Teo di Weierstrass : f continua $\Rightarrow$ $f(C)$ è compatta

- f continua e I intervallo $\subseteq$ dominio di f

  $\Rightarrow$ $f(I)$ intervallo

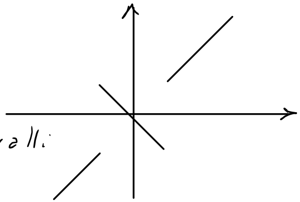
Def $f : A \longrightarrow Im(f)$ si dice invertibile $\Leftrightarrow$ è iniettiva

$f^{-1} : Im(f) \longrightarrow A$ t.c. $f \circ f^{-1} = id_{Im(f)}$

$f^{-1} \circ f = id_A$

- f strettamente monotona $\Rightarrow$ iniettiva

Per le funzioni continue questa diventa una condizione nec e suff ... sugli intervalli



Teo $f : I \to \mathbb{R}$ continua. Allora $f : I \to \mathrm{Im}(f)$
$f(I)$

è invertibile $\iff$ è strett monotona

$f(x)$

$f(z)$

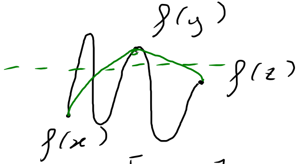
$f(y)$

Dim. ($\Leftarrow$) già visto

($\Rightarrow$) Supp non sia strett monotona

Per esempio $\exists\; x < y < z$ $\quad x, y, z \in I$ t.c.

$f(x) < f(y)$ poi $f(y) > f(z)$



Per il Teo dei valori intermedi

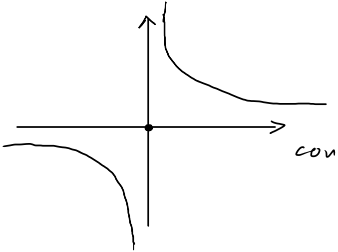
f assume tutti i valori tra $f(x)$ e $f(y)$ in $[x, y]$
f assume tutti i valori tra $f(y)$ e $f(z)$ in $[y, z]$
$\Rightarrow$ NON è iniettiva

$\boxed{X}$

Esempio $f(x) = \frac{1}{x}$ Dominio $(-\infty, 0) \cup (0, +\infty)$

non è un intervallo

f continua sul dominio



f è invertibile

$f^{-1}(x) = ?$ calcolare

Potete applicare il Teo

a $f: (-\infty, 0) \to (-\infty, 0)$ o a $f: (0, +\infty) \to (0, +\infty)$

- $g(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ $g: \mathbb{R} \to \mathbb{R}$ surg

invertibile (calcolare g^{-1})

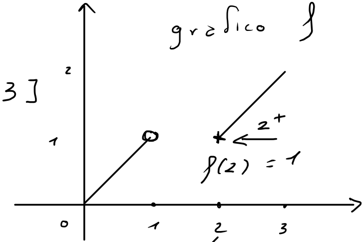
def su un intervallo $\mathbb{R}$

Teo non si applica perché g non è continua

Esempio

f cont
f^{-1} non cont

$$f(x) = \begin{cases} x & x \in [0,1) \\ \underline{x-1} & x \in [2,3] \end{cases}$$

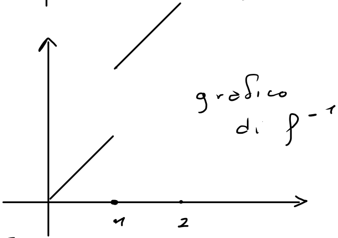


Dominio $[0,1) \cup [2,3]$

f monotona? sì crescente
f continua? sì
f invertibile? sì

Inversa $[0,1) \xrightarrow{f} [0,1)$

$[2,3] \xrightarrow{f} [1,2]$



$\mathrm{Im}(f) = \text{Dominio } f^{-1} = [0,1) \cup [1,2] = [0,2]$

$$f^{-1}(y) = \begin{cases} y & y \in [0,1) \\ y+1 & y \in [1,2] \end{cases}$$

$\lim_{y \to 1^-} f^{-1}(y) = 1$

$\lim_{y \to 1^+} f^{-1}(y) = 2$

salto

Teo Sia $f: I \longrightarrow f(I)$ continua e invertibile

Allora $f^{-1}: f(I) \longrightarrow I$ è continua

- **Oss** $f: C$ compatto $\longrightarrow f(C)$ continua e invertibile

$\Longrightarrow f^{-1}$ continua

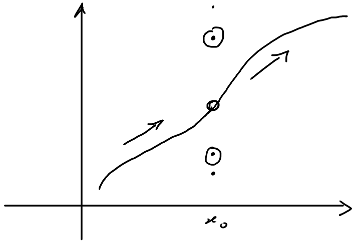
Dim f continua, def su int e invertibile $\xRightarrow{\text{Teo prec}}$ f st monotona

$\Longrightarrow$ (Teo funz monotone) f^{-1} st monotona.

