Funzioni potenza

Def x^a per $a \in \mathbb{Q}$

Def x^α $\alpha \in \mathbb{R}$ sfruttando la densità di $\mathbb{Q}$ in $\mathbb{R}$

È sempre possibile definire $A \subseteq \mathbb{Q}$ $B \subseteq \mathbb{Q}$

t.c. $\alpha = \operatorname{Sup}(A)$ $\alpha = \operatorname{Inf}(B)$

Per $x > 0$ $\alpha > 0$ $x^{m/n}$ $m/n \in \mathbb{Q}$ $m/n > 0$

è crescente

Posso definire $x^\alpha = \operatorname{Sup}(x^{m/n})$ $m/n \in A$

o $x^\alpha = \operatorname{Inf}(x^{m/n})$ $m/n \in B$

Funzioni esponenziali e logaritmiche

Voglio definire $f(x) = a^x$ a fissato $a \in \mathbb{R}$

- Voglio dominio $= \mathbb{R}$

Oss $x = 1/2 \Rightarrow a \geq 0$

$x = -1 \Rightarrow a \neq 0$

$\Big\}$ $f(x) = a^x$ posso def solo per $a > 0$

- $a = 1 \Rightarrow f(x) = 1^x$ è la funzione costante $= 1$ f_1

- $\underset{\sim\sim\sim}{0 < a < 1}$ $\underset{\sim\sim\sim}{a > 1}$

$a \longrightarrow \frac{1}{a}$ $(0,1) \longrightarrow (1,+\infty)$ biunivoca

$$f_a(x) = a^x = \left(\frac{1}{a}\right)^{-x} = f_{\frac{1}{a}}(-x)$$

Proprietà $\quad Im(f_a) \subseteq (0,+\infty)$

$f_a(x) = a^x$

Supponiamo $a > 1$: quindi $a^x > 1 \quad \forall x > 0$

$\forall x > y \in \mathbb{R} \quad a^{x-y} > 1$ Molt per a^y strettamente

$a^x > a^y \implies f_a$ crescente monotona

- $\sup(f_a) = +\infty$. Prendiamo $M > 0$ cerco $x \in \mathbb{R}$

t.c. $a^x > M$

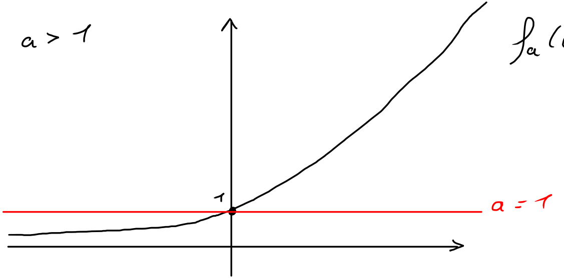
$a = 1 + b \quad b > 0 \qquad \exists n \in \mathbb{N}$ t.c. $1 + nb > M$

$a^n = (1+b)^n \underset{\text{Bernoulli}}{\geq} 1 + nb > M$ per ogni $x > n$ si ha $a^x > M$.

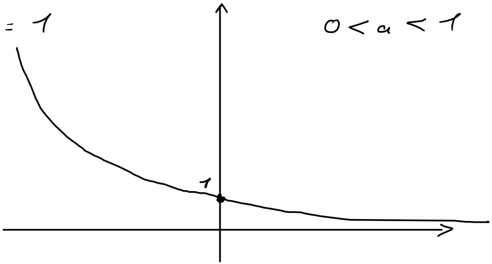
- $\inf(f_a) = 0 \qquad \forall \varepsilon > 0 \quad \exists x \in \mathbb{R}$ t.c. $a^x < \varepsilon$

$\varepsilon > 0 \implies \frac{1}{\varepsilon} > 0 \underset{\sup}{\implies} \exists y \in \mathbb{R}$ t.c. $a^y > \frac{1}{\varepsilon} \qquad a^{-y} < \varepsilon$

$f_a(\boxed{-y}) < \varepsilon$



$f_a(0) = 1$



$f_a : \mathbb{R} \longrightarrow (0, +\infty)$ surgettiva $\mathrm{Im}(f_a) = (0, +\infty)$

Se $a \neq 1$ f_a è biunivoca e posso definire

$f_a^{-1} : (0, +\infty) \longrightarrow \mathbb{R}$ $\quad y = (f_a \circ f_a^{-1})(y) = f_a(f_a^{-1}(y)) = a^{f_a^{-1}(y)}$

