Giovedì 6 no ricevimento

Venerdì 7 11:30 - 13:30

Algebra Lineare

Lunedì ore 16:00

ricevimento on-line

su TEAMS

COORDINATE RISPETTO ad una BASE

Teorema Sia V sp. vettoriale

Sia $\mathcal{B} = \{ v_1, \_, v_n \}$ BASE di V

ALLORA:

$\forall \; v \in V \quad \exists$ unici

$a_1, \_, a_n \in \mathbb{K}$ t.c.

$$v = a_1 \cdot v_1 + \_ + a_n \cdot v_n$$

Def. $a_1, \dots, a_n$ sono le coordinate di v rispetto alla BASE $\mathcal{B} = \{v_1, \dots, v_n\}$

dim (Teo): Esercizio

Esistenza: segue perché sono generatori

unicità:

suggerimento: Per assurdo

se $\exists$ 2 modi per scrivere v:

$$v = \begin{cases} a_1 v_1 + \dots + a_n v_n \\ b_1 v_1 + \dots + b_n v_n \end{cases}$$

allora $0_V = v - v = (a_1 - b_1) v_1 + \dots (a_n - b_n) v_n$

$\Rightarrow v_1, \dots, v_n$ sono lin. DIP.

poiché $(a_i - b_i) \neq 0$ ASSURDO $\square$

Esempi

① BASE CANONICA

$$V = \mathbb{R}^3 \qquad \mathcal{B} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

$$v = \begin{pmatrix}5\\-2\\3\end{pmatrix} = 5 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + (-2) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + 3 \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}$$

In questo caso: le componenti del vettore coincidono con le coordinate

2 $\quad V = \mathbb{R}^2$

$$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

n.b.: $\begin{cases} \text{i 2 vettori sono lin. IND.} \\ \dim \mathbb{R}^2 = 2 \end{cases}$

$\Rightarrow$ 2 vettori lin. IND. in $\mathbb{R}^2$ $\Rightarrow$ costituiscono una BASE

Prendiamo $\quad v = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

Le sue coordinate rispetto BASE canonica $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ sono $2, -6$

Cerchiamo le coordinate di

$v = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$ rispetto base $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Dobbiamo risolvere il sistema

$$a_1 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

$$\begin{cases} 3a_1 + a_2 = 2 \\ a_1 + 2a_2 = -6 \end{cases}$$

eq(2) $\longleftrightarrow$

eq(2) $- \frac{1}{3}$ eq(1): $\begin{cases} 3a_1 + a_2 = 2 \\ \frac{5}{3} a_2 = -\frac{20}{3} \end{cases}$

eq(2): $a_2 = -4$

eq(1): $a_1 = \frac{1}{3}(2 - a_2) = \frac{1}{3}(2+4) = 2$

Conclusione:

$$v = \begin{pmatrix} 2 \\ -6 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-4) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Le coordinate di v rispetto base $\mathcal{B}$ sono: $2, -4$

APPLICAZIONI LINEARI

DEF V, W spazi vettoriali su $\mathbb{K}$

$$f : V \to W$$ funzione

$$v \mapsto f(v)$$

si dice APPLICAZIONE LINEARE

SE

$$\forall\, v_1, v_2 \in V, \quad \forall\, \lambda_1, \lambda_2 \in \mathbb{K}$$

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \cdot f(v_1) + \lambda_2 f(v_2)$$

SPIEGAZIONE :

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \cdot \boxed{f(v_1)} + \lambda_2 \cdot \boxed{f(v_2)}$$

Equiv.$^{\text{mte}}$

$f : V \to W$ lineare

$$\Updownarrow$$

1. $\forall\, v_1, v_2 \in V$

$$f(v_1+v_2) = f(v_1) + f(v_2)$$

2. $\forall\, \lambda \in \mathbb{K},\ \forall\, v \in V$

$$f(\lambda \cdot v) = \lambda \cdot f(v)$$

ESEMPI

1. DERIVATA è appl. lineare

$$(g_1+g_2)' = g_1' + g_2'$$

$$(\lambda \cdot g)' = \lambda \cdot g'$$

der: $\{\text{funzioni derivabili}\} \to \{\text{funzioni continue}\}$

$$g \longmapsto g' = \mathrm{der}(g)$$

è LINEARE

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2 $\mathbb{R} = \mathbb{R}^1$ sp. vettoriale di dim $= 1$

$f(x)$ è lineare $\Longleftrightarrow$ $f(x) = c \cdot x$ c COSTANTE esempio $\quad c = 5$

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto 5 \cdot x \qquad \boxed{\text{è lineare}}$$

