathbb{R} \qquad \frac{d}{dx} f^2 = 2 f \cdot f'$

Sostituzione Cosa succede al differenziale

$\downarrow$

incremento della variabile

$dx \rightsquigarrow x_{i+1} - x_i$

Funzione composta $f(g(x))$ argomento di f è $g(x)$

incremento di $g(x)$ $\quad dg(x) \rightsquigarrow g(x_{i+1}) - g(x_i)$

$$\lim_{x_{i+1}-x_i \to 0} \frac{g(x_{i+1}) - g(x_i)}{x_{i+1}-x_i} = g'(x_i) \qquad \frac{dg(x)}{dx} \longrightarrow g'(x)$$

$$g(x) = g(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

$\rightsquigarrow$ Taylor $\quad g'(x)\,dx = dg(x)$

Esempio prec $\int \operatorname{sen}(x) \underbrace{\cos(x)\, dx}$

$\int \operatorname{sen}(x)\, (\operatorname{sen}(x))'\, dx$

Sost $\underbrace{y} = \operatorname{sen}(x)$

Sost

$\underbrace{dy} = d \operatorname{sen}(x) = \underbrace{\cos(x)\, dx}$

$\int y\, dy = y^2/2 + k \quad k \in \mathbb{R}$

$y = \operatorname{sen}(x)$

$\dfrac{\operatorname{sen}^2(x)}{2} + k$

Farlo con la sostituzione $y = \cos(x)$

Esempio $\displaystyle\int \frac{1}{(x^2+1)\cdot \operatorname{arctg}(x)}\, dx$

$\operatorname{arctg}(x)$ $\qquad \dfrac{1}{x^2+1} = (\operatorname{arctg}(x))'$

Sost $\operatorname{arctg}(x) = y$ $\qquad$ Per integrare rispetto a y

Sost. $\qquad$ mi serve $dy = d\operatorname{arctg}(x)$

$$= \frac{1}{1+x^2}\, dx$$

$$\int \frac{1}{y}\cdot dy = \ln|y| + k \quad k\in\mathbb{R}$$

$\hookrightarrow$ Primitiva $\ln|\operatorname{arctg}(x)| + k \quad k\in\mathbb{R}$

Esercizio $\displaystyle\int \operatorname{tg}(x)\, dx$

Integrale definito $\int_a^b f(x)\,dx$

$$\int_0^{\pi/2} \text{sen}(x)\underline{\cos(x)\,dx}$$

Sost $y = \text{sen}(x)$

Integro rispetto a y

$dy = \cos(x)\,dx$

$$\int_0^1 y\,dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2} - 0$$

Come "base" serve l'intervallo in cui varia y

$x = \pi/2 \longrightarrow y = \text{sen}(\pi/2) = 1$

$x = 0 \longrightarrow y = \text{sen}(0) = 0$

$$\frac{\text{sen}^2(x)}{2}\Big|_0^{\pi/2} = \frac{1}{2} - 0$$

Attenzione

$$\frac{y^2}{2}\Big|_0^{\pi/2} = \frac{\pi^2}{8} - 0$$

Esercizi

$$\int_0^{\pi/2} \frac{\cos^3(x)}{1+\operatorname{sen}^2(x)}\,dx = \int_0^{\pi/2} \frac{\cos(x)(\cos^2(x))}{1+\operatorname{sen}^2(x)}\,dx$$

$$= \int_0^{\pi/2} \cos(x)\,\frac{(1-\operatorname{sen}^2(x))}{(1+\operatorname{sen}^2(x))}\,dx$$