$x = (f_a^{-1} \circ f_a)(x) = f_a^{-1}(a^x)$

Def $\quad f_a^{-1} = \log_a$

Esempio di proprietà di $\log_a$

- Siano $x, y \in (0, +\infty)$

$$xy = \underbrace{a^{\log_a xy}} \qquad x \cdot y = a^{\log_a x} \cdot a^{\log_a y}$$
$$= \underbrace{a^{\log_a x + \log_a y}}$$

Esponenziale iniettiva
$$a^{x_1} = a^{x_2} \implies x_1 = x_2$$

Quindi $\log_a xy = \log_a x + \log_a y$

- Conseguenza immediata: $\log_a x^n = \log_a(\overbrace{x \cdot x \cdot x \cdots}^{n \text{ volte}}) =$
$$= n \log_a x \qquad n \in \mathbb{N}$$

$$\frac{1}{x} = a^{\log_a \frac{1}{x}} \qquad \frac{1}{x} = x^{-1} = (a^{\log_a x})^{-1} = a^{-\log_a x}$$
$$\implies \log_a \frac{1}{x} = -\log_a x$$

Analogamente $\log_a x^{m/n} = m/n \log_a x \qquad m/n \in \mathbb{Q}$

(conti ...)

Densità $\Rightarrow \log_a x^{\alpha} = \alpha \log_a x \qquad \forall \alpha \in \mathbb{R}$

- $a^0 = 1 \Rightarrow \log_a 1 = 0$

- $\underline{0 < a < 1}$ o $\underline{a > 1}$

$a^{-x} = \dfrac{1}{a^x}$

$$x = \underline{a^{\log_a x}}$$

$$\| $$

$$\left(\frac{1}{a}\right)^{\log_{1/a} x} = \underline{a^{-\log_{1/a} x}}$$

$$\Rightarrow \log_a x = -\log_{1/a} x$$

$b = 1/a \qquad \log_a \frac{1}{a} = -1$

- Cambio di base $\quad a, b$ in $(0,1) \cup (1, +\infty)$

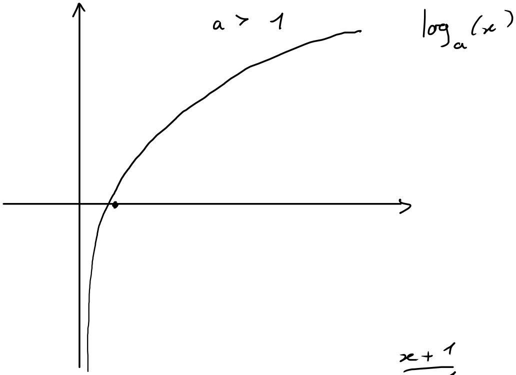
$$x = \underline{a^{\log_a x}}$$

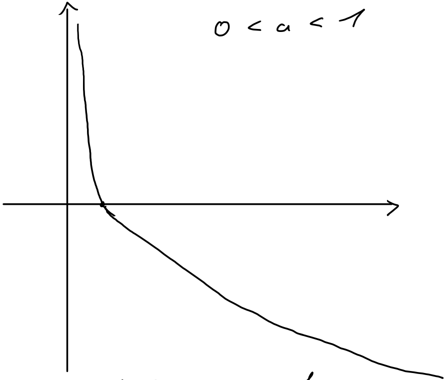
$$\| $$

$$b^{\log_b x} = \left(a^{\log_a b}\right)^{\log_b x} = \underline{a^{\log_a b \cdot \log_b x}}$$

$$b$$

$$\Rightarrow \log_a x = \log_a b \, \log_b x$$





Esempio

$$f(x) = e^{\frac{x+1}{x-1}}$$

$$g(x) = \frac{x+1}{x-1}$$

$$h(x) = e^{x}$$

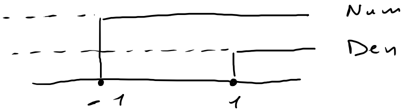
$$f(x) = (h \circ g)(x)$$

• Studio $g(x)$

Dominio $g(x) = \dfrac{x+1}{x-1}$ def $x \neq 1$

Dominio $(-\infty, 1) \cup (1, +\infty)$

Segno



$$g(x) \begin{cases} > 0 & (-\infty, -1) \cup (1, +\infty) \\ = 0 & x = -1 \\ < 0 & (-1, 1) \end{cases}$$

Monotonia $x_2 > x_1 \Longrightarrow g(x_2) - g(x_1) = \dfrac{x_2+1}{x_2-1} - \dfrac{x_1+1}{x_1-1}$