- $f(x+t) = 5(x+t) = 5x + 5t$
  $= f(x) + f(t)$

- $f(\lambda \cdot x) = 5 \cdot (\lambda \cdot x) = \lambda \cdot (5 \cdot x)$
  $= \lambda \cdot (f(x))$

oss $\quad f : \mathbb{R} \to \mathbb{R}$ lineare

per descrivere $f(x)$

è sufficiente conoscere $f(1)$

$$f(x) = f(x \cdot 1) = x \cdot f(1)$$

controesempi:

- $f: \mathbb{R} \to \mathbb{R}$

  $f(x) = \operatorname{sen}(x)$

  NON È LINEARE

$$\operatorname{sen}(x+t) \neq \operatorname{sen}(x) + \operatorname{sen}(t)$$

- $f: \mathbb{R} \to \mathbb{R}$

  $f(x) = x^2$

  NON E' LINEARE

$$f(x+t) = (x+t)^2 \neq x^2 + t^2 = f(x) + f(t)$$

$f(2\cdot 1) = 4 \neq 2 = 2\cdot f(1)$

Esempio 3

$$f: \mathbb{R}^2 \to \mathbb{R}$$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 5x_1 + 8x_2$$

è LINEARE

Infatti: preso $v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$v_2 = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$v_1 + v_2 = \begin{pmatrix} x_1 + t_1 \\ x_2 + t_2 \end{pmatrix}$$

$$f(v_1+v_2) = f\begin{pmatrix} x_1+t_1 \\ x_2+t_2 \end{pmatrix} =$$

$$5(x_1+t_1) + 8(x_2+t_2) =$$

$$(5x_1+8x_2) + (5t_1+8t_2) =$$

$$f(v_1) + f(v_2)$$

Seconda condizione:

$$f(\lambda \cdot v_1) = f\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} =$$

$$= 5(\lambda x_1) + 8(\lambda x_2) =$$

$$= \lambda \cdot (5x_1 + 8x_2)$$

$$= \lambda \cdot f(v_1)$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 5x_1 + 8x_2$$

calcoli espliciti

In questo caso, ad esempio:

$$f\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = 5 \cdot \overset{x_1}{1} + 8 \cdot \overset{x_2}{2} = 21$$

$$f\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) = 5 \cdot (\overset{x_1}{-2}) + 8 \cdot \overset{x_2}{3} = 14$$

Se consideriamo il vettore:

$$3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 7 \\ 0 \end{pmatrix} = 5 \cdot 7 + 8 \cdot 0 = 35$$

f LINEARE $\Rightarrow$

$$f\begin{pmatrix} 7 \\ 0 \end{pmatrix} = f\left(3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-2)\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right)$$

per linearità

$$= 3 \cdot \underbrace{f\begin{pmatrix} 1 \\ 2 \end{pmatrix}} + (-2) \cdot \underbrace{f\begin{pmatrix} -2 \\ 3 \end{pmatrix}}$$

$$= 3 \cdot 21 + (-2) \cdot 14$$

$$= 35$$

PROPRIETA' di essere lineare $\Rightarrow$

Se conosco $f(v_1)$ & $f(v_2)$

allora $f(v_1+v_2) = f(v_1) + f(v_2)$

Esercizio ①.

Sia $f: \mathbb{R}^2 \to \mathbb{R}^2$ lineare

t.c. $f\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$

$f\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$

Sapendo che f è lineare

calcolare $f\begin{pmatrix}4\\6\end{pmatrix}$.

SOL. $\begin{pmatrix}4\\6\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix}$

f lineare:

$\Longrightarrow f\begin{pmatrix}4\\6\end{pmatrix} = f\begin{pmatrix}1\\2\end{pmatrix} + f\begin{pmatrix}3\\4\end{pmatrix}$

$= \begin{pmatrix}5\\6\end{pmatrix} + \begin{pmatrix}-1\\2\end{pmatrix}$

$= \begin{pmatrix}4\\8\end{pmatrix}$

Esercizio 2.

