COHOMOLOGY OF QUASI-COHERENT SHEAVES

TAMÁS SZAMUELY

1. Definition of cohomology

To define cohomology, we first review some general facts about derived functors.

Facts 1.1. Let $\mathcal{C}$ be an abelian category. We say that $\mathcal{C}$ *has enough injectives* if every object can be embedded in an injective object. In this case every object A has an *injective resolution* $A \to I^\bullet$, i.e. an exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

with each I^j injective. Given a left exact functor F from $\mathcal{C}$ to another abelian cateory, one defines the i-th right derived functor R^iF of F by choosing an injective resolution $I^\bullet$ for each object A and setting

$$R^iF(A) := H^i(F(I^\bullet)).$$

One shows that $R^iF(A)$ does not depend on $I^\bullet$. Given a short exact sequence

$$0 \to A \to B \to C \to 0$$

of objects of $\mathcal{C}$ and a left exact functor F, one gets a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \cdots$$

where $R^0F \cong F$ as F is left exact. Note also that $R^iF(I) = 0$ for $i > 0$ and I injective, because then $0 \to I \to I \to 0$ is an injective resolution of I.

Now we apply the above to the category of sheaves of abelian groups on a topological space.

Lemma 1.2. *The category of sheaves of abelian groups on a topological space X has enough injectives.*

Proof. Given a sheaf $\mathcal{F}$ on X and a point $P \in X$, denote by $\mathcal{F}^P$ the skyscraper sheaf with given by $\mathcal{F}_P$ over open sets containing P and 0 elsewhere. There is a natural morphism of sheaves $\mathcal{F} \to \mathcal{F}^P$. Taking direct products we obtain a morphism $\mathcal{F} \to \prod_P \mathcal{F}_P$ which is injective by the first sheaf axiom. Now for each $P \in X$ choose an embedding $\mathcal{F}_P \to \mathcal{I}_P$ with $\mathcal{I}_P$ an injective abelian group (recall that the category of abelian groups has enough injectives). Consider the corresponding embedding $\mathcal{F}^P \to \mathcal{I}^P$ of skyscraper sheaves. Taking products we obtain an embedding $\mathcal{F} \to \prod_P \mathcal{I}^P$, so since a product of injectives is always injective, it is enough to show that each $\mathcal{I}^P$ is an injecive sheaf. This follows from the injectivity of $\mathcal{I}_P$ as an abelian group because every morphism $\mathcal{G} \to \mathcal{I}^P$ from another sheaf $\mathcal{G}$ factors through $\mathcal{G}^P$. □

By the lemma, we may define

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$$

where $\Gamma(X, \ )$ is the left exact functor $\mathcal{F} \mapsto \mathcal{F}(X)$.

By the above general facts, we have $H^0(X, \mathcal{F}) \cong \mathcal{F}(X)$, $H^i(X, \mathcal{I}) = 0$ for $\mathcal{I}$ injective and $i > 0$, and for every short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves a long exact sequence of abelian groups

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to \cdots$$

Similarly, for every morphism $\phi : X \to Y$ of schemes we can consider the right derived functors $R^i\phi_*$ of the left exact functor $\phi_* : \mathcal{F} \mapsto \phi_*\mathcal{F}$ from the category of sheaves on X to the category of sheaves on Y. The sheaves $R^i\phi_*\mathcal{F}$ for $i > 0$ are called the *higher direct images* of $\mathcal{F}$ by ϕ.

2. Flabby sheaves

Now to a concept particular to sheaves.

Definition 2.1. A sheaf $\mathcal{F}$ on a topological space X is *flabby* if the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are surjective for all inclusions of open sets $V \subset U$.

From now on we assume that the base space is locally connected.

Proposition 2.2. *Every injective sheaf is flabby.*

The proof requires some preparation. If U is an open subset of a locally connected topological space, there is a unique sheaf $\mathbf{Z}_U$ on X such that for a connected open subset $V \subset X$ we have $\mathbf{Z}_U(V) = \mathbf{Z}$ if $V \subset U$ and $\mathbf{Z}_U(V) = 0$ otherwise. (Indeed, it is straightforward to extend the above definition to non-connected open sets.)

Lemma 2.3. *For a sheaf $\mathcal{F}$ on X there are isomorphisms of abelian groups*

$$\mathrm{Hom}(\mathbf{Z}_U, \mathcal{F}) \cong \mathcal{F}(U)$$

for every open $U \subset X$.

Proof. Given a morphism $\mathbf{Z}_U \to \mathcal{F}$, we may consider the image of $1 \in \mathbf{Z}_U(U)$ in $\mathcal{F}(U)$. This defines a homomorphism $\mathrm{Hom}(\mathbf{Z}_U, \mathcal{F}) \to \mathcal{F}(U)$. Conversely, given $s \in \mathcal{F}(U)$, there is a unique morphism of sheaves $\mathbf{Z}_U \to \mathcal{F}$ that maps $1 \in \mathbf{Z}_V(V)$ for a connected $V \subset U$ to $s|_V$. The two constructions are inverse to each other. □

Proof of Proposition 2.2. Let $V \subset U$ be an inclusion of open subsets of X. We may naturally identify $\mathbf{Z}_V$ with a subsheaf of $\mathbf{Z}_U$. By the lemma above we may identify a section of an injective sheaf $\mathcal{I}$ over V with a morphism of sheaves $\mathbf{Z}_V \to \mathcal{I}$. By injectivity of $\mathcal{I}$ this morphism extends to a morphism $\mathbf{Z}_U \to \mathcal{I}$. This means precisely that the restriction map $\mathcal{I}(U) \to \mathcal{I}(V)$ is surjective. □

Proposition 2.4. *If $\mathcal{F}$ is a flabby sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

For the proof we need:

Lemma 2.5.
a) A short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves on a topological space X with $\mathcal{F}$ flabby induces an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

on global sections.
b) If in an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves $\mathcal{F}$ and $\mathcal{G}$ are flabby, then so is $\mathcal{H}$.

Proof. To prove *a*) the only issue is surjectivity of $\mathcal{G}(X) \to \mathcal{H}(X)$. Given $s \in \mathcal{H}(X)$, consider the system of pairs (U, s'), where $U \subset X$ is open and $s' \in \mathcal{G}(U)$ has image $s|_U$ in $\mathcal{H}(U)$. There is a natural partial order on this system in which $(U'', s'') \leq (U, s')$ if $U'' \subset U$ and $s'' = s'|_{U''}$. A standard application of Zorn's lemma shows that this partially ordered system has a maximal element. We show that for such a maximal element we must have $U = X$. Assume not, and let P be a point of $X \setminus U$. As the map $\mathcal{G} \to \mathcal{H}$ is surjective, we find an open neighbourhood V of P and a section $s'' \in \mathcal{G}(V)$ which maps to $s|_V$ in $\mathcal{H}(V)$. The section $s'|_{U\cap V} - s''|_{U\cap V}$ maps to 0 in $\mathcal{H}(U \cap V)$, so by shrinking V if necessary we find $t \in \mathcal{F}(U \cap V)$ that maps to $s'|_{U\cap V} - s''|_{U\cap V}$ in $\mathcal{G}(U \cap V)$. As $\mathcal{F}$ is flabby, we may extend t to a section in $\mathcal{F}(V)$ which we also denote by t. Changing s'' to $s'' + t$ (here we identify $\mathcal{F}$ with its image in $\mathcal{G}$) we obtain a section that still maps to $s|_V$ in $\mathcal{H}(V)$ but for which $s'|_{U\cap V} = s''|_{U\cap V}$. Therefore these sections patch together to a section in $\mathcal{G}(U \cup V)$ mapping to $s|_{U\cup V}$, contradicting the maximality of (U, s').

To prove b), let $V \subset U$ be an inclusion of open subsets. By part a) the map $\mathcal{G}(V) \to \mathcal{H}(V)$ is surjective as $\mathcal{F}|_V$ is flabby, and so is the map $\mathcal{G}(U) \to \mathcal{G}(V)$ as $\mathcal{G}$ is flabby. The composite map $\mathcal{G}(U) \to \mathcal{H}(V)$ is therefore surjective but it factors through the restriction $\mathcal{H}(U) \to \mathcal{H}(V)$ by definition of a morphism of sheaves. Hence the latter map is also surjective. □

Proof of Proposition 2.4. Embed $\mathcal{F}$ in an injective sheaf $\mathcal{I}$ and denote by $\mathcal{G}$ the quotient. By Proposition 2.2 in the exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$$

$\mathcal{F}$ and $\mathcal{I}$ are flabby, hence so is $\mathcal{G}$ by part b) of the above lemma. Part a) of the lemma therefore shows that in the long exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{I})$$

the map $\mathcal{I}(X) \to \mathcal{G}(X)$ is surjective. But $H^1(X, \mathcal{I}) = 0$, so $H^1(X, \mathcal{F}) = 0$ by the exact sequence. For $i > 1$ we use induction on i. In the part of the long exact sequence

$$H^{i-1}(X, \mathcal{G}) \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{I})$$

we have $H^i(X, \mathcal{I}) = 0$ by injectivity of $\mathcal{I}$ and $H^{i-1}(X, \mathcal{G}) = 0$ by the inductive assumption since $\mathcal{G}$ is also flabby. Therefore $H^i(X, \mathcal{F}) = 0$. □

The significance of Proposition 2.4 lies in the fact that it enables one to compute the groups $H^i(X, \mathcal{F})$ for an arbitrary sheaf $\mathcal{F}$ by means of flabby resolutions instead of injective ones. To see this, we need a lemma from homological algebra:

Lemma 2.6. *Let $\mathcal{C}$ be an abelian category with enoyugh injective, and $F : \mathcal{C} \to \mathcal{D}$ a left exact functor. Assume that A is an object of $\mathcal{C}$ for which there exists a resolution*

$$A \to B^0 \to B^1 \to B^2 \cdots$$

with $R^i F(B^j) = 0$ for all $i > 0$, $j \geq 0$. Then $R^i F(A) \cong H^i F(B^\bullet)$.