$= \cdots = \dfrac{2(x_1 - x_2)}{(x_2-1)(x_1-1)}$

Num sempre < 0

Den ....

$$g(x_2) - g(x_1) \begin{cases} > 0 & x_1 \in (-\infty, 1) \quad x_2 \in (1, +\infty) \\ < 0 & \text{se } x_1, x_2 \in (1, +\infty) \\ & x_1, x_2 \in (-\infty, 1) \end{cases}$$

non decrescente

Vediamo Inf e Sup per $x \in (1, +\infty)$

$x + 1 > x - 1 \qquad \frac{x+1}{x-1} > 1$

$\underset{(1,+\infty)}{\operatorname{Inf}}(g) = 1 \qquad \forall \varepsilon > 0$ cerco $x \in (1, +\infty)$ t.c.

$g(x) < 1 + \varepsilon \qquad \frac{x+1}{x-1} < 1 + \varepsilon$

$\cancel{x} + 1 < \cancel{x} - 1 + \varepsilon x - \varepsilon \qquad \varepsilon x > 2 + \varepsilon \qquad x > \underbrace{1 + \frac{2}{\varepsilon}}$

$\Rightarrow \underset{(1,+\infty)}{\operatorname{Inf}}(g) = 1$

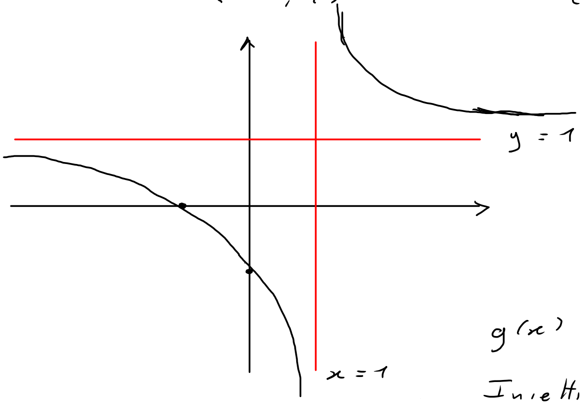
$\underset{(1,+\infty)}{\operatorname{Sup}}(g) = +\infty \qquad \forall M > 1$ cerco $x \in (1, +\infty)$ t.c.

$g(x) > M \qquad \frac{x+1}{x-1} > M$

$x + 1 > xM - M \qquad x(M - 1) < 1 + M \qquad x < \frac{1+M}{M-1}$

Esercizio

$\sup_{(-\infty,1)}(g(x)) = 1$ $\inf_{(-\infty,1)}(g(x)) = -\infty$



$g(-1) = 0$

$g(0) = -1$

In $(1,+\infty)$ mon dec

Sup $+\infty$ Inf 1

In $(-\infty,1)$ mon dec

Sup 1 Inf $-\infty$

$g(x) : (-\infty,1) \cup (1,+\infty) \longrightarrow (-\infty,1) \cup (1,+\infty)$

Iniettiva ? sí

Invertibile ... Esercizio : calcolare l'inversa

Grafico di $f(x) = e^{\frac{x+1}{x-1}} = e^{g(x)}$.

$(1, +\infty)$ e^x cresc $g(x)$ decresc $\Rightarrow$ decrescente

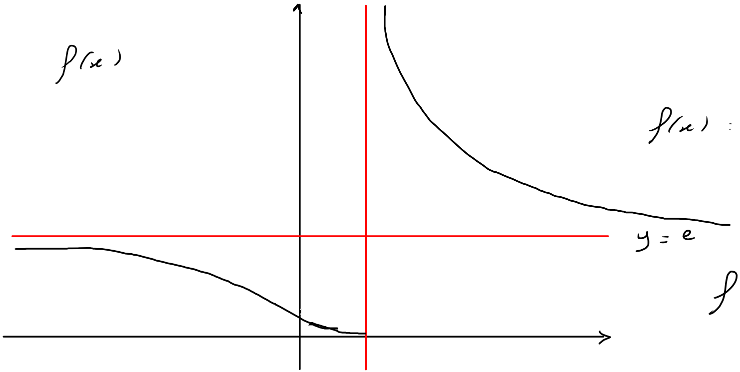
$\sup_{(1,+\infty)} g(x) = +\infty$ $\sup_{(1,+\infty)} (f(x)) = +\infty$