$\exists\, f : \mathbb{R}^2 \to \mathbb{R}^2$ lineare

tale che

$$f\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$$

$$f\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}7\\9\end{pmatrix} \quad ?$$

Sol. $\begin{pmatrix}2\\4\end{pmatrix} = 2\cdot\begin{pmatrix}1\\2\end{pmatrix}$

QUINDI se f è lineare

$$f\begin{pmatrix}2\\4\end{pmatrix} = f\left(2\cdot\begin{pmatrix}1\\2\end{pmatrix}\right) = 2\cdot\left[f\begin{pmatrix}1\\2\end{pmatrix}\right]$$

condizione necessaria per essere lineare

$$f\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix} \Rightarrow f\begin{pmatrix}2\\4\end{pmatrix}=2\cdot f\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}6\\8\end{pmatrix}$$

NOSTRO CASO: $f\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}7\\9\end{pmatrix}$

Poiché $\begin{pmatrix}7\\9\end{pmatrix}\neq\begin{pmatrix}6\\8\end{pmatrix}$

allora non $\exists$ f lineare con queste proprietà

$\square$

Esercizio 3

Sia $f: \mathbb{R}^2 \to \mathbb{R}^3$ lineare

Sapendo che $f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$f\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}$

Calcolare $f\begin{pmatrix}3\\2\end{pmatrix}$.

SOL. Cerchiamo $\lambda_1, \lambda_2 \in \mathbb{R}$

t.c. $\begin{pmatrix}3\\2\end{pmatrix} = \lambda_1\begin{pmatrix}1\\0\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\end{pmatrix}$

$$\Leftrightarrow \begin{cases} \lambda_1 + 2\lambda_2 = 3 \\ \lambda_2 = 2 \end{cases}$$

$$\Leftrightarrow \quad \lambda_2 = 2, \quad \lambda_1 = 3 - 4 = -1$$

CIOÈ:

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = (-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

f lineare

$$f\begin{pmatrix} 3 \\ 2 \end{pmatrix} = (-1) \cdot \left[f\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + 2 \cdot \left[f\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]$$

$$= (-1) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$\square$

Esercizio 4 $\exists f: \mathbb{R}^2 \to \mathbb{R}^2$

LINEARE

tale che: $f\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

$$f\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$f\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad ?$$

SOL. $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}$ sono lin. IND.

$\Rightarrow$ Costituiscono BASE di $\mathbb{R}^2$

Cerchiamo λ_1, λ_2 t.c.

$$\begin{pmatrix}3\\1\end{pmatrix} = \lambda_1\begin{pmatrix}1\\2\end{pmatrix} + \lambda_2\begin{pmatrix}2\\-1\end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + 2\lambda_2 = 3 \\ 2\lambda_1 - \lambda_2 = 1 \end{cases}$$

eq(2) $\leftarrow$ eq(2) $-$ 2 eq(1)

$$\begin{cases} \lambda_1 + 2\lambda_2 = 3 \\ -5\lambda_2 = -5 \end{cases}$$

Sol. $\lambda_2 = 1$, $\lambda_1 = 1$

CIOÈ : $\boxed{\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix}}$

f lineare $\Rightarrow f\begin{pmatrix}3\\1\end{pmatrix} = f\begin{pmatrix}1\\2\end{pmatrix} + f\begin{pmatrix}2\\-1\end{pmatrix}$

nostro caso:

$$f\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \neq \begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}$$

$$= f\begin{pmatrix}1\\2\end{pmatrix} + f\begin{pmatrix}2\\-1\end{pmatrix}$$

Quindi non $\exists$ f lineare con queste proprietà

VISTO:

$$\boxed{\begin{array}{c} f: \mathbb{R} \to \mathbb{R} \text{ lineare} \\ \Updownarrow \\ f(x) = c \cdot x \\ c \text{ costante} \end{array}}$$

Analogamente

$$\boxed{\begin{array}{c} f: \mathbb{R}^2 \to \mathbb{R} \text{ lineare} \\ \Updownarrow \\ f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \cdot x_1 + c_2 \cdot x_2 \\ \text{con } c_1, c_2 \text{ costanti} \end{array}}$$

TEOREMA di struttura per f lineari

$f: \mathbb{R}^n \to \mathbb{R}^k$

TEOREMA.

$f: \mathbb{R}^n \to \mathbb{R}^k$ LINEARE $\Longleftrightarrow$

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n \end{pmatrix}$$

con $a_{11}, a_{12}, \dots, a_{kn} \in \mathbb{R}$
COSTANTI

SPIEGAZIONE :

- **Dominio** = $\mathbb{R}^n$ $\Rightarrow$ calcolo f su vettori di n coordinate

incognite: $x_1, x_2, \dots, x_n$

- **Codominio** = $\mathbb{R}^k$ $\Rightarrow$ il valore di f è un vettore di k coordinate

- Ogni coordinata di $f(v)$ = riga dell'espressione di f

la scrivo come funzione di $x_1, \dots, x_n$

RIGA $i \longleftrightarrow$ $\boxed{\sum_{j=1}^{n} a_{ij} x_j}$

Esempi

(1) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ lineare

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 + 8x_2 \\ 3x_1 - 7x_2 \end{pmatrix}$$

