Variations on a theme by Ribet

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(joint work with Damian Rössler)

Introduction

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Theorem (Serre, Imai) If k is a number field, A/k an abelian variety, $A(k(\mu_{p^\infty}))_{\mathrm{tors}}$ is finite.

Even better:

Theorem (Ribet) Let k be a number field, and $K = k(\mu_\infty)$, the field obtained by adjoining all roots of unity. If A/k is an abelian variety, the torsion subgroup $A(K)_{\mathrm{tors}}$ is finite.

Remarks

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2. *Ribet's theorem is specific to the torsion subgroup.* The Mordell–Weil rank of an abelian variety can be infinite over the maximal cyclotomic extension of a number field obtained by adjoining *all* complex roots of unity (Rosen–Wong).

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Theorem. Let k and K be as above, and set $G := \mathrm{Gal}(\bar{k}|K)$. Let X be a smooth proper geometrically connected variety X defined over k.

For all **odd** i and all j the groups $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}/\mathbf{Z}(j))^G$ are finite.

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1. The twist j does not really play a role in the statement since G fixes all roots of unity.

2. Ribet's theorem is the special case $i = j = 1$ of the above statement.

Indeed, for $X = A^*$, the Kummer sequence induces a Galois-equivariant isomorphism

$$H^1_{\text{ét}}(\overline{A}^*, \mathbf{Q}/\mathbf{Z}(1)) \cong H^1_{\text{ét}}(\overline{A}^*, \mathbf{G}_m)_{\mathrm{tors}}$$

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But since the Néron–Severi group of an abelian variety is torsion free,

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3. The theorem is *not* true for even degree cohomology. Counterexamples: projective space $\mathbf{P}^n$, smooth complete intersections of dim. n in $\mathbf{P}^r$ (their cohomology is the same as that of $\mathbf{P}^n$ except in degree n where that of $\mathbf{P}^r$ is a direct summand).

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Remark. For X itself the Chow groups $CH^i(X)$ are conjecturally finitely generated (a consequence of the generalized Bass conjecture on the finite generation of motivic cohomology groups of regular schemes of finite type over $\mathbf{Z}$).

Evidence for the conjecture in codimension 2

Theorem. Assume moreover $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$. Then the torsion subgroup of $CH^2(X)$ has finite exponent.
It is finite if furthermore the ℓ-adic cohomology groups $H^3_{\text{ét}}(\overline{X}, \mathbf{Z}_\ell)$ are torsion free for all ℓ.

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1. All geometric assumptions of the theorem are satisfied, for instance, by smooth complete intersections of dimension > 2 in projective space.

2. For X/k satisfying $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$ the torsion part of $CH^2(X)$ is known to be finite *over k* (Colliot-Thélène, Raskind, Salberger).

Method of proof

Over $\bar{k}$ we have Bloch's Abel–Jacobi map

$$CH^i(\overline{X})_{\mathrm{tors}} \to H^{2i-1}(\overline{X}, \mathbf{Q}/\mathbf{Z}(i))$$

which is injective for $i = 2$.

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So for $G = \mathrm{Gal}(\bar{k}|K)$ we have an injection

$$CH^2(\overline{X})^G_{\mathrm{tors}} \hookrightarrow H^3(\overline{X}, \mathbf{Q}/\mathbf{Z}(2))^G$$

where the group on the right hand side is finite by our cohomological theorem. It therefore suffices to study the group $\ker(CH^2(X_K) \to CH^2(\overline{X}))$.

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Positive characteristic analogues

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The analogue of Ribet's theorem is plainly false for an abelian variety that is already defined over $\overline{\mathbf{F}}$. We have to impose a non-isotriviality condition.

Theorem (Lang–Néron) Let k and K be as above, A/k an abelian variety whose base change A_K has trivial $K|\overline{\mathbf{F}}$-trace. Then the torsion subgroup of $A(K)$ is finite.

In fact, the Lang–Néron theorem says that the group $A(K)$ is even finitely generated, but we only need the torsion part here.

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Definition. Let $k = \mathbf{F}(C)$ and $K = \overline{\mathbf{F}}(C)$ be as above, X/k a smooth proper geometrically connected variety, ℓ a prime different from $p = \mathrm{char}(\mathbf{F})$.

