Schemes: The Beginnings

We give the basic definitions and constructions concerning schemes. The word 'ring' will mean *commutative ring with unit.* Also, when referring to compact topological spaces, we do *not* assume that they are Hausdorff spaces.

1. Prime Spectra

Recall that a subset S of a ring A is called *multiplicatively closed* if $1 \in S$, $0 \notin S$ and for any $f, g \in S$ we have $fg \in S$. A *prime ideal* P is an ideal such that the set $A \setminus P$ is multiplicatively closed. An equivalent formulation of this is that the quotient ring A/P should be a (nontrivial) domain, i.e. it should have no zero-divisors. From this formulation it follows easily that any maximal ideal of A (i.e. an ideal contained in no other proper ideal of A than itself) is always a prime ideal since in this case the quotient ring is a field.

We now turn the set of prime ideals of an arbitrary ring A into a topological space.

Definition 1.1 The *prime spectrum* $\operatorname{Spec} A$ of A is the topological space whose points are prime ideals of A and a basis of open sets is given by the sets

$$D(f) := \{P : \ P \text{ is a prime ideal with} f \notin P\}$$

for all $f \in A$.

For this definition to be correct, we must verify that the system of the sets $D(f)$ is closed under finite intersections. But we have for all $f, g \in A$

$$D(f) \cap D(g) = D(fg) \qquad (1)$$

for by definition a prime ideal avoids fg if and only if it avoids f and g.

It follows from the definition that a closed subset in the topology of $\operatorname{Spec} A$ can be described as the set of prime ideals containing some fixed ideal I (generated by a system of elements $\{f_i : i \in J\}$ of A). Thus one-point sets given by maximal ideals are closed; in fact, maximal ideals give the only closed points of the prime spectrum since for any prime ideal P a closed subset containing P contains the maximal ideals containing P as well. This shows that in general the prime spectrum does not satisfy even the weakest of the separation axioms in topology. However, it enjoys a nice topological property:

Proposition 1.2 *For any ring A the prime spectrum* $\operatorname{Spec} A$ *is compact.*

First a lemma we shall also use later.

Lemma 1.3 *A system of elements $\{f_i : i \in I\}$ generates A if and only if the sets $D(f_i)$ give an open covering of* $\operatorname{Spec} A$.

Proof: Indeed, if the f_i generate A, there can be no prime ideal of A containing all of them, which is equivalent to the $D(f_i)$ covering $\operatorname{Spec} A$. If, however, they do not generate A, then they are all contained (by Zorn's Lemma) in some maximal ideal M which thus gives an element of $\operatorname{Spec} A$ not contained in any of the $D(f_i)$. □

Proof of Proposition 1.2: Let $\{U_i : i \in I\}$ be an open covering of $\operatorname{Spec} A$; we may assume that each U_i is in fact some basic open set $D(f_i)$. By the above lemma the f_i generate A. In particular, there is a relation of the form

$$a_1 f_1 + a_2 f_2 + \ldots + a_n f_n = 1 \tag{2}$$

with $a_i \in A$ and $f_1, \ldots, f_n$ chosen among the f_i above. This means, however, that already $f_1, \ldots, f_n$ generate A, i.e. the sets $D(f_1), \ldots, D(f_n)$ cover $\operatorname{Spec} A$. □

Equations of type (2) are sometimes referred to as *algebraic analogues of partitions of unity.*

Examples 1.4 We conclude this section by some easy examples.

1. The prime spectrum of a field consists of a single point, corresponding to the ideal (0).

2. The prime spectrum of $\mathbf{Z}$ is the space consisting of a closed point for each prime p, and a non-closed point corresponding to (0), called the *generic point*, whose closure is the whole space. Other closed subsets are only finite sets of primes; indeed, any ideal in $I \subset \mathbf{Z}$ is of the form $m\mathbf{Z}$ for some positive integer m; the prime ideals containing I are generated by the prime divisors of m.

3. The prime spectrum of $\mathbf{C}[x]$ consists of a closed point for each $a \in \mathbf{C}$, plus a non-closed generic point corresponding to (0). The closed subsets are again finite sets of closed points. Indeed, $\mathbf{C}[x]$ is a principal ideal ring with prime elements the polynomials $x - a$ ($a \in \mathbf{C}$), and we may argue as in the previous case.

4. If A is isomorphic to a finite direct sum $\bigoplus_{i=1}^{n} A_i$, then $\operatorname{Spec} A$ is a disjoint union of clopen sets each of which is homeomorphic to one of the $\operatorname{Spec} A_i$. To see this, observe first that if one writes e_i for the idempotent given by putting 1 at the i-th component and 0 elsewhere, any pairwise product $e_i e_j$ is 0 and hence no prime ideal P of A can avoid both e_i and e_j. However, P cannot contain all of the e_j since the sum of these is 1. Thus we conclude that P concludes all of the e_j except one, say e_i, which implies that P is of the form $A_1 \oplus \ldots A_{i-1} \oplus P_i \oplus A_{i+1} \ldots \oplus A_n$ with a prime ideal P_i of A_i. The required decomposition of $\operatorname{Spec} A$ is then induced by the map $P \mapsto P_i$.

2. Schemes – Mostly Affine

The prime spectrum of a ring is a rather coarse invariant: for instance, it cannot even distinguish between two fields. We shall remedy this by defining some additional structure on the prime spectrum. To motivate the construction to come, let us reconsider the third example from the last section.

Example 2.1 The ring $\mathbf{C}[x]$ is nothing but the ring of holomorphic functions on $\mathbf{C}$ having at worst a pole at infinity. The prime spectrum of this ring can be identified to $\mathbf{C}$ with a generic point (0) added; closed sets are finite sets not containing (0). Obviously one cannot recover $\mathbf{C}[x]$ from these data; we cannot even distinguish between constant functions. Remember, however, that we have seen in the previous chapter that a Riemann surface is uniquely determined by the underlying topological space plus the sheaf of holomorphic functions on it. If we restrict to the sheaf of holomorphic functions on $\mathbf{C}$ having at worst a pole at infinity, we can easily describe its sections over a set of the form $D(f)$ with the generic point removed (this is an open set in the complex topology). For instance, over $D(x)$ (which with the generic point thrown away identifies to $\mathbf{C}^*$) the sections are the rational functions whose denominator is a power of x, for thesections are meromorphic functions on $\mathbf{P}^1(\mathbf{C})$ and hence elements of $\mathbf{C}(T)$; moreover, any denominator other than the x^m has a zero elsewhere. We find an analogous result for $D(x - a)$ for ($a \in \mathbf{C}$); all other $D(f)$ are finite intersections of these, so the sections of the sheaf over $D(f)$ are just the restrictions of the sections over the $D(x-a)$ with $(x-a)$ dividing f.

If we wish to define something analogous to this for any ring A, we first have to extend the notion of a rational function, i.e. give a meaning to fractions of elements in an arbitrary ring A. So let S be a multiplicatively closed subset of A, i.e. a subset $S \subset A \setminus \{0\}$ containing 1 such that $x, y \in S \Rightarrow xy \in S$. We would like to define a ring A_S which is to be the "ring of fractions with numerator in A and denominator in S".

Example 2.2 When A is a domain, this is fairly easy to do since in this case A admits a fraction field K. Elements of K can be represented by fractions f/g with $f, g \in A$, $g \neq 0$, where $f/g = f_1/g_1$ whenever $fg_1 = f_1 g$. We may then take A_S to be the subring of those elements which can be written as fractions with denominators in S; this is indeed a subring as S is multiplicatively closed.

Now to treat the general case, observe first that just as the fraction field K can be defined as the object representing a certain functor, the ring A_S of the previous example is easily seen to represent the set-valued functor F given by

$$F(R) = \{\phi \in \mathrm{Hom}(A, R) : \phi(s) \text{ is a unit in } R \text{ for all } s \in S\}$$

on the category of rings. When A has zero-divisors, A has no fraction field, but the above functor F still exists.

Proposition 2.3 *The functor F is representable by a ring A_S for any ring A and multiplicatively closed subset S.*

The ring A_S is called the *localisation of A with respect to S*. By the Yoneda lemma, it is determined up to unique isomorphism. Moreover, it is equipped with a canonical homomorphism $\phi_S : A \to A_S$ sending elements of S to units which corresponds to the identity map $A_S \to A_S$.

Proof: Define A_S as a set to be the quotient of $A \times S$ by the equivalence relation:

$$(f, s) \sim (f', s') \quad \text{iff there is a } t \in S \text{ with } \quad (fs' - f's)t = 0.$$

One sees that this is indeed an equivalence relation; for transitivity, note that the equations $(fs' - f's)t = 0$ and $(f's'' - f''s')u = 0$ imply $(fs'' - f''s)s'tu = 0$

(multiply the first equation by $s''u$ and the second by st). Denote by f/s the image of (f,s) in A_S and define the addition and multiplication laws as for fractions; one checks that this is independent of the representatives chosen.

Now given a homomorphism $\phi : A \to R$ sending elements of S to units, define a homomorphism $A_S \to R$ by sending f/s to $\phi(f)\phi(s)^{-1}$ (note that units are never zero-divisors, so $\phi(s)^{-1}$ is a well-defined element of R). This is a well-defined map, for if (f', s') is another representative for f/s, we have

$$0 = \phi((fs' - f's)t) = (\phi(f)\phi(s') - \phi(f')\phi(s))\phi(t),$$

whence $\phi(f)\phi(s') = \phi(f')\phi(s)$ as $\phi(t)$ is a unit. Conversely, as any element of S maps to a unit in A_S by the map $\phi_S : A \to A_S$ sending s to $s/1$, homomorphisms $A_S \to R$ induce elements of $F(R)$ by composition with ϕ_S. Thus we have obtained a bijection between $F(R)$ and $\mathrm{Hom}(A_S, R)$ which is immediately seen to be functorial. □

We now wish to compare the prime spectra of A and A_S.

Lemma 2.4 *The map $P \mapsto \phi_S(P)A_S$ defines a canonical bijection between prime ideals P of A avoiding S and prime ideals of A_S.*

Proof: Let P be a prime ideal of A avoiding S. By this last condition, the ideal $\phi_S(P)A_S$ generated by $\phi_S(P)$ does not contain units and hence is different from A_S. Moreover, it is a prime ideal, for if $(f/s)(g/t) \in \phi_S(P)A_S$, then $ufg \in P$ for some $u \in S$, whence f or g is in P and thus (f/s) or (g/t) is in $\phi_S(P)A_S$. For surjectivity, note the easy fact that for any prime ideal Q of A_S the ideal $\phi_S^{-1}(Q)$ is a prime ideal of A avoiding S; the assertion then follows from the equality $\phi_S(\phi_S^{-1}(Q))A_S = Q$. Similarly, injectivity follows from $\phi_S^{-1}(\phi_S(P)A_S) = P$; the verification of these relations is left to the reader. □

Examples 2.5 The two key examples of localisation to be used in the sequel are the following.

1. Let S be the set $\{1, f, f^2, f^3, \ldots\}$ of all powers of f for some $f \in A$. In this case elements of A_S are represented by fractions with numerator in A and denominator a power of f; we shall use the notation A_f for this particular A_S. The previous lemma implies that $\mathrm{Spec}\, A_f$ is naturally homeomorphic to the open set $D(f)$.

2. Let P be a prime ideal of A and take S to be the complement of P; it is multiplicatively closed by primeness of P. Adopting a common

abuse of notation from the literature, we shall denote the localisation of A with respect to S by A_P instead of $A_{A\setminus P}$. The points of $\operatorname{Spec} A_P$ correspond to prime ideals of A contained in P; in particular, A_P has a unique maximal ideal generated by the image of P. Rings having a unique maximal ideal are usually called *local rings*.

This example contains the case of fraction fields: take P to be the ideal (0) in a domain.

Now we may turn to defining a sheaf of rings $\mathcal{O}_X$ on the prime spectrum X of any commutative ring A. In obvious analogy with the example of $\mathbf{C}[x]$ described above, we define $\mathcal{O}_X(D(f)) = A_f$ for all $f \in A$. To proceed further, we need an easy lemma.

Lemma 2.6 *If $f, g \in A$ are such that $D(f) \subset D(g)$, then the image of g in A_f is a unit.*

Proof: Indeed, if g did not give a unit in A_f, it would be contained in a maximal ideal Q. By Lemma 2.4 there is a unique prime ideal P of A whose image in A_f generates Q. This P contains g but not f, a contradiction. □

Combining the lemma with Proposition 2.3, we get for any inclusion $D(f) \subset D(g)$ of basic open sets a canonical restriction homomorphism $A_g \to A_f$. Clearly for a tower of inclusions $D(f) \subset D(g) \subset D(h)$ the map $A_h \to A_f$ thus obtained is the composition of the intermediate maps $A_h \to A_g$ and $A_g \to A_f$. So putting $\mathcal{O}_X(D(f)) = A_f$, we have obtained "something which behaves like a presheaf on basic open sets". That this indeed extends to a presheaf on X follows from the first statement of the following formal lemma (of which we advise the readers to skip the proof in a first reading).

Lemma 2.7 *Let X be a topological space and $\mathcal{V}$ a basis of open sets on X. Assume given for each $V \in \mathcal{V}$ a set (resp. abelian group, ring, etc.) $\mathcal{F}(V)$ and for each inclusion $V' \subset V$ of elements of $\mathcal{V}$ a map (resp. homomorphism) $\rho_{VV'} : \mathcal{F}(V) \to \mathcal{F}(V')$ satisfying $\rho_{VV} = \mathrm{id}_{\mathcal{F}(V)}$ and $\rho_{VV''} = \rho_{V'V''} \circ \rho_{VV'}$ for each tower $V'' \subset V' \subset V$ of elements of $\mathcal{V}$.*

1. *There exists a presheaf of sets (resp. abelian groups, rings, etc.) $\mathcal{F}$ on X whose sections and restriction maps over elements of $\mathcal{V}$ can be canonically identified to those given above.*

2. *Assume moreover that the $\mathcal{F}(V)$ above satisfy the sheaf axioms for all coverings of elements of $\mathcal{V}$ by elements of $\mathcal{V}$. Then there is a* unique

sheaf $\mathcal{F}$ on X whose sections and restriction maps over elements of $\mathcal{V}$ are those given above.