2 incognite

2 righe

(2) $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_1 + 8x_2 - 9x_3 \\ x_1 - x_2 + 3x_3 \end{pmatrix}$$

3 incognite 2 righe

③ $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 + 8x_2 \\ 3x_1 - x_2 \\ x_1 + 2x_2 \end{pmatrix}$$

2 incognite 3 righe

④ $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_1 + 8x_2 - x_3 \\ 2x_1 - 8x_2 + 7x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

3 incognite 3 righe

Esercizio

Data $f: \mathbb{R}^2 \to \mathbb{R}^2$ lineare

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_2 \\ 4x_1 + 2x_2 \end{pmatrix}$$

Calcolare $f\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 1 \end{pmatrix}, f\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Sol

$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 - 0 \\ 4 \cdot 1 + 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 0 - 1 \cdot 1 \\ 4 \cdot 0 + 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$f\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}3\cdot 3 - 4\\ 4\cdot 3 + 2\cdot 4\end{pmatrix} = \begin{pmatrix}5\\20\end{pmatrix}$$

$$= 3\cdot\begin{pmatrix}3\\4\end{pmatrix} + 4\cdot\begin{pmatrix}-1\\2\end{pmatrix}$$

$$= 3\cdot f\begin{pmatrix}1\\0\end{pmatrix} + 4\cdot f\begin{pmatrix}0\\1\end{pmatrix}$$

<u>Esercizio</u>. Data

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ lineare

$$f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1 \quad - 5x_3\\ 2x_2 + x_3\end{pmatrix}$$

1. Calcolare $f\begin{pmatrix}1\\0\\0\end{pmatrix}, f\begin{pmatrix}0\\1\\0\end{pmatrix}, f\begin{pmatrix}0\\0\\1\end{pmatrix}$

2. Risolvere l'eq.

$$f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

Sol. 1. $f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ $\leftarrow$ $x_1=1$, $x_2=0$, $x_3=0$

$f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$ $\leftarrow$ $x_1=0$, $x_2=1$, $x_3=0$

$f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-5\\1\end{pmatrix}$ $\leftarrow$ $x_1=0$, $x_2=0$, $x_3=1$

2.

L'eq. $f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\Updownarrow$$

SISTEMA LINEARE

$$\begin{cases} x_1 \qquad - 5x_3 = 1 \\ \qquad 2x_2 + x_3 = 2 \end{cases}$$

sistema 2 eq., 3 incognite

GRADINO DI LUNGHEZZA = 2

Poniamo $x_3 = t$

eq(2): $x_2 = \frac{1}{2}(2-t) = 1 - \frac{1}{2}t$

eq(1) : $x_1 = 1 + 5t$

SOLUZIONE : $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

PROP. $f: V \to W$ lineare

$\mathcal{B} = \{v_1, \ldots, v_n\}$ BASE di V

ALLORA: f è univocamente determinata da

$f(v_1), f(v_2), \ldots, f(v_n)$

DIM Dato $v \in V$

$\exists$ unici $\lambda_1, \ldots, \lambda_n$ t.c.

$v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$

Pertanto, f lineare

$$\Downarrow$$

$$f(v) = \lambda_1 \cdot f(v_1) + \lambda_2 \cdot f(v_2) + \dots + \lambda_n f(v_n)$$

$\square$

Esempio

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

BASE CANONICA di $\mathbb{R}^3$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$f\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}$$

$$f\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}$$

$$f\begin{pmatrix}1\\2\\3\end{pmatrix}=\ ?$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix}=1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+2\begin{pmatrix}0\\1\\0\end{pmatrix}+3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

f lineare $\Rightarrow$

$$f\begin{pmatrix}1\\2\\3\end{pmatrix}=1\cdot f\begin{pmatrix}1\\0\\0\end{pmatrix}+2\cdot f\begin{pmatrix}0\\1\\0\end{pmatrix}+3\cdot f\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= 1 \cdot \begin{pmatrix} 5 \\ 6 \end{pmatrix} + 2 \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 3 \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