$\inf_{(1,+\infty)} g(x) = 1$ $\inf_{(1,+\infty)} (f(x)) = e$

$(-\infty, 1)$ e^x cresc $g(x)$ decresc $\Rightarrow$ decresc

$\sup_{(-\infty,1)} (g(x)) = 1$ $\sup_{(-\infty,1)} (f(x)) = e$

$\inf_{(-\infty,1)} (g(x)) = -\infty$ $\inf_{(-\infty,1)} (f(x)) = 0$



$f(0) = e^{-1} = 1/e$

$f(x) : (-\infty, 1) \cup (1, +\infty)$

$\downarrow$

$(0, e) \cup (e, +\infty)$

f invertibile

Calcoliamo inversa $\quad f^{-1} : (0, e) \cup (e, +\infty) \longrightarrow (-\infty, 1) \cup (1, +\infty)$

$y = e^{\frac{x+1}{x-1}} \qquad \ln(y) = \ln\left(e^{\frac{x+1}{x-1}}\right) = \frac{x+1}{x-1} \qquad f^{-1}(y) = \frac{1+\ln(y)}{\ln(y)-1}$

$x\ln(y) - \ln(y) = x + 1 \qquad x(\ln(y) - 1) = \ln(y) + 1$

Esercizio Grafico di $f(x) = \dfrac{3x + 2\sqrt{x}}{2 - \sqrt{x}}$

Funzioni simmetriche / Funzioni trigonometriche
periodiche

$f : A \to \mathbb{R}$ si dice pari se $f(x) = f(-x)$
si dice dispari se $f(-x) = -f(x)$
si dice periodica di periodo r se
$f(x+r) = f(x) \quad \forall x$

Grafico : pari o dispari $\to$ studio solo per $A \cap [0, +\infty)$
periodica $\to$ studio solo per $A \cap [x, x+r)$

Ovviamente pari e periodiche non sono iniettive
Dispari può essere iniettiva e inoltre f dispari e $0 \in A$
$\Rightarrow f(0) = 0$ (perché $\underbrace{f(0)} = f(-0) = \underbrace{-f(0)}$ $\Rightarrow f(0) = 0$)

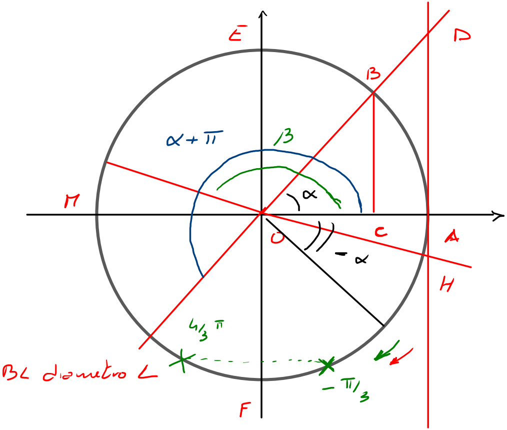
Esempi $f(x) = x^n$ n pari funzione pari
n dispari funzione dispari

$f(x) = |x|$ pari

- g funzione qualsiasi f pari $g \circ f$ pari ?
  " f dispari $g \circ f$ dispari ?
  " f pari $f \circ g$ pari ?
  " f dispari $f \circ g$ dispari ?

- g pari f pari h dispari ℓ dispari

$g \circ f$ $g \circ h$ $h \circ g$ $h \circ \ell$ pari o dispari ?



Funzioni trigonometriche

$\overline{OA} = 1 = \overline{OB}$

$AOA = 0$ senso antiorario

$AOE = 90° = \pi/2$ radianti

radiante = lunghezza dell'arco di circonferenza di raggio 1 che definisce l'angolo

$\alpha = \overset{\frown}{AB}$

Def $\text{sen}(\alpha) = \overline{BC}$ $\quad \cos(\alpha) = \overline{OC}$ $\quad tg(\alpha) = \overline{AD}$

Oss $\widehat{OBC} = \widehat{OAD}$ sono simili $\quad tg(\beta) = \overline{AH}$

$OC : OA = BC : AD$

$$\cos(\alpha)\, tg(\alpha) = 1 \cdot \text{sen}(\alpha)$$

$$tg(\alpha) = \frac{\text{sen}(\alpha)}{\cos(\alpha)}$$

Teo Pitagora su $\overset{\Delta}{OCB}$ $\quad \overline{OC}^2 + \overline{BC}^2 = \overline{OB}^2$

$$\cos(\alpha)^2 + \operatorname{sen}(\alpha)^2 = 1$$

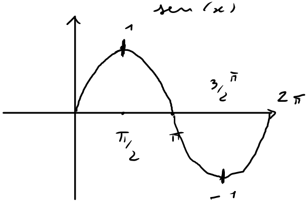
Per definizione $\cos(\alpha)$ $\operatorname{sen}(\alpha)$ $\operatorname{tg}(\alpha)$ sono periodiche di periodo 2π