(Teo sui limiti) f^{-1} ammette limite in ogni punto del dominio

$\Longrightarrow$ Uniche discontinuità possibili

1. Eliminabile
2. Salto



$f(x_0)$ "fuori" dal grafico

Non monotona : contraddizione

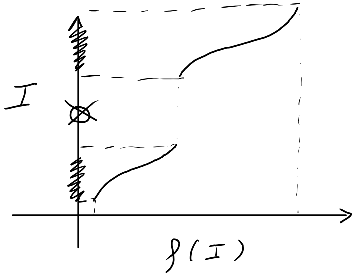


grafico f^{-1}

$\mathrm{Im}(f^{-1}) = I$
NON intervallo

contraddizione

Quindi f^{-1} non ha disc $\Rightarrow$ è continua.

Esercizio

$$f(x) = \begin{cases} 4 \operatorname{arctg}(2x^2) & x \in (0, +\infty) \\ x + a & x \in (-\infty, -1] \end{cases}$$

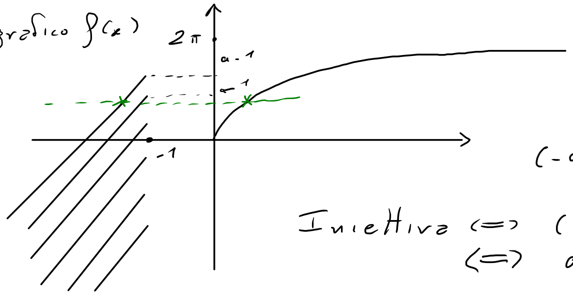
$a \in \mathbb{R}$

Al variare di a: f continua, invertibile, f^{-1} è continua?

$4 \operatorname{arctg}(2x^2)$ mon cresc su $(0, +\infty)$

$$(0, +\infty) \xrightarrow{2x^2} (0, +\infty) \xrightarrow{\operatorname{arctg}(x)} (0, \pi/2) \xrightarrow{4} (0, 2\pi)$$

grafico $f(x)$



$f(-1) = a - 1$

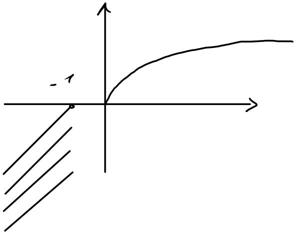
f continua $\forall a \in \mathbb{R}$

$$(-\infty, -1] \xrightarrow{x+a} (-\infty, a-1]$$

Iniettiva $\iff (-\infty, a-1] \cap (0, 2\pi) = \emptyset$

$\iff a - 1 \leq 0$ cioè $a \leq 1$

Supponiamo $a \le 1$



$$f^{-1}(y) = y - a \qquad y \in (-\infty, a-1]$$

$$y = 4\,\mathrm{arctg}(2x^2) \qquad \mathrm{arctg}(2x^2) = y/4$$

$$2x^2 = \mathrm{tg}(y/4) \qquad x = \cancel{\pm}\sqrt{\frac{1}{2}\,\mathrm{tg}(y/4)}$$

Oss $\sqrt{\ }$ ben def perché $y \in (0, 2\pi)$

$$y/4 \in (0, \pi/2) \Rightarrow \mathrm{tg}(y/4) > 0$$

$$f^{-1}(y) = \begin{cases} \sqrt{\frac{1}{2}\,\mathrm{tg}(y/4)} & y \in (0, 2\pi) \\ y - a & y \in (-\infty, a-1] \end{cases}$$

$$f^{-1}(a-1) = -1 \quad \forall a$$

Inversa continua $\forall a < 1$

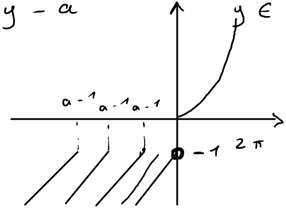


grafico f^{-1}

Se $a - 1 = 0$ cioè $\underline{a = 1}$

$$\lim_{y \to 0^+} f^{-1}(y) = 0$$

Salto

$$\lim_{y \to 0^-} f^{-1}(y) = -1$$

Esercizio

$$f(x) = \begin{cases} \cos(x+a) & x \in (-\infty, 0] \\ e^{-1/x} & x \in (0, +\infty) \end{cases} \qquad a \in \mathbb{R}$$