Data $f : V \longrightarrow W$ LINEARE

ad f si associano 2 sottospazi vettoriali:

$$Im(f) \subseteq W$$

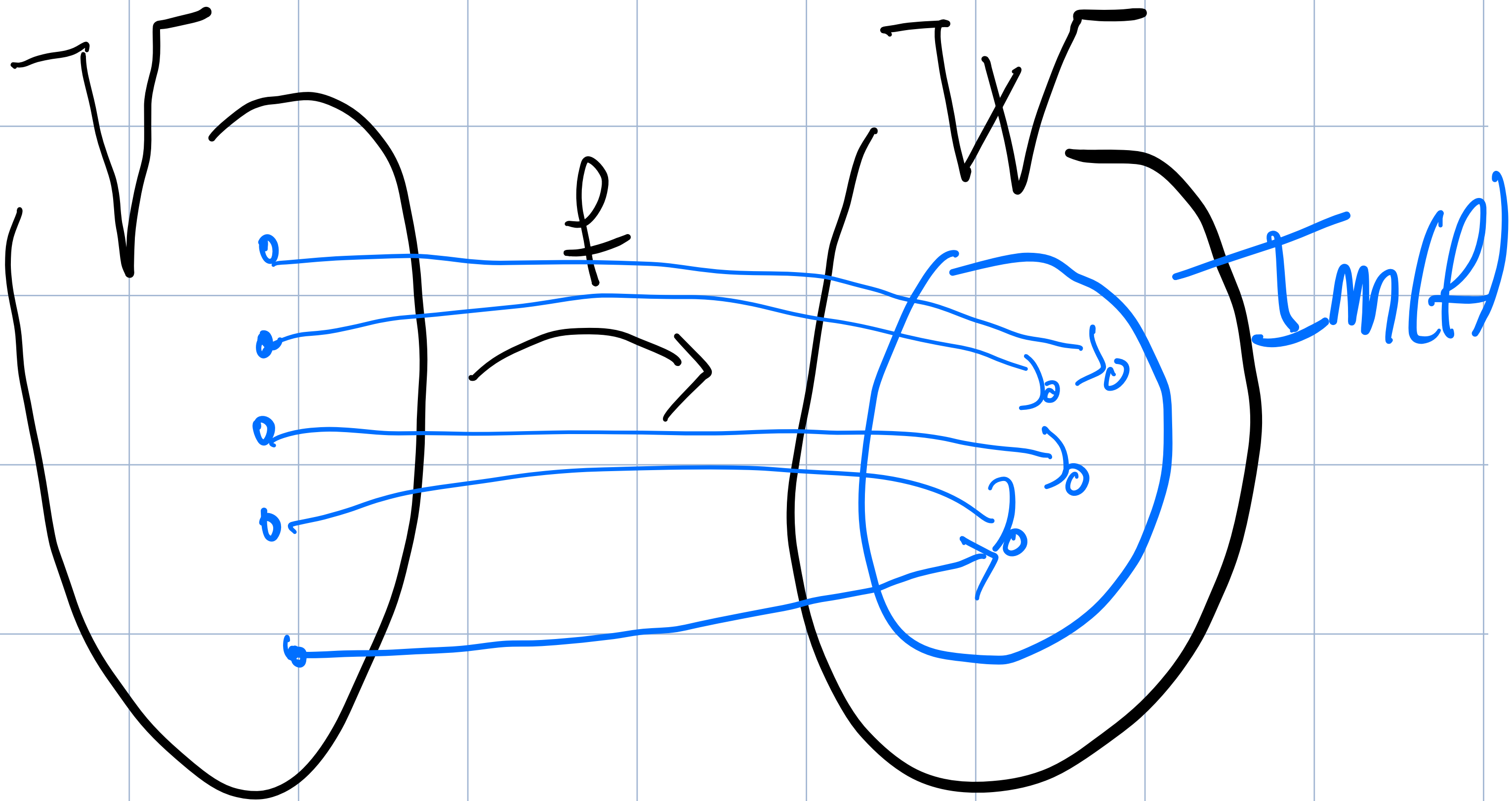
$$Ker(f) \subseteq V$$

Immagine di f =

$$= \mathrm{Im}(f) = \{ w \in W : \exists\, v \in V \text{ t.c. } f(v) = w \}$$

NUCLEO di f (oppure KERNEL di f)

$$= \mathrm{Ker}(f) = \{ v \in V : f(v) = 0_W \}$$



$Im(f)$ è l'immagine come funzione

ESEMPI: 1) $\cos : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto \cos(x)$