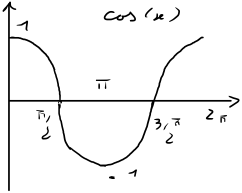
$$\operatorname{tg}(\alpha+\pi) = \frac{\operatorname{sen}(\alpha+\pi)}{\cos(\alpha+\pi)} = \frac{-\operatorname{sen}(\alpha)}{-\cos(\alpha)} = \operatorname{tg}(\alpha)$$

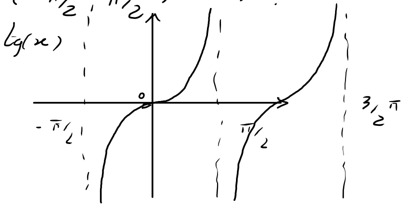
La tg è periodica di periodo π

sen, tg dispari $\quad$ cos pari $\quad$ cos, sen $[0, 2\pi] \longrightarrow ?$

tg $(-\pi/2, \pi/2) \longrightarrow ?$







$tg: (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ è biunivoca

Funzione inversa $tg^{-1} = arctg : \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$

$\star$ $sen : [-\pi/2, \pi/2] \longrightarrow [-1, 1]$ biunivoca

Funzione inversa $arcsen : [-1, 1] \longrightarrow [-\pi/2, \pi/2]$ $\star$

$\cos : [0, \pi] \longrightarrow [-1, 1]$ biunivoca

$arccos : [-1, 1] \longrightarrow [0, \pi]$

$$sen\left(\frac{10}{3}\pi\right) = sen\left(\frac{4}{3}\pi\right) = sen(-\pi/3)$$

Esempio Se arcsen def come in $\star$

$$arcsen\left(sen\left(\frac{10}{3}\pi\right)\right) \neq \frac{10}{3}\pi \in [-\pi/2, \pi/2]$$

$$= arcsen(sen(-\pi/3)) = -\pi/3$$

Esercizio   Dim che $\forall x > 0$

$$\operatorname{arctg}\left(\frac{1}{x}\right) + \operatorname{arctg}(x) = \pi/2 \qquad x, \frac{1}{x} \in (0, +\infty)$$

$$\Rightarrow \quad \operatorname{arctg}(x), \ \operatorname{arctg}\left(\frac{1}{x}\right) \in (0, \pi/2)$$

$$\left(y = \operatorname{arctg}\left(\frac{1}{x}\right) \right) \qquad \operatorname{tg}(y) = \operatorname{tg}\left(\operatorname{arctg}\left(\frac{1}{x}\right)\right) = \frac{1}{x}$$

$$\operatorname{tg}(y) = \frac{1}{x} \qquad \underbrace{x}_{} = \frac{1}{\operatorname{tg}(y)} = \frac{\cos(y)}{\operatorname{sen}(y)} \underset{\uparrow}{=} \frac{\operatorname{sen}(\pi/2 - y)}{\cos(\pi/2 - y)} =$$

grafico

$$= \operatorname{tg}(\pi/2 - y)$$

$$\underbrace{\operatorname{arctg}(x)}_{} = \operatorname{arctg}\left(\operatorname{tg}(\pi/2 - y)\right) = \pi/2 - y = \pi/2 - \underbrace{\operatorname{arctg}\left(\frac{1}{x}\right)}_{}$$

Esercizio

1. $x < 0$ $\quad \operatorname{arctg}(x) + \operatorname{arctg}\left(\frac{1}{x}\right) = -\pi/2$

2. Grafico di $\operatorname{arctg}(x)$ $\quad \operatorname{arcsen}(x)$ $\quad \arccos(x)$

3. f, g due funzioni $\mathbb{R} \to \mathbb{R}$

   Trovare relazioni tra $\operatorname{Inf}(f+g)$ e $\operatorname{Inf}(f)$ $\operatorname{Inf}(g)$

   $\operatorname{Sup}(f+g)$ e $\operatorname{Sup}(f)$ $\operatorname{Sup}(g)$

   $\operatorname{Sup}(f)$ $\quad \operatorname{Sup}(-f)$

   $\operatorname{Inf}(f)$ $\quad \operatorname{Inf}(-f)$