Proof. Split the resolution in short exact sequences

$$0 \to A \to B^0 \to K^0 \to 0, \ \ldots, \ 0 \to K^{i-1} \to B^i \to K^i \to 0, \ \ldots$$

The first one gives an exact sequence

$$0 \to F(A) \to F(B^0) \to F(K^0) \to R^1 F(A) \to 0$$

as F is left exact and $R^1 F(B^0) = 0$. We obtain

$$R^1 F(A) \cong \operatorname{coker}(F(B^0) \to F(K^0)) =$$
$$= \operatorname{coker}(F(B^0) \to \ker(F(B^1) \to F(B^2))) = H^1 F(B^\bullet).$$

Next, for $j > 0$ we have

$$R^j F(K^i) \cong R^{j+1} F(K^{i-1}), \ \ldots, \ R^j F(K^0) \cong R^{j+1} F(A).$$

This gives

$$R^{j+1}F(A) \cong R^jF(K^0) \cong R^{j-1}F(K^1) \cong \dots$$
$$\cong R^1F(K^{j-1}) \cong \operatorname{coker}(F(B^j) \to F(K^j)) = H^{j+1}F(B^\bullet).$$

□

Corollary 2.7. *Let $\mathcal{F}$ be a sheaf on a topological space X. Suppose there exists a resolution*

$$\mathcal{F} \to \mathcal{G}^\bullet$$

where the $\mathcal{G}^i$ are flabby sheaves. Then

$$H^i(X, \mathcal{F}) \cong H^i(\Gamma(X, \mathcal{G}^\bullet)).$$

Proof. Apply the lemma with $F = \Gamma(X, \_\_)$ and $B^\bullet = \mathcal{G}^\bullet$. The condition $R^iF(B^j) = 0$ for $i > 0$ is satisfied by Proposition 2.4. □

As a first application, we prove:

Proposition 2.8. *If $\phi : X \to Y$ is an morphism of topological spaces and $\mathcal{F}$ is any sheaf on X, the higher direct image sheaf $R^i\phi_*\mathcal{F}$ is the sheaf associated with the presheaf $V \mapsto H^i(\phi^{-1}(V), \mathcal{F}|_{\phi^{-1}(V)})$.*

Proof. Take an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^i\phi_*\mathcal{F}$ is the i-th cohomology sheaf of $\phi_*\mathcal{I}^\bullet$. This is the sheaf associated with the presheaf

$$V \mapsto H^i\Gamma(V, \phi_*\mathcal{I}^\bullet|_V) = H^i\Gamma(\phi^{-1}(V), \mathcal{I}^\bullet|_{\phi^{-1}(V)}).$$

But since $\mathcal{I}^j$ is injective, hence flabby for all j, so is $\mathcal{I}^j|_{\phi^{-1}(V)}$, and therefore $\mathcal{F}_{\phi^{-1}(V)} \to \mathcal{I}^\bullet|_{\phi^{-1}(V)}$ is a flabby resolution. Therefore by the previous corollary we have $H^i\Gamma(V, \phi_*\mathcal{I}^\bullet|_V) \cong H^i(\phi^{-1}(V), \mathcal{F}|_{\phi^{-1}(V)})$. □

3. Serre's vanishing theorem

We now prove:

Theorem 3.1. (Serre) *If X is an affine scheme and $\mathcal{F}$ a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

The proof is based on a general topological lemma.

Lemma 3.2. *Let X be a compact topological space, and $\mathcal{B}$ a basis of open sets of X. Given a sheaf $\mathcal{F}$ and an integer $i \geq 0$, we say that $\mathcal{F}$ has property (P_i) if for all $\alpha \in H^i(X, \mathcal{F})$ there exists a finite open covering $U_1, \dots, U_r$ of X by elements of $\mathcal{B}$ such that α maps to zero in $H^1(X, \mathcal{F}_j)$, where $\mathcal{F}_j := (u_j)_*(\mathcal{F}\ |_{U_j})$ for the open inclusion $u_j : U_j \to X$. Then:*
a) Property (P_1) holds for every sheaf $\mathcal{F}$.
b) If $i > 1$, assume that for all $U \in \mathcal{B}$ we have $H^p(U, \mathcal{F}) = 0$ for $0 < p < i$. Then (P_i) holds for $\mathcal{F}$.

[Recall that $\mathcal{F}_j(V) = \mathcal{F}(V \cap U_j)$ for all open sets $V \subset X$.]

Proof. To prove $a)$, we embed $\mathcal{F}$ in an injective sheaf $\mathcal{I}$, whence an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{C} \to 0.$$

The beginning of the associated long exact sequence reads

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{C}(X) \to H^1(X, \mathcal{F}) \to 0$$

as $\mathcal{I}$ is injective. Given $\alpha \in H^1(X, \mathcal{F})$, we may therefore lift it to $s \in \mathcal{C}(X)$. As the morphism of sheaves $\mathcal{I} \to \mathcal{C}$ is surjective and X is compact, we find an open covering $U_1, \ldots, U_r$ of X by elements of $\mathcal{B}$ such that each $s|_{U_j}$ lifts to $s_j \in I(U_j)$.

Set $\mathcal{I}_j = (u_j)_*(\mathcal{I}_{|U_j})$ and denote by $\overline{\mathcal{C}}_j$ the cokernel of the induced morphism $\mathcal{F}_j \to \mathcal{I}_j$. In particular, $\overline{\mathcal{C}}_j$ can be viewed as a subsheaf of $\mathcal{C}_j := (u_j)_*(\mathcal{C}|_{U_j}))$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & I & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_j & \longrightarrow & I_j & \longrightarrow & \overline{\mathcal{C}}_j & \longrightarrow & 0 \end{array} \tag{1}$$

and we know that the image s_j of $s \in \mathcal{C}(X)$ in $\overline{\mathcal{C}}_j(X) \subset \mathcal{C}_j(X) = \mathcal{C}(U_j)$ comes from $s_j \in \mathcal{I}_j(X) = \mathcal{I}(U_j)$. Therefore s_j maps to zero in $H^1(X, \mathcal{F}_j)$ and we obtain by functoriality of the long exact cohomology sequence that the image α of s in $H^1(X, \mathcal{F})$ maps to zero in $H^1(X, \mathcal{F}_j)$, which is what we wanted to prove.

To prove $b)$ we assume $i > 1$ and use induction on i. Take an arbitrary finite open covering $U_1, \ldots, U_r$ of X by elements of $\mathcal{B}$. If U is another open set in $\mathcal{B}$, then the sets $U \cap U_j$ are again in $\mathcal{B}$. By assumption $H^1((U \cap U_j), \mathcal{F}) = 0$, and therefore the sequence

$$0 \to \mathcal{F}_j(U) \to \mathcal{I}_j(U) \to \mathcal{C}_j(U) \to 0$$

is exact since $\mathcal{F}_j(U) = \mathcal{F}(U \cap U_j)$ (and similarly for $\mathcal{I}$ and $\mathcal{C}$). Similarly, the sequence

$$0 \to \mathcal{F}_j(U) \to \mathcal{I}_j(U) \to \overline{\mathcal{C}}_j(U) \to 0$$

is exact. As this holds for all $U \in \mathcal{B}$, we obtain $\overline{\mathcal{C}}_j \cong \mathcal{C}_j$.

We may therefore replace $\overline{\mathcal{C}}_j$ by $\mathcal{C}_j$ in diagram (1), and from the associated long exact cohomology sequence we obtain a commutative diagram

$$\begin{array}{ccc} H^{i-1}(X, \mathcal{C}) & \xrightarrow{\cong} & H^i(X, \mathcal{F}) \\ \downarrow & & \downarrow \\ H^{i-1}(X, \mathcal{C}_j) & \xrightarrow{\cong} & H^i(X, \mathcal{F}_j). \end{array}$$

Here the horizontal maps are isomorphisms by Proposition 2.4 because $\mathcal{I}$ is injective, hence flabby, and hence so is $\mathcal{I}_j$. As the sheaf $\mathcal{C}$ satisfies (P_1) by part $a)$, the diagram shows that $\mathcal{F}$ satisfies (P_2), whence the case $i = 2$. Assume now that the result holds for $p < i$. To show it for i, it is enough to see by the diagram that $\mathcal{C}$ satisfies (P_{i-1}). By the inductive assumption this will follow if we show $H^p(U, \mathcal{C}) = 0$ for all $U \in \mathcal{B}$ and $0 < p < i-1$. But this holds by the exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{C} \to 0$$

and the vanishing of $H^p(U, \mathcal{I})$ and $H^{p+1}(U, \mathcal{F})$ (the latter holding because $p+1 < i$). □

Proof of Theorem 5.1. Assume $X = \operatorname{Spec} A$ and $\mathcal{F}$ is a quasi-coherent sheaf on X. We prove the theorem by induction on i using the lemma above, where we take $\mathcal{B}$ to be the system of affine open subsets.