$Im(f) = [-1, 1] \subset \mathbb{R}$

2) $\exp: \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto e^x$

$\mathrm{Im}(f) = \mathbb{R}_{>0}$

CASO di APPLICAZIONI LINEARI

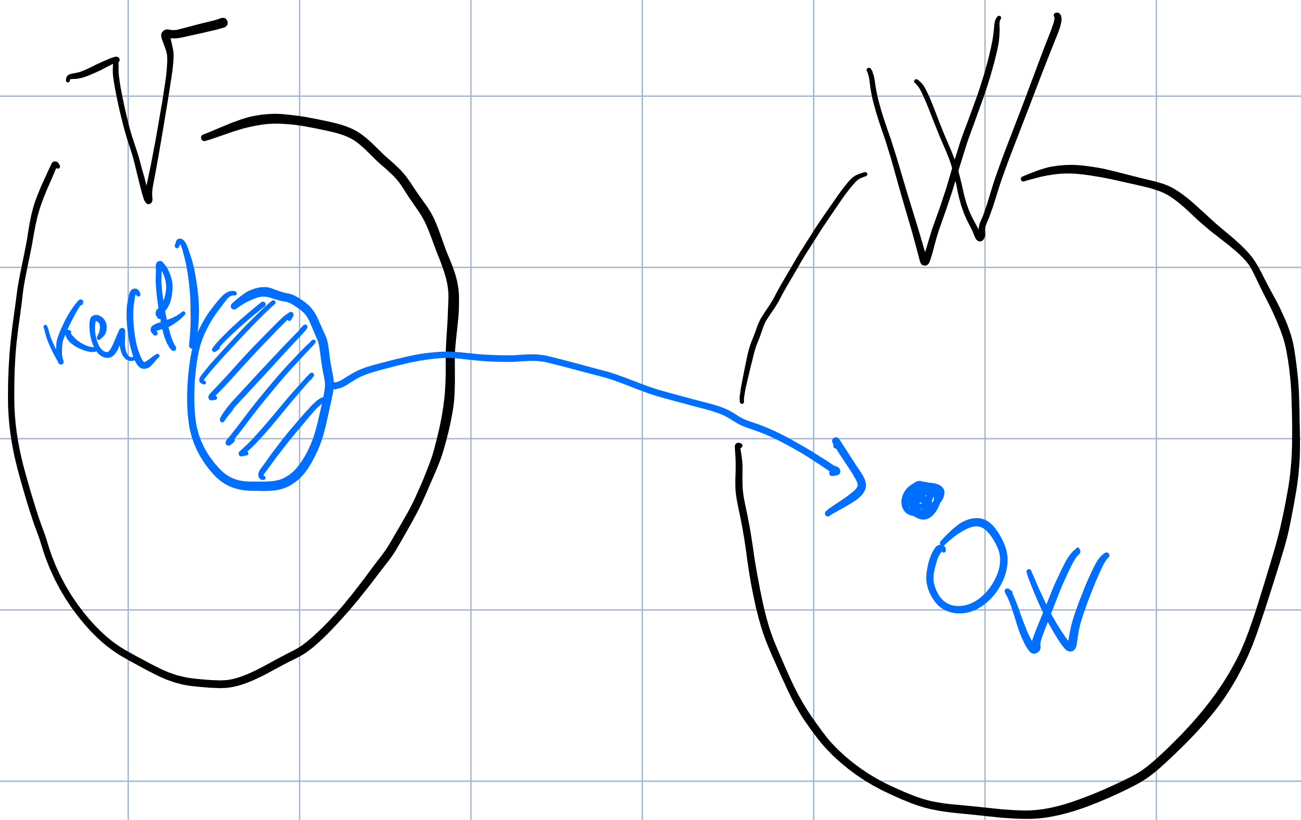
PROP. $f: \mathbb{R}^n \to \mathbb{R}^k$ lineare

$\mathcal{B} = \{v_1, \dots, v_n\}$ BASE di $\mathbb{R}^n$

ALLORA:

