Esercitazione online il 20-12-2021 ore 15
Sconnettersi da ogni account non istituzionale.

Compitino in corrispondenza del 2° appello invernale

Iscrivetevi sul portale esami.

Esercizio $\lim_{n\to+\infty} \frac{n!}{a^n} = +\infty \quad a > 1$ ///

$$\lim_{n\to+\infty} \frac{n!}{a^{2n}}$$

Rapporto $a_n = \frac{n!}{a^{2n}}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)}!}{a^{\cancel{2n}+2}} \cdot \frac{\cancel{a^{2n}}}{\cancel{n!}} \quad (n+1)$$

$$= \frac{n+1}{a^2} \xrightarrow[n\to+\infty]{} +\infty \qquad \lim_{n\to+\infty} \frac{n!}{a^{2n}} = +\infty$$

- $\lim_{n\to+\infty} \frac{n!}{a^{bn}} = +\infty \qquad \left(\frac{n!}{a^{bn}} = \frac{n!}{(a^b)^n} \qquad a > 1 \quad a^b > 1 \right)$

- $\lim_{n \to +\infty} \frac{n!}{a^{n^2}}$ Rapporto $\frac{(n+1)!}{a^{(n+1)^2}} \cdot \frac{a^{n^2}}{n!} =$

$$= \frac{\cancel{(n+1)!}^{(n+1)}}{a^{\cancel{n^2}+2n+1}} \cdot \frac{\cancel{a^{n^2}}}{\cancel{n!}} = \frac{(n+1)}{a^{2n+1}} \xrightarrow[n \to +\infty]{} 0$$

$b \in \mathbb{N}$

$$\lim_{n \to +\infty} \frac{n!}{a^{n^2}} = 0 \qquad \lim_{n \to +\infty} \frac{(2n)!}{a^{n^2}} = \lim_{n \to +\infty} \frac{(bn)!}{a^{n^2}} = 0$$

$$\lim_{n \to +\infty} \frac{(n^2)!}{a^{n^2}} = [\text{Sost } m = n^2]$$

$$= \lim_{m \to +\infty} \frac{m!}{a^m} = +\infty$$

Rapporto $\displaystyle\lim_{n \to +\infty} \frac{(n^2)!}{a^{n^2}} \qquad \frac{(n+1)^2!}{a^{(n+1)^2}} \cdot \frac{a^{n^2}}{(n^2)!}$

$$= \frac{(n^2+2n+1)!}{a^{\cancel{n^2}+2n+1}} \cdot \frac{\cancel{a^{n^2}}}{(n^2)!} = \frac{\overbrace{(n^2+2n+1)(n^2+2n)\cdots(n^2+1)}^{2n+1 \text{ fattori}} \cdot \cancel{(n^2)!}}{a^{2n+1} \cdot \cancel{(n^2)!}}$$

$$= \frac{(n^2)^{2n+1} + \text{ter di deg inferiore}}{a^{2n+1}} \xrightarrow[n \to +\infty]{} +\infty$$

Ter di deg max $\left(\frac{n^2}{a}\right)^{2n+1}$

- $\lim_{n \to +\infty} \frac{n!}{n^n}$ Rapporto

$$\frac{\cancel{(n+1)!}^{\,(n+1)}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{n!}} = \frac{\cancel{n+1}}{\cancel{(n+1)}} \cdot \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n+1}{n}\right)^{-n} = \left(\left(1+\frac{1}{n}\right)^n\right)^{-1} \xrightarrow[n \to +\infty]{} \frac{1}{e} < 1$$

Criterio del rapporto $\lim_{n \to +\infty} \frac{n!}{n^n} = 0$

- $\lim_{n \to +\infty} \frac{3^n \cdot n!}{n^n}$ Rapporto

$$\lim_{n \to +\infty} \frac{3^n \cdot n!}{n^n} = +\infty$$

$$\Uparrow$$

$$3^{\cancel{n}+1} \cdot \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{1}{\cancel{3^n}} \cdot \frac{n^n}{n!} \xrightarrow[n \to +\infty]{} 3 \cdot \frac{1}{e} > 1$$

- $\lim_{n \to +\infty} \dfrac{e^n \cdot n!}{n^n} = ?$

Teo (Stolz - Cesaro) Siano a_n, b_n due successioni con b_n monotona e divergente ($\lim_{n \to +\infty} b_n = \pm\infty$)

Allora se $\lim_{n \to +\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell \in \mathbb{R} \cup \{\pm\infty\}$

allora $\lim_{n \to +\infty} \dfrac{a_n}{b_n} = \ell$

Dim <u>NO</u> !