First assume $i = 1$ and pick $\alpha \in H^1(X, \mathcal{F})$. By part *a*) of the lemma we find a finite open covering $U_1, \ldots, U_r$ of X such that α maps to zero in all $H^1(X, \mathcal{F}_j)$. We know that $\mathcal{F}_j$ is quasi-coherent, hence so is the cokernel $\mathcal{G}$ of the morphism $\mathcal{F} \to \prod_{j=1}^{r} \mathcal{F}_j$ induced by the restriction maps $\mathcal{F}(U) \to \mathcal{F}(U \cap U_j)$). As X is affine and

$$0 \to \mathcal{F} \to \prod_{j=1}^{r} \mathcal{F}_j \to \mathcal{G} \to 0 \tag{2}$$

is an exact sequence of quasi-coherent sheaves, the sequence

$$0 \to \mathcal{F}(X) \to \prod_{j=1}^{r} \mathcal{F}_j(X) \to \mathcal{G}(X) \to 0$$

is again exact. Hence in the long exact cohomology sequence associated with (2) the map $H^1(X, \mathcal{F}) \to \prod_{j=1}^{r} H^1(X, \mathcal{F}_j)$ is injective. But α maps to zero in each $H^1(X, \mathcal{F}_j)$, hence $\alpha = 0$.

For $i > 1$ we use induction on i. By the inductive assumption for all affine open $U \subset X$ and $0 < p < i$ we have $H^p(U, \mathcal{F}|_U) = 0$, so by part *b*) of the lemma given $\alpha \in H^i(X, \mathcal{F})$, we find a finite open covering $U_1, \ldots, U_r$ of X such that α maps to zero in all $H^i(X, \mathcal{F}_j)$. By the inductive hypothesis applied to $\mathcal{G}$ we have $H^{i-1}(X, \mathcal{G}) = 0$, hence the long exact cohomology sequence associated with (2) shows that the map $H^i(X, \mathcal{F}) \to \prod_{j=1}^{r} H^i(X, \mathcal{F}_j)$ is injective. So again $\alpha = 0$, which completes the proof of the case $i = 1$. □

We now derive a consequence of Serre's theorem for the right derived functors $R^i\phi_*$ of an affine morphism $\phi : X \to Y$. Recall that a morphism $\phi : X \to Y$ is affine if for all $V \subset Y$ affine $\phi^{-1}(V)$ is affine as well. (In fact it is enough to require this for elements of a single affine open covering of X.) Examples of affine morphisms are given by closed immersions or, more generally, finite morphism, and also by the inclusion of an affine open subset in a separated scheme.

Theorem 3.3. *If $\phi : X \to Y$ is an affine morphism and $\mathcal{F}$ is a quasi-coherent sheaf on X, the sheaves $R^i\phi_*\mathcal{F}$ are 0 for all $i > 0$.*

In fact one may also view this statement as a generalization of Serre's theorem: the latter is equivalent to the special case where Y is a point.

Proof. Serre's vanishing theorem and Proposition 2.8 imply that the stalks $(R^i\phi_*\mathcal{F})_P$ must be 0 for all $P \in Y$. □

Corollary 3.4. *Under the assumptions of the theorem there are canonical isomorphisms $H^i(Y, \phi_*\mathcal{F}) \cong H^i(X, \mathcal{F})$ for all $i \geq 0$.*

Proof. Given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we obtain a flabby resolution $\phi_*\mathcal{F} \to \phi_*\mathcal{I}^\bullet$ after applying ϕ_*. Indeed, $\phi_*\mathcal{I}^j$ is flabby for all j because so is $\mathcal{I}^j$, and $\phi_*\mathcal{F} \to \phi_*\mathcal{I}^\bullet$ is a resolution because $R^i\phi_*\mathcal{F} = 0$ by the theorem. So we may compute the cohomology of $\phi_*\mathcal{F}$ using this resolution by Corollary 2.7. But $\Gamma(Y, \phi_*\mathcal{I}^j) = \Gamma(X, \mathcal{I}^j)$ for all j by definition, so the corollary follows by taking cohomology. □

Remark 3.5. There is a much simpler proof of Theorem 3.3 in the case when ϕ is a closed immersion, and in this case the theorem holds for an arbitrary sheaf $\mathcal{F}$. Namely, for a closed immersion $\phi : X \hookrightarrow Y$ the stalk of $\phi_*\mathcal{F}$ at $P \in Y$ is 0 if $P \notin X$ and equals $\mathcal{F}_P$ otherwise. Hence the functor ϕ_* is exact (checking exactness on stalks is immediate), and therefore $R^i\phi_* = 0$ for $i > 0$. It follows that the isomorphisms $H^i(Y, \phi_*\mathcal{F}) \cong H^i(X, \mathcal{F})$ hold for arbitrary sheaves in the case of a closed immersion.

4. A vanishing theorem for $\mathbf{P}^n$

By a theorem of Grothendieck (proven e.g. in Hartshorne's book) if X is a topological space in which every descending chain of proper irreducible closed subsets has length at most n, then $H^i(X, \mathcal{F}) = 0$ for $i > n$ and any sheaf $\mathcal{F}$ on X. We shall prove another result here which is a very special case of this theorem when A is a field:

Proposition 4.1. *If A is a ring and $\mathcal{F}$ is a quasi-coherent sheaf on $\mathbf{P}^n_A$, then $H^i(\mathbf{P}^n_A, \mathcal{F}) = 0$ for $i > n$.*

Instead of the proposition we shall prove the following more general result.

Theorem 4.2. *If X is a separated scheme that can be covered by $n + 1$ affine open subsets and $\mathcal{F}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > n$.*

Following an idea of Serre, we prove the theorem by a simplicial method for which we need some preliminaries. Let X be a topological space and $\mathcal{U} = \{U_i : i \in I\}$ an open covering of X. We assume that the index set I is well ordered; we shall only need the case where I is finite anyway. For a finite subset $\{i_0, ..., i_p\} \subset I$ we denote by $U_{i_0,...,i_p}$ the intersection $U_{i_0} \cap ... \cap U_{i_p}$.

For a sheaf $\mathcal{F}$ on X we define a complex $C^\bullet(\mathcal{U}, \mathcal{F})$ of abelian groups as follows. For $p \geq 0$ we set

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<...<i_p} \mathcal{F}(U_{i_0,...,i_p}).$$

An element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ is thus given by a system of $\alpha_{i_0,...,i_p} \in \mathcal{F}(U_{i_0,...,i_p})$ for all $(p+1)$-tuples $i_0 < ... < i_p$. We define a coboundary map $d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d^p\alpha)_{i_0,...,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k (\alpha_{i_0,...,\hat{i}_k,...,i_{p+1}}) \mid_{U_{i_0,...,i_{p+1}}}$$

(By convention $C^p(\mathcal{U}, \mathcal{F}) = 0$ if $|I| \leq p$.)

A straightforward computation shows that $d^{p+1} \circ d^p = 0$ for all p, so we indeed obtain a complex of abelian groups, called the *Čech complex* associated with $\mathcal{U}$ and $\mathcal{F}$.

Lemma 4.3. *Let $\mathcal{U}$ be an open covering of X such that $U_i = X$ for some i. Then the complex*

$$0 \to \mathcal{F}(X) \xrightarrow{\varepsilon} C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to ...$$

is exact, where ε is defined by restrictions $\mathcal{F}(X) \to \mathcal{F}(U_i)$.

Proof. We may assume $i = 1$. Exactness at $C^0(\mathcal{U}, \mathcal{F})$ follows from the sheaf axioms. We show exactness at the higher degree terms by proving that the identity map is homotopic to 0. This means that for $p > 0$ we define $k^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{U}, \mathcal{F})$ so that

$$d^{p-1} \circ k^p + k^{p+1} \circ d^p = \mathrm{id}_{C^p(\mathcal{U},\mathcal{F})},$$

which indeed implies $H^p(C^\bullet(\mathcal{U}, \mathcal{F})) = 0$ for $p > 0$. Given $\alpha_{i_0,...,i_p}$ in $\mathcal{F}(U_{i_0,...,i_p})$, to construct $k^p(\alpha_{i_0,...,i_p})$ it is enough to set for $k^p(\alpha_{i_0,...,i_p}) = 0$ if $i_0 \neq 1$ and $k^p(\alpha_{i_0,...,i_p}) = \alpha_{i_0,...,i_p}$ viewed as a section in $\mathcal{F}(U_{i_1,...,i_p})$ if $i_0 = 1$. □

We now define a sheafified version of the Čech complex. We set

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<...<i_p} j_*(\mathcal{F} \mid_{U_{i_0,...,i_p}})$$

where j is the inclusion map $U_{i_0,...,i_p} \to X$. The coboundary maps $d^p : \mathcal{C}^p\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}\mathcal{U}, \mathcal{F})$ are defined as above, and we obtain a complex $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ of sheaves on X satisfying $\Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F})$.

Proposition 4.4. *The sequence of sheaves*

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to ...$$

is exact.