$$\mathrm{Im}(f) = \mathrm{Span}(f(v_1), \dots, f(v_n))$$

NUCLEO di f



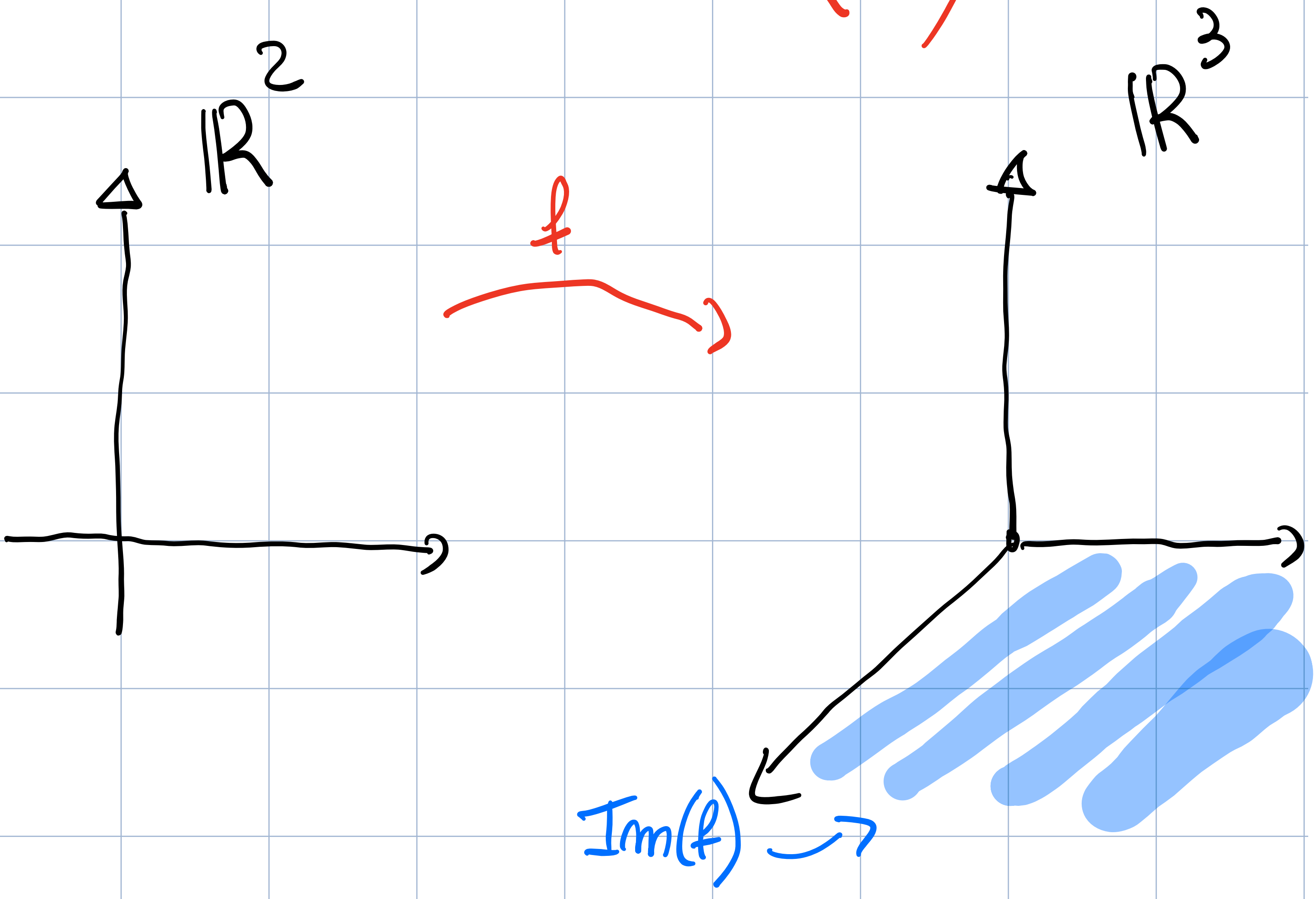
$Ker(f)$ = insieme dei vettori che vanno in 0_W