3. *Finally assume given two sheaves $\mathcal{F}$, $\mathcal{G}$ on X and for each $V \in \mathcal{V}$ a map $\phi_V : \mathcal{F}(V) \to \mathcal{G}(V)$ such that for each inclusion $V' \subset V$ of elements of $\mathcal{V}$ the diagram*

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ \rho^{\mathcal{F}}_{VV'}\downarrow & & \downarrow\rho^{\mathcal{G}}_{VV'} \\ \mathcal{F}(V') & \xrightarrow{\phi_{V'}} & \mathcal{G}(V') \end{array}$$

commutes. Then there is a unique morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ with the ϕ_V given as above.

Proof: For the first statement, consider for a given open set $U \subset X$ the set $\mathcal{V}_U$ of elements of $\mathcal{V}$ contained in U; this set is partially ordered by inclusion. The restriction maps $\phi_{VV'}$ for $V' \subset V \subset U$ turn the system of $\mathcal{F}(V)$ with $V \in \mathcal{V}_U$ into an inverse system. Note that this is a *non-filtered* inverse system. Define $\mathcal{F}(U)$ as the inverse limit of this system. By definition, $\mathcal{F}(U)$ consists of sequences (f_V) indexed by all $V \in \mathcal{V}_U$ with $f_V \in \mathcal{F}(V)$ having the property that $f_{V'} = \phi_{VV'}(f_V)$ whenever $V' \subset V$. If $U' \subset U$, define a restriction map $\rho_{UU'}$ by mapping the sequence (f_V) above to the sequence of those f_V for which $V \subset U'$. There is no difficulty in checking that we have thus defined a presheaf. Moreover, for $W \in \mathcal{V}$, the sections of $\mathcal{F}$ over W can be canonically identified with the elements of the prescribed set $\mathcal{F}(W)$ as in this case the sequences (f_V) defining the inverse limit are given by restrictions of elements of the prescribed $\mathcal{F}(W)$ to all elements of $\mathcal{V}$ contained in W. Thus for any U containing W, the restriction map ρ_{UW} can be identified to the map projecting a sequence (f_V) to f_W; in particular, a section in $\mathcal{F}(U)$ is uniquely determined by its restrictions to each $W \in \mathcal{V}_U$.

For the second statement, note first that unicity follows from the first sheaf axiom since each open $U \subset X$ can be covered by elements of $\mathcal{V}$. So it suffices to show that the presheaf $\mathcal{F}$ we have just defined satisfies the sheaf axioms. By construction of $\mathcal{F}$, for the first sheaf axiom it is enough to see that for any open cover $\{U_i : i \in I\}$ of U, two sections $(f_V), (g_V) \in \mathcal{F}(U)$ restricting to the same section over each U_i restrict to the same section over each $W \in \mathcal{V}_U$. Since $\mathcal{V}$ is a basis of open sets (hence in particular closed under finite intersections), we may write each U_i as a union of some $V_{ij} \in \mathcal{V}_U$ in such a way that W itself is a union of some of the V_{ij}. Now as (f_V) and (g_V) restrict to the same section over each V_{jk}, they must restrict to the same section over W by the assumption. The verification of the second sheaf axiom is similar and is left to the readers.

Finally, the last statement follows from the fact that the maps ϕ_V induce a morphism of the inverse systems defining $\mathcal{F}(U)$ and $\mathcal{G}(U)$ for a general U as above. The map $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is then obtained by passing to the limit: explicitly, it maps a sequence (f_V) to the sequence $(\phi_V(f_V))$. □

Now we are ready to prove:

Theorem 2.8 *For any ring A, there is a unique sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Spec} A$ for which $\mathcal{O}_X(D(f)) = A_f$ for all $f \in A$ and the restriction maps $\mathcal{O}_X(D(g)) \to \mathcal{O}_X(D(f))$ for $D(f) \subset D(g)$ are the natural maps $A_g \to A_f$ described above.*

Proof: We have to check the hypothesis of the previous proposition, i.e. the sheaf axioms over the basic open sets $D(f)$. Notice that since $\operatorname{Spec} A_f$ is naturally homeomorphic to $D(f)$ and $\mathcal{O}_X(D(f)) = A_f$, we may replace A by A_f and assume $f = 1$. Then for the first sheaf axiom we are given a covering of X by basic open sets $D(f_i)$; by compactness of X we may assume the covering is finite, say $X = D(f_1) \cup \ldots \cup D(f_n)$. To give a section of $\mathcal{O}_X(X) = A$ restricting to 0 over each $D(f_i)$ is to give an element $g \in A$ satisfying

$$f_i^{k_i} g = 0 \tag{3}$$

for all $1 \leq i \leq n$ with appropriate positive integers k_i. Now by the definition of prime ideals we have $D(f_i^{k_i}) = D(f_i)$ for all i, so the $D(f_i^{k_i})$ cover X as well and hence by Lemma 1.3 there exist $g_i \in A$ with

$$g_1 f_1^{k_1} + \ldots + g_n f_n^{k_n} = 1 \tag{4}$$

Thus if we multiply each equation in (3) by g_i and take the sum we get $g = 0$, as desired.

For the second sheaf axiom, assume again given a covering of X by basic open sets $D(f_i)$ $(1 \leq i \leq n)$ and elements $a_i/f_i^{k_i} \in A_{f_i}$ (viewed as sections of a would-be sheaf over $D(f_i)$) whose restrictions to the pairwise intersections $D(f_i) \cap D(f_j) = D(f_i f_j)$ coincide. This latter property can be written explicitly as $(a_i f_j^{k_j} - a_j f_i^{k_i})(f_i f_j)^{k_{ij}} = 0$ with some positive integer k_{ij}. By changing the a_i if necessary we may assume $k_i = k$ for all i and $k_{ij} = m$ for all i, j, where m is large enough. Thus

$$a_i f_j^k (f_i f_j)^m = a_j f_i^k (f_i f_j)^m \tag{5}$$

for all $1 \leq i, j \leq n$. Now apply (4) with $k_i = k + m$ for all i to get some g_i with $\sum_i g_i f_i^{k+m} = 1$ and define $a = \sum_i g_i a_i f_i^m$. Using equation (5) we get for all j a chain of equations

$$f_j^{k+m}a = \sum_{i=1}^{n} g_i a_i f_j^k (f_i f_j)^m = \sum_{i=1}^{n} g_i a_j f_i^k (f_i f_j)^m = a_j f_j^m \sum_i g_i f_i^{k+m} = a_j f_j^m$$

which means that the image of a in A_{f_j} coincides with a_j/f_j^k, as required. □

Definition 2.9 An *affine scheme* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X such that $X = \operatorname{Spec} A$ for some ring A and $\mathcal{O}_X$ is the sheaf occuring in the above theorem. We call $\mathcal{O}_X$ the *structure sheaf* of X.

By abuse of notation, we shall frequently write X or $\operatorname{Spec} A$ instead of the pair $(X, \mathcal{O}_X)$. Next an important fact:

Proposition 2.10 *If $X = \operatorname{Spec} A$ is an affine scheme, then the stalk $\mathcal{O}_{X,P}$ of $\mathcal{O}_X$ at any point P of X is canonically isomorphic to the localisation A_P; in particular, it is a local ring.*

Proof: Recall that the stalk at P is defined as the direct limit of the *filtered* direct system of the rings $\mathcal{O}_X(U)$, for U containing P. Since basic open sets $D(f)$ are cofinal in the index set of this direct system, we may restrict to the rings A_f. Then the proposition follows from the fact that the direct limit of these rings is obtained by "dividing out by all $f \notin P$". More precisely, it follows from the construction of $\varinjlim A_f$ that any $f \notin P$ is a unit in $\varinjlim A_f$, hence there is a homomorphism $A_P \to \varinjlim A_f$. If an element $g \in A_P$ maps to zero here, it means $f^n g = 0$ for some $f \notin P$ and $n \geq 0$ and thus $g = 0$ in A_P; surjectivity is equally obvious. □

Now some definitions.

Definition 2.11 A *ringed space* is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is a pair $(\phi, \phi^\sharp)$ consisting of a continuous map $\phi : X \to Y$ and a morphism $\phi^\sharp : \mathcal{G} \to \phi_* \mathcal{F}$ of sheaves of rings on Y.

Ringed spaces thus form a category with the morphisms just defined. Affine schemes are naturally objects of this category enjoying the additional property that the stalks of the structure sheaf are all local rings.

Next notice that given a morphism of ringed spaces $(\phi, \phi^\sharp)$ as above, for any $x \in X$ the morphism $\phi^\sharp$ induces a ring homomorphism $\mathcal{G}_{\phi(x)} \to \mathcal{F}_x$ on the stalks, for by definition $\mathcal{G}_{\phi(x)}$ is the (filtered) direct limit of $\mathcal{G}(U)$ for the open sets U containing $\phi(x)$, whereas $\phi_*\mathcal{F}(U) = \mathcal{F}(\phi^{-1}(U))$ and there is a natural map from the direct limit of the latter sets to $\mathcal{F}_x$, for $\mathcal{F}_x$ is defined as the direct limit of *all* open neighbourhoods containing x.

Definition 2.12 A *locally ringed space* is a ringed space $(X, \mathcal{F})$ such that the stalk $\mathcal{F}_x$ is a local ring for all $x \in X$. A *morphism* of locally ringed spaces is to be a morphism of ringed spaces for which the induced maps $\mathcal{G}_{\phi(x)} \to \mathcal{F}_x$ on stalks described above are *local* homomorphisms, which means that the preimage of the maximal ideal of $\mathcal{F}_x$ is the maximal ideal of $\mathcal{G}_{\phi(x)}$. Thus the category of locally ringed spaces is a subcategory of that of ringed spaces.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that X admits an open covering $\{U_i : i \in I\}$ such that for all i the locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ are isomorphic (in the category of locally ringed spaces) to affine schemes. The category of schemes is defined as the *full* subcategory of that of locally ringed spaces whose objects are schemes.

Construction 2.13 As we have already remarked, for any commutative ring A the affine scheme $X = \operatorname{Spec} A$ is naturally an object of the category of schemes. We now show that the map $A \mapsto \operatorname{Spec} A$ is in fact a *contravariant functor* from the category of rings to that of schemes. For this we have to assign to any ring homomorphism $\phi : A \to B$ a morphism $(\operatorname{Spec}(\phi), \operatorname{Spec}(\phi)^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$ of schemes. The definition of $\operatorname{Spec}(\phi)$ is obvious: it maps a prime ideal $P \in \operatorname{Spec} B$ to $\phi^{-1}(P)$ which is immediately seen to be a prime ideal of A. The map is continuous since the inverse image of a basic open set $D(f)$ is just $D(\phi(f))$. Now for defining $\operatorname{Spec}(\phi)^\sharp$ note that by the third statement of Lemma 2.7 it suffices to consider sections over the basic open sets $D(f)$. By construction of the structure sheaves, over $D(f)$ our task is to define a ring homomorphism $A_f \to B_{\phi(f)}$. But there is a canonical such homomorphism according to Lemma 2.3: it corresponds to the composite $A \to B \to B_{\phi(f)}$. Finally, by passing to stalks we verify that $\operatorname{Spec}(\phi)^\sharp$ is a morphism of *locally* ringed spaces.

Now consider the natural question: given an affine scheme $X = \operatorname{Spec} A$, how can we recover A from X? The answer is easy: we have $A = \mathcal{O}_X(X)$. Moreover, the rule $X \mapsto \mathcal{O}_X(X)$ is also a contravariant functor: given a morphism $\phi : X \to Y$ of affine schemes, we have in particular a morphism of sheaves $\phi^\sharp : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$, whence a homomorphism $\mathcal{O}_Y(Y) \to \phi_*\mathcal{O}_X(Y) = \mathcal{O}_X(X)$.

Theorem 2.14 *The functors $A \mapsto \operatorname{Spec} A$ and $X \mapsto \mathcal{O}_X(X)$ are inverse to each other. Thus the category of affine schemes is* isomorphic *to the opposite category of the category of commutative rings with unit.*

Proof: If $Y = \mathrm{Spec}\, B$ and the scheme morphism $X \to Y$ comes from a homomorphism $\lambda : A \to B$, by construction the map $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is none but λ.