Proof. We show exactness on stalks. Given $P \in X$, it suffices to show that there exists an open neighbourhood V of P such that the sections of the sequence of the proposition over V give an exact sequence of abelian groups. But this results from the previous lemma if we take $V \subset U_i$ for some U_i containing P. □

Lemma 4.5. *If $\mathcal{U} = \{U_1, \dots, U_r\}$ is a finite affine open covering of a separated scheme X and $\mathcal{F}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = 0$ for all p and $i > 0$.*

Proof. It is enough to show that $H^i(X, \mathcal{F}_{i_0,\dots,i_p}) = 0$ for all $i_0 < \dots < i_p$ and $i > 0$, where $\mathcal{F}_{i_0,\dots,i_p} = j_*(\mathcal{F}|_{U_{i_0,\dots,i_p}})$ and j is the inclusion map $U_{i_0,\dots,i_p} \to X$ as before. By separatedness of X the subset $U_{i_0,\dots,i_p}$ is affine, hence so is the morphism j. Thus by Corollary 3.4 we reduce to showing $H^i(U_{i_0,\dots,i_p}, \mathcal{F}|_{U_{i_0,\dots,i_p}}) = 0$. Since $\mathcal{F}|_{U_{i_0,\dots,i_p}}$ is still quasi-coherent, we may conclude by Serre's vanishing theorem. □

Proof of Theorem 4.2. By the preceding proposition and lemma $\mathcal{F}$ has a resolution $\mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ with $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > n$ and $H^i(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = 0$ for all p and $i > 0$. Therefore the statement follows from Lemma 2.6. □

Remark 4.6. For a sheaf $\mathcal{F}$ on a topological space X Serre defined the Čech cohomology groups of X with coefficients in $\mathcal{F}$ by

$$\check{H}^p(X, \mathcal{F}) := \varinjlim H^p(C^\bullet(\mathcal{U}, \mathcal{F}))$$

where the direct limit is taken over all open coverings $\mathcal{U}$ with respect to a natural partial order. Grothendieck proved using Serre's vanishing theorem and a spectral sequence lemma of Cartan that for X a scheme and X quasi-coherent the Čech cohomology groups agree with the cohomology groups defined using derived functors. The arguments in this section show that in the situation of Lemma 4.5 we already have $H^p(C^\bullet(\mathcal{U}, \mathcal{F})) \cong H^p(X, \mathcal{F})$.

5. Serre's finiteness theorems

Let A be a ring, and $\phi : X \to \mathrm{Spec}\,(A)$ a projective morphism. Recall that this means that ϕ is the composite of a closed immersion $i : X \to \mathbf{P}^n_A$ for some n with the natural projection $\mathbf{P}^n_A \to \mathrm{Spec}\,(A)$.

If $\mathcal{F}$ is a quasi-coherent sheaf on X, for all $i \geq 0$ the groups $H^i(X, \mathcal{F})$ are in fact modules over A. This is obvious from the definition of $\mathcal{O}_X$-modules if $i = 0$. For $i > 0$ one may see this as follows: as $\mathcal{F}$ is an $\mathcal{O}_X$-module, one shows by the same argument as in Lemma 1.2 that $\mathcal{F}$ has a resolution by injective $\mathcal{O}_X$*-modules.* Again by a similar argument as before, such $\mathcal{O}_X$-modules are flabby sheaves, so they may be used to compute the groups $H^i(X,\mathcal{F})$. But then it follows from the construction that each $H^i(X,\mathcal{F})$ is a module over $\mathcal{O}_X(X)=A$.

Let $\mathcal{O}(m)$ be the m-th twisting sheaf on $\mathbf{P}^n_A$, and set

$$\mathcal{O}_X(m) := i^*\mathcal{O}(m), \qquad \mathcal{F}(m) := \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_X(m).$$

Note that for $X=\mathbf{P}^n_A$ one may define $\mathcal{F}(m)$ by setting $\mathcal{F}(m)|_{D_+(x_i)} = \mathcal{F}|_{D_+(x_i)}$ for all i and then patching over the intersections using the isomorphisms $f\mapsto (x_j/x_i)^m f$.

Theorem 5.1. (Serre) *Assume moreover that A is a Noetherian ring and $\mathcal{F}$ is a coherent sheaf. Then*

(1) *$H^i(X,\mathcal{F})$ is a finitely generated A-module over A for all $i\geq 0$.*
(2) *$H^i(X,\mathcal{F}(m))$=0 for $i>0$ and m sufficiently large.*

Corollary 5.2. *If $\phi : X\to Y$ is a projective morphism with Y Noetherian, then for every coherent sheaf $\mathcal{F}$ on X the sheaves $R^i\phi_*\mathcal{F}$ are also coherent for $i\geq 0$.*

Proof. For $P\in Y$ consider the natural morphism $\mathrm{Spec}\,(\mathcal{O}_{Y,P})\to Y$. (For an affine open neighbourhood $V=\mathrm{Spec}\,(B)$ of P it is given by the composite of the natural maps $\mathrm{Spec}\,(B_P)\to\mathrm{Spec}\,(B)\to Y$.) Then ϕ induces a natural map $X\times_Y\mathrm{Spec}\,(\mathcal{O}_{Y,P})\to\mathrm{Spec}\,(\mathcal{O}_{Y,P})$. Denoting by ${}^P\mathcal{F}$ the pullback of $\mathcal{F}$ to $X\times_Y\mathrm{Spec}\,(\mathcal{O}_{Y,P})$ the theorem tells us that the $\mathcal{O}_{X,P}$-modules $H^i(X\times_Y\mathrm{Spec}\,(\mathcal{O}_{Y,P}),{}^P\mathcal{F})$ are finitely generated. But Proposition 2.8 implies that these are exactly the stalks $(R^i\phi_*\mathcal{F})_P$, so the corollary follows because Y is Noetherian. □

Remark 5.3. Grothendieck has extended Serre's theorem to the case of an arbitrary proper morphism. His proof is by reduction to the projective case.

We prove the theorem in three steps.

Step 1: *Reduction to the case $X=\mathbf{P}^n_A$.*

We need a *projection formula*:

Lemma 5.4. *If $i : X\to Y$ is an affine morphism, $\mathcal{F}$ a quasi-coherent sheaf on X, $\mathcal{G}$ a quasi-coherent sheaf on Y, there is a natural isomorphism*

$$i_*(\mathcal{F}\otimes_{\mathcal{O}_X} i^*\mathcal{G})\cong (i_*\mathcal{F})\otimes_{\mathcal{O}_Y}\mathcal{G}.$$

Proof. Assume first $X=\mathrm{Spec}\,(B)$ and $Y=\mathrm{Spec}\,(A)$ are affine, and $\mathcal{F}=\widetilde{M}$, $\mathcal{G}=\widetilde{N}$ for a B=module M and an A-module N. Then $i_*\mathcal{F}=\widetilde{M}$ with M viewed as an A-module via the morphism $A\to B$ induced by i, and $i^*\mathcal{G}=\widetilde{N\otimes_A B}$ by definition. So

$$i_*(\mathcal{F}\otimes_{\mathcal{O}_X} i^*\mathcal{G})\cong M\otimes_B\widetilde{(N\otimes_A B)}\cong\widetilde{M\otimes_A N}\cong(i_*\mathcal{F})\otimes_{\mathcal{O}_Y}\mathcal{G}.$$

The general case follows by patching as i is affine. □

Apply the lemma with $Y = \mathbf{P}^n_A$, $\mathcal{G} = \mathcal{O}(m)$. It gives an isomorphism $i_*(\mathcal{F}(m)) \cong (i_*\mathcal{F})(m)$. Hence by Corollary 3.4 we have isomorphisms

$$H^i(X, \mathcal{F}(m)) \cong H^i(\mathbf{P}^n_A, i_*(\mathcal{F}(m))) \cong H^i(\mathbf{P}^n_A, i_*(\mathcal{F})(m))$$

and we conclude by observing that $i_*\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^n_A$.

Step 2: *Reduction to the case* $\mathcal{F} = \mathcal{O}(m)$.

The proof of this step is based on the following proposition, also due to Serre.

Proposition 5.5. *Given a coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_A$, there is a surjective morphism of sheaves*

$$\mathcal{O}(m)^{\oplus r} \twoheadrightarrow \mathcal{F}$$

with suitable $m \in \mathbf{Z}$, $r \geq 0$.

The proposition should be compared with the following fact: every coherent sheaf on an affine scheme $X = \mathrm{Spec}\,(A)$ is the quotient of $\mathcal{O}^r_X$ for suitable r. This follows from the fact that every finitely generated module over A is a quotient of A^r for some r.

To prove the proposition, it is enough to find a surjection $\mathcal{O}^{\oplus r}_{\mathbf{P}^n_A} \to \mathcal{F}(m)$ for some $r > 0$ and $m \in \mathbf{Z}$, for then we may conclude by tensoring with $\mathcal{O}(-m) = \mathcal{H}om(\mathcal{O}(m), \mathcal{O}_{\mathbf{P}^n_A})$. Therefore we must find global sections $s_1, \dots, s_r \in \mathcal{F}(m)(X)$ such that for all $P \in \mathbf{P}^n_A$ some of the $(s_i)_P$ generate the stalk $\mathcal{F}(m)_P$ as an $\mathcal{O}_{\mathbf{P}^n_A,P}$-module. Indeed, if we have such global sections, then

$$(f_1, \dots, f_r) \in \mathcal{O}^{\oplus r}_{\mathbf{P}^n_A}(U) \mapsto f_1 s_1|_U + \dots + f_r s_r|_U \in \mathcal{F}(m)(U)$$

defines a surjection of sheaves as required.