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Definition. Let $k = \mathbf{F}(C)$ and $K = \overline{\mathbf{F}}(C)$ be as above, X/k a smooth proper geometrically connected variety, ℓ a prime different from $p = \mathrm{char}(\mathbf{F})$.
The cohomology group $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell(j))$ has *large variation* if after finite extension of the base field $\mathbf{F}$ there exists a proper flat morphism $\mathcal{X} \to C$ of finite type with generic fibre X and two $\mathbf{F}$-rational points $c_1, c_2 \in C$ such that the fibres $\mathcal{X}_{c_1}$, $\mathcal{X}_{c_2}$ are smooth and the associated Frobenius elements $\mathrm{Frob}_{c_1}, \mathrm{Frob}_{c_2}$ act on $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell(j))$ with coprime characteristic polynomials.

Remarks

1. As usual, the action of the Frob_{c_r} $(r = 1, 2)$ is to be understood as follows. We pick a decomposition group $D_r \subset \mathrm{Gal}(\bar{k}|k)$ attached to c_r; it is defined only up to conjugacy but this does not affect the definition. The smoothness condition on $\mathcal{X}_{c_r}$ implies that the inertia subgroup $I_r \subset D_r$ acts trivially on cohomology, hence we have an action of $D_r/I_r = \langle \mathrm{Frob}_{c_r} \rangle$ on $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell(j))$.

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2. It is a consequence of the Weil conjectures that the above definition is independent of the prime ℓ.

3. In the case where X is an abelian variety and $\mathcal{X}$ its Néron model over C, the large variation assumption with respect to the model $\mathcal{X}$ has the following geometric reformulation: there exist two closed points $c_1, c_2 \in C$ whose associated geometric fibres are abelian varieties over $\overline{\mathbf{F}}$ having no common simple isogeny factor.

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If the $k|\mathbf{F}$-trace of X is nontrivial, it is not hard to check that all geometric fibres must have a common simple isogeny factor. It would be nice to know whether the converse holds.

Cohomological theorem

Based on the above definition, we have

Theorem. Let k, K and X be as in the previous definition. Assume moreover that $i > 0$ and $j \in \mathbf{Z}$ are such that the cohomology group $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell(j))$ has large variation. Then the group $H^i_{\text{ét}}(\overline{X}, (\mathbf{Q}/\mathbf{Z})'(j))^G$ is finite, where

$$G = \mathrm{Gal}(\bar{k}|K)$$

and

$$(\mathbf{Q}/\mathbf{Z})'(j) = \bigoplus_{\ell \neq p} \mathbf{Q}_\ell/\mathbf{Z}_\ell(j).$$

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Conjecture. Let k, K and X be as in the above definition. Given $i > 0$, assume that the the cohomology group $H^{2i-1}_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell(i))$ has large variation for $\ell \neq p$, where $p = \mathrm{char}(k)$.

Then the prime-to-p torsion subgroup of $CH^i(X_K)$ is finite.

Evidence in codimension 2

Theorem. Assume moreover that X is a projective **surface** which is liftable to characteristic 0 and for which the coherent cohomology group $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X)$ vanishes. Under the large variation assumption for $i = 2$, the prime-to-p torsion subgroup of $CH^2(X_K)$ is finite.

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The liftability assumption holds for smooth complete intersections or for surfaces satisfying the condition $H^2_{\mathrm{Zar}}(X, \mathcal{T}_{X/k}) = 0$ in addition to $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X)$, where $\mathcal{T}_{X/k}$ denotes the tangent sheaf. We have to restrict to dimension 2 in order to ensure the vanishing of $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X)$ for the lifting as well.

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Statement 2. If i is odd, we have

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for all but finitely many primes p.

Proof of Statement 1

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Lemma. For every p the largest subextension of $K|k$ unramified outside the primes dividing p and infinity is obtained as the composite of $k(\mu_{p^\infty})$ with the largest subextension of $K|k$ unramified at all finite primes (which is a finite extension).

Proof of Statement 1

Assume p is such that $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_p(j))^G \neq 0$. The Galois group $\Gamma := \mathrm{Gal}(\overline{k}|k)$ acts on $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_p(j))^G$ via its quotient Γ/G.

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– Then the above Lemma implies that, again up to replacing k by a finite extension, the action of Γ on W factors through $\Gamma_p := \mathrm{Gal}(k(\mu_{p^\infty})|k)$.