Sost $y = \operatorname{sen}(x)$

$dy = \cos(x)\,dx$

$\downarrow$ Sost

$$\int_0^1 \frac{1-y^2}{1+y^2}\,dy = \int_0^1 \frac{1+y^2-2y^2}{1+y^2}\,dy$$

$x = \pi/2 \longrightarrow y = 1$

$x = 0 \longrightarrow y = 0$

$$= \int_0^1 1 - \frac{2y^2}{1+y^2}\,dy$$

//// $\rightsquigarrow$ VD DOPO

$$\int_0^1 \frac{1-y^2}{1+y^2}\,dy = \int_0^1 \frac{2-1-y^2}{1+y^2}\,dy = \int_0^1 \frac{2}{1+y^2} - \overset{1}{\frac{1+y^2}{1+y^2}}\,dy$$

$$= 2\,\operatorname{arctg}(y)\Big|_0^1 - y\Big|_0^1 = \frac{\pi}{2} - 1$$

- $\int_0^{\pi/2} \frac{\cos(x)}{2+\cos^2(x)}\,dx = \int_0^{\pi/2} \frac{\cos(x)}{2+1-\mathrm{sen}^2(x)}\,dx$

$= \int_0^{\pi/2} \frac{\cos(x)}{3-\mathrm{sen}^2(x)}\,dx$

Sost $\quad y = \mathrm{sen}(x)$

$dy = \cos(x)\,dx$

$x = \pi/2 \longrightarrow y = 1$

$x = 0 \longrightarrow y = 0$

$\downarrow$ Sost

$\int_0^1 \frac{1}{3-y^2}\,dy$

$3-y^2 = (\sqrt{3}+y)(\sqrt{3}-y)$

$\frac{1}{3-y^2} = \frac{A}{\sqrt{3}+y} + \frac{B}{\sqrt{3}-y}$

$\int \frac{1}{\sqrt{3}+y}\,dy = \ln|\sqrt{3}+y|$

$\frac{1}{3-y^2} = \frac{A(\sqrt{3}-y)+B(\sqrt{3}+y)}{3-y^2}$

Num $(B-A)y + (B+A)\sqrt{3} = 1$

$\begin{cases} B-A=0 \\ (B+A)\sqrt{3}=1 \end{cases}$

$A = B \qquad B = \frac{1}{2\sqrt{3}}$

$$\int_0^1 \frac{1}{3-y^2}\,dy = \int_0^1 \frac{\frac{1}{2\sqrt{3}}}{\sqrt{3}-y} + \frac{\frac{1}{2\sqrt{3}}}{\sqrt{3}+y}\,dy =$$

$$\frac{1}{2\sqrt{3}}\left[\int_0^1 \frac{1}{\sqrt{3}-y} + \frac{1}{\sqrt{3}+y}\,dy\right] = \frac{1}{2\sqrt{3}}\Big[-\ln|\sqrt{3}-y| + \ln|\sqrt{3}+y|\Big]_0^1$$

$$\frac{1}{2\sqrt{3}}\left(-\ln(\sqrt{3}-1) + \ln(\sqrt{3}+1) + \cancel{\ln(\sqrt{3})} - \cancel{\ln(\sqrt{3})}\right]$$

$$= \frac{1}{2\sqrt{3}}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

disparì

- $\int_0^1 x^3 \sqrt{4-x^2}\, dx$

$x^3 \dfrac{(4-x^2)}{\sqrt{4-x^2}} dx$

Sost $\quad y = \sqrt{4-x^2}$ ✱

$y^2 = 4 - x^2 \qquad x^2 = 4 - y^2$

$x = \sqrt{4-y^2} \qquad dx = \dfrac{1}{2\sqrt{4-y^2}} \cdot (-2y)\, dy$

✱ $dy = \dfrac{1}{2\sqrt{4-x^2}} (-2x)\, dx$

$\cos(x)\, dx$

Sost

$\int_2^{\sqrt3} \left(\sqrt{4-y^2}\right)^3 y \dfrac{(-2y)}{2\sqrt{4-y^2}}\, dy$

$x = 0 \longrightarrow y = 2$

$x = 1 \longrightarrow y = \sqrt3$

$\parallel$

$\int_2^{\sqrt3} (4-y^2)\cdot(-y^2)\, dy = \int_2^{\sqrt3} -4y^2 + y^4\, dy = -\dfrac{4}{3} y^3 + \dfrac{y^5}{5} \Big|_2^{\sqrt3}$

$= -\dfrac{4}{3} 3\sqrt3 + \dfrac{9\sqrt3}{5} + \dfrac{4}{3}\cdot 8 - \dfrac{243}{5} \ldots$

