On the Albanese map for smooth quasi-projective varieties

By Michael Spieß and Tamás Szamuely

1 Introduction

Consider an algebraically closed field k of characteristic $p \geq 0$ and a smooth connected quasi-projective k-variety X. When X is in fact projective, a famous theorem due to A. A. Roitman ([Roi], see also [Bl]) asserts that the Albanese map

$$alb_X : CH_0(X)^0 \longrightarrow Alb_X(k) \tag{1}$$

from the Chow group of zero-cycles of degree 0 on X to the group of k-points of the Albanese variety induces an isomorphism on prime-to-p torsion subgroups (later J. S. Milne proved that the isomorphism holds for p-primary torsion subgroups as well, cf. [Mi]). As a well-known counter-example of Mumford shows, in dimensions greater than one the map alb_X itself is not an isomorphism in general. Still, Kato and Saito ([KS], Section 10) have established the bijectivity of alb_X in the case when k is the algebraic closure of a finite field (in fact, in this case both groups are torsion). Moreover, bijectivity over $k = \overline{\mathbb{Q}}$ has been conjectured by Bloch and Beilinson, as a consequence of some expected standard features of the conjectural category of mixed motives over $\overline{\mathbb{Q}}$.

In the present paper we present a new conceptual approach to the theorem of Roitman which at the same time yields a generalisation to the case when X is not necessarily projective but admits a smooth compactification. Here the natural target for the Albanese map is the generalised Albanese variety introduced by Serre [Se1]. If X is a curve, this variety is a generalised Jacobian in the sense of Rosenlicht [Ros] and for X proper it is the usual Albanese. In general, it is a semi-abelian variety universal for morphisms of X into semi-abelian varieties; it is related to the Picard variety by a duality theorem (see sections 3 and 4 for more details). The generalisation of alb_X

to this context is a map

$$alb_X : h_0(X)^0 \longrightarrow Alb_X(k) \tag{2}$$

where the group on the left is the degree zero part of Suslin's 0-th algebraic singular homology group defined in [SV]; it coincides with $CH_0(X)^0$ when X is proper. The map (2) first appeared in the 1998 preprint version of N. Ramachandran's paper [Ra]; we give a simple proof for the "reciprocity law" implying its existence in Section 3.

Now we can state our main result.

Theorem 1.1 *Let k be an algebraically closed field of characteristic $p \geq 0$ and let X be a smooth connected quasi-projective variety over k. Assume that there exists a smooth projective connected k-variety $\mathfrak{X}$ containing X as an open subscheme. Then the Albanese map (2) induces an isomorphism on prime-to-p torsion subgroups.*

Note that the required smooth compactification $\mathfrak{X}$ exists if k is of characteristic 0 or if X is of dimension ≤ 3 and $p \geq 5$, by virtue of the desingularisation theorems of Hironaka and Abhyankar.

Our method for proving Theorem 1.1 is new even in the proper case and is (at least to our feeling) more conceptual than the previous ones. The proof exploits the comparison maps $h^i(X, \mathbb{Z}/n\mathbb{Z}) \to H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$ relating algebraic singular cohomology to étale cohomology according to Suslin and Voevodsky [SV] for any n prime to p; by the main result of *loc. cit.* these maps are isomorphisms. We reduce the proof of our theorem to the case $i = 1$ of this fundamental result by showing that, in that case, taking the dual of the inverse map for all n and passing to the direct limit one obtains the restriction of the map (2) to the prime-to-p torsion subgroup of $h_0(X)$. One of the basic observations for proving this identification, which may be of independent interest, is that thanks to its functoriality and homotopy invariance properties, the (generalised) Albanese variety can be regarded as an object of Voevodsky's triangulated category of effective motivic complexes $DM_-^{eff}(k)$ and in fact for smooth varieties the Albanese map can be interpreted as a morphism in this category.

Over the algebraic closure of a finite field we can prove somewhat more:

Theorem 1.2 *We keep the hypotheses of Theorem 1.1 and assume moreover that k is the algebraic closure of a finite field. Then (2) is an isomorphism of torsion groups.*

The proof of this theorem is more traditional: in fact, it is a direct generalisation of the argument given in ([KS], section 10), using the "tamely ramified class field theory" developed in [SS].

Finally it should be mentioned that during recent years fruitful efforts have been made for generalising Roitman's theorem to singular complex projective varieties (see [BS] and the references quoted there). Our generalisation seems to be unrelated to this theory except perhaps in the case when X is the complement of the singular locus of a complex projective variety.

We are grateful to the referee whose careful reading enabled us to improve the exposition considerably. The second author was partially supported by OTKA grants Nos. T-031984 and F-032325.

A word on notation: For an abelian group A and a nonzero integer n we denote by ${}_nA$ the n-torsion subgroup of A and we write A/n as a shorthand for A/nA. For a prime number ℓ we let $A\{\ell\}$ be the ℓ-primary component of the torsion subgroup of A.

2 Review of Algebraic Singular Homology

This section and the next are devoted to the definition of the map (2) and the groups involved. We begin by recalling the definition of the algebraic singular homology groups introduced in [SV]. In this section k may stand for an arbitrary perfect field.

For an integer $n \geq 1$ consider the algebraic n-simplex

$$\Delta^n = \operatorname{Spec} k[T_0, \ldots, T_n]/(T_0 + \ldots + T_n - 1).$$

If X is a k-variety (i.e. an integral separated scheme of finite type over k), denote by $C_n(X)$ the free abelian group generated by those integral closed subschemes Z of $X \times \Delta^n$ for which the projection $Z \to \Delta^n$ is finite and surjective. Any nondecreasing map $\alpha : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ induces a morphism $\Delta^m \to \Delta^n$ and thus a homomorphism $\alpha^* : C_n(X) \to C_m(X)$ via pull-back of cycles. These maps endow the set of the $C_n(X)$ with the structure of a simplicial abelian group; we denote by $C_\bullet(X)$ the associated chain complex. For an abelian group A the n-th algebraic singular homology group $h_n(X, A)$ of X with coefficients in A is defined as the n-th homology of the complex $C_\bullet(X) \otimes A$ and the n-th algebraic singular cohomology $h^n(X, A)$ as the n-th cohomology of $\operatorname{Hom}(C_\bullet(X, \mathbb{Z}), A)$. For $A = \mathbb{Z}$ we shall simply write $h_n(X)$ for $h_n(X, \mathbb{Z})$ etc.