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– Therefore, by a theorem of Tate, some open subgroup of D_v, and hence of Γ, acts on W via the direct sum of integral powers of the p-adic cyclotomic character χ_p. Replacing k by a finite extension for the last time, we may assume that the whole of Γ acts in this way.

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– A Frobenius element F_w at a prime w of good reduction thus acts with eigenvalues that are integral powers of $\chi_p(F_w) = Nw$ (the cardinality of the residue field of w). But by the Weil conjectures as proven by Deligne, these eigenvalues should have absolute value $(Nw)^{i/2-j}$, a contradiction for odd i. □

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– Up to replacing k by a finite extension, we may assume that for *all* primes p, every inertia subgroup in $\mathrm{Gal}(\overline{k}|k')$ associated with a prime not lying above p acts unipotently on $H^i_{\text{ét}}(\bar{X}, \mathbf{Q}_p(j))$. This uses the strong form of the local monodromy theorem (de Jong–Deligne)

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– Next, we replace k by its maximal extension contained in K in which no finite prime ramifies.

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1. p is unramified in k.
2. $\mu_p \not\subset k$.
3. X has good reduction at the primes dividing p.

Assume now that there exist infinitely may primes p satisfying the conditions above for which $H^i_{\text{ét}}(\overline{X}, \mathbf{Z}/p\mathbf{Z}(j))^G \neq 0$. We shall derive a contradiction.

Proof of Statement 2

– By arguments similar to the above, for each such p the restriction of the action of Γ/G to a simple Γ-submodule $W_p \subset H^i_{\text{ét}}(\overline{X}, \mathbf{Z}/p\mathbf{Z}(j))^G$ factors through $\mathrm{Gal}(k(\mu_p)|k) \cong \mathbf{F}_p^\times$. Since W_p is simple for the action of Γ, it must be 1-dimensional over $\mathbf{F}_p$, with Γ acting by a power $\bar{\chi}_p^{n(p)}$ of the mod p cyclotomic character $\bar{\chi}_p$.

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– By Serre's tame inertia conjecture (proven in our case by Fontaine–Laffaille, and by Caruso in general), there exists a bound N *independent of* p such that the integer $n(p)$ appearing in the above action satisfies $n(p) \leq N$.

Proof of Statement 2

– By arguments similar to the above, for each such p the restriction of the action of Γ/G to a simple Γ-submodule $W_p \subset H^i_{\text{ét}}(\overline{X}, \mathbf{Z}/p\mathbf{Z}(j))^G$ factors through $\mathrm{Gal}(k(\mu_p)|k) \cong \mathbf{F}_p^\times$. Since W_p is simple for the action of Γ, it must be 1-dimensional over $\mathbf{F}_p$, with Γ acting by a power $\bar{\chi}_p^{n(p)}$ of the mod p cyclotomic character $\bar{\chi}_p$.

– By Serre's tame inertia conjecture (proven in our case by Fontaine–Laffaille, and by Caruso in general), there exists a bound N *independent of* p such that the integer $n(p)$ appearing in the above action satisfies $n(p) \leq N$.

Proof of Statement 2

– Choose a place w of k not dividing p where X has good reduction, and let Nw be the cardinality of its residue field. The Frobenius at w acts on W_p as multiplication by $(Nw)^{n(p)}$. If Q is its characteristic polynomial on $H^i_{\text{ét}}(\bar{X}, \mathbf{Q}_p(j))$ (which has $\mathbf{Z}$-coefficients), we thus have $Q((Nw)^{n(p)}) \equiv 0$ modulo p.

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– By the previous paragraph, this congruence holds for infinitely many p but with $n(p)$ varying between 0 and a fixed bound N. Hence for some integer $0 \leq n(p) \leq N$ we must have $Q((Nw)^{n(p)}) = 0$. But by the Weil conjectures proven by Deligne, we must then have $(Nw)^{n(p)} = (Nw)^{i/2-j}$, which is impossible for odd i.

Concluding remarks

1. In his proof, Ribet used the Oort–Tate classification of finite group schemes at the point where we invoked Serre's tame inertia conjecture. On the other hand, instead of our final weight argument (which nicely parallels the proof of Statement 1) he exploited the finiteness of global torsion on abelian varieties over k. In the general case we do not have such deep global information at our disposal.

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2. Although the two statements are different in nature, there are remarkable similarities between the above proof and that of Faltings for his theorem that the height of abelian varieties over number fields is bounded in an isogeny class. Compare especially with the rendition by Deligne in his Bourbaki seminar.