$2 > \sqrt3$

- $\displaystyle\int_1^2 \frac{e^x}{1+e^{2x}}\,dx$

Sost $y = e^x$ $\quad dy = \underbrace{e^x\,dx}$ 1

$x = \ln(y)$ $\quad dx = \frac{1}{y}\,dy$ 2

$x = 2 \longrightarrow y = e^2$

$x = 1 \longrightarrow y = e$

Sost 1

$$\int_e^{e^2} \frac{1}{1+y^2}\,dy$$

Sost 2

$$\int_e^{e^2} \frac{\cancel{y}}{1+y^2}\,\frac{1}{\cancel{y}}\,dy$$

$$\int_e^{e^2} \frac{1}{1+y^2}\,dy = \operatorname{arctg}(y)\Big|_e^{e^2} = \operatorname{arctg}(e^2) - \operatorname{arctg}(e)$$

Integrazione per parti

Formula di derivazione del prodotto $(f \cdot g)' = f' \cdot g + f \cdot g'$

Integrando $\underbrace{\int (f \cdot g)' \, dx} = \int f' \cdot g \, dx + \int f \cdot g' \, dx$

$$f \cdot g + k = \int f' \cdot g \, dx + \int f \cdot g' \, dx$$

Formula $\int f' \cdot g \, dx = f \cdot g - \int f \cdot g' \, dx \quad (+k) \qquad k \in \mathbb{R}$

Esempio $\int 1 \cdot \ln(x) \, dx = x \ln(x) - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$ $\qquad g(x) = \ln(x) \to g'(x) = \frac{1}{x}$

$= x \ln(x) - x + k \qquad k \in \mathbb{R}$ $\qquad f'(x) = 1 \longrightarrow f(x) = x$

- $\int x^n \ln(x)\, dx$ $\qquad f' = x^n \Rightarrow f = \frac{x^{n+1}}{n+1}$

$n \neq -1$ $\qquad g = \ln(x) \Rightarrow g' = \frac{1}{x}$

$$\int x^n \ln(x)\, dx = \frac{x^{n+1}}{n+1} \ln(x) - \int \frac{x^{n+\cancel{1}}}{n+1} \cdot \frac{1}{\cancel{x}}\, dx$$

$$= \frac{x^{n+1}}{n+1} \ln(x) - \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} + k$$

$n = -1$ $\qquad \int \frac{\ln(x)}{x}\, dx$ $\qquad$ Già risolta con sostituzione

$f' = \frac{1}{x} \Rightarrow f = \ln(x)$

$g = \ln(x) \Rightarrow g' = \frac{1}{x}$

$$= \ln^2(x) - \int \ln(x) \cdot \frac{1}{x}\, dx$$

$$2 \int \frac{\ln(x)}{x}\, dx = \ln^2(x) \longrightarrow \int \frac{\ln(x)}{x}\, dx = \frac{\ln^2(x)}{2} + k \qquad k \in \mathbb{R}$$

- $\displaystyle\int 1 \cdot \operatorname{arctg}(x)\, dx =$

$g(x) = \operatorname{arctg}(x) \longrightarrow g' = \dfrac{1}{1+x^2}$

$f'(x) = 1 \longrightarrow f = x$

$$= x \operatorname{arctg}(x) - \int \frac{x}{1+x^2}\, dx = \star$$

Notiamo che $(1+x^2)' = 2x$

$$\int \frac{x}{1+x^2}\, dx =$$

Sost $\quad y = 1+x^2 \qquad dy = 2x\, dx$

$x\,dx = \dfrac{dy}{2}$

Sost

$$\int \frac{1}{y} \frac{dy}{2} = \frac{1}{2} \ln(y) \longrightarrow \frac{1}{2} \ln(1+x^2)$$

$$= \star \quad x \operatorname{arctg}(x) - \frac{1}{2} \ln(1+x^2) + k \qquad k \in \mathbb{R}$$