The group $h_0(X)$ has the following concrete description. Let $\mathcal{Z}(X)$ be the free abelian group with basis the set X_0 of closed points of X. Then $h_0(X)$ is the quotient of $\mathcal{Z}(X)$ by the submodule $\mathcal{R}$ generated by $i_0^*(Z) - i_1^*(Z)$, where $i_\nu : X \to X \times \mathbb{A}^1$ ($\nu = 0, 1$) stand for the inclusions $x \mapsto (x, \nu)$ and Z runs through all closed integral subschemes of $X \times \mathbb{A}^1$ such that the projection $Z \to \mathbb{A}^1$ is finite and surjective. There is a natural degree map $\mathcal{Z}(X) \longrightarrow \mathbb{Z}$ given by the formula

$$\sum_i n_i P_i \mapsto \sum_i n_i [k(P_i) : k],$$

of which we denote the kernel by $\mathcal{Z}(X)^0$. Using the fact that the projections $Z \to \mathbb{A}^1$ are finite and flat, one checks that $\mathcal{Z}(X)^0$ contains $\mathcal{R}$; the quotient $\mathcal{Z}(X)^0/\mathcal{R}$ will be denoted by $h_0(X)^0$.

For the proofs we shall also need a sheafified version of the above construction. For this, denote by Sm/k the category of smooth schemes of finite type over k. Let $\mathcal{F}$ be an abelian presheaf on Sm/k, i.e. a contravariant functor from Sm/k to the category of abelian groups. For any $m \geq 0$ we may define a presheaf $\mathcal{F}_m$ by the rule $\mathcal{F}_m(X) = \mathcal{F}(X \times \Delta^m)$. Together with the operations induced from the cosimplicial scheme $\Delta^\bullet$ these presheaves assemble to form a simplicial presheaf whose associated chain complex we denote by $C_\bullet(\mathcal{F})$. If $\mathcal{F}$ is *homotopy invariant*, i.e. if the natural map $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$ is an isomorphism for all $X \in Sm/k$, then the augmentation map $C_\bullet(\mathcal{F}) \to \mathcal{F}$ given by the identity in degree 0 is a map of complexes and in fact a quasi-isomorphism (here we view $\mathcal{F}$ as a complex concentrated in degree 0). Indeed, in view of the canonical isomorphism $\Delta^n \cong \mathbb{A}^n$ in this case $C_\bullet(\mathcal{F})$ is none but the complex associated to the constant simplicial presheaf defined by $\mathcal{F}$.

We also recall the notion of *presheaves with transfers* from Section 2 of [Vo]. These are contravariant additive functors with values in abelian groups from the category $SmCor(k)$ whose objects are smooth schemes of finite type over k and where a morphism from an object X to an object Y is a *finite correspondence,* i.e. an element of the free abelian group $c(X, Y)$ generated by those integral closed subschemes Z of $X \times Y$ for which the projection $Z \to X$ is finite and surjective over a component of X. (*Note:* This definition of presheaves with transfers differs from the one in the earlier paper [SV] whose results we shall use in the sequel, but the two definitions are equivalent.) Now the link with the algebraic singular complex is the following. For a separated k-scheme X the rule $U \mapsto c(U, X)$ defines a presheaf with transfers on which we denote by $\mathbb{Z}_{tr}(X)$; actually it is a sheaf for the étale topology on $SmCor(k)$. Then by definition $C_\bullet(\mathbb{Z}_{tr}(X))(k) = C_\bullet(X)$.

3 The Generalised Albanese Map

In this section we explain the construction of the generalised Albanese maps on two levels of generality: first, in order to keep technicalities to a minimum, we construct the map (2) over an algebraically closed k as stated in the introduction, and then we explain a sheafified version over an arbitrary perfect field.

So we begin by working over an algebraically closed field k and recalling the notion of the generalised Albanese variety Alb_X of a variety X, as introduced in [Se1]. It is a semiabelian variety satisfying the following universal property: for every k-point P of X there is a morphism $\iota_P : X \to Alb_X$ such that $\iota_P(P) = 0$ and if (B, f) is a pair consisting of a semiabelian variety B and a morphism $f : X \to B$ mapping P to 0_B there is a unique morphism $g : Alb_X \to B$ of group schemes with $g \circ \iota_P = f$. Note that the maps ι_P satisfy the formula

$$\iota_P(Q) = \iota_P(R) + \iota_R(Q) \tag{3}$$

for any k-points P, Q, R of X.

If X is proper, then Alb_X is the Albanese variety in the classical sense. If X is a curve, it coincides with Rosenlicht's generalised Jacobian for the modulus defined by the sum of points at infinity.

The assignment $X \mapsto Alb_X$ is a covariant functor for arbitrary morphisms of varieties. Moreover, there is also a contravariant functoriality of Alb_X with respect to finite flat morphisms $f : X \to Y$ which we now briefly explain. Mapping a closed point Q of Y to the pull-back zero-cycle $f^*(Q)$ defines a morphism of Y into the d-fold symmetric product $Sym^d(X)$, where d is the degree of f. On the other hand, for a fixed k-point P of Y the zero-cycle $f^*(P) = P_1 + \ldots + P_d$ defines a morphism $Sym^d(X) \to Alb_X$ via the sum of the maps ι_{P_i} $(1 \leq i \leq d)$. The composite of these two maps sends P to 0 in Alb_X, hence by definition of Alb_Y factors as the composite of ι_P with a morphism $f^* : Alb_Y \to Alb_X$. Using formula (3) one checks that f^* is independent of the choice of P; it is the map we were looking for.

We denote by

$$a_X : \mathcal{Z}(X)^0 \longrightarrow Alb_X(k)$$

the homomorphism

$$\sum_i n_i P_i \mapsto \sum_i n_i(\iota_P(P_i))$$

for some $P \in X(k)$; again the map is independent of the choice of P by formula (3). For a morphism $f : X \to Y$ of varieties the diagram

$$(4)\qquad \begin{array}{ccc} \mathcal{Z}(X)^0 & \xrightarrow{a_X} & Alb_X(k) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ \mathcal{Z}(Y)^0 & \xrightarrow{a_Y} & Alb_Y(k) \end{array}$$

commutes where the vertical maps are induced by f through covariant functoriality. Similarly, for a finite flat $f : X \to Y$ the diagram

$$(5)\qquad \begin{array}{ccc} \mathcal{Z}(Y)^0 & \xrightarrow{a_Y} & Alb_Y(k) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} \\ \mathcal{Z}(X)^0 & \xrightarrow{a_X} & Alb_X(k) \end{array}$$

commutes.