- Per quali $a \in \mathbb{R}$ $f(x)$ è continua

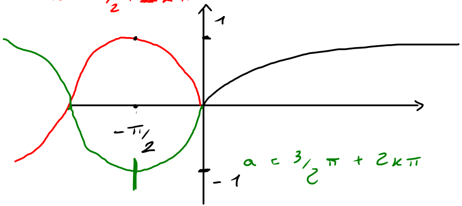
Basta controllare in 0 $\quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{-1/x} = 0$

$$\lim_{x \to 0^-} f(x) = \cos(a) = f(0)$$

Continua $\iff$ $\cos(a) = 0$ $\qquad a = \pi/2,\ 3/2\,\pi,\ 5/2\,\pi \ldots$

$$= \frac{\pi}{2} + k\pi \qquad k \in \mathbb{Z}$$

$a = \pi/2 + 2k\pi$



$a = \frac{\pi}{2}$ $\quad \cos(x + \pi/2)$ $\quad$ Maiuscola [sic: "Maiuiettiva"]

$a = \frac{3}{2}\pi$ $\quad \cos(x + \frac{3}{2}\pi)$

Per $a = \frac{3}{2}\pi$ la funzione è continua e invertibile

$$f : [-\pi/2, +\infty) \longrightarrow [-1, 1)$$

$$\exists\, f^{-1} : [-1, 1) \longrightarrow [-\pi/2, +\infty)$$

$$f(x) = \begin{cases} \cos(x + 3/2\,\pi) & x \in [-\pi/2, 0] \\ e^{-1/x} & x \in (0, +\infty) \end{cases}$$

$y = \cos(x + 3/2\,\pi) \qquad x = \arccos(y) - 3/2\,\pi \qquad [-1, 0]$

$y = e^{-1/x} \qquad -\frac{1}{x} = \ln(y) \qquad x = -\frac{1}{\ln(y)} \qquad (0, 1)$

$$f^{-1}(y) = \begin{cases} \arccos(y) - 3/2\,\pi & y \in [-1, 0] \xrightarrow{f^{-1}} \underline{[-\pi/2, 0]} \\ -\frac{1}{\ln(y)} & y \in (0, 1) \xrightarrow{f^{-1}} (0, +\infty) \end{cases}$$

Abbiamo def $\text{arccos}: [-1,1] \longrightarrow [0,\pi]$

$\arccos(y) - \frac{3}{2}\pi$ in $y=-1$ $\quad \pi - \frac{3}{2}\pi = -\frac{\pi}{2}$

in $y=0$ $\quad \frac{\pi}{2} - \frac{3}{2}\pi = -\pi$

Con tale def $\arccos(y) - \frac{3}{2}\pi : [-1,0] \longrightarrow [-\pi, -\frac{\pi}{2}]$

Bisogna usare $\text{arccos}: [-1,1] \longrightarrow [\pi, 2\pi]$

Con tale def $\arccos(y) - \frac{3}{2}\pi$ in -1 vale $\pi - \frac{3}{2}\pi = -\frac{\pi}{2}$

$\arccos(y) - \frac{3}{2}\pi$ in 0 vale $\frac{3}{2}\pi - \frac{3}{2}\pi = 0$

$[-1,0] \longrightarrow [-\frac{\pi}{2}, 0]$.

Compitino del 26-01-2022

Esercizio 1 Ipotesi: Per giocare in NBA bisogna essere abilissimi con la palla o alti almeno 2.15.