Esempio $\lim_{n \to +\infty} \left(\sum_{i=1}^{n} i^k \right) \cdot \frac{1}{n^{k+1}}$ $\quad k \in \mathbb{N} - \{0\}$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Per $k = 1$ $\quad \left(\sum_{i=1}^{n} i \right) \frac{1}{n^2} = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} \xrightarrow[n \to +\infty]{} \frac{1}{2}$

$a_n = \sum_{i=1}^{n} i^k \qquad b_n = n^{k+1}$

monotona crescente divergente

$$\lim_{n \to +\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

$$a_{n+1} - a_n = \sum_{i=1}^{n+1} i^k - \sum_{i=1}^{n} i^k = (n+1)^k$$

$$\lim_{n \to +\infty} \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}} \quad (*)$$

$$b_{n+1} - b_n = (n+1)^{k+1} - n^{k+1}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} n^i - n^{k+1}$$

$$b_{n+1} - b_n = \sum_{i=0}^{k+1} \binom{k+1}{i} n^i - n^{k+1}$$

$$= \underset{i=k+1}{\cancel{n^{k+1}}} + \underset{i=k}{\binom{k+1}{k} n^k} + \underset{i=k-1}{\binom{k+1}{k-1} n^{k-1}} + \dots \text{ termini di } \deg < k-1 \; \cancel{- n^{k+1}}$$

$$= (k+1) \cdot n^k + \text{termini di } \deg < k$$

$$\star \quad \lim_{n \to +\infty} \frac{(n+1)^k}{(k+1)\, n^k + \text{ter di } \deg < k} = \frac{1}{k+1} = \lim_{n \to +\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

Teo S-C $\Rightarrow \displaystyle \lim_{n \to +\infty} \frac{a_n}{b_n} = \lim_{n \to +\infty} \frac{1}{n^{k+1}} \cdot \sum_{i=1}^{n} i^k = \frac{1}{k+1}$

Non vale il viceversa del Teo S-C

Esempio $a_n = (-1)^n + 1 \qquad b_n = n$

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{(-1)^n+1}{n} = 0$$

Ma $$\lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to+\infty} \frac{[(-1)^{n+1}+1]-[(-1)^n+1]}{(n+1)-n} =$$

$$= \lim_{n\to+\infty} 2(-1)^{n+1}$$ irregolare / non ha limite

Applicazioni

A Se $\lim_{n\to+\infty} a_{n+1} - a_n = \ell \in \mathbb{R} \cup \{\pm\infty\}$ allora $\lim_{n\to+\infty} \frac{a_n}{n} = \ell$

- Se $\ell \neq 0$ a_n non è Cauchy $\Rightarrow$ non converge

- Se $\ell > 0$, dire che $\lim_{n \to +\infty} a_{n+1} - a_n = \ell$

vuol dire $|a_{n+1} - a_n| > \ell - \varepsilon$ per n abbastanza grande

$\exists \varepsilon$ t.c. $\ell - \varepsilon > 0$

Per n abb grande $a_{n+1} > a_n + \ell - \varepsilon$

$a_{n+2} > a_{n+1} + \ell - \varepsilon > a_n + 2(\ell - \varepsilon)$

$\vdots$

$a_{n+m} > a_{n+m-1} + \ell - \varepsilon > \dots > a_n + m(\ell - \varepsilon) \xrightarrow[m \to +\infty]{} +\infty$

- Se $\ell < 0$ $\lim_{n \to +\infty} a_n = -\infty$
- Se $\ell = 0$ a_n di Cauchy converge $\Rightarrow \lim_{n \to +\infty} \frac{a_n}{n} = 0$

Banale

Se $\ell \neq 0$ Teo S-C applicato con $a_n = a_n$

e $b_n = n$

$$\lim_{n \to +\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \to +\infty} \frac{a_{n+1} - a_n}{1} = \ell \text{ per ipotesi}$$

$$\overset{\text{Teo}}{\Longrightarrow} \lim_{n \to +\infty} \frac{a_n}{b_n} = \lim_{n \to +\infty} \frac{a_n}{n} = \ell .$$

(B) Media aritmetica Sia a_n t.c. $\lim_{n \to +\infty} a_n = \ell \in \mathbb{R} \cup \{\pm\infty\}$

Allora $\lim_{n \to +\infty} \left(\sum_{i=0}^{n-1} a_i \right) \frac{1}{n} = \ell$

Dim. Esercizio Provare a farlo applicando (A)

S-C $A_n = \sum_{i=0}^{n-1} a_i$ $b_n = n$ monotona divergente

$$\lim_{n \to +\infty} \frac{A_{n+1} - A_n}{b_{n+1} - b_n} = \lim_{n \to +\infty} \frac{\sum_{i=0}^{n} a_i - \sum_{i=0}^{n-1} a_i}{(n+1) - n} =$$

$= \lim_{n \to +\infty} \frac{a_n}{1} = \ell$ per ipotesi

$\Rightarrow$ Per S-C $\lim_{n \to +\infty} \frac{A_n}{b_n} = \lim_{n \to +\infty} \frac{\sum_{i=0}^{n-1} a_i}{n} = \ell$.