Now since $D_+(x_i)$ is affine, the $\mathcal{O}_{D_+(x_i)}$-module $\mathcal{F}|_{D_+(x_i)}$ is generated by some global sections $t_1, \dots, t_N$ by the remark just made. Therefore to prove the proposition it is enough to verify the following lemma.

Lemma 5.6. *Given $t_i \in \mathcal{F}(D_+(x_i))$ for some i, there is a section $\tilde{t} \in \mathcal{F}(m)(\mathbf{P}^n_A)$ with $\tilde{t}|_{D_+(x_i)} = t_i$ for m suitably large.*

Proof. We may assume $i = 0$. Pick some j and consider the restriction of t_0 to $D_+(x_0) \cap D_+(x_j)$. As $D_+(x_0) \cap D_+(x_j)$ is the affine open set defined by the non-vanishing of $x_0 x_j^{-1}$ inside the affine scheme $D_+(x_j)$, there exists $m \in \mathbf{Z}$ and a global section t'_j of $\mathcal{O}_{D_+(x_j)}$ such that $t'_j = (x_0 x_j^{-1})^m t_0$ on $D_+(x_0) \cap D_+(x_j)$. We may choose the same m for all j if we make m sufficiently large (even for $j = 0$ where we set $t'_0 = t_0$). It may still happen that the restrictions of t'_j and $(x_k x_j^{-1})^m t'_k$ to $D_+(x_j) \cap D_+(x_k)$ do not coincide for $j, k \neq 0$. But we know that on $D_+(x_j) \cap D_+(x_j) \cap D_+(x_k)$ we have $t'_j - (x_k x_j^{-1})^m t'_k = 0$ by comparison with t_0. As this is the affine open subscheme of $D_+(x_j) \cap D_+(x_k)$ given by the non-vanishing of $x_0 x_j^{-1}$, we get $(x_0 x_j^{-1})^p (t'_j - (x_k x_j^{-1})^m t'_k) = 0$ for some $p > 0$ on $D_+(x_j) \cap D_+(x_k)$. We may choose the same p for all

j, k, and then $\tilde{t}_j := (x_0 x_j^{-1})^p t'_j$ satisfies $\tilde{t}_j = (x_k x_j^{-1})^{m+p} \tilde{t}_k$ for all j, k. These therefore patch to a global section of $\mathcal{F}(m+p)$. □

Given Proposition 5.5, we can handle Step 2 as follows. By Theorem 4.2 we know $H^i(\mathbf{P}^n_A, \mathcal{F}) = 0$ for $i > n$. We now employ *descending* induction on i. The proposition gives an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}(m)^{\oplus r} \to \mathcal{F} \to 0$$

with some coherent sheaf $\mathcal{K}$. Part of the long exact cohomology sequence reads

$$H^i(\mathbf{P}^n_A, \mathcal{O}(m)^{\oplus r}) \to H^i(\mathbf{P}^n_A, \mathcal{F}) \to H^{i+1}(\mathbf{P}^n_A, \mathcal{K}).$$

The A-module on the left is finitely generated by assumption and that on the right by the inductive hypothesis. Since A is Noetherian, statement 1 of the theorem follows. Statement 2 is proven similarly.

Step 3: *The case $X = \mathbf{P}^n_A$, $\mathcal{F} = \mathcal{O}(m)$.*

We prove the following more precise statement.

Theorem 5.7. *We have $H^i(\mathbf{P}^n_A, \mathcal{O}(m)) = 0$ unless $i = 0$ or n. Moreover,*

$$H^0(\mathbf{P}^n_A, \mathcal{O}(m)) = \text{ degree } m \text{ part of } A[x_0, \ldots, x_n]$$

and

$$H^n(\mathbf{P}^n_A, \mathcal{O}(m)) = \text{ submodule of degree } m \text{ part of } A[x_0^{-1}, \ldots, x_n^{-1}]$$

generated by monomials $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with all $\alpha_i < 0$.

We begin with a lemma.

Lemma 5.8. *Consider the morphism $\pi : \mathbf{A}^{n+1}_A \setminus \{0\} \to \mathbf{P}^n_A$ given by patching the natural morphisms $D(x_i) \to D_+(x_i)$ corresponding to $A[x_0/x_i, \ldots, x_n/x_i] \to A[x_0, \ldots, x_n]_{x_i}$ together. There is a natural isomorphism*

$$\pi_* \mathcal{O}_{\mathbf{A}^{n+1}_A \setminus \{0\}} \cong \bigoplus_{m \in \mathbf{Z}} \mathcal{O}(m).$$

Proof. The restriction of $\pi_* \mathcal{O}_{\mathbf{A}^{n+1}_A \setminus \{0\}}$ to the affine open set $D_+(x_i)$ is given by the $A[x_0/x_i, \ldots, x_n/x_i]$-module $A[x_0, \ldots, x_n]_{x_i}$. This module decomposes as the direct sum of submodules

$$A^i_m := A[x_0/x_i, \ldots, x_n/x_i] x_i^m$$

for all $m \in \mathbf{Z}$. Over $D_+(x_i) \cap D_+(x_j)$ we have isomorphisms

$$\widetilde{A^j_m} \cong (x_j/x_i)^m \widetilde{A^i_m},$$

so the $\widetilde{A^i_m}$ patch together to an invertible sheaf isomorphic to $\mathcal{O}(m)$. □

Corollary 5.9. *There are natural isomorphisms*

$$H^i(\mathbf{A}^{n+1}_A \setminus \{0\}, \mathcal{O}_{\mathbf{A}^{n+1} \setminus \{0\}}) \cong \bigoplus_{m \in \mathbf{Z}} H^i(\mathbf{P}^n_A, \mathcal{O}(m)).$$

Proof. As π is an affine morphism, the lemma together with Corollary 3.4 yield isomorphisms

$$H^i(\mathbf{A}_A^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbf{A}^{n+1}\setminus\{0\}}) \cong H^i(\mathbf{P}_A^n, \bigoplus_{m\in\mathbf{Z}} \mathcal{O}(m)).$$

It remains to observe that cohomology commutes with direct sums. This can be seen by choosing a flabby resolution for each $\mathcal{O}(m)$, and then taking the direct sum of these resolutions as a flabby resolution of the direct sum. □

In view of the corollary above the theorem follows from:

Proposition 5.10. *We have* $H^i(\mathbf{A}_A^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbf{A}_A^{n+1}\setminus\{0\}}) = 0$ *unless* $i = 0$ *or* n*. Moreover,*

$$H^0(\mathbf{A}_A^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbf{A}_A^{n+1}\setminus\{0\}}) = A[x_0, \dots, x_n]$$

except for $n = 0$ *where it is* $A[x_0, x_0^{-1}]$*, and for* $n > 0$

$$H^n(\mathbf{A}_A^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbf{A}_A^{n+1}\setminus\{0\}}) = \textit{ submodule of } A[x_0^{-1}, \dots, x_n^{-1}]$$

generated by monomials $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ *with all* $\alpha_i < 0$*.*

Proof. The case $i = 0$ is well known from elementary algebraic geometry. Also, for $n = 0$ the scheme $\mathbf{A}_A^1 \setminus \{0\}$ is affine and therefore

$$H^i(\mathbf{A}_A^1 \setminus \{0\}, \mathcal{O}_{\mathbf{A}_A^1\setminus\{0\}}) = 0$$

for $i > 0$ by Serre's vanishing theorem. So $n = 0$ is also known and we may use induction on i and n. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{A}_A^{n+1}\setminus\{0\}} \to j_*\mathcal{O}_{D(x_n)} \to \bigoplus_{l=1}^{\infty} \mathcal{O}_{\mathbf{A}_A^n\setminus\{0\}} x_n^{-l} \to 0 \tag{3}$$

of sheaves on $\mathbf{A}_A^{n+1} \setminus \{0\}$. For $i \geq 1$ we have $H^i(\mathbf{A}_A^{n+1} \setminus \{0\}, j_*\mathcal{O}_{D(x_n)}) = 0$ by Corollary 3.4 and Serre's vanishing theorem because j is affine, so the long exact cohomology sequence yields

$$H^1(\mathbf{A}_A^{n+1}\setminus\{0\}, \mathcal{O}_{\mathbf{A}_A^{n+1}\setminus\{0\}}) \cong \operatorname{coker}(H^0(D(x_n), \mathcal{O}_{D(x_n)}) \to \bigoplus_{l=1}^{\infty} H^0(\mathbf{A}_A^n\setminus\{0\}, \mathcal{O}_{\mathbf{A}_A^n\setminus\{0\}}) x_n^{-l}).$$

For $n > 1$ the right hand side is

$$\operatorname{coker}(A[x_0, \dots, x_n, x_n^{-1}] \to \bigoplus_{l=1}^{\infty} A[x_0, \dots, x_{n-1}] x_n^{-l}) = 0.$$

For $n = 1$ it is

$$\operatorname{coker}(A[x_0, x_1, x_1^{-1}] \to \bigoplus_{l=1}^{\infty} A[x_0, x_0^{-1}] x_1^{-l}) \cong \bigoplus_{l,m=1}^{\infty} A x_0^{-m} x_1^{-l}.$$

This completes the proof of the case $i = 1$ via induction on n. For $i > 1$ we derive from exact sequence (3) and the vanishing of $H^i(\mathbf{A}^{n+1}_A \setminus \{0\}, j_*\mathcal{O}_{D(x_n)})$ isomorphisms

$$H^i(\mathbf{A}^{n+1}_A \setminus \{0\}, \mathcal{O}_{\mathbf{A}^{n+1}_A \setminus \{0\}}) \cong \bigoplus_{l=1}^{\infty} H^{i-1}(\mathbf{A}^n_A \setminus \{0\}, \mathcal{O}_{\mathbf{A}^n_A \setminus \{0\}}) x_n^{-l}$$

where we use again that cohomology commutes with direct sums. Thus we may conclude using induction (on i and n). □

6. Serre's GAGA theorems

Serre's famous theorems give a comparison between coherent sheaves on complex projective schemes and coherent sheaves on the associated analytic space. To state them, we first have to review some constructions in the analytic theory.