We are left to prove that given a morphism $(\phi, \phi^\sharp) : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ of schemes, if $\lambda : A \to B$ is the ring homomorphism induced by taking global sections, then $(\phi, \phi^\sharp) = (\mathrm{Spec}\,(\lambda), \mathrm{Spec}\,(\lambda)^\sharp)$. For this, we have to show first that for $P \in \mathrm{Spec}\, B$ we have $\phi(P) = \lambda^{-1}(P)$. Indeed, $\phi^\sharp$ induces a map on the stalks $\phi^\sharp_P : A_{\phi(P)} \to B_P$ which by definition makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \downarrow & & \downarrow \\ A_{\phi(P)} & \xrightarrow{\phi^\sharp_P} & B_P \end{array}$$

commute. But $\phi^\sharp_P$ is a *local* homomorphism, i.e. $\phi(P)A_{\phi(P)} = (\phi^\sharp_P)^{-1}(PB_P)$; on the other hand, the vertical maps are localisation maps, whence the assertion. The equality $\phi^\sharp = \mathrm{Spec}\,(\lambda)^\sharp$ follows from the analogous commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \downarrow & & \downarrow \\ A_f & \xrightarrow{\phi^\sharp_{D(f)}} & B_{\lambda(f)} \end{array}$$

for sections over basic open sets. □

Corollary 2.15 *Given a scheme X and a ring A, the functor $Y \mapsto \mathcal{O}_Y(Y)$ induces a bijection*

$$\mathrm{Hom}(X, \mathrm{Spec}\, A) \cong \mathrm{Hom}(A, \mathcal{O}_X(X)).$$

Proof: We construct an inverse to the map $\mathrm{Hom}(X, \mathrm{Spec}\, A) \to \mathrm{Hom}(A, \mathcal{O}_X(X))$. Assume given a homomorphism $A \to \mathcal{O}_X(X)$ and an affine open subset $U = \mathrm{Spec}\, B$ in X. Composing with the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ we obtain a map $A \to B$ which corresponds to a morphism $\phi^U : U \to \mathrm{Spec}\, A$ by the theorem. Moreover, given a basic open set $W = D(f) \subset U$, the composite map $A \to B \to B_f$ corresponds to the composite of ϕ^U with the inclusion map $W \hookrightarrow U$. If W is contained in another affine open subset $V = \mathrm{Spec}\, C$ in A, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_X(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) & \longrightarrow & \mathcal{O}_X(W) \end{array}$$

commutes, which implies that the restrictions of the maps $\phi^U : U \to \operatorname{Spec} A$ and $\phi^V : V \to \operatorname{Spec} A$ to W coincide. We may thus patch them together to a map $X \to \operatorname{Spec} A$ as required. □

Remark 2.16 The corollary may be rephrased by saying that the affine scheme $\operatorname{Spec} A$ represents the contravariant functor $X \mapsto \operatorname{Hom}(A, \mathcal{O}_X(X))$ on the category of schemes.

3. First Examples of Schemes

It is now time for some examples. Let us first take a new look of those of the first section, but this time considering the structure sheaves as well.

Examples 3.1

1. For k a field, the underlying topological space of $\operatorname{Spec} k$ is a single point. The stalk of the structure sheaf at this point is k.

2. The generic stalk of $\operatorname{Spec} \mathbf{Z}$, i.e. the stalk of $\mathcal{O}_{\operatorname{Spec} \mathbf{Z}}$ at the generic point (0) is $\mathbf{Q}$. The inclusion $\mathbf{Z} \to \mathbf{Q}$ corresponds to a morphism $\operatorname{Spec} \mathbf{Q} \to \operatorname{Spec} \mathbf{Z}$, identifiable as the inclusion of the generic point into $\operatorname{Spec} \mathbf{Z}$. At a closed point corresponding to the prime ideal (p) the stalk $\mathcal{O}_{\operatorname{Spec} \mathbf{Z},(p)}$ is isomorphic to the subring of $\mathbf{Q}$ formed by fractions whose denominator is not divisible by p. The maximal ideal of this ring is generated by p; we have $\mathcal{O}_{\operatorname{Spec} \mathbf{Z},(p)}/p\mathcal{O}_{\operatorname{Spec} \mathbf{Z},(p)} \cong \mathbf{F}_p$. The natural projection $\mathbf{Z} \to \mathbf{F}_p$ corresponds to a map $\operatorname{Spec} \mathbf{F}_p \to \operatorname{Spec} \mathbf{Z}$, the inclusion of the closed point (p).

3. The generic stalk of $\operatorname{Spec} \mathbf{C}[x]$ is the rational function field $\mathbf{C}(x)$. At the closed point $(x - a)$ the stalk consists of those elements of $\mathbf{C}[x]$ whose denominator does not vanish at a; the maximal ideal of $\mathcal{O}_{\operatorname{Spec} \mathbf{C}[x],(x-a)}$ is generated by functions vanishing at a. The quotient by this maximal ideal is isomorphic to $\mathbf{C}$; the image of a function by the projection $\mathcal{O}_{\operatorname{Spec} \mathbf{C}[x],(x-a)} \to \mathbf{C}$ is its value at a. Here again a map $\mathbf{C}[x] \to \mathbf{C}[x]/(x - a) \cong \mathbf{C}$ corresponds to the inclusion of the point $a \in \mathbf{C}$.

Thus by comparing the two last examples, we may think of elements of $\mathbf{Q}$ as functions on the space $\operatorname{Spec} \mathbf{Z}$. If the denominator of $f \in \mathbf{Q}$ is not divisible by a prime p, then f is "defined" in a neighbourhood of (p); its image in $\mathbf{F}_p$ is its "value" at p. This is the coarsest analogy one may observe; it will be considerably refined later.

Example 3.2 *Affine spaces.* For a field k we define *affine n-space over k* as the affine scheme $\mathbf{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ (with $k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k). An explanation for this name is provided by a form of a classical theorem of Hilbert's called the *Nullstellensatz* (see e.g. Lang [1], Chapter IX.1): this says that if k is algebraically closed, then any maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with some $a_i \in k$. Thus in this case we may identify the set of closed points of $\mathbf{A}^n_k$ with elements of k^n. Note that the above statement is false even for $n = 1$ if k is not algebraically closed: for instance, the polynomial $x^2 + 1$ generates a maximal ideal of $\mathbf{R}[x]$ not of the above form.

We next give the basic example of a non-affine scheme. Before discussing it, an easy definition.

Definition 3.3 An *open subscheme* of a scheme X is the ringed space consisting of an open subset U and the restriction $O_X|_U$ of the structure sheaf of X to U.

Indeed, one checks that U admits an open covering by affine schemes (use basic open sets, for instance).

Example 3.4 It is possible to define projective spaces $\mathbf{P}^n_A$ over any commutative ring A (not to mention any scheme...) using a patching construction (see Construction 4.3 for details). We may patch together the affine schemes

$$D_+(x_i) \cong \operatorname{Spec} A[(x_0/x_i), \ldots, (x_{i-1}/x_i), (x_{i+1}/x_i), \ldots, (x_n/x_i)]$$

over the isomorphic open subschemes

$$D(x_j/x_i) \cong \operatorname{Spec} A[(x_0/x_i), \ldots, (x_{i-1}/x_i), (x_{i+1}/x_i), \ldots, (x_n/x_i)]_{x_j/x_i}$$

of $D_+(x_i)$ and

$$D_+(x_i/x_j) \cong \operatorname{Spec} A[(x_0/x_j), \ldots, (x_{j-1}/x_j), (x_{j+1}/x_j), \ldots, (x_n/x_j)]_{x_i/x_j}$$

of $D_+(x_j)$ by remarking that

$$A[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}]_{\frac{x_j}{x_i}} = A[\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}]_{\frac{x_i}{x_j}}$$

as subrings of $A[x_0, \ldots, x_n, x_0^{-1}, \ldots, x_n^{-1}]$, so we may use the identity maps as patching isomorphisms.

The next definition enables us to define the basic objects of study in algebraic geometry, namely loci of zeros of systems of polynomials in affine or projective space.

Definition 3.5 A morphism $\phi : Y \to X$ of schemes is a *closed immersion* if the underlying continuous map is the inclusion of a closed subset of X and moreover there is a covering of X by affine open subschemes $U_i = \operatorname{Spec} A_i$ such that for all i the open subscheme of Y defined by $\phi^{-1}(U_i)$ is isomorphic to an affine scheme $\operatorname{Spec} B_i$ with the induced maps $A_i \to B_i$ surjections. We say that Y is a *closed subscheme* of X if there is a closed immersion of Y into X.

Remark 3.6 It can be shown that any closed subscheme of an affine scheme $X = \operatorname{Spec} A$ is of the form $\operatorname{Spec} A/I$ with some ideal I. However, the reader should be warned that in general it is possible to give several different closed subscheme structures on a given closed subset of the underlying topological space of a scheme.

Example 3.7 An (irreducible) *affine hypersurface* of dimension $n-1$ over a field k is the closed subscheme of $\mathbf{A}^n_k$ given by the quotient of the polynomial ring $k[x_1, \ldots, x_n]$ by the principal ideal generated by an irreducible polynomial f (here the covering of $\mathbf{A}^n_k$ is just the one-element covering by the whole space). Affine hypersurfaces of dimension 1 are also called *plane curves*. For instance, the quotient of $k[x_1, x_2]$ modulo the principal ideal generated by the polynomial $x_1x_2 - 1$ defines an affine plane curve: the conic of equation $x_1x_2 = 1$.

A *projective hypersurface* is a closed subscheme Y of some $\mathbf{P}^n_k$ which restricts to an affine hypersurface on each canonical open subset $D_+(x_i)$ via the isomorphisms $D_+(x_i) \cong \mathbf{A}^n_k$. As above, in dimension 1 we get *projective plane curves*. For instance, the locus of zeros of the homogenous polynomial $X_0X_1 - X_2^2$ in $\mathbf{P}^2_k$ defines a projective plane curve given on $D_+(X_0)$ by the affine plane curve of equation $x_1 = x_2^2$, on $D_+(X_1)$ that of equation $x_0 = x_2^2$ and on $D_+(X_2)$ that of equation $x_0x_1 = 1$ (notice that different types of affine conics arise from the same projective conic).

4. Fibres of a Morphism

We next define the fibre of a morphism of schemes *as a scheme* and not just a point set. Motivated by the situation in the topological setting, we introduce more generally the notion of a *fibre product* of schemes and get the definition of fibres as a special case.

Given topological spaces $Y \to X$, $Z \to X$ over the same space X, their fibre product can be defined as the space representing the functor.

$$S \mapsto \{(\phi, \psi) \in \operatorname{Hom}(S, Y) \times \operatorname{Hom}(S, Z) : p \circ \phi = q \circ \psi\}$$

on the category of topological spaces over X. We can adopt the same definition in the context of schemes if we show that the similarly defined functor on the category of schemes equipped with morphisms to a fixed base scheme X is representable. We first prove representability in the category of affine schemes.

Proposition 4.1 *Assume given affine schemes* $Y = \mathrm{Spec} A$ *and* $Z = \mathrm{Spec} B$ *equipped with morphisms* $p : Y \to X$, $q : Z \to X$ *into an affine scheme* $X = \mathrm{Spec}\, R$. *Then the contravariant functor*

$$S \mapsto \{(\phi, \psi) \in \mathrm{Hom}(S, Y) \times \mathrm{Hom}(S, Z) : p \circ \phi = q \circ \psi\}$$

on the category of affine schemes is representable by $Y \times_X Z := \mathrm{Spec}\, (A \otimes_R B)$.

Proof: Indeed, by Theorem 2.14 the statement of the proposition is equivalent to saying that given ring homomorphisms $\mu : R \to A$ and $\nu : R \to B$, the ring $A \otimes_R B$ represents the functor

$$C \mapsto \{(\kappa, \lambda) \in \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C) : \kappa \circ \mu = \lambda \circ \nu\}$$

on the category of commutative rings with unit. But this is precisely the defining property of the tensor product of R-algebras. Indeed, for $(a, b) \in A \times B$ the map $(a, b) \mapsto \kappa(a)\lambda(b)$ is R-bilinear, hence induces an R-module homomorphism $A \otimes_R B \to C$. When $A \otimes_R B$ is equipped with its ring structure, it is moreover an R-algebra homomorphism. □

Remark 4.2 It is not true that the underlying topological space of a fibre product of affine schemes is the topological fibre product of the underlying topological spaces of the schemes. As an easy example, take $X = \mathrm{Spec}\, k$ with some field k, $Y = \mathrm{Spec}\, L$, with $L|k$ a finite separable extension of k, $Z = \mathrm{Spec}\, \bar{k}$. Then the topological fibre product of Y and Z over X is a one-element set, whereas $L \otimes_k \bar{k}$ is a direct sum of $[L : k]$ copies of $\bar{k}$ and hence its prime spectrum consists of $[L : k]$ points.

To extend this construction to arbitrary schemes, the idea is of course to cover them with open affine subschemes and then to "patch" the fibre products of these affine schemes together. How this can be done precisely is explained next.

Construction 4.3 Assume given a family of schemes $\{X_i : i \in I\}$ and for each $(i, j) \in I^2$ an open subscheme $U_{ij} \subset X_i$ such that $U_{ii} = X_i$. Assume moreover we are given isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$ satisfying the *cocycle condition*

$$\phi_{jk} \circ \phi_{ij} = \phi_{ik}$$

on $U_{ij} \cap U_{ik}$ for all $i, j.k$ (here we tacitly assume that $\phi_{ij}(U_{ij} \cap U_{ik}) \subset U_{jk}$). Note that the cocycle condition for $i = j = k$ implies that ϕ_{ii} is the identity and then for $i = k$ that $\phi_{ji} = \phi_{ij}^{-1}$.

We now construct a scheme X having an open covering $\{U_i : i \in I\}$ such that each U_i is isomorphic to X_i as a scheme and via these isomorphisms the U_{ij} correspond to the intersections $U_i \cap U_j$. The above compatibility relations for the ϕ_{ij} are thus necessary conditions for such a scheme X to exist.