Using these functoriality properties we can give an easy proof of the following "reciprocity law" which immediately yields the existence of the map alb_X as in (2).

Lemma 3.1 *With notations as above, the subgroup $\mathcal{R} \subset \mathcal{Z}(X)^0$ is contained in the kernel of the map a_X.*

Proof. Let $Z \subseteq X \times \mathbb{A}^1$ be a closed integral subscheme such that the projection $q : Z \to \mathbb{A}^1$ is finite and surjective (hence also flat, its target being a regular integral scheme of dimension 1) and denote by $p : Z \to X$ the other projection. By the commutativity of (4) and (5) we have

$$a_X(i_0^*(Z) - i_1^*(Z)) = a_X(p_*(q^*((0) - (1)))) = p_*(q^*(a_{\mathbb{A}^1}((0) - (1)))).$$

Since the generalised Albanese of $\mathbb{A}^1$ is trivial (any map of $\mathbb{A}^1$ into a semi-abelian variety being constant), it follows that the left hand side is 0. □

Now we treat the sheafification of the above construction, over an arbitrary perfect base field k. For this purpose it is convenient to replace Alb_X by the "Albanese scheme" $\widetilde{Alb}_X$ considered in [Ra] (where it is denoted by A_X; for the following facts, see his Definition 1.5 and the subsequent discussion). For geometrically connected X it is a smooth commutative group scheme which is an extension of the constant group scheme $\mathbb{Z}$ by a semi-abelian variety $\widetilde{Alb}_X^0$. Any k-point of X (if exists) defines a splitting, i.e. an isomorphism $\mathbb{Z} \times \widetilde{Alb}_X^0 \cong \widetilde{Alb}_X$. For k algebraically closed, the variety $\widetilde{Alb}_X^0$ is none but our Alb_X considered above. The scheme $\widetilde{Alb}_X$ comes equipped with a canonical morphism $\iota : X \to \widetilde{Alb}_X$ satisfying an appropriate universal property.

Now for simplicity we restrict to the case when X is *smooth*, which is sufficient for the applications we have in mind. Consider the abelian presheaf on Sm/k represented by the group scheme $\widetilde{Alb}_X$ which we also denote by $\widetilde{Alb}_X$. It is a sheaf for the étale (even the $fppf$) topology.

Lemma 3.2 *The étale sheaf $\widetilde{Alb}_X$ is a homotopy invariant presheaf with transfers.*

Proof. Homotopy invariance is again a consequence of the fact that there is no non-constant map $\mathbb{A}^1 \to \widetilde{Alb}_X$. To construct transfer maps, we can work more generally with an arbitrary commutative group scheme G. Take $X, Y \in Sm/k$ and let $Z \subset X \times Y$ be a closed integral subscheme finite and surjective over a component of X. As explained before Theorem 6.8 of [SV], to X one can associate a canonical map $\alpha_Z : X \to Sym^d(Y)$, where d is the degree of the projection $Z \to X$. Now given a map $Y \to G$, it induces a map $Sym^d(Y) \to Sym^d(G)$, whence we obtain the required map $X \to G$ by composing by α_Z on the left and by the summation map on the right. $\square$

The lemma implies that there is a unique map of presheaves

$$\mathbb{Z}_{tr}(X) \to \widetilde{Alb}_X \tag{6}$$

which maps the correspondence associated to the identity map $id : X \to X$ to the Albanese map $\iota \in \widetilde{Alb}_X(X)$. By applying the functor $C_\bullet(\ )$ we get a map $C_\bullet(\mathbb{Z}_{tr}(X)) \to C_\bullet(\widetilde{Alb}_X)$. Composing it with the augmentation map on the right (existing by homotopy invariance of $\widetilde{Alb}_X$; see the previous section) yields the map of complexes of étale sheaves with transfers

$$C_\bullet(\mathbb{Z}_{tr}(X)) \longrightarrow \widetilde{Alb}_X. \tag{7}$$

Here we again consider $\widetilde{Alb}_X$ as a complex concentrated in degree 0. Since this is a morphism of complexes, it factors through the 0-th homology presheaf $H_0(C_\bullet(\mathbb{Z}_{tr}(X)))$; as $\widetilde{Alb}_X$ is an étale sheaf, it even factors through the associated étale sheaf $H_0(C_\bullet(\mathbb{Z}_{tr}(X)))_{\text{ét}}$. We remark for later use that the map

(6) can be obtained as a composite of (7) with the natural morphism of complexes $\mathbb{Z}_{tr}(X) \to C_\bullet(\mathbb{Z}_{tr}(X))$ (again with $\mathbb{Z}_{tr}(X)$ placed in degree 0 on the left).

Passing to sections over k and taking homology, we get a map $h_0(X) \to \widetilde{Alb}_X(k)$, and, in the presence of a k-point, a map as in (2) which agrees with the previous one for k algebraically closed. In other words, the existence of the map (7) subsumes a sheafified version of the reciprocity law (perceptive readers have already noted the similarity of the argument with the proof of Lemma 3.1). The existence of the map (2) over a perfect base field, as demonstrated here, will be used in the proof of Theorem 1.2.

Remark 3.3 In the terminology of [Vo], Lemma 3.2 states that the sheaf $\widetilde{Alb}_X$ defines an object in the category $DM_-^{eff}(k)$ of effective motivic complexes; on the other hand, $C_\bullet(\mathbb{Z}_{tr}(X))$ is precisely the motivic complex that Voevodsky associates to the smooth variety X. Therefore the map (7), which was shown above to be a morphism in $DM_-^{eff}(k)$, can be regarded as the "motivic interpretation" of the Albanese map.

4 Relation to Tame Abelian Covers

Assume now we are in the situation of Theorem 1.1. In this case Alb_X has been described by Serre in his exposé [Se2] as an extension of the abelian variety $Alb_{\mathfrak{X}}$ by a torus T whose rank is equal to the rank of the subgroup B_X of divisors on $\mathfrak{X}$ which are algebraically equivalent to zero and whose support is contained in $\mathfrak{X} - X$. As a consequence of this result one gets for any n prime to p, just as in the proper case (see [KL], Lemma 5 or [Mi], p. 273), an injection of the dual group of ${}_nAlb_X(k)$ into $H^1_{\text{ét}}(X, \mathbb{Z}/n)$ with a finite cokernel of order bounded independently of n.