Esercizio $\displaystyle\lim_{n \to +\infty} \frac{\ln(n!)}{n}$

$a_n = \ln(n)$

$\displaystyle\lim_{n \to +\infty} a_n = +\infty$

$$\left(\sum_{i=1}^{n} a_n\right) / n = \frac{\ln(n) + \ln(n-1) + \dots + \ln(1)}{n} = \frac{\ln(n!)}{n}$$

$\downarrow$ Media Aritmetica

$+\infty$

$\displaystyle\Rightarrow \lim_{n \to +\infty} \frac{\ln(n!)}{n} = +\infty$

Oss 1. $\displaystyle\frac{\ln(n!)}{n}$ F. I. $\displaystyle\frac{+\infty}{+\infty}$

$$\frac{\ln(n) + \ln(n-1) + \dots + \ln(1)}{n} = \frac{\ln(n)}{n} + \frac{\ln(n-1)}{n} + \dots + \frac{\ln(1)}{n}$$

F. I. $0 \cdot +\infty$

Oss 2 Rapporto $\lim_{n \to +\infty} \frac{\ln(n!)}{n}$

$$\frac{\ln((n+1)!)}{(n+1)} \cdot \frac{n}{\ln(n!)} = \frac{n}{n+1} \cdot \left(\frac{\ln(n+1) + \ln(n!)}{\ln(n!)} \right) =$$

$$= \frac{n}{n+1} \left(1 + \frac{\ln(n+1)}{\ln(n!)} \right) \xrightarrow[n \to +\infty]{} 1$$

Perchè $\frac{\ln(n+1)}{\ln(n!)} \to 0$? Visto in precedenza $\frac{3^n \cdot n!}{n^n} \to +\infty$

Quindi $\frac{3^n \cdot n!}{n^n} > 1$ definitivamente

$$\Longrightarrow n! > n^n \cdot 3^{-n} \qquad \frac{\ln(n+1)}{\ln(n!)} < \frac{\ln(n+1)}{\ln(n^n \cdot 3^{-n})} =$$

$$= \frac{\ln(n+1)}{n\ln(n) - n\ln(3)} \xrightarrow[n \to +\infty]{} 0$$

(C) **Media geometrica** Sia a_n t.c. $a_n > 0 \quad \forall n$

e $\lim_{n \to +\infty} a_n = \ell$. Allora $\lim_{n \to +\infty} \sqrt[n]{\prod_{i=0}^{n-1} a_i} = \ell$

Dim. Strategia: con il log ci riconduciamo alla media aritmetica

Def $b_n = \ln(a_n) \implies \lim_{n \to +\infty} b_n = \ln(\ell)$

(B) $\implies \lim_{n \to +\infty} \left(\sum_{i=0}^{n-1} b_i \right) \cdot \frac{1}{n} = \ln(\ell)$ ☆

$$\left(\sum_{i=0}^{n-1} b_i \right) \frac{1}{n} = \frac{\ln(a_{n-1}) + \ln(a_{n-2}) + \dots + \ln(a_0)}{n} =$$

$$= \frac{1}{n} \ln \left(\prod_{i=0}^{n-1} a_i \right) = \ln \left(\prod_{i=0}^{n-1} a_i \right)^{1/n} \xrightarrow[n \to +\infty]{☆} \ln(\ell)$$

Componendo con l'esponenziale $\lim_{n \to +\infty} \left(\prod_{i=0}^{n-1} a_i \right)^{1/n} = \ell$

D) Legame tra rapporto e radice n-esima

Sia a_n t.c. $a_n > 0$ (definitivamente) se $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \ell \in \mathbb{R}\cup\{\pm\infty\}$

a_n $\quad a_n \neq 0$ (definitivamente) $\quad \left|\frac{a_{n+1}}{a_n}\right|$

allora $\lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$.

$\sqrt[n]{|a_n|}$

Dim. Definiamo $b_n = \frac{a_{n+1}}{a_n}$ dato che $\lim_{n\to+\infty} b_n = \ell$ per ipotesi

C) Media geometrica $\Rightarrow$ $\lim_{n\to+\infty} \sqrt[n]{\prod_{i=0}^{n-1} b_i} = \ell$

$$\sqrt[n]{\prod_{i=0}^{n-1} b_i} = \Big(\underbrace{\frac{\cancel{a_1}}{a_0}}_{i=0} \cdot \underbrace{\frac{\cancel{a_2}}{\cancel{a_1}}}_{i=1} \cdot \underbrace{\frac{\cancel{a_3}}{\cancel{a_2}}}_{i=2} \cdot \ldots \cdot \underbrace{\frac{a_n}{\cancel{a_{n-1}}}}_{i=n-1}\Big)^{1/n} = \left(\frac{a_n}{a_0}\right)^{1/n} = \frac{\sqrt[n]{a_n}}{\sqrt[n]{a_0}}$$

$n \to +\infty$ $\qquad 1$

$\Rightarrow \lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$.