Define the underlying set of X to be the disjoint union of those of the X_i modulo the equivalence relation which identifies points of U_{ij} with those of U_{ji} via ϕ_{ij}. The compatibility conditions for the ϕ_{ij} ensure that this is indeed an equivalence relation; we endow X with the quotient topology. The composite maps $p_i : X_i \to \coprod X_i \to X$ map each X_i homeomorphically onto an open subset $U_i \subset X$. Now to define the structure sheaf $\mathcal{O}_X$ of X it suffices by Lemma 2.7 to define its sections over a basis of open sets in X in a compatible fashion. The open sets U which are contained in one of the U_i clearly form a basis. For such a U one is tempted to define $\mathcal{O}_X(U)$ as $\mathcal{O}_{X_i}(p_i^{-1}(U))$, but the problem is that U may be contained in the intersection of several U_i. However, the rings obtained for each choice of U_i are all isomorphic via the ϕ_{ij}, so to remedy this one defines $\mathcal{O}_X(U)$ to be the subring of $\prod_{U \subset U_i} \mathcal{O}_{X_i}(p_i^{-1}(U))$ consisting of sequences of sections mapped to each other by the ϕ_{ij}. More precisely, we take those sequences (s_i) with $\phi_{ij}^\sharp(s_j) = s_i$ for all (i, j) (here s_i is viewed as a section in $\phi_{ij*}(\mathcal{O}_{X_i}|_{U_{ij}})(p_j^{-1}(U))$). One defines restriction maps for subsets $V \subset U$ as induced by the product of the restriction maps of the $\mathcal{O}_{X_i}$; in fact, any element of $\mathcal{O}_X(V)$ is determined by its components indexed by the sets U_i containing U. It is now straightforward to check the sheaf axioms over U as well as the fact that the ringed space thus obtained is a scheme.

Armed with this patching construction, we may now construct fibre products of arbitrary schemes. Just as in topology, let us refer to a morphism $p : Y \to X$ of schemes as a *scheme over X*.

Proposition 4.4 *Given two schemes $p : Y \to X$, $q : Z \to X$ over the same scheme X, the contravariant functor*

$$S \mapsto F_{YZ}(S) := \{(\phi, \psi) \in \mathrm{Hom}(S, Y) \times \mathrm{Hom}(S, Z) : p \circ \phi = q \circ \psi\}$$

the category of schemes is representable by a scheme $Y \times_X Z$.

The scheme $Y \times_X Z$ is called the *fibre product of* Y *and* Z *over* X. It is equipped with two canonical morphisms into Y and Z making the diagram

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ Y & \xrightarrow{p} & X \end{array}$$

commute (they correspond to the identity morphism of $Y \times_X Z$).

Proof: We show first that if Y, Z and X are all affine, then the scheme $Y \times_X Z$ defined in Proposition 4.1 is indeed a fibre product *in the category of schemes*. For this we have to see that for an arbitrary scheme S any element of $F_{YZ}(S)$ factors as a composite $(\pi_1, \pi_2) \circ \phi$ with some morphism $\phi : S \to Y \times_S Z$. Choosing an affine open cover $\{S_i : i \in I\}$ of S, for each i we dispose of a morphism $\phi_i : S_i \to Y \times_X Z$ with the above property according to Proposition 4.1. Since by definition for any affine open subset $U \subset S_i \cap S_j$ the elements of $F_{YZ}(U)$ are in bijection with $\mathrm{Hom}(U, Y \times_X Z)$, we see that the restrictions of ϕ_i and ϕ_j to the open subschemes $S_i \cap S_j$ are the same for all (i, j). Hence there is a unique morphism ϕ agreeing with ϕ_i over S_i (the existence of the underlying continuous map is straightforward; for the existence of $\phi^\sharp$ use Lemma 2.7 (3)).

Still assuming X affine, choose affine open coverings $\{Y_i : i \in I\}$ and $\{Z_j : j \in J\}$ of Y and Z, respectively. First fix some $l \in J$. We then dispose of affine shemes $Y_i \times_X Y_l$ for each $i \in I$. Now note that quite generally if $Y \times_X Z$ represents the functor F_{YZ} and $U \subset Y$ is an open subscheme, then the open subscheme $\pi_1^{-1}(U) \subset Y \times_X Z$ represents the functor F_{UZ} and as such is unique up to unique isomorphism by the Yoneda lemma. Applying this remark with Z_l in place of Z, Y_i (resp. Y_j) in place of Y and $Y_i \cap Y_j$ in place of U we see that there exist unique isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$, where U_{ij} (resp. U_{ji}) is the inverse image of $Y_i \cap Y_j$ by the projection $Y_i \times_X Z_l \to Y_i$ (resp. $Y_j \times_X Z_l \to Y_j$). The uniqueness of the ϕ_{ij} implies that the compatibility conditions in Construction 4.3 are satisfied, so we may patch the $Y_i \times_X Z_l$ together along the U_{ij} to obtain a scheme $Y \times_X Y_l$. The projections $Y \times_X Z_l \to Y$ and $Y \times_X Z_l \to Z_l$ are defined by patching the projections from the elements of the open covering $\{Y_i \times_X Z_l : i \in I\}$ of $Y \times_X Z_l$ as in the previous paragraph. To show that $Y \times_X Z_l$ represents F_{YZ_l} one considers for a pair $(\phi, \psi) \in F_{YZ_l}(S)$ the restrictions $(\phi|_{\phi^{-1}(Y_i)}, \psi) \in F_{Y_i Z_l}$ and patches the corresponding morphisms $S \to Y_i \times_X Z_l$ together, again arguing as in the previous paragraph.

Now by exactly the same method one shows that the schemes $Y \times_X Z_l$ patch together to give a scheme $Y \times_X Z$ representing F_{YZ}. Finally one extends the construction to arbitary X by choosing a covering of X by affine open subschemes X_k and noting that the open subschemes $Y_k = p^{-1}(X_k)$ form an open covering of Y such that the fibre products $Y_k \times_{X_k} q^{-1}(X_k)$ represent the functors $F_{Y_k Z}$ where the Y_k are viewed as schemes *over* X (indeed, given $(\phi, \psi) \in F_{Y_k Z}(S)$ we must have $\psi(S) \subset q^{-1}(X_k)$), so one may repeat the previous procedure to patch the schemes $Y_k \times_X Z = Y_k \times_{X_k} q^{-1}(X_k)$ together. □

Now if we imitate the situation in topology, to define the fibre of a morphism $Y \to X$ at some point P of X we first need to define the inclusion morphism $\{P\} \to X$. This is achieved as follows. Take an affine open neighbourhood $U = \operatorname{Spec} A$. Then P is identified with a prime ideal of A and we dispose of a morphism $A \to A_P$ which we may compose with the natural projection $A_P \to A_P/PA_P =: \kappa(P)$. We get a morphism $\operatorname{Spec} \kappa(P) \to U$, whence by composition with the inclusion map $U \to X$ a morphism $i_P : \operatorname{Spec} \kappa(P) \to X$.

Lemma 4.5 *The morphism* $i_P : \operatorname{Spec} \kappa(P) \to X$ *just defined does not depend on the choice of* U.

Proof: If $V = \operatorname{Spec} B$ is another affine open neighbourhood of P, then there is some affine open subscheme $W \subset V \cap U$. We may assume that W as a subscheme of U is of the form $D(f)$ with some $f \in A \setminus P$. But the localisation map $A \to A_P$ factors through A_f (since f is a unit in A_P), which means that the map $\operatorname{Spec} A_P \to U$ factors through W. By symmetry, we get the same conclusion for V. □

Definition 4.6 Given a morphism $\phi : Y \to X$ and a point P of X, the *fibre of* ϕ *at* P is the scheme $Y_P := Y \times_X \operatorname{Spec} \kappa(P)$, the fibre product being taken with respect to the maps ϕ and i_P.

We saw in Remark 4.2 that the underlying topological space of a fibre product is not a topological fibre product in general. However, the good news is:

Proposition 4.7 *Given a morphism* $\phi : Y \to X$ *and a point* P *of* X, *the underlying topological space of the fibre* Y_P *is homeomorphic to the subspace* $\phi^{-1}(P)$ *of the underlying space of* Y.

Proof: We may assume we are dealing with affine schemes $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. We first show there is a bijection between $\phi^{-1}(P)$ and $\operatorname{Spec} B \otimes_A \kappa(P)$ as sets, the homomorphism $\lambda : A \to B$ defining the A-module structure of B being the one corresponding to ϕ by Theorem 2.14. Now the above λ induces a map $\overline{\lambda} : A/P \to B/\lambda(P)B$ and a point $Q \in \operatorname{Spec} B$ is in $\phi^{-1}(P)$ if and only if $Q \supset \lambda(P)$ and its image $\overline{Q}$ in $B/\lambda(P)B$ satisfies $\overline{\lambda}^{-1}(\overline{Q}) = (0)$. This is the same as saying that $\overline{\lambda}(A/P) \cap \overline{Q} = \{0\}$, or else, putting $S = \overline{\lambda}(A/P) \setminus \{0\}$, that $\overline{Q}$ defines a prime ideal of the localisation $(B/\lambda(P)B)_S$. But the latter ring is none but $B \otimes_A \kappa(P)$. To see this, note first the isomorphism $B/\lambda(P)B \cong B \otimes_A (A/P)$ coming from the exact sequence

$$B \otimes_A P \to B \otimes_A A \to B \otimes_A (A/P) \to 0$$

coming from tensoring with B the short exact sequence

$$0 \to P \to A \to A/P \to 0$$

of A-modules. Here we have $B \otimes_A A \cong B$ coming from the multiplication map $b \otimes a \mapsto ba$ and so the image of $B \otimes_A P$ in B is exactly $\lambda(P)B$ (since B is an A-module via λ). Now the natural map $A/P \to B \otimes_A (A/P) = B/\lambda(P)B$ is given by $a \mapsto 1 \otimes a$ and the localisation of $B \otimes_A (A/P)$ by the subset $\{1 \otimes a : a \in (A/P) \setminus \{0\}\}$ is exactly $B \otimes_A \kappa(P)$.

In the above procedure we identified $Y_P = \operatorname{Spec} B \otimes_A \kappa(P)$ with a subset of $\operatorname{Spec} B$; by looking at basic open sets $D(f)$ one sees easily that the topology of Y_P corresponds to the subspace topology. □

The fibre product enables us to introduce an important class of morphisms.

Definition 4.8 *Let $X \to S$ be a scheme over S. The* diagonal *of X is the morphism*

$$\Delta = \Delta_{X/S} : X \to X \times_S X$$

induced by the pair of S-morphisms $(\mathrm{id}_X : X \to X, \mathrm{id}_X : X \to X)$.

The scheme X is separated over S *if $\Delta_{X/S}$ is a closed immersion. In the special case $S = \operatorname{Spec} \mathbf{Z}$ we say that X is a* separated scheme.

Remark 4.9 Separatedness is the analogue of the Hausdorff property in topology: a topological space T is Hausdorff if and only if the diagonal morphism $T \to T \times T$ is a closed embedding.

Examples 4.10

1. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is always separated. Indeed, the diagonal morphism is the morphism $\operatorname{Spec} B \to \operatorname{Spec}(B \otimes_A B)$ induced by the multiplication map $B \otimes_A B \to B$. The latter is a surjective ring homomorphism, hence it defines a closed immersion.

2. Let k be a field. Take two copies of the affine line $\mathbf{A}^1_k = \operatorname{Spec} k[x]$ and patch them together over the open set $\operatorname{Spec} k[x, x^{-1}]$ using the identity map. The resulting scheme is *the affine line with the origin doubled;* one checks that it is not separated over k.

Proposition 4.11 *The following are equivalent for a scheme X.*

1. *X is separated.*

2. *The intersection $U \cap V$ of two affine open subsets U, V is always affine, and the natural morphism*

$$\mathcal{O}_X(U) \otimes_{\mathbf{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$$

   is surjective.

3. *There exists an open covering of X be affine open subsets U_i such that the intersections $U_i \cap U_j$ have the property in (2).*

Proof: The implication (1) $\Rightarrow$ (2) holds because $U \cap V$ is naturally isomorphic to the closed subscheme $\Delta(X) \cap (U \times V)$ of the affine scheme $U \times V$. (Here the intersection is the closed subscheme defined by restricting the sheaf of ideals $\mathcal{I}_{\Delta_X} \subset \mathcal{O}_{X \times X}$ to the open subscheme $U \times V$; see the next section). As (3) is a special case of (2), it remains to show that (3) implies (1). By assumption (3), the morphism $U_i \cap U_j \to U_i \times U_j$ given by $(\mathrm{id}_{U_i \cap U_j}, \mathrm{id}_{U_i \cap U_j})$ is a closed immersion, from which (1) follows as the $U_i \times U_j$ form an affine open covering of $X \times X$. $\square$

Corollary 4.12 *For any ring A the projective space $\mathbf{P}^n_A$ is separated over* $\operatorname{Spec} A$.

Proof: In the case $A = \mathbf{Z}$ property (3) of the proposition follows from the patching construction defining $\mathbf{P}^n_{\mathbf{Z}}$. In the general case, note that $\mathbf{P}^n_A \cong \mathbf{P}^n_{\mathbf{Z}} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} A$, and it is straightforward to check that a base change of a separated morphism is again separated (more generally, the base change of a closed immersion is again a closed immersion). $\square$

5. Quasi-coherent Sheaves

In Section 2 we saw that any ring A defines an affine scheme $X = \operatorname{Spec} A$. Here we study how to associate sheaves on X to modules over the ring A, a construction that will be very useful in the next chapter.

As the construction of affine schemes makes one expect, sheaves associated to A-modules should be, in some sense, modules over the structure sheaf $\mathcal{O}_X$. The following definition makes this precise.

Definition 5.1 Let X be any scheme. A *sheaf of $\mathcal{O}_X$-modules* or an *$\mathcal{O}_X$-module* for short is a sheaf of abelian groups $\mathcal{F}$ on X such that for each open $U \subset X$ the group $\mathcal{F}(U)$ is equipped with an $\mathcal{O}_X(U)$-module structure $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ making the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commute for each inclusion of open sets $V \subset U$. In the special case when $\mathcal{F}(U)$ is an ideal in $\mathcal{O}_X(U)$ for all U we speak of a *sheaf of ideals* on X.