The construction of this injection is completely analogous to the proper case, only technically a bit more involved. Consider the group C_X of irreducible divisors of $\mathfrak{X}$ supported in $\mathfrak{X} \setminus X$. Composing the projection $\mathrm{Pic}(\mathfrak{X}) \to NS(\mathfrak{X})$ to the Néron-Severi group by the natural map

$$C_X \to \mathrm{Pic}(\mathfrak{X}) \tag{8}$$

associating to a divisor its class one gets an exact sequence

$$0 \to B_X \to C_X \to S \to 0$$

with the appropriate subgroup S of $NS(\mathfrak{X})$. Denote by $M^*(X)$ the complex of smooth commutative group schemes (concentrated in degrees 0 and 1) associated to (8). By restriction to B_X we get another complex $[B_X \to \mathrm{Pic}^0(\mathfrak{X})]$ which we denote by $M^1(X)$. The above considerations give a distinguished triangle in the derived category of bounded complexes of smooth commutative group schemes

$$M^1(X) \longrightarrow M^*(X) \longrightarrow S[-1] \longrightarrow M^1(X)[1].$$

By taking k-valued points (this is an exact functor since k is algebraically closed), tensoring with $\mathbb{Z}/n$ in the derived sense and passing to cohomology we obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(M^1(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n) \longrightarrow H^0(M^*(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n) \longrightarrow \mathrm{Tor}(S, \mathbb{Z}/n) \\ \longrightarrow H^1(M^1(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n). \end{aligned} \tag{9}$$

Here the last term vanishes by the following easy lemma:

Lemma 4.1 *Let k be an algebraically closed field and M a complex of commutative k-group schemes concentrated in degrees 0 and 1 whose degree 1 term is smooth and connected. Then $H^1(M(k) \otimes^{\mathbb{L}} \mathbb{Z}/n) = 0$ for all integers $n \neq 0$.*

Proof. This boils down to the divisibility of the group of rational points of a smooth connected commutative group scheme over an algebraically closed field. We leave the details to the reader. □

Now assume for a moment that X is proper or the complement of a divisor in $\mathfrak{X}$. As remarked by Ramachandran (in (2-30) of the preprint version of [Ra]), the same argument as that for curves given on p. 70 of [De] gives a canonical isomorphism

$$H^0(M^*(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n) \xrightarrow{\cong} H^1_{\text{ét}}(X, \mu_n). \tag{10}$$

For the convenience of the reader we recall the definition of (10). The target can be identified with group of isomorphism classes of pairs $(\mathcal{L}, \psi)$ consisting of a line bundle $\mathcal{L}$ on X and an isomorphism $\psi : \mathcal{L}^{\otimes n} \to \mathcal{O}_X$. On the other $H^0(M^*(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n)$ can be described as the group of equivalence classes of pairs $(\bar{\mathcal{L}}, D)$ where $\bar{\mathcal{L}}$ is a line bundle on $\mathfrak{X}$ and $D \in C_X$ with $\bar{\mathcal{L}}^{\otimes n} \cong \mathcal{O}(D)$, two such pairs $(\bar{\mathcal{L}}_1, D_1)$, $(\bar{\mathcal{L}}_2, D_2)$ being equivalent if there exist $D_3 \in C_X$

such that $\bar{\mathcal{L}}_1 \otimes \bar{\mathcal{L}}_2^{-1} \cong \mathcal{O}(D_3)$ and $D_1 - D_2 = nD_3$. Given a pair $(\bar{\mathcal{L}}, D)$ we choose an isomorphism $\bar{\psi} : \bar{\mathcal{L}}^{\otimes n} \to \mathcal{O}(D)$. The map (10) is given by sending the class of $(\bar{\mathcal{L}}, D)$ to the isomorphism class of $(\bar{\mathcal{L}} \mid_X, \bar{\psi} \mid_X)$.

As explained by ([Ra], Theorem 2.3), the main result of [Se2] can be reinterpreted by saying that the Cartier dual of the complex $M^1(X)$ regarded as a 1-motive (cf. [De], Chapter 10 for this terminology) is the 1-motive $[0 \to Alb_X]$; in particular, the toric part T of Alb_X has character group B_X. Since the construction of ([De], 10.2.5 and 10.2.11) puts into duality the "n-adic realisations" $T_{\mathbb{Z}/n\mathbb{Z}}M$ and $T_{\mathbb{Z}/n\mathbb{Z}}M^\vee$ of a 1-motive M and of its Cartier dual $M^\vee$ (this is a generalisation of the classical fact that the duality between an abelian variety and its dual induces a duality on n-torsion points), as a consequence we get a canonical isomorphism

$$H^0(M^1(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n) \xrightarrow{\cong} \mathrm{Hom}({}_nAlb_X(k), \mu_n). \tag{11}$$

Hence we have almost proven

Proposition 4.2 *Let X and $\mathfrak{X}$ be as in Theorem 1.1. For every integer n prime to p the above construction gives an exact sequence*

$$0 \longrightarrow Hom({}_nAlb_X(k), \mathbb{Z}/n) \longrightarrow H^1_{\acute{e}t}(X, \mathbb{Z}/n) \longrightarrow Hom(\mu_n, S) \longrightarrow 0. \tag{12}$$

Proof. When $X = \mathfrak{X}$ or the complement of a divisor, this follows from the above considerations after twisting by μ_n. In the general case, we may find an open subscheme X' of $\mathfrak{X}$ containing X which is the complement of a divisor in $\mathfrak{X}$ and such that the codimension of $X' - X$ in X' is at least 2. Then we have canonical isomorphisms $Alb_X \cong Alb_{X'}$ (see [Ra], Corollary 2.4) and $H^1_{\text{ét}}(X, \mathbb{Z}/n) \cong H^1_{\text{ét}}(X', \mathbb{Z}/n)$ (a consequence of Zariski-Nagata purity; see [SGA2], exposé X for an exposition in the language of étale covers) and therefore the construction of the exact sequence for X reduces to that for X' using contravariant functoriality. □

Corollary 4.3 *The dual of the first map of the proposition induces a canonical isomorphism*

$$Hom(H^1_{\acute{e}t}(X, \mathbb{Z}_\ell), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong Alb_X(k)\{\ell\}$$

for any prime number $\ell \neq p$.