The affine space $\mathbf{A}^N_{\mathbf{C}}$ comes equipped with a natural sheaf $\mathcal{O}^{\text{an}}$ given by holomorphic functions in n variables over complex open subsets of $\mathbf{C}^N$. The pair $(\mathbf{A}^N(\mathbf{C}), \mathcal{O}^{\text{an}})$ is a locally ringed space. Given holomorphic functions $f_1, \ldots, f_m \in \mathcal{O}^{\text{an}}(\mathbf{A}^N(\mathbf{C}))$, their restrictions define an ideal sheaf $\mathcal{I} \subset \mathcal{O}^{\text{an}}$. Let $Z \subset \mathbf{A}^N(\mathbf{C})$ be the support of $\mathcal{O}^{\text{an}}/\mathcal{I}$ (i.e. the closed subset of points where the stalk is nonzero). The restriction of $\mathcal{O}^{\text{an}}/\mathcal{I}$ to Z defines a locally ringed space we denote by $(Z, \mathcal{O}^{\text{an}}_Z)$. We call such ringed spaces *closed analytic subspaces* of $\mathbf{A}^N(\mathbf{C})$. An *analytic subspace* of $\mathbf{A}^N(\mathbf{C})$ is an open subset in a closed analytic subspace, equipped with the restriction of the structure sheaf.

Definition 6.1. A *complex analytic space* is a locally ringed space $(X, \mathcal{O}^{\text{an}}_X)$ such that there exists an open covering $\mathcal{U}$ of X such that for $U \in \mathcal{U}$ the locally ringed space $(U, \mathcal{O}^{\text{an}}_X|_U)$ is isomorphic to an analytic subspace of $\mathbf{A}^N(\mathbf{C})$.

A *coherent sheaf* on a complex analytic space is an $\mathcal{O}^{\text{an}}_X$-module $\mathcal{F}$ such that there exists an open covering $\mathcal{U}$ of V such that for $U \in \mathcal{U}$ the restriction $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism $(\mathcal{O}^{\text{an}}_U)^{\oplus r} \to (\mathcal{O}^{\text{an}}_U)^{\oplus s}$ of finitely generated free $\mathcal{O}^{\text{an}}_U$-modules.

Now we can associate a complex analytic space to a scheme of finite type over $\mathbf{C}$ as follows.

Proposition 6.2. *Let X be a scheme of finite type over $\mathbf{C}$. There exists a complex analytic space X^{an} equipped with a morphism $\varepsilon : X^{\text{an}} \to X$ of ringed spaces such that every morphism of locally ringed spaces $Y \to X$ with Y a complex analytic space factors uniquely through ε.*

In the above proposition the structure sheaves of the ringed spaces are all $\mathbf{C}$-algebras; we require morphisms to preserve this $\mathbf{C}$-algebra structure.

Proof. For $X = \mathbf{A}^N_{\mathbf{C}}$ we define X^{an} to be $\mathbf{C}^N = \mathbf{A}^N(\mathbf{C})$ with its usual holomorphic structure sheaf $\mathcal{O}^{\mathrm{an}}$ recalled above. The morphism ε is given by identifying $\mathbf{C}^N$ with complex points of $\mathbf{A}^N_{\mathbf{C}}$ and considering algebraic functions as holomorphic functions. Let us check that it satisfies the above universal property. First consider the case $N = 1$. In this case to give a morphism of locally ringed spaces $\phi : Y \to \mathbf{A}^1_{\mathbf{C}}$ for an analytic space $(Y, \mathcal{O}^{\mathrm{an}}_Y)$ is equivalent to specifying a global section $\mathcal{O}^{\mathrm{an}}_Y$, i.e. a global holomorphic function on Y. Indeed, ϕ induces a $\mathbf{C}$-algebra map $\mathbf{C}[t] = \Gamma(\mathbf{A}^1_C, \mathcal{O}_{\mathbf{A}^1_{\mathbf{C}}}) \to \Gamma(Y, \mathcal{O}^{\mathrm{an}}_Y)$ which is uniquely determined by the image f of t. Conversely, such a holomorphic function f induces a morphism $\phi : Y \to \mathbf{A}^1_{\mathbf{C}}$ which manifestly factors through ε as defined above. The case of general N then follows inductively from a formal observation: given two separated schemes X_1, X_2 of finite type over $\mathbf{C}$ such that X_i^{an} exists for $i = 1, 2$, the product $X_1^{\mathrm{an}} \times_{\mathbf{C}} X_2^{\mathrm{an}}$ satisfies the universal property required of $(X_1 \times_{\mathbf{C}} X_2)^{\mathrm{an}}$.

Next, consider the case where X is a closed subscheme of $\mathbf{A}^N_{\mathbf{C}}$ corresponding to an ideal $I = (f_1, \dots, f_r) \subset \mathbf{C}[t_1, \dots, t_N]$. Viewing the f_i as holomorphic functions, they give rise to a sheaf of ideal $\mathcal{I} \subset \mathcal{O}^{\mathrm{an}}$, and we define X^{an} to be the associated closed analytic subset of $\mathbf{A}^N(\mathbf{C})$. To show that X^{an} satisfies the required universal property, one first observes that X^{an} is none but the fibre product $X \times_{\mathbf{A}^N_{\mathbf{C}}} \mathbf{A}^N(\mathbf{C})$ in the category of locally ringed spaces. Afterwards the argument is formal: for a morphism $Y \to X$ of locally ringed spaces with Y analytic the composite $Y \to X \to \mathbf{A}^N_{\mathbf{C}}$ factors through a morphism $Y \to \mathbf{A}^N(\mathbf{C})$ by the case $X = \mathbf{A}^N_{\mathbf{C}}$, whence also a map into the above fibre product.

Now let X be a closed subscheme of $\mathbf{A}^N_{\mathbf{C}}$ and $U \subset X$ an open subscheme. By the previous paragraph $\varepsilon : X^{\mathrm{an}} \to X$ exists for X. We define U^{an} to be $\varepsilon^{-1}(U)$ equipped with the restriction of $\mathcal{O}_{X^{\mathrm{an}}}$; it is an analytic subspace of $\mathbf{A}^N(\mathbf{C})$ by construction. Moreover, it is none but the fibre product $U \times_X X^{\mathrm{an}}$ in the category of locally ringed spaces, hence it satisfies the universal property by a similar argument as above.

Finally, in the general case X has an open covering by closed subschemes in affine space. By the previous two steps we know that the associated analytic space exists for elements of the cover as well as for their intersections, hence we may patch them together as locally ringed spaces by means of the isomorphisms resulting from the universal property. □

Remark 6.3. We have worked with Grothendieck's definition of analytic spaces. Serre's original definition in GAGA was more restrictive: he required the topology of the analytic space to be Hausdorff and elements of the open covering to be *closed* analytic subspaces of $\mathbf{A}^N_{\mathbf{C}}$. However, by following through the steps of the above construction one sees that for *separated* schemes of finite type over $\mathbf{C}$ the space X^{an} will be an analytic space in Serre's sense as well. The GAGA theorems concern projective (or more generally proper) schemes over $\mathbf{C}$, so the separatedness assumption always holds.

Definition 6.4. *Given an algebraic coherent sheaf $\mathcal{F}$ on X, we define*

$$\mathcal{F}^{\mathrm{an}} := \varepsilon^* \mathcal{F}.$$

Here the pullback is in the sense of ringed spaces, so it is given by the tensor product $\varepsilon^{-1}\mathcal{F} \otimes_{\varepsilon^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$. Note that by definition $(\mathcal{O}_X)^{\mathrm{an}} = \mathcal{O}_{X^{\mathrm{an}}}$, so by exactness of ε^{-1} and right exactness of the tensor product we obtain that $\mathcal{F}^{\mathrm{an}}$ is a coherent analytic sheaf.

Proposition 6.5. *The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is exact.*

Proof. Exactness can be checked on stalks. By construction the stalk of $\mathcal{F}^{\mathrm{an}}$ at $P \in X(\mathbf{C})$ is isomorphic to $\mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{O}_{X^{\mathrm{an}},P}$. Here the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{X^{\mathrm{an}},P}$ have the same formal completions along the maximal ideal. This is obvious for the case $X = \mathbf{A}^N_{\mathbf{C}}$ where the completion in both cases is the ring of formal power series in N variables; the general case results by following the steps in the construction of X^{an}. The proposition then follows from the purely algebraic lemma below, applied to the homomorphism $\mathcal{O}_{X,P} \to \mathcal{O}_{X^{\mathrm{an}},P}$ (in fact, only flatness is used). □

Recall that an A-module B is flat if the functor $M \mapsto M \otimes_A B$ is exact on the category of A-modules; it is faithfully flat if moreover $M \otimes_A B = 0$ implies $M = 0$. A fundamental fact (see e.g. Matsumura, Commutative Ring Theory, §8) is that the completion $\widehat{A}$ of a Noetherian local ring along its maximal ideal is a faithfully flat A-module.

