Geometry of Severi–Brauer varieties (after Kollár)

Let k be a field. A Severi–Brauer variety is a k-variety P such that $P_{\overline{k}} \cong \mathbf{P}^d_{\overline{k}}$ for some d.

Observation: Up to scaling by an element of $k^\times$ there is a unique nonsplit extension

$$0 \to O_P \to F(P) \to T_P \to 0 \tag{1}$$

of vector bundles on P, where T_P is the tangent bundle.

Proof: $\mathrm{Ext}^1(T_P, O_P) = \mathrm{Ext}^1(O_P, Hom(T_P, O_P)) = H^1(P, \Omega^1_{P/k})$. To compute the latter, we may pass to $\overline{k}$ and compute $H^1(\mathbf{P}^d, \Omega^1_{\mathbf{P}^d/\overline{k}})$. Recall from Hartshorne the exact sequence:

$$0 \to \Omega^1_{\mathbf{P}^d/\overline{k}} \to O_{\mathbf{P}^d}(-1)^{\oplus(d+1)} \to O_{\mathbf{P}^d} \to 0$$

whose long exact sequence gives $H^1(\mathbf{P}^d, \Omega^1_{\mathbf{P}^d/\overline{k}}) \cong H^0(\mathbf{P}^d, O_{\mathbf{P}^d}) = \overline{k}$. □

The dual of the above exact sequence is

$$0 \to O_{\mathbf{P}^d} \to O_{\mathbf{P}^d}(1)^{\oplus(d+1)} \to T_{\mathbf{P}^d} \to 0$$

so comparison with (1) shows

$$F(P)_{\overline{k}} \cong O_{\mathbf{P}^d}(1)^{\oplus(d+1)}.$$

Corollary. $A(P) := \mathrm{End}\, F(P)^{\mathrm{opp}}$ is a central simple algebra of degree $d+1$. This is the central simple algebra associated with P.

Remark: Quillen has constructed before the bundle $F(P)$ by applying descent to the PGL_{d+1}-equivariant sheaf $O_{\mathbf{P}^d}(1)^{\oplus(d+1)}$ on $\mathbf{P}^d_{\overline{k}}$ along the PGL_{d+1}-torsor P, and defined $A(P)$ as above. The above method is more explicit.

Quite generally, given a proper geometrically connected k-variety X and a line bundle $\bar{L}$ on $\bar{X}$, consider the category $T(\bar{L})$ of vector bundles F on X whose pullback to $\bar{X}$ is isomorphic to a direct sum of copies of $\bar{L}$.

Observation: The category $T(\bar{L})$ is abelian semisimple and if $E(\bar{L})$ is an object of minimal rank, then every object is isomorphic to a direct sum of copies of $E(\bar{L})$. Moreover, $\mathrm{End}\, E(\bar{L})$ is a division algebra over k ('geometric Schur Lemma'), and $E(\bar{L})$ is the only object with this property.

Corollary (Geometric version of Wedderburn's theorem) For $X = P$ and $\bar{L} = O_{\mathbf{P}^d}(1)$ we have $F(P) \cong E^{\oplus r}$ for some $r > 0$ and a vector bundle E on P such that $\operatorname{End} E$ a division algebra.

Observation: In the general case, given a morphism $p : Y \to X$ of k-varieties, we have $p^*E(\bar{L}) \cong E(p^*\bar{L})$ for a line bundle $\bar{L}$ on Y.

Proof: p induces an algebra morphism $\operatorname{End} E(\bar{L}) \to \operatorname{End} p^*E(\bar{L})$. The source is a division algebra over k and the target has the same dimension. Thus the map is an isomorphism, and $\operatorname{End} p^*E(\bar{L})$ is a division algebra, so that $p^*E(\bar{L}) \cong E(p^*\bar{L})$.

Corollary: If $X(k) \neq \emptyset$, the case $Y = \operatorname{Spec} k$ implies that $E(\bar{L})$ has rank 1, so that $\bar{L}$ descends to a line bundle on X.

For $X = P$ and $\bar{L} = O_{\mathbf{P}^d}(1)$ this is **Châtelet's theorem:** if $P(k) \neq \emptyset$, then $O_{\mathbf{P}^d}(1)$ descends to a line bundle on P and induces a morphism $P \to \mathbf{P}^d$ that is an isomorphism over $\overline{k}$, hence over k.

Remark: Châtelet's theorem implies that every Severi–Brauer variety is split by a finite separable field extension. Henceforth we denote by $\overline{k}$ a separable closure of k.