Giocare in NBA $\Longrightarrow$ ab palla o alti 2.15

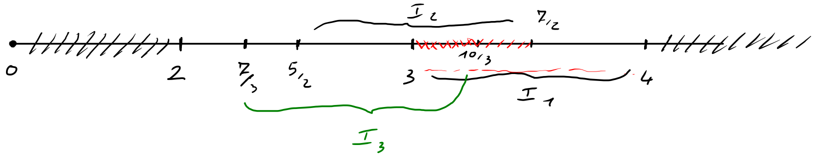
- Shaquille gioca in NBA ma no ab palla $\overset{\Longrightarrow}{\text{quindi}}$ è alto almeno 2.15 V
- Hakeem 2.20 e ab palla $\overset{\Longrightarrow}{\text{quindi}}$ gioca in NBA F
- Charles no ab palla e alto 2.12 $\overset{\Longrightarrow}{\text{quindi}}$ non gioca in NBA V
- Kobe ab palla e alto solo 2.08 $\overset{\Longrightarrow}{\text{quindi}}$ non può giocare in NBA F

Esercizio 2

$$I_n = \left[2+\frac{1}{n},\ 3+\frac{1}{n}\right] \qquad n \in \mathbb{N} - \{0\}$$

$$A = \bigcup_n I_n \qquad B = \bigcap_n I_n$$

$$I_1 = [3, 4] \qquad I_2 = [5/2, 7/2] \qquad I_3 = \left[\frac{7}{3}, \frac{10}{3}\right] \ \dots$$



$\forall\, a \in (2,3) \quad \exists\, n$ t.c. $\quad a > 2 + 1/n \implies a \in I_n$

$\implies A = (2, 4] \qquad \sup A = \max A = 4 \quad \inf A = 2 \quad \min A$ N.E.

Intersezione $\quad \forall\, b \in (3, 4] \quad \exists\, n$ t.c. $\quad b > 3 + 1/n \implies b \notin I_n$

$B = \{3\} \qquad \sup B = \max B = \inf B = \min B = 3.$

Esercizio 3

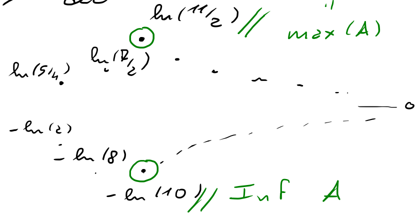
$$A = \left\{ (-1)^n \ln\left(\frac{n+5}{|n-4|}\right) : n \in \mathbb{N} - \{4\} \right\}$$

$n - 4 < 0$ per $n = 0, 1, 2, 3$ • $n = 0$ $\ln\left(5/4\right)$ $n = 2$ $\ln\left(7/2\right)$

$n - 4 > 0$ per $n \geq 5$. $n = 1$ $-\ln(2)$ $n = 3$ $-\ln(8)$

Supp $n \geq 5$ $\quad \dfrac{(n+1)+5}{(n+1)-4} - \dfrac{n+5}{n-4} = \dfrac{\cancel{n^2} + \cancel{2n} - 24 - \cancel{n^2} - \cancel{2n} + 15}{(n-3)(n-4)}$

$= -\dfrac{9}{(n-3)(n-4)} < 0$ $\quad$ Arg dec $\xrightarrow[\ln]{}$ dec



Sup A = max(A)

Inf A = min A

n pari $\Rightarrow$ succ dec

n dispari $\Rightarrow$ succ cres

n pari Parte da $n = 6$ $\ln\left(\dfrac{11}{2}\right)$

Arriva a $\lim\limits_{n \to +\infty} \ln\left(\dfrac{n+5}{n-4}\right) = 0$

n dispari Parte da $n = 5$ $-\ln(10)$

Arriva a $\lim\limits_{n \to +\infty} -\ln\left(\dfrac{n+5}{n-4}\right) = 0$

Esercizio 4 • $f(x) = \operatorname{arctg}\left(e^{1/x+2}\right)$

$\operatorname{arctg}(x)$ e e^x mon crescenti $\quad \frac{1}{x+2}$

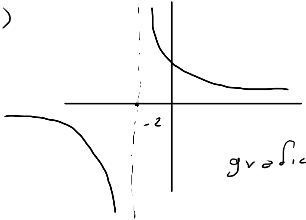


grafico $\frac{1}{x+2}$

Dominio $(-\infty, -2) \cup (-2, +\infty)$

$\downarrow$ analogo

cr o cr o dec $\longrightarrow$ dec

$\operatorname{arctg}(x)$ e^x $1/x+2$

Immagine $(-\infty, -2) \cup (-2, +\infty) \xrightarrow{1/x+2} (-\infty, 0) \cup (0, +\infty)$