Proof. The group S being finitely generated, its Tate module is trivial. Therefore passing to the inverse limit by making n run over powers of ℓ in

(12) yields an isomorphism between the limit of the first two terms. The corollary follows by dualising. □

In the remainder of this section, which will only be needed for the proof of Theorem 1.2, we strengthen the result of the proposition to obtain a description of the abelianised tame fundamental group $\pi_1^{t,ab}(X)$ of X. By definition, this group classifies finite abelian Galois covers of $\mathfrak{X}$ which are étale over X and tamely ramified at codimension 1 points of $\mathfrak{X} \setminus X$ (i.e. the ramification index at such a point is prime to p and the extension of its residue field is separable). One has a direct sum decomposition

$$\pi_1^{t,ab}(X) \cong \pi_1^{ab}(X)(p') \oplus \pi_1^{ab}(\mathfrak{X})(p)$$

where the symbols (p') and (p) stand for the maximal profinite prime-to-p (resp. p) quotients of the groups in question. Indeed, any finite abelian Galois cover of X of order prime to p extends to a tamely ramified cover of $\mathfrak{X}$ by normalisation; for the p-part, notice that any abelian cover of $\mathfrak{X}$ which is of p-power degree, étale over X and tamely ramified in codimension 1 must be étale in codimension 1, hence étale by Zariski-Nagata purity. Since the fundamental group is a birational invariant of projective varieties, the above decomposition shows that $\pi_1^{t,ab}(X)$ depends only on X but not on the compactification $\mathfrak{X}$. Needless to say, all these notions and facts are valid more generally over any perfect base field in place of k.

Proposition 4.4 *Under the assumptions of the previous proposition, there is an exact sequence*

$$0 \longrightarrow T \longrightarrow \pi_1^{t,ab}(X) \longrightarrow T(Alb_X) \longrightarrow 0 \tag{13}$$

where $T(Alb_X)$ denotes the full Tate module of Alb_X and T is a finite abelian group whose twisted dual can be described as follows: its prime-to-p part is isomorphic to that of the finite torsion subgroup of the group S considered above and its p-part is isomorphic to the p-primary torsion subgroup of $NS(\mathfrak{X})$.

Proof. We use the decomposition of $\pi_1^{t,ab}(X)$ recalled above. The assertion for the prime-to-p part follows from the previous proposition by dualising and passing to the limit. For the p-part we note that $T_p(Alb_X) \cong T_p(Alb_{\mathfrak{X}})$, the toric part of Alb_X having no p-primary torsion, so the result follows from the analogous statement for $\mathfrak{X}$ proven in ([KL], Lemma 5). □

5 The Generalisation of Roitman's Theorem

Keeping the assumptions of the previous section, we now prove Theorem 1.1. The proof involves the verification of some delicate compatibilities (Proposition 5.1 and Lemma 5.3) which occupy much of this section. We therefore offer alternative arguments in Remarks 5.4 and 5.5. In the first of these we explain how Lemma 5.3 can be avoided by using a counting argument. In Remark 5.5 we give a second shorter proof of the theorem – based on a hypersurface section argument – which circumvents the use of both 5.1 and 5.3. We note, however, our firm belief that from the conceptual point of view the optimal proof passes through the checking of compatibilities and not through the shortcuts.

We begin with some preliminary observations. For any positive integer n prime to p the long exact sequence

$$\ldots \longrightarrow h_i(X) \xrightarrow{n} h_i(X) \longrightarrow h_i(X, \mathbb{Z}/n) \longrightarrow h_{i-1}(X) \xrightarrow{n} \ldots$$

yields a surjection

$$h_1(X, \mathbb{Z}/n) \longrightarrow {}_n h_0(X). \tag{14}$$

On the other hand, we have a chain of isomorphisms

$$h_1(X, \mathbb{Z}/n) \cong \mathrm{Hom}(h^1(X, \mathbb{Z}/n), \mathbb{Z}/n) \cong \mathrm{Hom}(H^1_{\text{ét}}(X, \mathbb{Z}/n), \mathbb{Z}/n), \tag{15}$$

the first by the very definition of the groups in question (note that $\mathbb{Z}/n$ is injective as a $\mathbb{Z}/n$-module) and the second by the comparison theorem of Suslin and Voevodsky (see [SV], Corollary 7.8 for the argument in characteristic 0; for the modifications in positive characteristic using de Jong's work on alterations, cp. [Ge], Theorem 3.2).

Let

$$\mathrm{Hom}({}_n Alb_X(k), \mathbb{Z}/n) \to H^1_{\text{ét}}(X, \mathbb{Z}/n) \tag{16}$$

be the map given by "pulling back covers from Alb_X to X". Indeed, each $\phi : {}_n Alb_X(k) \to \mathbb{Z}/n$ gives an étale $\mathbb{Z}/n$-cover of Alb_X by pushing out the extension

$$0 \to {}_n Alb_X \to Alb_X \xrightarrow{n} Alb_X \to 0$$

via the map of group schemes associated to ϕ, hence defines a class in $H^1_{\text{ét}}(Alb_X, \mathbb{Z}/n)$. Whence a map $\mathrm{Hom}({}_n Alb_X, \mathbb{Z}/n) \to H^1_{\text{ét}}(Alb_X, \mathbb{Z}/n)$, which by composition with the map induced on cohomology by a canonical map $X \to Alb_X$ yields the map (16).

Now we can state the key result:

Proposition 5.1 *For any positive integer n prime to p we have a commutative diagram*

$$
\begin{array}{ccc}
h_1(X, \mathbb{Z}/n) & \longrightarrow & {}_n h_0(X) \\
\cong \Big\downarrow & & \Big\downarrow alb_X \\
Hom(H^1_{\acute{e}t}(X, \mathbb{Z}/n), \mathbb{Z}/n) & \longrightarrow & {}_n Alb_X(k)
\end{array} \tag{17}
$$

where the upper horizontal, left vertical and bottom horizontal maps are respectively (14), (15) and the dual of (16).

For the proof of the proposition we need the following technical statements about abelian groups whose formal proof will be left to the reader.

Lemma 5.2

1. *For any abelian group A and integer $n > 0$ there is a canonical isomorphism*
$$Hom({}_n A, \mathbb{Z}/n) \cong Ext^1(A, \mathbb{Z}/n).$$

2. *Let $(C_\bullet, d)$ be a homological complex of free abelian groups concentrated in nonnegative degrees. Then the natural map*
$$H_1(C_\bullet \otimes \mathbb{Z}/n) \to {}_n H_0(C_\bullet)$$
coming from tensoring by the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ can be identified with the natural map
$$H_1(C_\bullet \otimes \mathbb{Z}/n) \to Tor(H_0(C_\bullet), \mathbb{Z}/n) \tag{18}$$
coming from computing the Tor-group by means of the free resolution $d(C_1) \to C_0$ of $H_0(C_\bullet)$.