• Siano $\alpha, \beta \in \mathbb{R}$ $\quad \alpha, \beta \neq 0$

$$\int e^{\alpha x} \operatorname{sen}(\beta x)\, dx = \qquad (1) \quad f' = e^{\alpha x} \longrightarrow f = \frac{e^{\alpha x}}{\alpha} \qquad \alpha \neq 0$$

$$g = \operatorname{sen}(\beta x) \longrightarrow g' = \beta \cos(\beta x)$$

$$= \frac{e^{\alpha x}}{\alpha} \operatorname{sen}(\beta x) - \int \frac{e^{\alpha x}}{\alpha} \cdot \beta \cos(\beta x)\, dx = \star\star$$

Attenzione !!! Ripetendo l'integrazione per parti fare attenzione a mantenere i "ruoli" della 1^a integrazione

Esempio di scambio (1) integro exp derivo trig

(2) integro trig der exp $\quad f' = \cos(\beta x) \longrightarrow \frac{\operatorname{sen}(\beta x)}{\beta} = f$

$$g = e^{\alpha x} \longrightarrow g' = \alpha e^{\alpha x}$$

$$\star\star = \frac{e^{\alpha x}}{\alpha}\,\mathrm{sen}(\beta x) - \frac{\beta}{\alpha}\left[\frac{\mathrm{sen}(\beta x)}{\beta}\cdot e^{\alpha x} - \int \frac{\mathrm{sen}(\beta x)}{\beta}\cdot \alpha e^{\alpha x}\,dx\right]$$

$$= \cancel{\frac{e^{\alpha x}}{\alpha}\,\mathrm{sen}(\beta x)} - \cancel{\frac{\mathrm{sen}(\beta x)}{\alpha}}\, e^{\alpha x} + \int \mathrm{sen}(\beta x)\, e^{\alpha x}\,dx$$

Quindi mantengo i ruoli di 1 $\quad f' = e^{\alpha x} \longrightarrow f = \frac{e^{\alpha x}}{\alpha}$

$g = \cos(\beta x) \longrightarrow g' = -\beta\,\mathrm{sen}(\beta x)$

$$\star\star = \frac{e^{\alpha x}}{\alpha}\,\mathrm{sen}(\beta x) - \frac{\beta}{\alpha}\left[\frac{e^{\alpha x}}{\alpha}\cos(\beta x) - \int \frac{e^{\alpha x}}{\alpha}\cdot(-\beta\,\mathrm{sen}(\beta x))\,dx\right]$$

$$= \frac{e^{\alpha x}}{\alpha}\,\mathrm{sen}(\beta x) - \frac{\beta}{\alpha^2}\, e^{\alpha x}\cos(\beta x) - \frac{\beta^2}{\alpha^2}\int e^{\alpha x}\,\mathrm{sen}(\beta x)\,dx$$

porto dall'altro lato dell'uguaglianza

$$\left(1+\frac{\beta^2}{\alpha^2}\right)\int e^{\alpha x}\,\mathrm{sen}(\beta x)\,dx = \frac{e^{\alpha x}}{\alpha}\left[\mathrm{sen}(\beta x) - \frac{\beta}{\alpha}\cos(\beta x)\right] + k$$

Esercizio Rifarlo con $f' = \text{sen}(\beta x)$

$g = e^{\alpha x}$

Esercizio $\int \text{sen}(\alpha x)\cos(\beta x)\,dx \qquad \alpha \neq \beta$

Risultato $\dfrac{\beta^2}{\beta^2-\alpha^2} \cdot \left(\dfrac{\text{sen}(\alpha x)\,\text{sen}(\beta x)}{\beta} + \dfrac{\alpha}{\beta^2}\cos(\alpha x)\cos(\beta x) \right)$

Farlo per $\alpha = \beta \rightsquigarrow \text{sen}(2\alpha x)$

Stesso metodo per $\int \text{sen}(\alpha x)\,\text{sen}(\beta x)\,dx$

$\alpha = \beta \qquad \text{sen}^2(\alpha x) \rightsquigarrow \cos(2\alpha x)$

Esercizio $\int (x^2-2x)^2 e^{2x}\,dx$