$\xrightarrow{e^x} (0, 1) \cup (1, +\infty) \xrightarrow{\operatorname{arctg}} (0, \pi/4) \cup (\pi/4, \pi/2)$

$\operatorname{Im}(f) = \text{Dominio}(f^{-1})$

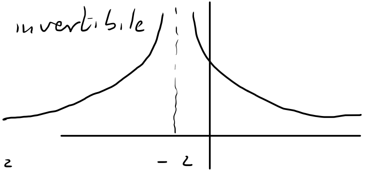
Iniettiva (comp di iniettive è iniettiva) $\Rightarrow$ invertibile

$y = \operatorname{arctg}\left(e^{1/x+2}\right) \qquad tg(y) = e^{1/x+2} \qquad \ln(tg(y)) = \frac{1}{x+2}$

$x = \frac{1}{\ln(tg(y))} - 2 = f^{-1}(y)$

- $g(x) = \operatorname{arctg}\left(e^{-\frac{1}{|x+2|}}\right)$ Dominio $(-\infty, -2) \cup (-2, +\infty)$

Non è iniettiva $\Rightarrow$ non invertibile



$\frac{1}{|x+2|}$

$(-\infty, -2) \cup (-2, +\infty) \xrightarrow{1/(x+2)} (-\infty, 0) \cup (0, +\infty) \xrightarrow{|x|} (0, +\infty)$

$\xrightarrow{-x} (-\infty, 0) \xrightarrow{e^x} (0, 1) \xrightarrow{\operatorname{arctg}} (0, \pi/4) = Im(f)$

Esercizio 5.

$$\lim_{x \to 0^+} \frac{2^{2/x} - x^{-1000} + e^{1/x}}{\ln(x^{2000}) - 3(5^x + 4^{1/x})} \qquad \text{F.I. } \frac{\infty}{\infty} \qquad x \to \frac{1}{y}$$

$$\lim_{y \to +\infty} \frac{4^y - y^{1000} + e^y}{\ln(y^{-2000}) - 3(5^{1/y} + 4^y)} = \lim_{y \to +\infty} \frac{\cancel{4^y}}{\cancel{4^y}} \frac{\left(1 - \frac{y^{1000}}{4^y} + \left(\frac{e}{4}\right)^y\right)}{\frac{\ln(y^{-2000})}{4^y} - 3\left(\frac{5^{1/y}}{4^y} + 1\right)}$$

$$= \frac{1}{-3} = -\frac{1}{3}$$

$$1^{+\infty}$$

$$\lim_{x \to +\infty} \left(\frac{5x^4 - 2x^2 + 2}{5x^4 + 3x^4 + 2}\right)^{\frac{2x^2-1}{3x+2}}$$

$$\lim_{x \to +\infty} \left[\left(1 - \frac{3x^3 + 2x^2}{5x^4 + 3x^3 + 2}\right)^{-\frac{5x^4 + 3x^3 + 2}{3x^3 + 2x^2}}\right]^{\underbrace{-\frac{3x^3 + 2x^2}{5x^4 + 3x^3 + 2} \cdot \frac{2x^2 - 1}{3x + 2}}_{\to -2/5}}$$

$$= e^{-2/5}$$

$$\lim_{n \to +\infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!\,(5n)!}{((3n)!)^2}}$$

Criterio rapporto $\Rightarrow$ radice

Se $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \ell$ allora $\lim_{n \to +\infty} \sqrt[n]{a_n} = \ell$

$$\sqrt[n]{\underbrace{\frac{1}{n^n} \frac{(2n)!\,(5n)!}{((3n)!)^2}}_{a_n}}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)^{n+1}} \frac{(2n+2)!\,(5n+5)!}{((3n+3)!)^2} \cdot \frac{n^n((3n)!)^2}{(2n)!\,(5n)!}$$

$$= \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \cdot \frac{(2n+2)(2n+1)(5n+5)(5n+4)\cdots(5n+1)}{(3n+3)^2(3n+2)^2(3n+1)^2}$$

$$\xrightarrow[n \to +\infty]{} \frac{1}{e} \cdot \frac{2^2 \cdot 5^5}{3^6} = \ell$$

$$\left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(1 + \frac{1}{n}\right)^{-n} \longrightarrow e^{-1}$$