3. *With the previous notations, the natural map*
$$Ext^1(H_0(C_\bullet), \mathbb{Z}/n) \to Ext^1(C_\bullet, \mathbb{Z}/n)$$
induced by the truncation map $C_\bullet \to H_0(C_\bullet)$ can be identified (using statement 1. and the self-injectivity of the ring $\mathbb{Z}/n$) with the image of the map (18) under the functor $Hom(\ , \mathbb{Z}/n)$.

Proof of Proposition 5.1. We prove the commutativity of the dual diagram

$$\begin{array}{ccc} \mathrm{Hom}({}_n Alb_X(k), \mathbb{Z}/n) & \longrightarrow & H^1_{\text{ét}}(X, \mathbb{Z}/n) \\ \downarrow & & \downarrow \cong \\ \mathrm{Hom}({}_n h_0(X), \mathbb{Z}/n) & \longrightarrow & h^1(X, \mathbb{Z}/n) \end{array}$$

which, using Lemma 5.2 (1), can be rewritten as

$$\begin{array}{ccc} \mathrm{Ext}^1(\widetilde{Alb}_X(k), \mathbb{Z}/n) & \longrightarrow & H^1_{\text{ét}}(X, \mathbb{Z}/n) \\ \downarrow & & \downarrow \cong \\ \mathrm{Ext}^1(h_0(X), \mathbb{Z}/n) & \longrightarrow & \mathrm{Ext}^1(C_\bullet(X), \mathbb{Z}/n) \end{array}$$

where the Ext-groups are taken with respect to the category of abelian groups (there was no harm in replacing Alb_X by $\widetilde{Alb}_X$ since $\mathbb{Z}$ is torsion-free). Using Lemma 5.2 the bottom horizontal map can then be identified as coming from the natural truncation map. Now we apply the rigidity theorem of Suslin-Voevodsky ([SV], Theorem 4.5) to the three Ext-groups and a standard comparison theorem to the fourth group to obtain a diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\text{ét}}(\widetilde{Alb}_X, \mathbb{Z}/n) & \longrightarrow & \mathrm{Ext}^1_{\text{ét}}(\mathbb{Z}(X), \mathbb{Z}/n) \\ \downarrow & & \downarrow \cong \\ \mathrm{Ext}^1_{\text{ét}}(H_0(C_\bullet(\mathbb{Z}_{tr}(X)))_{\text{ét}}, \mathbb{Z}/n) & \longrightarrow & \mathrm{Ext}^1_{\text{ét}}(C_\bullet(\mathbb{Z}_{tr}(X)), \mathbb{Z}/n) \end{array} \tag{19}$$

where the Ext-groups are now taken on the étale site of Sm/k, the subscript *ét* means sheafification for the étale topology and $\mathbb{Z}(X)$ is the étale sheaf whose sections over a smooth k-scheme Y are given by the free abelian group with basis $\mathrm{Hom}(Y, X)$. Note that the rigidity theorem was applicable to the upper left group by virtue of Lemma 3.2 and to the two lower ones by ([SV], Corollary 7.5). Now to finish the proof, we claim that the above diagram is induced by applying the functor $\mathrm{Ext}^1_{\text{ét}}(\ \ , \mathbb{Z}/n)$ to the commutative diagram of complexes of sheaves

$$\begin{array}{ccc} C_\bullet(\mathbb{Z}_{tr}(X)) & \longrightarrow & H_0(C_\bullet(\mathbb{Z}_{tr}(X)))_{\text{ét}} \\ \uparrow & & \downarrow \\ \mathbb{Z}_{tr}(X) & \longrightarrow & \widetilde{Alb}_X \end{array}$$

whose existence was established in Section 3 (the map on the left inducing the inverse of the isomorphism marked in (19)) .

The identification of the bottom horizontal and left vertical arrows in (19) follows from the functoriality of the rigidity isomorphism. As for the upper horizontal map, note first that it is well known to be induced by the map of étale sheaves $\mathbb{Z}(X) \to \widetilde{Alb_X}$ which factors through the natural inclusion $\mathbb{Z}(X) \to \mathbb{Z}_{tr}(X)$ by Lemma 3.2. Now by ([SV], Corollary 10.10) the natural map $\mathrm{Ext}^1_{\text{ét}}(\mathbb{Z}_{tr}(X), \mathbb{Z}/n) \to \mathrm{Ext}^1_{\text{ét}}(\mathbb{Z}(X), \mathbb{Z}/n)$ can be identified with the map $\mathrm{Ext}^1_{qfh}(\mathbb{Z}_{tr}(X)_{qfh}, \mathbb{Z}/n) \to \mathrm{Ext}^1_{qfh}(\mathbb{Z}(X)_{qfh}, \mathbb{Z}/n)$, where the subscript qfh denotes sheafification for the so-called qfh-topology introduced in *loc. cit.* , which is finer than the étale topology. But the latter map is an isomorphism, for by ([SV], Theorem 6.7) $\mathbb{Z}_{tr}(X)$ can be identified, after localisation by the characteristic p, with $\mathbb{Z}(X)_{qfh}$. This finishes the identification of the upper horizontal map, and for the right vertical map one first uses the same argument to pass from $\mathbb{Z}(X)$ to $\mathbb{Z}_{tr}(X)$, whereupon the result follows from the construction of the isomorphism $h^1(X, \mathbb{Z}/n) \to H^1_{\text{ét}}(X, \mathbb{Z}/n)$ in the proof of Theorem 7.5 in [SV] (from which one sees that it can be identified with the map induced by $\mathbb{Z}_{tr}(X) \to C_*(\mathbb{Z}_{tr}(X))$ on Ext^1-groups). □

Proof of Theorem 1.1. It is enough to consider ℓ-primary torsion for a prime $\ell \neq p$. It then suffices to see that by making n vary among powers of ℓ and passing to the direct limit we get a diagram whose bottom horizontal map is an isomorphism. Indeed, since the left vertical map is also an isomorphism and the upper horizontal map is surjective, by commutativity all maps in the diagram (17) must become isomorphisms in the limit.

The most natural way to prove that the bottom horizontal map induces an isomorphism in the limit is to identify it with the map between the first two terms of exact sequence (12) and apply Corollary 4.3. This identification is well known in the proper case and we give now a detailed sketch of checking it in general. Alternatively, one can avoid checking this compatibility by arguing as in Remark 5.4 below.