ESEMPI

1. Immersione canonica

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$



$$\mathrm{Im}(f) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \right\}$$

$$= \text{PIANO ORIZZONTALE} : \{x_3 = 0\}$$

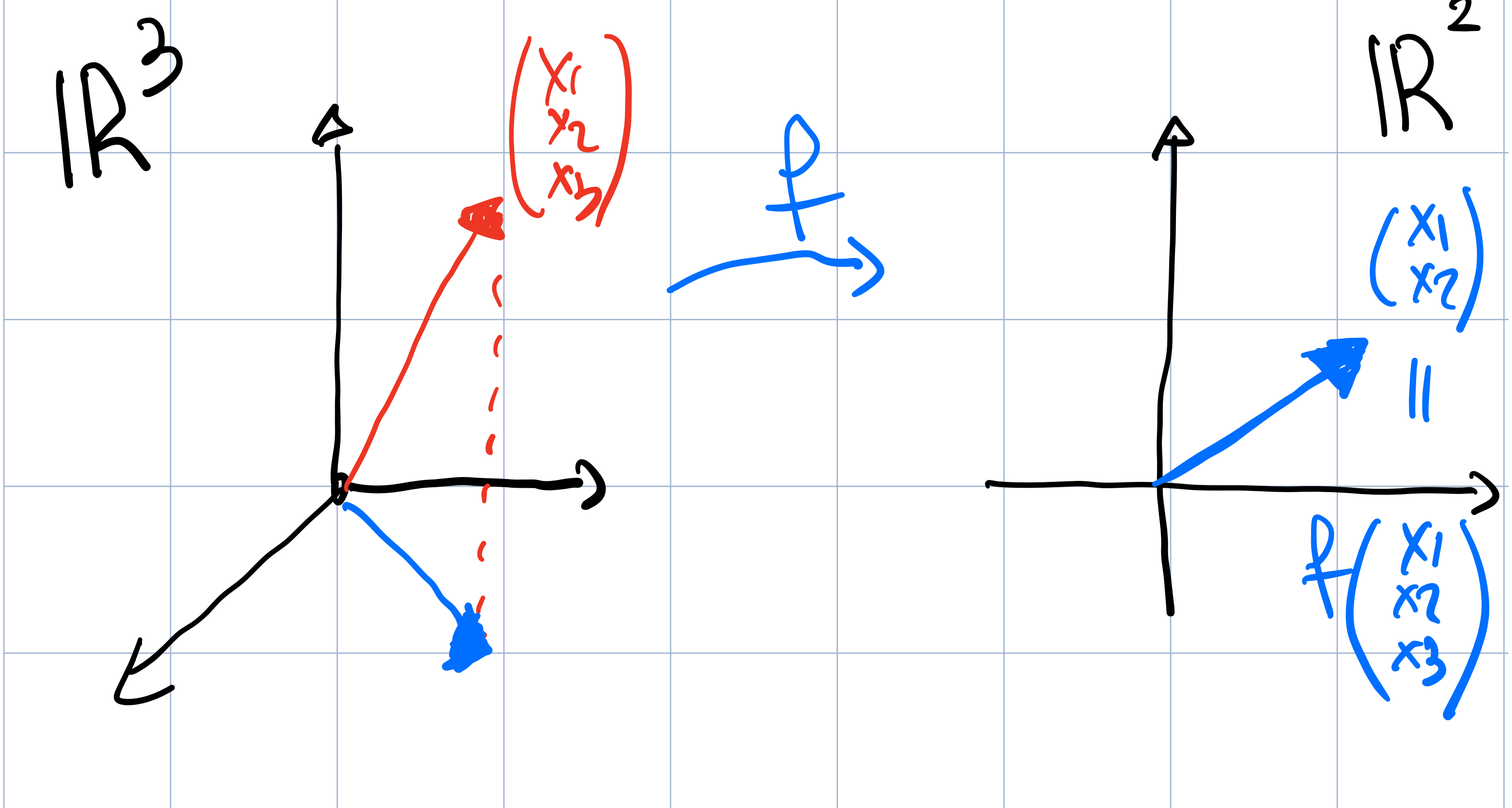
$$= \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$= \mathrm{Span}\left(f\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

2 PROIEZIONE CANONICA

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

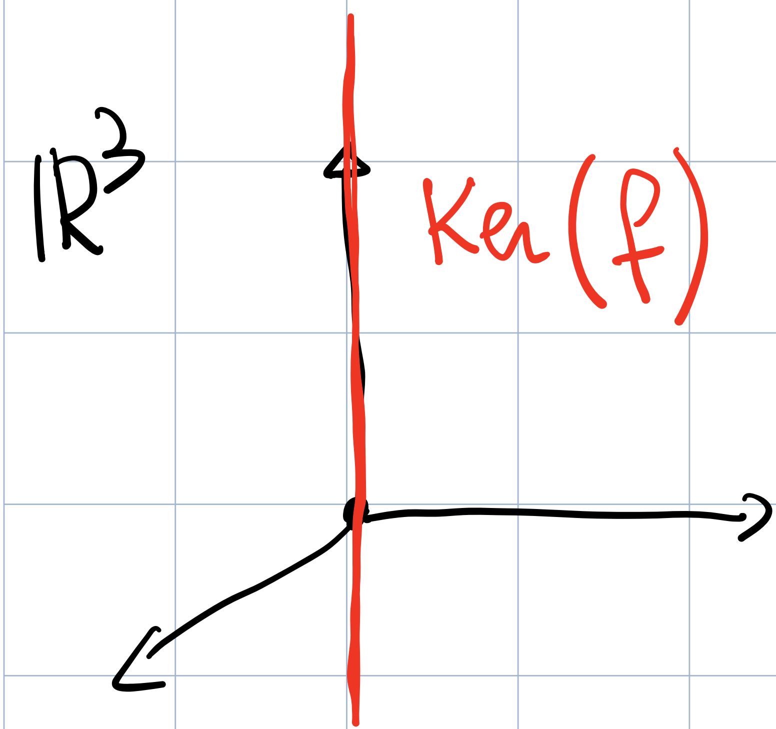


$$\mathrm{Im}(f) = \mathbb{R}^2$$

$$\mathrm{Ker}(f) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$



$\longleftrightarrow$ vettori perpendicolari al piano orizzontale

③ Immersione non canonica

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_1 \\ 5x_2 \end{pmatrix}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$ BASE di $\mathbb{R}^2$

$$Im(f) = Span\left(f\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= Span\left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \right)$$