Esempi

1 $\quad f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ lineare

Teo. dim. $\Rightarrow \quad 4 = \dim(\mathrm{Im}) + \dim(\mathrm{Ker})$

$\mathrm{Im}(f) \subseteq \mathbb{R}^3 \Longrightarrow \dim(\mathrm{Im}) \leq 3$

Infatti $\quad rg(f) = \dim(\mathrm{Im}) \leq \min\{4,3\}$

QUINDI:

$$\dim(\mathrm{Ker}(f)) = 4 - \dim(\mathrm{Im}(f)) \geq 4-3 = 1$$

CIOÈ $\ker(f) \neq \{0_V\}$

$\Rightarrow$ f non è iniettiva

In questo caso:

f surietiva $\Leftrightarrow \dim(\mathrm{Im}) = 3$

Esempio concreto:

$$f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 + x_3 + x_4 \\ 2x_2 + 2x_3 + 2x_4 \\ 3x_3 + 3x_4 \end{pmatrix}$$

$$\mathrm{Im}(f) = \left\langle f\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \right\rangle$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle$$

I 3 vettori sono lin. IND.

$$\Rightarrow \quad \dim(\mathrm{Im}(f)) = rg(f) = 3$$

f è suriettiva

Teo. dimensione :

$$\dim(\mathrm{Ker}(f)) = 4 - rg(f) = 4 - 3 = 1$$

$\square$

② $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$3 = \dim(\operatorname{Im}(f)) + \dim(\operatorname{Ker}(f))$$

ALLORA:

$\dim(\operatorname{Im}(f)) = 3 \iff f$ suriettive

$$\Updownarrow$$

$\dim(\operatorname{Ker}(f)) = 0 \iff f$ iniettiva

Esempio esplicito:

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 5x_2 - 3x_3 \\ 2x_2 - 4x_3 \\ 3x_1 + 9x_2 + 9x_3 \end{pmatrix}$$

$$\mathrm{Im}(f) = \left\langle f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 9 \end{pmatrix}, \begin{pmatrix} -3 \\ -4 \\ 9 \end{pmatrix} \right\rangle$$

$\dim(\mathrm{Im}) = 3 \iff$ I 3 vettori sono lin. IND.

$\Longleftrightarrow \ \mathrm{Ker}(f) = \{0_V\}$

$\Longleftrightarrow \ \dim(\mathrm{Ker}(f)) = 0$

$$\mathrm{Ker}(f): \begin{cases} x_1 + 5x_2 - 3x_3 = 0 \\ 2x_2 - 4x_3 = 0 \\ 3x_1 + 9x_2 + 9x_3 = 0 \end{cases}$$

Alg. Gauss: $(3) \longleftrightarrow (3) - 3(1)$

$$\begin{cases} x_1 + 5x_2 - 3x_3 = 0 \\ 2x_2 - 4x_3 = 0 \\ -6x_2 + 18x_3 = 0 \end{cases}$$

$(3) \longleftrightarrow (3) + 3\cdot(2)$ usiamo eq.(2)

$$\begin{cases} x_1 + 5x_2 - 3x_3 = 0 \\ 2x_2 - 4x_3 = 0 \\ 6x_3 = 0 \end{cases}$$

eq 3: $x_3 = 0$

$\rightarrow$ ep.2: $x_2 = 0$

$\rightarrow$ eq. 1: $x_1 = 0$

conclusione: $\mathrm{Ker}(f) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

eq.$^{\text{nte}}$: i tre vettori che generano e' immagine sono lin. IND.

Pertanto: f è iniettiva & suriettiva

3 $f: \mathbb{R}^5 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 - 3x_3 + 7x_4 + 9x_5 \\ x_2 + 4x_3 + 8x_4 - x_5 \\ x_3 + 6x_4 + 7x_5 \end{pmatrix}$$

$$f : \mathbb{R}^5 \to \mathbb{R}^3$$

$5 > 3 \Rightarrow$ f non può essere iniettiva

$$rg(f) \leq \min\{3, 5\} = 3$$

$$\Rightarrow \dim(Ker(f)) \geq 5 - 3 = 2$$

$$Im(f) = \langle f\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, f\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, f\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, f\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, f\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

$$Im(f) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 6 \end{pmatrix}, \begin{pmatrix} 9 \\ -1 \\ 7 \end{pmatrix} \right\rangle$$

$\dim(Im) \geq 2$ poiché $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

sono lin. IND.

$\dim(Im) \leq 3$ poiché $Im \subseteq \mathbb{R}^3$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$ sono lin. IND.

poiché

$$\lambda_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = \underline{0}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + 2\lambda_2 - 3\lambda_3 = 0 \\ \lambda_2 + 4\lambda_3 = 0 \\ \lambda_3 = 0 \end{cases}$$

conclusione: ho trovato 3 vettori di Im lin. IND.

$$\Rightarrow \dim(\mathrm{Im}) = 3$$

In particolare: f è suriettiva

$$\dim(\mathrm{Ker}) = 5 - 3 = 2$$