Lemma 5.3 *The map (16) coincides with the map between the first two terms in (12) given in the last section.*

Proof. We may again assume that X is proper or the complement of a divisor and argue about the map (16) twisted by μ_n. The map (16) associates to $\phi \in \mathrm{Hom}({}_nAlb_X(k), \mu_n)$ an extension of the group scheme Alb_X by μ_n and hence also an extension E of Alb_X by $\mathbb{G}_m$, corresponding to a line bundle $\mathcal{L}$ with an isomorphism $\psi : \mathcal{L}^{\otimes n} \to \mathcal{O}_{Alb_X}$ (the latter is a consequence of the fact that the n-fold sum of the extension $0 \to \mathbb{G}_m \to E \to Alb_X \to 0$ is canonically isomorphic to the trivial extension). Pulling back $\mathcal{L}$ and ψ to X we get a pair $(\mathcal{L}_X, \psi_X)$ which defines an element ξ of $H^1(X, \mu_n)$, the image of ϕ by (16). Now since the natural map $Ext^1(Alb_{\mathfrak{X}}, \mathbb{G}_m) \to Ext^1(Alb_X, \mathbb{G}_m)$ is surjective (the toric part T of Alb_X having no non-trivial extensions by $\mathbb{G}_m$), there is some extension $\bar{E}$ of $Alb_{\mathfrak{X}}$ by $\mathbb{G}_m$ (defining a line bundle $\bar{\mathcal{L}}$) of which E is the pullback to Alb_X. Since $Ext^1(Alb_{\mathfrak{X}}, \mathbb{G}_m) \cong Pic^0(Alb_{\mathfrak{X}})$, the isomorphism class of the pullback $\bar{\mathcal{L}}_{\mathfrak{X}}$ of $\bar{\mathcal{L}}$ to $\mathfrak{X}$ lies in $Pic^0(\mathfrak{X})$ and hence $\bar{\mathcal{L}}_{\mathfrak{X}}^{\otimes n} \cong \mathcal{O}_{\mathfrak{X}}(D)$ with some $D \in B_X$. As in the previous section, the pair $(\bar{\mathcal{L}}_{\mathfrak{X}}, D)$ defines an element of $H^0(M^1(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n) \subset H^0(M^*(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n)$ which, by construction, is mapped to ξ under (10). On the other hand, one sees by going through Serre's duality construction that the element of $H^0(M^1(X)(k) \otimes^{\mathbb{L}} \mathbb{Z}/n)$ represented by $(\bar{\mathcal{L}}_{\mathfrak{X}}, D)$ is exactly the image of ϕ under (11). This completes the proof of the lemma and thereby that of Theorem 1.1. □

Remark 5.4 Alternatively, one may prove that the bottom horizontal map in (17) induces an isomorphism in the limit as follows. Consider the dual map $\pi_1^{ab}(X)/n \to {}_nAlb_X$ which is known to be surjective by ([Se1], Théorème 10). For $n = \ell^m$ this is none but the surjection $\pi_1^{ab}(X)(\ell) \to T_\ell Alb_X$ tensored by $\mathbb{Z}/\ell^m$ (where (ℓ) denotes the maximal pro-ℓ quotient). This latter surjection must have finite kernel for by Corollary 4.3 its domain and target are finitely generated $\mathbb{Z}_\ell$-modules of the same rank. Hence the modulo n map has a finite kernel of order bounded independently of n, from which we conclude by the same argument as in Corollary 4.3.

Remark 5.5 One can give a quicker, albeit less conceptual proof of Theorem 1.1 which avoids the verification of commutativity in (17), in the spirit of the simplified version of Bloch's approach to Roitman's theorem given in [CT]. Indeed, using the horizontal and left vertical maps in (17) and passing to the limit one gets a surjection $Alb_X(k)\{\ell\} \to h_0(X)\{\ell\}$. Since Alb_X is a semiabelian variety, both groups here must be isomorphic to some finite direct power of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, so that for any $m > 0$ the groups ${}_{\ell^m}Alb_X(k)$ and ${}_{\ell^m}h_0(X)$ are finite and the order of the second one doesn't exceed that of the first. So by comparing orders, we are done once we show the surjectivity of alb_X on ℓ^m-torsion. This is achieved by induction on dimension starting from the case of curves treated in [Li] and ([SV], Theorem 3.1). For the inductive step, taking into account the covariant functoriality of alb_X, it suffices to prove the surjectivity of ${}_{\ell^m}Alb_Y(k) \to {}_{\ell^m}Alb_X(k)$ for an appropriate smooth closed subvariety $Y \subsetneq X$, or else, using the injectivity part of Proposition 4.2, the injectivity of $H^1_{\text{ét}}(X, \mathbb{Z}/\ell^m) \to H^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m)$. To choose Y, we may assume as before that the complement Z of X in $\mathfrak{X}$ is empty or has pure codimension one. Then by the Bertini theorems we may find a smooth connected hyperplane section $\mathfrak{Y}$ of $\mathfrak{X}$ that cuts each component of Z smoothly and away from the intersections. Putting $Y = \mathfrak{Y} \cap X$ and $W = \mathfrak{Y} \cap Z$, the claim then follows from the injectivity of the first and third vertical maps in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\text{ét}}(\mathfrak{X}, \mathbb{Z}/\ell^m) & \longrightarrow & H^1_{\text{ét}}(X, \mathbb{Z}/\ell^m) & \longrightarrow & H^0_{\text{ét}}(Z, \mathbb{Z}/\ell^m(-1)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{\text{ét}}(\mathfrak{Y}, \mathbb{Z}/\ell^m) & \longrightarrow & H^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m) & \longrightarrow & H^0_{\text{ét}}(W, \mathbb{Z}/\ell^m(-1)) \end{array}$$

whose exact rows are Gysin sequences. Indeed, the injectivity of the first arrow is classical (it follows e.g. by Poincaré duality from the weak Lefschetz Theorem), and that of the third follows from the choice of $\mathfrak{Y}$, each component of Z containing at least one component of W.

6 Proof of Theorem 1.2

Assume now that k is the algebraic closure of a finite field $\mathbb{F}$ and denote by G the Galois group $Gal(k|\mathbb{F})$. Before embarking on the proof of Theorem 1.2, we remark that, as the perceptive reader has surely noticed, in this case one can immediately show by reduction to the case of curves that the groups whose isomorphism we are to establish are both torsion. Hence in this case the prime-to-p part of Theorem 1.2 is equivalent to Theorem 1.1. But in the proof below we shall use a different method (and thus give another proof of Theorem 1.1 in this special case which works also for the p-part) originating in an argument of [KS].