A subvariety $Q \subset P$ in a Severi–Brauer variety is **twisted linear** if over $\overline{k}$ it becomes the inclusion of a linear subspace in $\mathbf{P}^d$.

Châtelet correspondence: There is a 1-1 correspondence between

- twisted-linear subvarieties $Q \subset P$ of dimension r
- direct summands in $F(P)$ of rank $r + 1$
- left ideals in $A(P)$ of dimension $(r + 1)(d + 1)$.

(Can be checked after passing to $\overline{k}$.)

This can be used to recover P from $A(P)$ by Châtelet's method: P is the closed subvariety in $\mathrm{Grass}(d + 1, (d + 1)^2)$ corresponding to left ideals of dimension $d + 1$ in $A(P)$.

Geometric Brauer equivalence: P and P' are said to be Brauer equivalent if they contain isomorphic twisted-linear subvarieties.

Equivalently, they are Brauer equivalent if there is a twisted-linear rational map $\phi : P \rightsquigarrow P'$. [Indeed, the locus of indeterminacy Z of ϕ is a twisted-linear subvariety in P, and if Q is a complement to Z given by the Châtelet correspondence, then ϕ induces an inclusion $Q \hookrightarrow P'$.]

Dual Severi–Brauer variety: Quite generally, assume given a geometrically connected k-variety and a line bundle $\bar{L}$ on $X_{\overline{k}}$. For $\sigma \in \mathrm{Gal}(\overline{k}|k)$ denote also by $\sigma : X_{\overline{k}} \to X_{\overline{k}}$ the induced automorphism.

Assume $\bar{L}$ is such that $\sigma^*\bar{L} \cong \bar{L}$ for all σ. This does not define a descent datum on $\bar{L}$; however, the isomorphism $\sigma^*\bar{L} \cong \bar{L}$ is unique up to multiplication by a scalar. It follows that the projective space $\mathbf{P}(H^0(X_{\overline{k}}, \bar{L})^\vee)$ descends to a Severi–Brauer variety $|\bar{L}|$ over k.

We may apply the above to $X = P$ and $L = O_{\mathbf{P}^d}(1)$: indeed,

$$O_{\mathbf{P}^d}(1)^{\oplus(d+1)} \cong F(P)_{\overline{k}} \cong \sigma^* F(P)_{\overline{k}} \cong (\sigma^* O_{\mathbf{P}^d}(1))^{\oplus(d+1)}$$

implies $\sigma^* O_{\mathbf{P}^d}(1) \cong O_{\mathbf{P}^d}(1)$ for all $\sigma \in \mathrm{Gal}(\overline{k}|k)$. We get a Severi–Brauer variety $P^\vee := |O_{\mathbf{P}^d}(1)|$.

Product structure: For Severi–Brauer varieties P, Q we define

$$P \cdot Q := |\pi^*_{P_{\overline{k}}} O_{P_{\overline{k}}}(1) \otimes_{O_{P_{\overline{k}} \times Q_{\overline{k}}}} \pi^*_{Q_{\overline{k}}} O_{Q_{\overline{k}}}(1)|^\vee.$$

One quickly checks that this respects Brauer equivalence and induces an abelian group structure on equivalence classes with unit $[\{\mathrm{pt}\}]$ and inverse $P^\vee$. We get the Brauer group $\mathrm{Br}(k)$.

Lemma. Assume $\bar{L}_1$, $\bar{L}_2$ are line bundles on $P_{\overline{k}} \cong \mathbf{P}^d_{\overline{k}}$ is such that $\sigma^*\bar{L}_i \cong \bar{L}_i$ for all σ. There is a Brauer equivalence of Severi–Brauer varieties

$$|\bar{L}_1 \otimes \bar{L}_2| \sim |\bar{L}_1| \cdot |\bar{L}_2|$$

Proof: Consider the diagonal map $\Delta : P \to P \times P$. Then

$$\bar{L}_1 \otimes \bar{L}_2 \cong \Delta^*_{\overline{k}}(\pi^*_{P_{\overline{k}}}\bar{L}_1 \otimes \pi^*_{P_{\overline{k}}}\bar{L}_2).$$