Esercizio $\lim_{n \to +\infty} \sqrt[n]{n}$ $\qquad e^{\ln(\sqrt[n]{n})} = e^{\frac{1}{n}\ln(n)}$

D $\quad a_n = n \qquad \lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \to +\infty} \frac{n+1}{n} = 1 \implies$

$\implies \lim_{n \to +\infty} \sqrt[n]{a_n} = \lim_{n \to +\infty} \sqrt[n]{n} = 1$

Esercizio $\lim_{n \to +\infty} \sqrt[n]{n!} \qquad a_n = n! \qquad \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} =$

$= n+1 \xrightarrow[n \to +\infty]{} +\infty \implies \lim_{n \to +\infty} \sqrt[n]{n!} = +\infty$

Esercizio $\lim_{n \to +\infty} \sqrt[n]{\sqrt[n]{n!}} = \lim_{n \to +\infty} (n!)^{1/n^2}$

$\underbrace{a_n}_{} = \sqrt[n]{n!}$ $\quad \frac{a_{n+1}}{a_n} = \underbrace{\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}}_{} = \frac{\sqrt[n+1]{(n+1)} \cdot \sqrt[n+1]{n!}}{\sqrt[n]{n!}} < \sqrt[n+1]{n+1}$ ✶ $\quad /// \quad \to 1$

$\frac{\sqrt[n+1]{n!}}{\sqrt[n]{n!}} = \frac{(n!)^{1/n+1}}{(n!)^{1/n}} < 1$ ✶ $\qquad \frac{1}{n} > \frac{1}{n+1} \qquad (n!)^{1/n} > (n!)^{1/n+1}$ ✶

Cerco l'altro lato del confronto

$(\sqrt[n]{n!})^{n+1} = n! \cdot \sqrt[n]{n!} < n! \cdot (n+1) = (n+1)!$ (Ricordare $n^n > n!$ $\quad n = \sqrt[n]{n^n} > \sqrt[n]{n!}$)

Prendo la radice $(n+1)$-esima

$\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!} \qquad \underbrace{1 < \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}}_{///}$ $\Longrightarrow$ Confronto $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = 1$

Quindi $\lim\limits_{n\to+\infty} \sqrt[n]{a_n} = \lim\limits_{n\to+\infty} \sqrt[n]{\sqrt[n]{n!}} = 1$

Esercizio $\lim\limits_{n\to+\infty} \dfrac{n}{\sqrt[n]{n!}}$ ///

$a_n =$ deve essere t.c.

$\sqrt[n]{a_n} = \dfrac{n}{\sqrt[n]{n!}} \Rightarrow a_n = \dfrac{n^n}{n!}$

$$\lim_{n\to+\infty} \frac{(n+1)^{n+1}}{\underbrace{(n+1)!}_{(n+1)\cancel{n!}}} \cdot \frac{\cancel{n!}}{n^n} = \lim_{n\to+\infty} \frac{\cancel{(n+1)}}{\cancel{n+1}} \cdot \left(\frac{n+1}{n}\right)^n = e$$

$$\Rightarrow \lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \frac{n}{\sqrt[n]{n!}} = e$$

Oss Per $n \to +\infty$ $\quad n \sim e\sqrt[n]{n!} \Rightarrow n! \sim n^n e^{-n}$

Formula Stirling $\quad n! \sim n^n e^{-n} \sqrt{2\pi n}$

Esercizio $\lim_{n \to +\infty} \frac{n^n}{e^{n^2}}$

Rapporto
$\Downarrow$
Radice

Radice n-esima (criterio)

$\sqrt[n]{a_n} = \frac{n^n}{e^{n^2}}$ $a_n = \frac{n^{n^2}}{e^{n^3}}$

Rapporto (criterio)

- Rapporto $\frac{(n+1)^{n+1}}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n^n} = (n+1) \cdot \left(\frac{n+1}{n}\right)^n \cdot \frac{\cancel{e^{n^2}}}{e^{\cancel{n^2}+2n+1}}$

$= \underbrace{\frac{n+1}{e^{2n+1}}}_{\downarrow \, 0} \cdot \underbrace{\left(\frac{n+1}{n}\right)^n}_{\downarrow \, e} \xrightarrow[n \to +\infty]{} \underbrace{\begin{matrix} 0 \\ \ell < 1 \end{matrix}} \Longrightarrow \lim_{n \to +\infty} \frac{n^n}{e^{n^2}} = 0$