Examples 5.2 Here are two natural situations where O_X-modules arise.

1. Let $\phi : X \to Y$ be a morphism of schemes. We know that on the level of structure sheaves ϕ is given by a morphism $\phi^\sharp : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$, whence an O_Y-module structure on $\phi_*\mathcal{O}_X$.

2. In the previous situation the kernel $\mathcal{I}$ of the morphism $\phi^\sharp : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ (defined by $\mathcal{I}(U) = \ker(\mathcal{O}_Y(U) \to \phi_*\mathcal{O}_X(U))$) is a sheaf of ideals on Y. This is particularly interesting when X is a closed subscheme of Y and ϕ is the natural inclusion. In this case we call $\mathcal{I}$ the *sheaf of ideals defining X*.

3. More generally, given any $\mathcal{O}_X$-module $\mathcal{F}$ one can define its *annihilator* as the ideal sheaf whose sections over an open set U consist of those $f \in \mathcal{O}_X(U)$ for which $fs = 0$ for all $s \in \mathcal{F}(U)$. For instance, the annihilator of the $\mathcal{O}_X$-module 0 is $\mathcal{O}_X$.

We may now proceed to construct $\mathcal{O}_X$-modules over affine schemes from modules over rings. For this we first need an algebraic concept.

Definition 5.3 Let A be a ring, $S \subset A$ a multiplicatively closed subset and M an A-module. The *localisation of M by S* is the A_S-module M_S given by $M \otimes_A A_S$.

As in the case of rings, given an element $f \in A$ or a prime ideal P, we shall use the notations M_f for $M \otimes_A A_f$ and M_P for $M \otimes_A A_P$.

Construction 5.4 Let A be a ring and M an A-module. For any multiplicatively closed $S \subset A$ there is a natural map $M \to M_S$ obtained by tensoring the natural map $A \to A_S$ by M and similarly there is a natural map $M_g \to M_f$ for any inclusion $D(f) \subset D(g)$. The sheaf axioms for $\mathcal{O}_X$ imply that the M_f satisfy the sheaf axioms over basic open sets, so that Lemma 2.7 may be applied to get a sheaf $\tilde{M}$ over X which is an $\mathcal{O}_X$-module by construction.

The rule $M \to \tilde{M}$ is naturally a functor from the category of A-modules to the category of $\mathcal{O}_X$-modules and it is easy to check that it is fully faithful.

One cannot expect, however, that in this way an equivalence of categories arises, as the following simple counterexample shows.

Example 5.5 Let A be the local ring of the affine line $\mathbf{A}^1_k$ in 0, i.e. the localisation of the polynomial ring $k[x]$ by the ideal (x). Then $X = \operatorname{Spec} A$ consists only of two points: a closed point coming from (x) and a so-called generic point η coming from the ideal (0). The stalks of $\mathcal{O}_X$ are A in the closed point and $k(x)$ in the generic point. Now define an $\mathcal{O}_X$-module $\mathcal{F}$ on $X = \operatorname{Spec} A$ by putting $\mathcal{F}(X) = A$ and $\mathcal{F}(\eta) = 0$, the restriction $\mathcal{F}(X) \to \mathcal{F}(\eta)$ being the zero map. As the only nonempty open subsets of X are η and X itself, these data indeed define an $\mathcal{O}_X$-module whose A-module of global sections is A. But this $\mathcal{O}_X$-module is not isomorphic to $\tilde{A}$ as the stalks at η are different.

Definition 5.6 Let X be a scheme. A *quasi-coherent sheaf* on X is an $\mathcal{O}_X$-module $\mathcal{F}$ for which there is an open affine cover $\{U_i : i \in I\}$ of X such that the restriction of $\mathcal{F}$ to each $U_i = \operatorname{Spec} A_i$ is isomorphic to an $\mathcal{O}_{U_i}$-module of the form $\tilde{M}_i$ with some A_i-module M_i. If moreover each M_i is finitely generated over A_i, then $\mathcal{F}$ is called a *coherent sheaf.*

Remark 5.7 The functor $M \to \tilde{M}$ is an *exact* functor, i.e. takes exact sequences of A-modules to exact sequences of sheaves. This follows from the construction of $\tilde{M}$ together with the general fact that given a multiplicatively closed subset S in a ring A, the A-module A_S is *flat* (see e.g. Matsumura [1], Theorem 4.5).

Proposition 5.8 *The following are equivalent for an $\mathcal{O}_X$-module $\mathcal{F}$.*

1. *The sheaf $\mathcal{F}$ is quasi-coherent.*

2. *For all affine open sets $U = \operatorname{Spec} A$ contained in X we have $\mathcal{F}|_U \cong \tilde{M}$ for some A-module M.*

3. *There is an open covering $\{U_i : i \in \Lambda\}$ of X such that over each U_i*

$$\mathcal{F}|_{U_i} \cong \operatorname{coker}(\mathcal{O}_{U_i}^{\oplus I} \xrightarrow{\phi} \mathcal{O}_{U_i}^{\oplus J})$$

for some index sets I, J and morphism ϕ.

Proof: To show (1) $\Rightarrow$ (2), let first $U_i = \operatorname{Spec} A_i$ be as in the definition of quasi-coherence such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. For $f \in A_i$ we then have

$$\mathcal{F}|_{D(f)} \cong (M_i \otimes_{A_i} (A_i)_f)^{\sim}.$$

Indeed, we see using Lemma 2.6 that these sheaves have the same sections over a basic open set $D(g) \subset D(f)$. Thus (2) holds over basic open sets. Given an affine open $U = \operatorname{Spec} A \subset X$, choose an open covering of U by basic open sets $D(g_i)$. Define sheaves $\mathcal{F}_i$, $\mathcal{F}_{ij}$ on U by setting

$$\mathcal{F}_i(V) := \mathcal{F}(V \cap D(g_i)), \quad \mathcal{F}_{ij}(V) := \mathcal{F}(V \cap D(g_i g_j))$$

for all open $V \subset U$. As $\mathcal{F}|_{D(g_i)} \cong \tilde{N}_i$ for some A_{g_i}-module N_i, we have $\mathcal{F}_i \cong \tilde{N}_i$ with N_i viewed as an A-module via the map $A \to A_{g_i}$. Similarly, $\mathcal{F}_{ij} \cong \tilde{N}_{ij}$ for some A-module N_{ij}. The restriction maps $\mathcal{F}(D(g_i)) \to \mathcal{F}(D(g_{ij}))$ correspond to A-module maps $N_i \to N_{ij}$, hence we may set

$$N := \ker(\prod_i N_i \to \prod_{i \neq j} N_{ij}).$$

By exactness of the functor $M \mapsto \tilde{M}$ we have

$$\tilde{N} = \ker(\prod_i \mathcal{F}_i \to \prod_{i \neq j} \mathcal{F}_{ij}).$$

But it follows from the sheaf axioms that the right hand side is none but the sheaf $\mathcal{F}|_U$, whence (2).

To show (2) $\Rightarrow$ (3), consider M such that $\mathcal{F}|_U \cong \tilde{M}$ over $U = \operatorname{Spec} A$, and choose a presentation $M = \operatorname{coker}(A^{\oplus I} \to A^{\oplus J})$ for some index sets I, J. It follows that $\mathcal{F}|_U \cong \operatorname{coker}(\mathcal{O}_U^{\oplus I} \to \mathcal{O}_U^{\oplus J})$ over U. Finally, for (3) $\Rightarrow$ (1) we may assume the $U_i = \operatorname{Spec} A_i$ are affine by refining the covering if necessary, and write $M_i := \operatorname{coker}(A_i^{\oplus I} \to A_i^{\oplus J})$ for the module map corresponding to ϕ. By exactness of the functor $M \mapsto \tilde{M}$ we conclude $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. □

Corollary 5.9 *On an affine scheme $X = \operatorname{Spec} A$ the functor $M \to \tilde{M}$ induces an equivalence of categories between A-modules M and quasi-coherent sheaves on X.*

Proof: Fully faithfulness of $M \to \tilde{M}$ follows from the fact that a morphism $\tilde{M} \to \tilde{N}$ of sheaves induces a module homomorphism $M \to N$ by taking global sections. Essential surjectivity follows from applying (1) $\Rightarrow$ (2) of the proposition with $U = X$. □

Using this fact we can prove:

Proposition 5.10 *If $X = \operatorname{Spec} A$ is an affine scheme, then an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of quasi-coherent sheaves on X induces an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

of abelian groups.

Proof: The only nontrivial statement is surjectivity of the map $\mathcal{G}(X) \to \mathcal{H}(X)$. Write $M := \mathcal{G}(X)$, $N := \mathcal{H}(X)$. Then by the previous corollary we have $\mathcal{G} = \tilde{M}$, $\mathcal{H} = \tilde{N}$. By taking global sections we have a morphism of A-modules $M \to N$; let P be its cokernel. Since $M \to \tilde{M}$ is an exact functor (Remark 5.7), we have an exact sequence

$$\tilde{M} \to \tilde{N} \to \tilde{P} \to 0.$$

Here the first morphisms identifies with $\mathcal{G} \to \mathcal{H}$ which is surjective by assumption, hence $\tilde{P} = 0$. Thus $P = \tilde{P}(X) = 0$. □

Coherent sheaves are usually well behaved on locally noetherian schemes. Here is the definition.

Definition 5.11 A scheme X is *locally noetherian* if it has a covering by affine open subschemes of the form Spec A with A a noetherian ring. If it has a finite such covering, it is called *noetherian.*

Remark 5.12 It can be shown that *any* affine open subset of a locally noetherian scheme is of the form form Spec A with A a noetherian ring. See Hartshorne [1], Proposition II.3.2.

We then have the following version of Proposition 5.8:

Corollary 5.13 *If X is locally noetherian, the following are equivalent for an $\mathcal{O}_X$-module $\mathcal{F}$.*

1. *The sheaf $\mathcal{F}$ is coherent.*
2. *There is an open covering $\{U_i : i \in \Lambda\}$ of X such that over each U_i*

$$\mathcal{F}|_{U_i} \cong \operatorname{coker}(\mathcal{O}_{U_i}^{\oplus s} \xrightarrow{\phi} \mathcal{O}_{U_i}^{\oplus r})$$

for some integers r, s and morphism ϕ.

Proof: Over a noetherian ring finitely generated modules are finitely presented, so in Proposition 5.8 we may take both index sets to be finite. □

6. Examples of Quasi-coherent Sheaves

We first return to the first example in 5.2 and investigate the question of determining whether a morphism $\phi : X \to Y$ yields a quasi-coherent sheaf $\phi_*\mathcal{O}_X$ in Y. Unfortunately, this is not true in general but Section II.5 of Hartshorne [1] contains several sufficient conditions. For our purposes the following easy condition on ϕ will suffice.

Definition 6.1 A morphism $\phi : X \to Y$ of schemes is *affine* if Y has an covering by affine open subsets $U_i = \operatorname{Spec} A_i$ such that for each i the open subscheme $\phi^{-1}(U_i)$ of X is affine as well.

Example 6.2 Any morphism of affine schemes is obviously affine. Another important class of affine morphisms is that of *finite* morphisms where we require in addition that $\phi^{-1}(U_i)$ is a finite A_i-module for each i. This contains as a special case the class of closed immersions (closedness of the image follows from the going-up theorem in commutative algebra).

Lemma 6.3 *If $\phi : X \to Y$ is an affine morphism, then $\phi_*\mathcal{O}_X$ and the ideal sheaf defined by the kernel of $\phi^\sharp$ are quasi-coherent sheaves on Y.*

Proof: Assume first $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine schemes. Then $\phi_*\mathcal{O}_X$ is just $\tilde{B}$ with B regarded as an A-module via the map $\lambda : A \to B$ inducing ϕ. Indeed, it is enough to check this over basic open sets $D(f)$ for which we may argue in the same way as in the second half of the proof of Theorem 2.14. Moreover, a similar reasong shows that the ideal sheaf on Y defined by the kernel of $\phi^\sharp$ is just $\tilde{I}$ with $I = \ker(\lambda)$. Once we have these results at hand, the general case of the lemma follows from the definition of affine morphisms and quasi-coherent sheaves. □

The lemma applies in particular to a closed immersion $i : X \to Y$ of schemes which is affine by definition. Thus to any closed subscheme of Y we may associate a quasi-coherent sheaf of ideals. We now prove the converse.

Proposition 6.4 *The above construction gives a bijection between closed subschemes $X \subset Y$ and quasi-coherent sheaves of ideals on Y.*

Proof: Given a quasi-coherent sheaf of ideals $\mathcal{I}$ on Y, we may take a covering of Y by affine open subschemes $U_j = \operatorname{Spec} A_j$ as in the definition of quasi-coherence. Define for each j a closed immersion $i_j : X_j \to U_j$ as the map induced by the projection $A_j \to A_j/I_j$, where I_j is the ideal for which $\mathcal{I}|_{U_j} \cong \tilde{I}_j$. To see that $X = \bigcup X_j$ is closed in X, note first that $X \cap U_j = X_j$ for all j (look at the restriction of $\mathcal{I}$ to basic open sets contained in the intersections $U_i \cap U_j$). But then any point of $U_j \cap (Y \setminus X)$ has an open neighbourhood contained in $U_j \setminus (X \cap U_j)$, whence the claim. Finally, the i_j endow X with the structure of a closed subscheme. It is manifest that the two constructions are inverse to each other. □

The previous proposition may be generalized as follows. A *quasi-coherent sheaf of $\mathcal{O}_X$-algebras* is a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}(U)$ carries an $\mathcal{O}_X(U)$-algebra structure for each open $U \subset X$ and the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are algebra homomorphisms. Equivalently, for each affine open $U = \operatorname{Spec} A$ in X we have $\mathcal{F}|_U \cong \tilde{M}$ for an A-algebra M.