For any line bundle $\bar{L}$ on $P_{\overline{k}} \times P_{\overline{k}}$ we have an induced map $H^0(P_{\overline{k}} \times P_{\overline{k}}, \bar{L}) \to H^0(P_{\overline{k}}, \Delta^*_{\overline{k}}\bar{L})$. if moreover $\bar{L}$ satisfies $\sigma^*\bar{L} \cong \bar{L}$ for all σ, it descends to a rational map $|\bar{L}| \rightsquigarrow |\Delta^*_{\overline{k}}\bar{L}|$ which is twisted-linear. In particular,

$$|\bar{L}_1 \otimes \bar{L}_2| \sim |\pi^*_{P_{\overline{k}}}\bar{L}_1 \otimes \pi^*_{P_{\overline{k}}}\bar{L}_2|.$$

The RHS is $|\bar{L}_1| \cdot |\bar{L}_2|$ by construction (we'll need this for $\bar{L}_i = O_{\mathbf{P}^d}(1)$ where it is easy.)

Given a Severi–Brauer variety P, define its **period** $\mathrm{per}(P)$ to be its order in $\mathrm{Br}(k)$. The lemma implies:

Corollary. The period of P is the smallest r for which $O_{\mathbf{P}^d}(r)$ descends to a line bundle on P.

Since $O_{\mathbf{P}^d}(d+1)$ descends to the anticanonical bundle of P, we have:

Corollary. $\mathrm{per}(P) \mid \dim P + 1$, and hence $\mathrm{Br}(k)$ is torsion.

Now define the **index** $\mathrm{ind}(P)$ to be the greatest common divisor of degrees of zero-cycles on P.

This is an invariant of the Brauer class (Brauer equivalence implies stable birationality which implies invariance by Lang–Nishimura).

Proposition. $\mathrm{ind}(P) = \dim(P^{\min}) + 1$, where $P^{\min}$ is a twisted-linear subvariety in P of minimal dimension.

Remark: Note that by the Châtelet correspondence $P^{\min}$ corresponds to a minimal direct summand in $F(P)$ and hence to the division algebra in the Brauer class of $A(P)$.

Proof of proposition: Write $m := \dim P^{\min}$. First, a general section of the tangent bundle of $\mathbf{P}^m$ has $m+1$ zeros, and therefore a general section of the tangent bundle of $P^{\min}$ gives a zero-cycle of degree $m+1$ on $P^{\min} \subset P$. It thus remains to prove that the degree of every zero-cycle on P is divisible by $m+1$.

By the Châtelet correspondence every twisted-linear subvariety of dimension r corresponds to a direct summand in $F(P)$ of rank $r+1$ which must be divisible by $m+1$, the dimension of the minimal direct summand. Thus it suffices to associate with every effective zero-cycle Z of degree $r+1$ on P a twisted-linear subvariety of dimension r on some Severi–Brauer variety Brauer equivalent to P. To do so, choose $n > 0$ so that $n(m+1) > r$. Then on $P_{\overline{k}} \cong \mathbf{P}^d$ there exists a hypersurface of degree $nm+n+1$ passing through the support of $Z_{\overline{k}}$, i.e. there is a surjection

$$H^0(P_{\overline{k}}, O(nm+n+1)) \twoheadrightarrow H^0(Z_{\overline{k}}, O(nm+n+1)|_{Z_{\overline{k}}}) \cong H^0(Z_{\overline{k}}, O_{Z_{\overline{k}}}) \cong \overline{k}^{r+1}.$$

Its kernel defines a twisted-linear subvariety $P_Z \subset |O(nm+n+1)|$ of codimension r. By the previous corollary $\mathrm{per}(P) = \mathrm{per}(P^{\min})$ divides $m+1$ and hence $O(n(m+1))$ descends to a line bundle on P. But then $|O(nm+n+1)| = |O(nm+m)| \cdot |O(1)| \sim |O(1)| = P^\vee$. Now dualize.

Corollary (Brauer's theorem) We have

$$\mathrm{per}(P) \mid \mathrm{ind}(P) \mid \mathrm{per}(P)^d$$

where $d = \dim P$. Consequently, $\mathrm{per}(P)$ and $\mathrm{ind}(P)$ have the same prime divisors.

Proof: By the previous corollary and proposition,

$$\mathrm{per}(P) = \mathrm{per}(P^{\min}) \mid (\dim(P^{\min}) + 1) = \mathrm{ind}(P).$$

On the other hand, the line bundle $O_{\mathbf{P}^d}(\mathrm{per}(P))$ descends to a line bundle on P. Intersecting d general sections gives a zero-cycle of degree $\mathrm{per}(P)^d$ on P.