Lemma 6.6. *Assume $A \to B$ is a local homomorphism of Noetherian local rings inducing an isomorphism $\widehat{A} \xrightarrow{\sim} \widehat{B}$. Then B is faithfully flat over A.*

Proof. Assume M is an A-module with $M \otimes_A B = 0$. Tensoring with $\widehat{B}$ gives

$$0 = M \otimes_A B \otimes_B \widehat{B} \cong M \otimes_A \widehat{B} \cong M \otimes_A \widehat{A},$$

in view of $\widehat{B} \cong \widehat{A}$. As $\widehat{A}$ is faithfully flat over A, we get $M = 0$.

Similarly, if $M \hookrightarrow N$ is an injective map of A-modules, tensoring the induced map $M \otimes_A B \to N \otimes_A B$ with $\widehat{B}$ over B, we get the map $M \otimes_A \widehat{A} \to N \otimes_A \widehat{A}$ which is injective by flatness of $\widehat{A}$ over A. As $\widehat{B}$ is flat over B, this is only possible if $M \otimes_A B \to N \otimes_A B$ is injective. □

Construction 6.7. The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ induces homomorphisms

$$H^i(X, \mathcal{F}) \to H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

for all $i \geq 0$ as follows. Choose a projective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Since the functor ε^* is exact, the complex $\varepsilon^*\mathcal{F} \to \varepsilon^*\mathcal{I}^\bullet$ is still a resolution.

Choose an injective resolution $\varepsilon^*\mathcal{F} \to \mathcal{J}^\bullet$. By a general lemma of homological algebra (which is easy to prove by an inductive construction), the identity morphism of $\varepsilon^*\mathcal{F}$ extends to a morphism of complexes

$$\varepsilon^*\mathcal{I}^\bullet \to \mathcal{J}^\bullet.$$

Applying the functor ε_* gives a map

$$\varepsilon_*\varepsilon^*\mathcal{I}^\bullet \to \varepsilon_*\mathcal{J}^\bullet$$

that we may compose with the adjunction map $\mathcal{I}^\bullet \to \varepsilon_*\varepsilon^*\mathcal{I}^\bullet$. Taking global sections we get

$$\Gamma(X, \mathcal{I}^\bullet) \to \Gamma(X, \varepsilon_*\varepsilon^*\mathcal{I}^\bullet) \to \Gamma(X, \varepsilon_*\mathcal{J}^\bullet) = \Gamma(X^{\mathrm{an}}, \mathcal{J}^\bullet).$$

The required maps result by passing to cohomology.

We may now state Serre's GAGA theorems.

Theorem 6.8 (Serre)**.** *Let X be a closed subscheme of $\mathbf{P}^N_{\mathbf{C}}$ for some $N > 0$.*

(1) *The maps $H^i(X, \mathcal{F}) \to H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ defined above are isomorphisms for all $i \geq 0$.*
(2) *The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ induces an equivalence of categories between coherent algebraic sheaves on X and coherent analytic sheaves on X^{an}.*

Remark 6.9. Grothendieck has generalized both GAGA theorems to arbitrary proper schemes over $\mathbf{C}$.

The proof of Theorem 6.8 uses the following analogue of Theorem 5.1 for analytic sheaves (which was actually proven a few years earlier, using techniques from functional analysis).

Theorem 6.10 (Cartan–Serre)**.** *Let $\mathcal{F}$ be a coherent analytic sheaf on $\mathbf{P}^N(\mathbf{C})$. Then*

(1) *$H^i(\mathbf{P}^N(\mathbf{C}), \mathcal{F})$ is a finite-dimensional $\mathbf{C}$-vector space for $i \geq 0$.*
(2) *$H^i(\mathbf{P}^N(\mathbf{C}), \mathcal{F}(m)) = 0$ for $i > 0$ and m sufficiently large.*

Here $\mathcal{F}(m)$ for analytic sheaves is defined in the same way as in the algebraic setting, by patching together the restrictions $\mathcal{F}|_{D_+(x_i)}$ along the isomorphisms $f \mapsto (x_i/x_j)^m f$.

We shall only need the second statement, but its proof is based on the first one, which is the most difficult result of the analytic theory. Let us note the following consequence:

Corollary 6.11. *If $\mathcal{F}$ is a coherent analytic sheaf of $\mathbf{P}^N(\mathbf{C})$, then $\mathcal{F}(m)$ is generated by global sections for m sufficiently large.*

Proof. Let $\mathcal{I}_P$ be the ideal sheaf of a point $P \in \mathbf{P}^N(\mathbf{C})$. We have an exact sequence of sheaves

$$0 \to \mathcal{I}_P\mathcal{F}(m) \to \mathcal{F}(m) \to \mathcal{F}(m)/\mathcal{I}_P\mathcal{F}(m) \to 0.$$

A piece of its associated long exact sequence reads

$$H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{F}(m)) \to H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{F}(m)/\mathcal{I}_P\mathcal{F}(m)) \to H^1(\mathbf{P}^N(\mathbf{C}), \mathcal{I}_P\mathcal{F}(m)).$$

As $\mathcal{I}_P\mathcal{F}$ is coherent (being a quotient of $\mathcal{I}_P \otimes \mathcal{F}$), the last group here is trivial for m sufficiently large by part (2) of the theorem. Since $\mathcal{O}^{\mathrm{an}}/\mathcal{I}_P$ is a skyscraper sheaf concentrated at P, we have an isomorphism $H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{F}(m)/\mathcal{I}_P\mathcal{F}(m)) \cong \mathcal{F}(m)_P/\mathcal{I}_P\mathcal{F}(m)_P$, where we have identified $\mathcal{I}_P$ with the maximal ideal in $\mathcal{O}_{\mathbf{P}^N(\mathbf{C}),P}$. The exact sequence gives a surjection $H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{F}(m)) \twoheadrightarrow \mathcal{F}(m)_P/\mathcal{I}_P\mathcal{F}(m)_P$, whence a surjection $H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{F}(m)) \twoheadrightarrow \mathcal{F}(m)_P$ by Nakayama's lemma. As $\mathcal{F}$ is coherent, such a surjection holds for all points in an open neighbourhood of P. As $\mathbf{P}^N(\mathbf{C})$ is compact, we may cover it with finitely many such open neighbourhoods, so for m sufficiently large $\mathcal{F}(m)_P$ is generated by global sections for all points P. □

Remark 6.12. An analogous result holds for coherent algebraic sheaves on a projective scheme. The proof is the same, using Theorem 5.1 (2) instead of Theorem 6.10 (2).

Proof of Theorem 6.8 (1). Using the isomorphisms

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(\mathbf{P}^N_{\mathbf{C}}, \phi_*\mathcal{F}), \quad H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) \xrightarrow{\sim} H^i(\mathbf{P}^N(\mathbf{C}), \phi_*\mathcal{F}^{\mathrm{an}})$$

(where the second isomorphism follows as in Remark 3.5), we reduce to the case $X = \mathbf{P}^N(\mathbf{C})$.

In this case, we first prove the theorem for $\mathcal{F} = \mathcal{O}(m)$ by induction on N, the case $N = 0$ being obvious. We first treat the case $i = 0$. Consider the exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0 \tag{4}$$

where H is the hyperplane of equation $x_0 = 0$, and the first map is given by multiplication by x_0 (and we have written $\mathcal{O}_H$ in place of $i_{H*}\mathcal{O}_H$, where $i_H : H \hookrightarrow \mathbf{P}^N(\mathbf{C})$ is the inclusion map). Tensoring by the invertible sheaf $\mathcal{O}(m+1)$ gives an exact sequence

$$0 \to \mathcal{O}(m) \to \mathcal{O}(m+1) \to \mathcal{O}_H(m+1) \to 0 \tag{5}$$

where the last term is the sheaf $\mathcal{O}(m+1)$ on $H \cong \mathbf{P}^{N-1}$ by the projection formula. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^N_{\mathbf{C}}, \mathcal{O}(m)) & \longrightarrow & H^0(\mathbf{P}^N_{\mathbf{C}}, \mathcal{O}(m+1)) & \longrightarrow & H^0(H, \mathcal{O}_H(m+1)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{O}^{\mathrm{an}}(m)) & \longrightarrow & H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{O}^{\mathrm{an}}(m+1)) & \longrightarrow & H^0(H(\mathbf{C}), \mathcal{O}_H^{\mathrm{an}}(m+1)) \end{array}$$

Here the last vertical map is an isomorphism by induction on N. Moreover, the map $H^0(\mathbf{P}^N_{\mathbf{C}}, \mathcal{O}) \to H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{O}^{\mathrm{an}})$ is also an isomorphism (both groups equal $\mathbf{C}$). Therefore the diagram implies that we have isomorphisms $H^0(\mathbf{P}^N_{\mathbf{C}}, \mathcal{O}(m)) \to H^0(\mathbf{P}^N(\mathbf{C}), \mathcal{O}^{\mathrm{an}}(m))$ for all $m \leq 0$ by descending induction on m starting from the case $m = 0$. In the case $m > 0$ the last upper horizontal map is surjective because $H^1(\mathbf{P}^N_{\mathbf{C}}, \mathcal{O}(m)) = 0$ by Theorem 5.7. Thus we may apply the snake lemma to the diagram, and prove the statement by ascending induction on m, again starting from the case $m = 0$.