Proposition 6.5 *Assume given a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-algebras. The functor on the category of schemes over X sending $\phi : Y \to X$ to $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \phi_*\mathcal{O}_Y)$ is representable by a scheme $\rho_{\mathcal{F}} : \operatorname{Spec} \mathcal{F} \to X$. Moreover, the morphism $\rho_{\mathcal{F}}$ is affine.*

Sketch of proof: Assume first both $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine, and $\mathcal{F} = \tilde{M}$ for an A-module M. Then we have isomorphisms

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \phi_*\mathcal{O}_Y) \cong \operatorname{Hom}_A(M, B) \cong \operatorname{Hom}_X(Y, \operatorname{Spec} M)$$

where the first two Hom-sets consist of algebra homomorphisms and the last one of scheme morphisms. The first isomorphism follows from the quasi-coherence of $\mathcal{F}$ and the second from Theorem 2.14. The general case is done by a patching construction as in the proof of Theorem 4.4. □

Next we turn to another important example.

Definition 6.6 A *locally free sheaf* on a scheme X is an $\mathcal{O}_X$-module $\mathcal{F}$ for which there exists an open covering $\mathcal{U} = \{U_i : i \in I\}$ of X such that the restriction of $\mathcal{F}$ to each U_i is isomorphic to $\mathcal{O}_{U_i}^{n_i}$ for some positive integer n_i. A *trivialisation* of $\mathcal{F}$ is a covering $\mathcal{U}$ as above and a system of isomorphisms $\mathcal{O}_{U_i}^{n_i} \cong \mathcal{F}|_{U_i}$.

If X is connected, then the n_i are all equal to the same number n called the *rank* of $\mathcal{F}$. A locally free sheaf of rank 1 is called an *invertible sheaf.*

Lemma 6.7 *Any locally free sheaf is coherent. Moreover, if X is noetherian and connected, a coherent sheaf $\mathcal{F}$ on X is locally free of rank n if and only if it stalk $\mathcal{F}_P$ at each point P is a free $\mathcal{O}_{X,P}$-module of rank n.*

Proof: For the first statement, take any affine open subset $V = \operatorname{Spec} A$ contained in one of the U_i as in the definition. Then by the assumption the restriction of $\mathcal{F}$ to V is isomorphic to the coherent sheaf defined by the free A-module $A \oplus \ldots \oplus A$ (with A repeated n_i times).

In the second statement necessity follows from the definitions by taking the direct limit. For sufficiency, assume $\mathcal{F}_P$ is freely generated over $\mathcal{O}_{X,P}$ by some generators $t_1, \ldots, t_n$. We may view the t_i as sections generating $\mathcal{F}(U)$ for some sufficiently small open neighbourhood U of P. By shrinking U if necessary we may assume $U = \operatorname{Spec} A$ and $\mathcal{F} = \tilde{M}$ for some A-module M generated by the t_i. Since X is noetherian, M is the quotient of the free A-module of rank n by a submodule generated by *finitely many* relations among the t_i. By assumption, any of the finitely many coefficients occuring in these relations vanishes when restricted to some open neighbourhood of P contained in U. Denoting by V the intersection of these neighbourhoods, the elements $t_i|_V$ generate $\mathcal{F}|_V$ freely over $\mathcal{O}_V$. □

Remark 6.8 A similar (but easier) argument as in the second part of the above proof shows that if $\mathcal{F}$ is a coherent sheaf on any scheme X and P is a point for which $\mathcal{F}_P = 0$ then there is some open neighbourhood V of P with $\mathcal{F}|_V = 0$.

For a locally free sheaf $\mathcal{F}$ and point $P \in X$ with residue field $\kappa(P)$ the group $\mathcal{F}_P \otimes \kappa(P)$ is a finite dimensional $\kappa(P)$-vector space. So we may think of a locally free sheaf as a family of $\kappa(P)$-vector spaces which is locally trivial. We now make this notion precise.

Definition 6.9 A *vector bundle* of rank n on X is a morphism of schemes $p : V \to X$ such that there exists an open covering $\{U_i : i \in I\}$ of X by affine open subsets $U_i = \operatorname{Spec} A_i$ together with isomorphisms

$$\phi_i : V \times_X U_i \xrightarrow{\sim} \mathbf{A}^n_{U_i} := \operatorname{Spec} A_i[T_1, \ldots, T_n]$$

of schemes over U_i for all i, and moreover for all $U = \operatorname{Spec} A$ contained in $U_i \cap U_j$ the automorphism of $\mathbf{A}_U^n$ given by $c_i^{-1}|_U \circ c_j|_U$ corresponds to an A-*linear* automorphism of $A[T_1, \ldots, T_n]$.

Given an arbitrary open $U \subset X$, a *section* of V is a morphism $s : U \to V$ such that $p \circ s = \mathrm{id}_U$.

Proposition 6.10 *Given a rank n vector bundle $p : V \to X$, the sheaf of sets on X given by*

$$U \mapsto S_V(U) := \{\text{sections of } p \text{ over U}\}$$

is a locally free sheaf of rank n on X.

Conversely, given a locally free sheaf $\mathcal{F}$ of rank n on X, there exists a vector bundle $p : V \to X$ such that $S_V \cong \mathcal{F}$.

Proof: If $U = \operatorname{Spec} A$ is such that $U \times_X V \cong \mathbf{A}_U^n$, then

$$S_V(U) \cong \operatorname{Hom}_A(A[T_1, \ldots, T_n], A) \cong A^{\oplus n}.$$

It follows that $S_V|_U \cong \mathcal{O}_U^n$. This also implies that S_V is an $\mathcal{O}_X$-module by an argument similar to that in Lemma 2.7, whence the first statement.

For the second statement, assume first $X = \operatorname{Spec} A$ is affine and $\mathcal{F} = \mathcal{O}_X^n$. Then $V = \operatorname{Spec} A[T_1, \ldots, T_n]$ is a good choice. The general case is done by a patching construction, noting that if U_i and U_j are small enough to have isomorphisms $\phi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^n$ and $\phi_j : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_j}^n$, then for all $U = \operatorname{Spec} A$ contained in $U_i \cap U_j$ the automorphism $\phi_i^{-1}|_U \circ \phi_j|_U$ induces an A-linear automorphism of $A[T_1, \ldots, T_n]$. □

Remark 6.11 There is a more conceptual approach to the above proposition. Note first that given a ring A and an A-module M, the functor on the category of A-algebras given by $B \mapsto \operatorname{Hom}_A(M, B)$ (module homomorphisms!) is representable by an A-algebra $Sym(M)$. To construct $Sym(M)$, one divides the tensor algebra

$$T(M) := \bigoplus_{i=0}^{\infty} M^{\otimes i}$$

by the two-sided ideal generated by elements of the form $m \otimes n - n \otimes m$ $(m, n \in B)$.

The defining property implies that for every A-algebra C we have a canonical isomorphism $Sym(M) \otimes_A C \cong Sym(M \otimes_A C)$. This allows us to associate a quasi-coherent sheaf of $\mathcal{O}_X$-algebras on $X = \operatorname{Spec} A$ to every quasi-coherent sheaf $\mathcal{F} = \tilde{M}$ on A by setting $Sym(\mathcal{F}) = Sym(M)^{\sim}$. It also shows that we may globalize the construction to associate a quasi-coherent sheaf $Sym(\mathcal{F})$ of $\mathcal{O}_X$-algebras to a quasi-coherent sheaf $\mathcal{F}$ on an arbitrary scheme X. It represents the functor $\mathcal{G} \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ on the category of quasi-coherent $\mathcal{O}_X$-algebras.

Finally, given a rank n locally free sheaf $\mathcal{F}$ on X, consider the dual sheaf $\mathcal{F}^\vee$ given by $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{O}_U)$. It is also locally free of rank n. The affine morphism $\mathrm{Spec}\,(Sym(\mathcal{F}^\vee)) \to X$ given by Proposition 6.5 then defines a vector bundle with sheaf of sections $\mathcal{F}$. Indeed, over an affine open $U = \mathrm{Spec}\,A$ over which $\mathcal{F}$ is trivial, $Sym(\mathcal{F}^\vee) \cong A[T_1, \dots, T_n]$ and all patching isomorphisms are linear.

The case of invertible sheaves is particularly important. First a general definition.

Definition 6.12 A scheme X is called *integral* if for all open subsets $U \subset X$ the ring $\mathcal{O}_X(U)$ is an integral domain.

This algebraic notion has a strong consequence for the underlying topological space of the scheme. Namely, call a topological space X *irreducible* if it cannot be written as a union of two closed subsets properly contained in X, or, equivalently, if any two open subsets have a nonempty intersection. Now the basic fact is:

Lemma 6.13 *The underlying topological space of an integral scheme is irreducible.*

Proof: Indeed, if U_1 and U_2 are nonempty disjoint open subsets of a scheme X, then the sheaf axioms imply that $\mathcal{O}_X(U_1 \cup U_2)$ is isomorphic to the direct sum $\mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2)$,which is not an integral domain. $\square$

Remark 6.14 In fact, a scheme is integral if and only if its underlying space is irreducible and if the rings $\mathcal{O}_X(U)$ contain no nilpotent elements. See Hartshorne [1], Proposition II.3.1.

Proposition 6.15 *Let X be an integral scheme.*

1. *There is a unique point $\eta \in X$ whose closure is the underlying space of X.*

2. *The stalk $\mathcal{O}_{X,\eta}$ is a field K which is naturally isomorphic to the fraction field of any local ring of X.*

Proof: We begin with the first statement. For uniqueness, assume η_1, η_2 both have the required property. Then any affine open subset $U = \operatorname{Spec} A$ contains both η_1 and η_2: they correspond to prime ideals P and Q of A with the property $V(P) = V(Q) = U$. Since A is an integral domain, this is only possible for $P = Q = (0)$. This argument also shows the existence of η: indeed, define it as the point corresponding to the ideal (0) of A. Its closure in X contains U, hence it must be the whole of X by the previous lemma. The second statement is obvious from this construction: $\mathcal{O}_{X,\eta}$ is none but the fraction field of A which is the common fraction field of all local rings of U; for the points of $X \setminus U$ we work with other affine open subsets which all have a non-empty intersection with U by irreducibility of X and hence have a local ring in common. □

Definition 6.16 The point η of the proposition is called the *generic point* of X and the field K the *function field* of X.

Now let X be an integral scheme, and denote by $\mathcal{K}$ the constant abelian sheaf on X defined by *the additive group* of K. It has an $\mathcal{O}_X$-module structure coming from the natural embedding of $\mathcal{O}_X$ into $\mathcal{K}$ but is not a quasi-coherent sheaf.

Proposition 6.17 *Every invertible sheaf $\mathcal{L}$ on an integral scheme X is isomorphic to a sub-$\mathcal{O}_X$-module of the constant $\mathcal{O}_X$-module $\mathcal{K}$ defined above.*

Proof: Choose a nonempty (hence dense) open set $U \subset X$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Denoting by $j : U \to X$ the inclusion map, we have a natural map $\mathcal{O}_X \to j_*\mathcal{O}_U$ sending $s \in \mathcal{O}_X(V)$ to $s|_{U\cap V}$. This map is injective because if $V = \operatorname{Spec} A$ is affine and $D(f)$ is a basic open set contained in $U \cap V$, the composite map $s \mapsto s|_{U\cap V} \mapsto s|_{D(f)}$ corresponds to the localization map $A \to A_f$ which is injective as X is integral. It follows that the natural map $\mathcal{L} \to j_*\mathcal{L}|_U$ is injective. Indeed, this may be checked by restricting to an open covering trivializing $\mathcal{L}$, where it follows from the case $\mathcal{L} = \mathcal{O}_X$ treated above. Finally, we have a sequence of injective maps

$$\mathcal{L} \hookrightarrow j_*\mathcal{L}|_U \xrightarrow{\sim} j_*\mathcal{O}_U \hookrightarrow j_*\mathcal{K}|_{\mathcal{U}} = \mathcal{K}$$

where the last equality holds because $\mathcal{K}$ is a constant sheaf and U is dense in X. □

Definition 6.18 Let X be an integral scheme. A *Cartier divisor* on X is given by an open covering $\{U_i : i \in I\}$ of X together with rational functions $f_i \in \mathcal{K}(U_i) \setminus \{0\}$ such that $f_i f_j^{-1}|_{U_i\cap U_j}$ and $f_j f_i^{-1}|_{U_i\cap U_j}$ both lie in

$\mathcal{O}_X(U_i \cap U_j)$ for all pairs $i \neq j$ in I. Two such systems $\{(U_i, f_i) : i \in I\}$ and $\{(V_j, g_j) : j \in J\}$ define the same Cartier divisor if $f_i g_j^{-1}|_{U_i \cap V_j}$ and $g_j f_i^{-1}|_{U_i \cap V_j}$ both lie in $\mathcal{O}_X(U_i \cap U_j)$ for all pairs $i \neq j$.

Construction 6.19 Given a Cartier divisor D on X, we construct an invertible sheaf $\mathcal{L}(D)$ as follows. If D is represented by a system $\{(U_i, f_i) : i \in I\}$, we define $\mathcal{L}(D)$ as the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ such that $\mathcal{L}(D)|_{U_i}$ is the free $\mathcal{O}_{U_i}$-module generated by f_i^{-1}. The definition of Cartier divisors shows that $\mathcal{L}(D)$ is well defined and does not depend on the representative $\{(U_i, f_i) : i \in I\}$.