By extending $\mathbb{F}$ if necessary we may assume that there are varieties $X_\mathbb{F} \subset \mathfrak{X}_\mathbb{F}$ defined over $\mathbb{F}$ such that $X_\mathbb{F}$ has an $\mathbb{F}$-rational point and $X_\mathbb{F} \times_\mathbb{F} k \cong X$, $\mathfrak{X}_\mathbb{F} \times k \cong \mathfrak{X}$. Similarly to section 10 of [KS], the key to the proof of Theorem 1.2 is the exact sequence (13) which in this case is in fact an exact sequence of G-modules.

Recall from Section 4 that the abelianised tame fundamental group can be defined for $X_\mathbb{F}$ as well. Moreover, there is a natural projection $\pi_1^{t,ab}(X_\mathbb{F}) \to G \cong \widehat{\mathbb{Z}}$ whose kernel $\pi_1^{t,ab}(X_\mathbb{F})^0$ can be identified with the coinvariants of $\pi_1^{t,ab}(X)$ under the action of G. Therefore taking coinvariants under Frobenius in the exact sequence (13) yields the sequence

$$0 \to T_G \longrightarrow \pi_1^{t,ab}(X_\mathbb{F})^0 \longrightarrow Alb_{X_\mathbb{F}}(\mathbb{F}) \longrightarrow 0 \qquad (20)$$

(for exactness note that a semi-abelian variety over a finite field has only finitely many rational points and therefore the Frobenius acting on its Tate module has no eigenvalue 1). There are similar exact sequences over each finite extension $\mathbb{F}'$ of $\mathbb{F}$ which naturally form a direct system. (One way to see this is that coinvariants under Frobenius form the first Galois cohomology group over a finite field, so the maps in the direct system are just the restriction maps.) The direct limit of the finite groups $T_{Gal(k|\mathbb{F}')}$ is trivial (this is a general fact for the first cohomology of a finite Galois module over any field; over a sufficiently large extension such a module becomes isomorphic to a sum of $\mathbb{Z}/m$'s and one may conclude e.g. by using Kummer and Artin-Schreier theory).

Now by the main result of [SS] the middle group in (20) is isomorphic to $h_0(X_{\mathbb{F}})^0$ by means of a reciprocity map $h_0(X_{\mathbb{F}})^0 \to \pi_1^{t,ab}(X_{\mathbb{F}})$ which sends the class of a closed point of X to the class of its Frobenius. Using this isomorphism and taking the direct limit over finite extensions of $\mathbb{F}$ as above we thus get an isomorphism $h_0(X)^0 \cong Alb_X(k)$.

It remains to see that it is induced by the Albanese map. For this it suffices to consider the image of the class of a zero-cycle of the form $P_1 - P_2$ and we may replace k by the finite extension of $\mathbb{F}$ over which both P_1 and P_2 are defined. Moreover, by using the covariant functoriality of the Albanese map and of the reciprocity map we may assume that P_1 and P_2 both lie on some smooth curve C and check the required compatibility for C, but this is a well-known property of Lang's class field theory (see [Se3]).

Finally we note that the above proof has the following interesting by-product, generalising the similar statement proved in [KS] for the proper case:

Corollary 6.1 *With notations as above, the natural map $h_0(X_{\mathbb{F}}) \to h_0(X)$ has a finite kernel isomorphic to the group T introduced in Proposition 4.4.*

References

[Bl] S. Bloch, *Torsion algebraic cycles and a theorem of Roitman.* Compos. Math. **39** (1979), 107–127.

[BS] J. Biswas, V. Srinivas, *Roitman's theorem for singular complex varieties.* Compos. Math. **119** (1999), 213–237.

[CT] J.-L. Colliot-Thélène, *Cycles algébriques de torsion et K-théorie algébrique.* In: Arithmetic Algebraic Geometry. Lect. Notes in Math. **1553**, Springer Verlag 1993, 1–49.

[De] P. Deligne, *Théorie de Hodge III.* Publ. Math. IHES **44** (1974), 6–77.

[Ge] T. Geisser, *Applications of de Jong's theorem on alterations.* In: Resolution of Singularities. Progress in Math. **181**, Birkhäuser 2000, 299–314.

[KL] N. Katz and S. Lang, *Finiteness theorems in geometric class field theory.* L'Enseign. Math. **27** (1981), 285–314.

[KS] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces.* Ann. of Math. **118** (1983), 241–275.

[Li] S. Lichtenbaum, *Suslin homology and Deligne 1-motives*, In: Algebraic K-Theory and Algebraic Topology (P. G. Goerss and J. F. Jardine, eds.). NATO ASI Series (1993), 189–197.

[Mi] J. S. Milne, *Zero cycles on algebraic varieties in nonzero characteristic: Rojtman's theorem.* Compos. Math. **47** (1982), 271–287.

[Ra] N. Ramachandran, *Duality of Albanese and Picard 1-motives*, K-theory **21** (2001), 271–301. Longer preprint version under the title: *Albanese and Picard motives of schemes* (1998).

[Roi] A. A. Roitman, *The torsion of the group of 0-cycles modulo rational equivalence.* Ann. of Math. **111** (1980), 553–569.

[Ros] M. Rosenlicht, *Generalized Jacobian varieties.* Ann. of Math. **59** (1954), 505–530.

[SS] A. Schmidt and M. Spieß, *Singular homology and class field theory for varieties over finite fields.* J. reine angew. Math. **527** (2000), 13–36.

[Se1] J.-P. Serre, *Morphismes universels et variété d'Albanese.* Séminaire Chevalley, année 1958/59, exposé 10.

[Se2] J.-P. Serre, *Morphismes universels et différentielles de troisième espèce.* Séminaire Chevalley, année 1958/59, exposé 11.

[Se3] J.-P. Serre, *Groupes algébriques et corps de classes.* Hermann, 1959.

[SV] A. Suslin and V. Voevodsky, *Singular homology of abstract algebraic varieties*. Invent. Math. **123** (1996), 61–94.

[Vo] V. Voevodsky *Triangulated categories of motives over a field*. In V. Voevodsky, A. Suslin, E. M. Friedlander: Cycles, Transfers, and Motivic Homology Theories. Annals of Math. Studies **143**, Princeton University Press, 2000, 188–238.

[SGA2] A. Grothendieck, *Cohomologie des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2)*. North-Holland, 1968.

Michael Spieß
School of Mathematical Sciences
University of Nottingham
University Park
NG7 2RD Nottingham
United Kingdom

Tamás Szamuely
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
PO Box 127
H-1364 Budapest
Hungary

e-mail: mks@maths.nott.ac.uk
szamuely@renyi.hu