In the case $i > 0$ both groups are trivial for m sufficiently large by Theorems 5.1 (2) and 6.10 (2), so it will suffice to prove that the theorem holds for $\mathcal{O}(m)$ if it holds for $\mathcal{O}(m+1)$. Now the maps

$$H^i(\mathbf{P}^N_{\mathbf{C}}, \mathcal{O}_H(m+1)) \to H^i(\mathbf{P}^N(\mathbf{C}), \mathcal{O}^{\mathrm{an}}_H(m+1))$$

are isomorphisms for all i and m by induction on N, and the maps

$$H^i(\mathbf{P}^N_{\mathbf{C}}, \mathcal{O}(m+1)) \to H^i(\mathbf{P}^N(\mathbf{C}), \mathcal{O}^{\mathrm{an}}(m+1))$$

are isomorphisms for all i by descending induction on m. Writing out the long exact sequences in cohomology for (5) in both the analytic and the algebraic setting, the statement for $\mathcal{O}(m)$ follows from the above isomorphisms and the five lemma.

To treat the case of arbitrary $\mathcal{F}$, apply Proposition 5.5 to obtain an exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(m)^{\oplus r} \to \mathcal{F} \to 0.$$

Applying the exact functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ gives an exact sequence

$$0 \to \mathcal{G}^{\mathrm{an}} \to \mathcal{O}^{\mathrm{an}}(m)^{\oplus r} \to \mathcal{F}^{\mathrm{an}} \to 0.$$

Writing out the associated long exact sequences and using the case $\mathcal{F} = \mathcal{O}(m)$, we can proceed by a similar descending induction as above, but this time on i. To start the induction, use Theorem 4.2 in the algebraic setting that shows that all groups are 0 in degree $> N$. The proof of that theorem goes through in the analytic setting as well, except that instead of Serre's vanishing theorem one should use the analogous theorem for coherent analytic sheaves on affine spaces (more generally, Stein spaces) which is called Cartan's 'Theorem A'. (Alternatively, one can apply Grothendieck's general theorem mentioned at the beginning of Section 4.) When proving the inductive step, the case $\mathcal{F} = \mathcal{O}(m)$ and the degree $i+1$ case first imply the surjectivity of the map $H^i(\mathbf{P}^N_{\mathbf{C}}, \mathcal{F}) \to H^i(\mathbf{P}^N(\mathbf{C}), \mathcal{F}^{\mathrm{an}})$. Since this holds for all coherent sheaves on $\mathbf{P}^N_{\mathbf{C}}$, it also holds for $\mathcal{G}$. Surjectivity of the map for $\mathcal{G}$ together with the already known isomorphisms then implies injectivity for $\mathcal{F}$ by a diagram chase. □

For the proof of full faithfulness in statement (2) we use a lemma.

Lemma 6.13. *Given coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, we have a canonical isomorphism*

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})^{\mathrm{an}} \cong \mathcal{H}om(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$$

of sheaves on X^{an}.

Proof. First we construct a morphism

$$\mathcal{H}om(\mathcal{F},\mathcal{G})^{\mathrm{an}} \to \mathcal{H}om(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}}). \tag{6}$$

A section of $\varepsilon^{-1}\mathcal{H}om(\mathcal{F},\mathcal{G})$ over an open subset $U \subset X^{\mathrm{an}}$ induces a morphism of sheaves $\varepsilon^{-1}\mathcal{F}|_U \to \varepsilon^{-1}\mathcal{G}|_U$. Tensoring by $\mathcal{O}_U^{\mathrm{an}}$ over $\varepsilon^{-1}\mathcal{O}_X|_U$ gives a morphism of sheaves $\varepsilon^*\mathcal{F}|_U \to \varepsilon^*\mathcal{G}|_U$, i.e. a section of $\mathcal{H}om(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$ over U. We thus obtain a morphism of $\varepsilon^{-1}\mathcal{O}_X$-modules $\varepsilon^{-1}\mathcal{H}om(\mathcal{F},\mathcal{G}) \to \mathcal{H}om(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$ which extends to a morphism of $\mathcal{O}_{X^{\mathrm{an}}}$-modules $\varepsilon^{-1}\mathcal{H}om(\mathcal{F},\mathcal{G}) \otimes_{\varepsilon^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{H}om(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$. This is the map (6).

To show that (6) is an isomorphism, we consider the induced map on stalks

$$\mathrm{Hom}(\mathcal{F}_P,\mathcal{G}_P)\otimes_{\mathcal{O}_{X,P}}\mathcal{O}_{X^{\mathrm{an}},P} \to \mathrm{Hom}(\mathcal{F}_P\otimes_{\mathcal{O}_{X,P}}\mathcal{O}_{X^{\mathrm{an}},P},\mathcal{G}_P\otimes_{\mathcal{O}_{X,P}}\mathcal{O}_{X^{\mathrm{an}},P}).$$

Since $\mathcal{O}_{X^{\mathrm{an}},P}$ is flat over $\mathcal{O}_{X,P}$, we conclude by a purely algebraic fact: given a ring A, a flat A-algebra B and A-modules M, N, the natural map

$$\mathrm{Hom}_A(M,N) \otimes_A B \to \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$$

is an isomorphism. (This is obvious for $M = A$, hence also for M free over A. In the general case, write A as a cokernel of a morphism of free modules, and use flatness of B.) □

Proof of Theorem 6.8 (2). We first prove that the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is fully faithful. Given coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, part (1) of the theorem applied to H^0 of the sheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$ induces an isomorphism

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) \xrightarrow{\sim} H^0(X^{\mathrm{an}},\mathcal{H}om(\mathcal{F},\mathcal{G})^{\mathrm{an}}).$$

By the lemma above, the right hand side identifies with $\mathrm{Hom}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$.

It remains to prove that the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is essentially surjective, i.e. every coherent analytic sheaf $\mathcal{G}$ on X^{an} is isomorphic to $\mathcal{F}^{\mathrm{an}}$ for a coherent algebraic sheaf $\mathcal{F}$ on X. First we reduce to the case $X = \mathbf{P}^N_{\mathbf{C}}$. Consider the inclusion map $i : X \hookrightarrow \mathbf{P}^N_{\mathbf{C}}$. By the case $X = \mathbf{P}^N_{\mathbf{C}}$ we have $i^{\mathrm{an}}_*\mathcal{G} = \mathcal{F}^{\mathrm{an}}$ for some coherent sheaf $\mathcal{F}$ on $\mathbf{P}^N_{\mathbf{C}}$. If $\mathcal{I}$ is the ideal sheaf defining X in $\mathbf{P}^N_{\mathbf{C}}$, we have $(\mathcal{I}\mathcal{F})^{\mathrm{an}} = \mathcal{I}^{\mathrm{an}}(i_*\mathcal{G}) = 0$ as $\mathcal{I}^{\mathrm{an}}$ is the ideal sheaf defining X^{an} in $\mathbf{P}^N(\mathbf{C})$. But then $\mathcal{I}\mathcal{F} = 0$. Indeed, this can be checked on stalks, where it follows from *faithful* flatness of $\mathcal{O}_{X^{\mathrm{an}},P}$ over $\mathcal{O}_{X,P}$ (Lemma 6.6). But $\mathcal{I}\mathcal{F} = 0$ means that $\mathcal{F}$ is the pushforward of a coherent sheaf $\mathcal{F}_X$ on X, and we have $\mathcal{F}_X^{\mathrm{an}} = \mathcal{G}$ as analytification commutes with pushforward along closed immersions.

In the case $X = \mathbf{P}^N_{\mathbf{C}}$, apply Corollary 6.11 to obtain a surjection

$$\mathcal{O}^{\mathrm{an}}(-m)^{\oplus s} \twoheadrightarrow \mathcal{G}$$

for suitable m and s. Repeating the argument for the kernel of this map (which is again a coherent analytic sheaf) we obtain an exact sequence

$$\mathcal{O}^{\mathrm{an}}(-n)^{\oplus r} \to \mathcal{O}^{\mathrm{an}}(-m)^{\oplus s} \to \mathcal{G} \to 0.$$

By full faithfulness here the morphism $\mathcal{O}^{\mathrm{an}}(-n)^{\oplus r} \to \mathcal{O}^{\mathrm{an}}(-m)^{\oplus s}$ comes from a morphism $\mathcal{O}(-n)^{\oplus r} \to \mathcal{O}(-m)^{\oplus s}$ via the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$. As this functor is moreover (right) exact, we obtain an isomorphism

$$\mathcal{G} \cong \operatorname{coker}(\mathcal{O}(-n)^{\oplus r} \to \mathcal{O}(-m)^{\oplus s})^{\mathrm{an}}.$$

Thus $\mathcal{F} := \operatorname{coker}(\mathcal{O}(-n)^{\oplus r} \to \mathcal{O}(-m)^{\oplus s})$ is a good choice. □