Conversely, given an invertible sub-$\mathcal{O}_X$-module $\mathcal{L}$ of $\mathcal{K}$, choosing an open covering $\{U_i : i \in I\}$ such that $\mathcal{L}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module of rank 1 defines a Cartier divisor D on X by fixing generators. By construction, we have $\mathcal{L} = \mathcal{L}(D)$, so the two maps are inverse to each other.

Assume finally that $\mathcal{L} \subset \mathcal{K}$ is an invertible sheaf such that $\mathcal{L} \cong \mathcal{O}_X$. The image of $1 \in \mathcal{O}(X)$ under this isomorphism gives a rational function $f \in \mathcal{L}(X) \subset \mathcal{K}(X)$. By construction, f generates $\mathcal{L}$ as a free $\mathcal{O}_X$-module, hence $\mathcal{L} = \mathcal{L}(D)$ for the Cartier divisor D represented by the pair (X, f). Such a divisor is called a *principal divisor*. Plainly, if D is principal, then $\mathcal{L}(D) \cong \mathcal{O}_X$.

The set of Cartier divisors on X inherits an abelian group structure from the multiplicative structure of K. Principal divisors form a subgroup. The quotient group is called the *Picard group* of X and is denoted by $\mathrm{Pic}(X)$.

On the other hand the set of invertible sheaves on X is also equipped with a group operation induced by tensor product. We first need the notion of the *tensor product* $\mathcal{F} \otimes \mathcal{G}$ of two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$. We define it as follows. By Lemma 2.7 it is enough to define $(\mathcal{F} \otimes \mathcal{G})(U)$ for $U \subset X$ affine open. If $U = \operatorname{Spec} A$ and $\mathcal{F}|_U = \tilde{M}$, $\mathcal{G}|_U = \tilde{N}$, we set $(\mathcal{F} \otimes \mathcal{G})(U) := (M \otimes_A N)^{\sim}(U)$. If $D(f) \subset U$ is a basic affine open set, we have $(\mathcal{F} \otimes \mathcal{G})(D(f)) = M_f \otimes_{A_f} N_f$. This implies that the tensor product is well defined. Moreover, by passing to the direct limit over basic affine open neighbourhoods of a point P, we obtain natural isomorphisms

$$(\mathcal{F} \otimes \mathcal{G})_P \cong \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P.$$

Proposition 6.20 *For any scheme X, tensor product of $\mathcal{O}_X$-modules induces an abelian group structure on the set of isomorphism classes of invertible sheaves on X. The unit element of this group is $\mathcal{O}_X$ and the inverse of a class represented by an invertible sheaf $\mathcal{L}$ is the class of the sheaf $\mathcal{L}^\vee$ given by $U \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U)$.*

Proof: First, notice that if $\mathcal{L}$, $\mathcal{L}'$ are invertible sheaves, then so is their tensor product. Indeed, given an affine open subset $U = \operatorname{Spec} A$ where both invertible sheaves are trivial, we have isomorphisms $\mathcal{L}|_U \cong \mathcal{L}'|_U \cong \mathcal{O}_U$. Therefore over U the tensor product is isomorphic to

$$(A \otimes_A A)^{\sim} \xrightarrow{\sim} \tilde{A} = \mathcal{O}_U. \tag{6}$$

The group law is well defined since the tensor product of modules respects isomorphisms. The abelian group axioms concerning commutativity, associativity and the unit element follow from the corresponding properties of the tensor product. So only the axiom concerning the inverse remains. For each open set U define a morphism $\mathcal{L}(U) \times \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{L}(U), \mathcal{O}_X(U)) \to \mathcal{O}_X(U)$ by the natural evaluation map $(s, \phi) \mapsto \phi(s)$. For $U = \operatorname{Spec} A$ affine it comes from the evaluation map on modules $ev : M \times \operatorname{Hom}_A(M, A) \to A$ which is A-bilinear, hence induces a map $M \otimes \operatorname{Hom}_A(M, A) \to A$. It follows that there is a morphism of invertible sheaves $\mathcal{L} \otimes \mathcal{L}^{\vee} \to \mathcal{O}_X$. To check that this map is an isomorphism, we may pass to an affine open covering trivializing $\mathcal{L}$. There it follows from the fact that ev is an isomorphism for $M = A$. □

Proposition 6.21 *The rule $D \mapsto \mathcal{L}(D)$ induces an isomorphism between the Picard group of X and the group of isomorphism classes of invertible sheaves on X.*

Proof: First we check that $D \mapsto \mathcal{L}(D)$ induces a group homomorphism. For this notice first that in formula (6) the first isomorphism is induced by the multiplication map $A \otimes A \to A$. It follows that if $U = \operatorname{Spec} A$ is an affine open set over which $\mathcal{L}(D_i)$ is generated over $\mathcal{O}_U$ by a section s_i for $i = 1, 2$, then the tensor product is generated by $s_1 s_2$.

The kernel of the map $D \mapsto \mathcal{L}(D)$ consists of the principal divisors by the last remark in Construction 6.19. To conclude the proof it remains to apply Proposition 6.17. □

7. Modules and Sheaves of Differentials

In differential geometry, the tangent space at a point P on some variety is defined to consist of so-called *linear derivations*, i.e. linear maps that associate a scalar to each function germ at P and satisfy the Leibniz rule. We begin by an algebraic generalisation of this notion.

Definition 7.1 Let B be a ring and M a B-module. A *derivation* of B into M is a map $d : B \to M$ subject to the two conditions:

1. (Additivity) $d(x+y) = dx + dy$;
2. (Leibniz rule) $d(xy) = xdy + ydx$.

Here we have written dx for $d(x)$ to emphasise the analogy with the classical derivation rules. If moreover B is an A-algebra for some ring A (for example $A = \mathbf{Z}$), an A-linear derivation is called an *A-derivation.* The set of A-derivations of B to M is equipped with a natural B-module structure via the rules $(d_1 + d_2)x = d_1x + d_2x$ and $bdx = dbx$. This B-module is denoted by $Der_A(B, M)$.

Note that applying the Leibniz rule to the equality $1 \cdot 1 = 1$ gives $d(1) = 0$ for all derivations; hence all A-derivations are trivial on the image of A in B.

In the example one encounters in (say) real differential geometry we have $A = M = \mathbf{R}$, and B is the ring of germs of differentiable functions at some point; $\mathbf{R}$ is a B-module via evaluation of functions. Now comes a purely algebraic example.

Example 7.2 Assume given an A-algebra B which decomposes *as an A-module* into a direct sum $B \cong A \oplus I$, where I is an ideal of B with $I^2 = 0$. Then the natural projection $d: B \to I$ is an A-derivation of B into I. Indeed, A-linearity is immediate; for the Leibniz rule we take elements $x_1, x_2 \in B$ and write $x_i = a_i + dx_i$ with $a_i \in k$ for $i = 1, 2$. Now we have

$$d(x_1x_2) = d[(a_1+dx_1)(a_2+dx_2)] = d(a_1a_2+a_2dx_1+a_1dx_2) = x_2dx_1+x_1dx_2$$

where we used several times the facts that $I^2 = 0$ and $d(A) = 0$.

In fact, given any ring A and A-module I, we can define an A-algebra B as above by defining a product structure on the A-module $A \oplus I$ by the rule $(a_1, i_1)(a_2, i_2) = (a_1a_2, a_1i_2 + a_2i_1)$. So the above method yields plenty of examples of derivations.

Now notice that for fixed A and B the rule $M \to Der_A(B, M)$ defines a functor on the category of B-modules; indeed, given a homomorphism $\phi: M_1 \to M_2$ of B-modules, we get a natural homomorphism $Der_A(B, M_1) \to Der_A(B, M_2)$ by composing derivations with ϕ.

Proposition 7.3 *The functor $M \to Der_A(B, M)$ is representable by a B-module $\Omega^1_{B/A}$.*

Proof: The construction is done in a similar way to that of the tensor product of two modules. Define $\Omega^1_{B/A}$ to be the quotient of the free B-module generated by symbols dx for each $x \in B$ modulo the relations given by the additivity and Leibniz rules as in Definition 7.1 as well as the relations $d(\lambda(a)) = 0$ for all $a \in A$, where $\lambda : A \to B$ is the map defining the A-module structure on B. The map $x \to dx$ is an A-derivation of B into $\Omega^1_{B/A}$. Moreover, given any B-module M and A-derivation $\delta \in Der_A(B, M)$, the map $dx \to \delta(x)$ induces a B-module homomorphism $\Omega^1_{B/A} \to M$ whose composition with d is just δ. This implies that $\Omega^1_{B/A}$ represents the functor $M \to Der_A(B, M)$; in particular, d is the universal derivation corresponding to the identity map of $\Omega^1_{B/A}$. □

We call $\Omega^1_{B/A}$ the module of *relative differentials* of B with respect to A. We shall often refer to the elements of $\Omega^1_{B/A}$ as *differential forms.*

Next we describe how to compute relative differentials of a finitely presented A-algebra.

Proposition 7.4 *Let B be the quotient of the polynomial ring $A[x_1, \dots, x_n]$ by an ideal generated by finitely many polynomials $f_1, \dots, f_m$. Then $\Omega^1_{B/A}$ is the quotient of the free B-module on generators $dx_1, \dots, dx_n$ modulo the B-submodule generated by the elements $\sum_j (\partial_j f_i) dx_j$ $(i = 1, \dots, m)$, where $\partial_j f_i$ denotes the j-th (formal) partial derivative of f_i.*

Proof: First consider the case $B = A[x_1, \dots, x_n]$. As B is the free A-algebra generated by the x_i, one sees that for any B-module M there is a bijection between $Der_A(B, M)$ and maps of the set $\{x_1, \dots, x_n\}$ into B. This implies that $\Omega^1_{B/A}$ is the free A-module generated by the dx_i.

The general case follows from this in view of the easy observation that given any M, composition by the projection $A[x_1, \dots, x_n] \to B$ induces an isomorphism of $Der_A(B, M)$ onto the submodule of $Der_A(A[x_1, \dots, x_n], M)$ consisting of derivations mapping the f_i to 0. □

Next some basic properties of modules of differentials.

Lemma 7.5 *Let A be a ring and B an A-algebra.*

1. *(Direct sums) For any A-algebra B'*
$$\Omega^1_{(B \oplus B')/A} \cong \Omega^1_{B/A} \oplus \Omega^1_{B'/A}.$$
2. *(Exact sequence) Given a map of A-algebras $\phi : B \to C$, there is an exact sequence of C-modules*
$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0.$$

In particular, if ϕ is surjective, we have a surjection $\Omega^1_{B/A}\otimes_B C \to \Omega^1_{C/A}$.

3. *(Base change) Given a ring homomorphism $A \to A'$, denote by B' the A'-algebra $B \otimes_A A'$. There is a natural isomorphism*

$$\Omega^1_{B/A} \otimes_B B' \cong \Omega^1_{B'/A'}.$$

4. *(Localisation) For any multiplicatively closed subset $S \subset B$ there is a natural isomorphism*

$$\Omega^1_{B_S/A} \cong \Omega^1_{B/A} \otimes_B B_S.$$

Proof: The first property is easy and left to the readers. For the second, note that for any C-module M we have a natural exact sequence

$$0 \to Der_B(C, M) \to Der_A(C, M) \to Der_A(B, M)$$

of C-modules isomorphic to

$$0 \to \mathrm{Hom}_C(\Omega^1_{C/B}, M) \to \mathrm{Hom}_C(\Omega^1_{C/A}, M) \to \mathrm{Hom}_B(\Omega^1_{B/A}, M).$$

The claim follows from this in view of the formal Lemma **??** of Chapter 0 and the isomorphism $\mathrm{Hom}_B(\Omega^1_{B/A}, M) \cong \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, M)$. This isomorphism is obtained by mapping a homomorphism $\Omega^1_{B/A} \to M$ to the composite $\Omega^1_{B/A} \otimes_B C \to M \otimes_B C \to M$ where the second map is multiplication; an inverse is given by composition with the natural map $\Omega^1_{B/A} \to \Omega^1_{B/A} \otimes_B C$. If the map $B \to C$ is onto, then any B-derivation is a C-derivation as well, so $\Omega^1_{B/C} = 0$ and the first map in the exact sequence is onto.

For base change, note first that the universal derivation $d : B \to \Omega^1_{B/A}$ is an A-module homomorphism and so tensoring it by A' we get a map

$$d' : B' \to \Omega^1_{B/A} \otimes_A A' \cong \Omega^1_{B/A} \otimes_B B \otimes_A A' \cong \Omega^1_{B/A} \otimes_B B'$$

which is easily seen to be an A'-derivation. Now any A'-derivation $\delta' : B' \to M'$ induces an A-derivation $\delta : B \to M'$ by composition with the natural map $B \to B'$. But δ factors as $\delta = \phi \circ d$, with a B-module homomorphism $\phi : \Omega^1_{B/A} \to M'$, whence a map $\phi' : \Omega^1_{B/A} \otimes_B B' \to M'$ constructed as above. Now one checks that $\delta' = \phi' \circ d'$ which means that $\Omega^1_{B/A} \otimes_B B'$ represents the functor $M' \mapsto Der_{A'}(B', M')$.

For the localisation property, given an A-derivation $\delta : B \to M$, we may extend it uniquely to an A-derivation $\delta_S : B_S \to M \otimes_B B_S$ by setting $\delta_S(b/s) = (\delta(b)s - b\delta(s)) \otimes (1/s^2)$. (We leave it to the reader to check that for $b'/s' = b/s$ we get the same result – this is much simpler in the case when there are no zero-divisors in S which is the only case we shall need.) This applies in particular to the universal derivation $d : B \to \Omega^1_{B/A}$, and one argues as in the previous case to show that any A-derivation $B_S \to M_S$ factors uniquely through d_S. □

As an application of differentials we give a criterion for a one-dimensional closed subscheme of affine or projective space to be a smooth curve. For this it is enough to check that all local rings at closed points are discrete valuation rings. Since the proof works more generally for regular local rings, we state the result in this context.

Proposition 7.6 *Let k be a perfect field and let A be a localisation of a finitely generated n-dimensional k-algebra at some closed point P. Then A is a regular local ring if and only if $\Omega^1_{A/k}$ is a free A-module of rank n. In particular, if $n = 1$, A is a discrete valuation ring if and only if $\Omega^1_{A/k}$ is free of rank 1.*

Remark 7.7 Explicitly, if A is a localisation of the k-algebra

$$B = k[x_1, \dots, x_d]/(f_1, \dots, f_m),$$

then Proposition 7.4 and the localisation property of differentials imply that the proposition amounts to saying that among the relations $\sum_j (\partial_j f_i) dx_j = 0$ there should be exactly $d - n$ linearly independent ones, which in turn is equivalent by linear algebra to the fact that the $k \times m$ "Jacobian" matrix $J = [\partial_j f_i]$ should have rank $d - n$. In fact, for $k = \mathbf{C}$ reducing the entries of J modulo the maximal ideal of A gives just the classical Jacobian matrix of the closed subscheme of $\mathbf{C}^d$ defined by the equations $f_i = 0$ at the point P corresponding to A and the condition says that some open neighbourhood of P should be a complex manifold of dimension n.

For the proof of the proposition we need two lemmas from algebra. The first of these is a form of Hilbert's Nullstellensatz (which implies the one used in the previous chapter).

Lemma 7.8 *Let k be a field and let P be a maximal ideal in a finitely generated k-algebra A. Then the field A/P is a finite extension of k.*

For a proof, see Lang [1], Chapter IX, Corollary 1.2. See also Atiyah-Macdonald [1] for four different proofs.

The other lemma is from field theory.

Lemma 7.9 *Let k be a perfect field and let $K|k$ be a finitely generated field extension of transcendence degree n. Then there exist algebraically independent elements $x_1, \ldots, x_n \in K$ such that the finite extension $K|k(x_1, \ldots, x_n)$ is separable.*

For a proof, see Lang [1], Chapter VIII, Corollary 4.4.

Corollary 7.10 *In the situation of the lemma, the K-vector space $\Omega^1_{K/k}$ is of dimension n, a basis being given by the dx_i.*

Proof: We may write the field K as the fraction field of the quotient A of the polynomial ring $k[x_1, \ldots, x_n, x]$ by a single polynomial relation f. Here f is the minimal polynomial of a generator of the extension $K|k(x_1, \ldots, x_n)$ multiplied with a common denominator of its coefficients. Now according to Proposition 7.4 the A-module $\Omega^1_{A/k}$ has a presentation with generators $dx_1, \ldots, dx_n, dx$ and a relation in which dx has a nontrivial coefficient because $f' \neq 0$ by the lemma. The corollary now follows using Lemma 7.5 (4). $\square$

Proof of Proposition 7.6: We give the proof under the additional assumption that there exists a subfield $k \subset k' \subset A$ that maps isomorphically onto the residue field $\kappa(P) = A/P$ by the projection $A \to A/P$. (Lemma 7.8 implies that this condition is trivially satisfied if k is algebraically closed.) In the remark below we shall explain how one can reduce the general case to this one.

Notice that since k is perfect and $k'|k$ is a finite extension by Lemma 7.8, we have $\Omega^1_{k'|k} = 0$ by the previous corollary. Hence by applying Lemma 7.5 (2) (with our k in place of A, k' in place of B and A in place of C) we get $\Omega^1_{A/k} \cong \Omega^1_{A/k'}$, so we may as well assume $k = k' \cong \kappa(P)$.

In this case the k-module P/P^2 is canonically isomorphic to $\Omega^1_{A/k}/P\Omega^1_{A/k}$. Indeed, the latter k-vector space is immediately seen to represent the functor $M \to Der_k(A, M)$ for any k-vector space M viewed as an A-module via the quotient map $A \to A/P \cong k$. On the other hand, the above functor is also represented by P/P^2. To see this, note first that the Leibniz rule implies that any k-derivation $\delta : A \to M$ is trivial on P^2, hence we may as well assume $P^2 = 0$. But then we are in the situation of Example 7.2 and we may observe that δ factors uniquely as $\delta = \phi \circ d$, with d as in the quoted example and $\phi \in \mathrm{Hom}_k(P, N)$.

Now if $\Omega^1_{A/k}$ is free of rank n, then $\Omega^1_{A/k}/P\Omega^1_{A/k} \cong P/P^2$ has dimension n. For the converse, observe first that the previous isomorphism and the corollary to Nakayama's lemma (Corollary **??**) gives that $\Omega^1_{A/k}$ can be generated as an A-module by n elements $dt_1, \ldots, dt_n$. Were there a nontrivial relation

$\sum f_i dt_i = 0$ in $\Omega^1_{A/k}$, by the localisation property of differentials this relation would survive in $\Omega^1_{K/k}$, contradicting Corollary 7.10. This implies that $\Omega^1_{A/k}$ is free. □

Remark 7.11 To reduce the general case of the proposition to the one discussed above it is convenient to use the completion $\hat{A}$ of A. This is the inverse limit of of the natural inverse system formed by the quotients A/P^n of A. There is a natural map $A \to \hat{A}$ which is injective for A noetherian by Corollary **??**. The image of P gives a maximal ideal $\hat{P}$ of $\hat{A}$ with $\hat{P}^i/\hat{P}^{i+1} \cong P^i/P^{i+1}$ for all $i > 0$. If A is of dimension 1, the case $i = 1$ of this isomorphism together with Corollary **??** implies that A is a discrete valuation ring if and only if $\hat{A}$ is. In general, we get that $\hat{A}$ is regular if and only if A is regular, for one can prove (see Atiyah-Macdonald [1], Corollary 11.19) that the Krull dimension of A is the same as that of $\hat{A}$. Also, the base change property of differentials implies that $\Omega^1_{\hat{A}/k}$ is free of rank n if and only if $\Omega^1_{A/k}$ is.

Therefore it remains to see that $\hat{A}$ satisfies the condition at the beginning of the above proof. For this, let $f \in k[x]$ be the minimal polynomial of a (separable) generator α of the extension $\kappa(P)|k$; it is enough to lift α to a root of f in $\hat{A}$. This can be done by means of Hensel's lemma (see Chapter 7, Section 4).

In the remaining of this section we discuss quasi-coherent sheaves associated to modules of differentials. Namely, we shall define *sheaves of relative differentials* $\Omega^1_{Y/X}$ for certain classes of morphisms of schemes $Y \to X$. In fact, one may define these for any morphism $Y \to X$ but since we did not develop the necessary background we refer the interested readers to the excellent treatment in Mumford's notes [1] or to Section II.8 of Hartshorne [1]. What we propose instead is a more down-to-earth discussion in two special cases.

Construction 7.12 First, if $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are both affine, we define $\Omega^1_{Y/X}$ as the quasi-coherent sheaf $\tilde{\Omega}^1_{B/A}$. Notice that according to the localisation property of differentials, over a basic open set $D(g) = \operatorname{Spec} B_g$ of X the sheaf $\Omega^1_{Y/X}$ is given by the B_g-module $\Omega^1_{B_g/A}$.

Construction 7.13 Next assume we have a morphism $X \to \operatorname{Spec} k$ with an arbitrary scheme X; we shall use the abusive notation $\Omega^1_{X/k}$ for the corresponding sheaf of differentials which we now construct. For any affine open covering of X by subsets $U_i = \operatorname{Spec} A_i$ the rings A_i are all k-algebras and the sheaf $\Omega^1_{U_i/k} = \tilde{\Omega}^1_{A_i/k}$ is defined on U_i. Moreover, any basic open subset contained in $U_i \cap U_j$ is canonically isomorphic to both $(A_i)_{f_i}$ and $(A_j)_{f_j}$, whence an isomorphism $\Omega^1_{(A_i)_{f_i}/k} \cong \Omega^1_{(A_j)_{f_j}/k}$. These isomorphisms are compatible for inclusions of basic open sets, so the third statement of Chapter 5, Lemma 2.7 applies to give an isomorphism $(\Omega^1_{U_i/k})|_{U_i \cap U_j} \cong (\Omega^1_{U_j/k})|_{U_i \cap U_j}$. These latter isomorphisms in turn are compatible over triple intersections $U_i \cap U_j \cap U_k$ so we may patch the $\Omega^1_{U_i/k}$ together by the method of Chapter 5, Construction 4.3 (which adapts to the construction of quasi-coherent sheaves) to get $\Omega^1_{X/k}$. Finally one checks that if we use a different open covering we get an $\mathcal{O}_X$-module isomorphic to $\Omega^1_{X/k}$.

Remark 7.14 Let X be an affine or a projective variety of dimension n. Then Proposition 7.6 may be rephrased by saying that X is a regular scheme if and only if the the sheaf $\Omega^1_{X/k}$ is locally free of rank n. In this case, we say that X is *smooth* over k.

Construction 7.15 Finally, the other case where we can easily define relative differentials is that of an *affine* morphism $\phi : Y \to X$. In this case X is covered by affine open subsets $U_i = \operatorname{Spec} A_i$ whose inverse images $V_i = \operatorname{Spec} B_i$ form an open covering of Y and the B_i are A_i-modules via the maps $\lambda_i : A_i \to B_i$ arising from ϕ. Take $f_i \in A_i$ and put $g_i = \lambda_i(f_i)$. Then the inverse image of the basic open set $D(f_i) = \operatorname{Spec}(A_i)_{f_i}$ is none but $D(g_i)$ which in turn is isomorphic to $\operatorname{Spec}(B_i \otimes_{A_i} (A_i)_{f_i})$; indeed, one checks easily that $(B_i \otimes_{A_i} (A_i)_{f_i})$ represents the functor defining the localisation $(B_i)_{g_i}$. Hence by the base change property of differentials we have canonical isomorphisms

$$\Omega^1_{V_i/U_i}(D(g_i)) = \Omega^1_{B_i/A_i} \otimes_{B_i} (B_i)_{g_i} \cong \Omega^1_{(B_i)_{g_i}/(A_i)_{f_i}} = \Omega^1_{D(g_i)/D(f_i)},$$

so we may patch the sheaves $\Omega^1_{V_i/U_i}$ together over inverse images of basic affine open subsets contained in $U_i \cap U_j$ by the same method as in the previous case.

8. Dimension

We now introduce the notion of dimension for schemes. Of course, we would like affine and projective n-space to be n-dimensional, a point to be 0-dimensional, a plane curve 1-dimensional, a surface 2-dimensional, etc. One heuristic approach is the following inductive “argument”: a curve should be of dimension 1 because its irreducible proper closed subsets are only points, a surface should have dimension 2 as it contains only curves and points as proper closed subsets etc. This approach is summarised in the following definition.

Definition 8.1 The *dimension* of a scheme X is the supremum of the integers n for which there exists a strictly increasing chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subsets properly contained in X.

Remark 8.2 The dimension is either a positive integer or infinite. It is mainly interesting for noetherian schemes because noetherian rings have no infinite ascending chains of prime ideals. However, there exist noetherian rings whose associated affine scheme has infinite dimension; see Atiyah-Macdonald [1], Exercise 11.4 for an example due to Nagata.

In order to be able to give examples in the affine case, we first prove an easy lemma.

Lemma 8.3 *Let $X = \operatorname{Spec} A$ be an affine scheme. Then any irreducible closed subset of X is of the form $V(P)$, with P a prime ideal of A.*

Proof: Let $Z = V(I)$ be a closed subset of X. We may and do assume that I is the intersection of the prime ideals corresponding to the points of Z. Assume $fg \in I$ for some $f, g \in A$. Then any prime ideal containing I must contain f or g, hence the union of the closed subsets $V(I + (f))$ and $V(I + (g))$ is Z. Therefore Z is irreducible if and only if one of them, say $V(I + (f))$ equals Z. By our assumption on I this is equivalent to $f \in I$, whence the claim. □

By the lemma, the dimension of $\operatorname{Spec} A$ is the supremum of the lengths of chains of prime ideals in A. In ring theory this is called the *Krull dimension of A* and is usually denoted by $\dim A$.

Thus for instance, the Krull dimension of a field is 0, that of $\mathbf{Z}$ is 1. But in general with the above definition the dimension is hard to determine in practice. It is not even clear that affine or projective spaces have the dimension we expect. Fortunately, this can be remedied by means of a criterion for wich we need to recall a definition first.

Definition 8.4 The transcendence degree of a field extension $K|k$ is the maximal number of elements of K algebraically independent over k; the transcendence degree of an integral domain A containing k is defined as the transcendence degree of its fraction field over k and is denoted by $tr.deg_k\, A$.

Now comes the criterion which we only quote from the literature.

Proposition 8.5 *Let k be a field and A an integral domain which is a finitely generated k-algebra. Then the Krull dimension of A is equal to its transcendence degree over k.*

For a proof, see e.g. Matsumura [1], Theorem 5.6.

Example 8.6 As immediate applications of the proposition, we get that $\mathbf{A}_k^n$ and $\mathbf{P}_k^n$ both have dimension n as expected, and that affine and projective plane curves have dimension 1. In general, affine or projective varieties of dimension 1 are called *curves*, those of dimension 1 *surfaces.*