EXISTENCE OF MINIMAL CLUSTERS

Let us consider the *cluster isoperimetric problem*. That is, we are given $M \in \mathbb{N}$ and a vector $m \in (\mathbb{R}^+)^M$. A *cluster of volume* m is a family $\mathcal{E} = \{E_1, E_2, \dots, E_M\}$ where $E_i \subseteq \mathbb{R}^N$ are pairwise disjoint Borel sets, with volumes $|E_i| = m_i$ for each $1 \leq i \leq M$. We will write for brevity $|\mathcal{E}| = m$. We say that $\mathcal{E}$ is a *cluster of finite perimeter* if each E_i is a set of finite perimeter, and we call *perimeter of the cluster* $\mathcal{E}$ the quantity

$$P(\mathcal{E}) = \mathscr{H}^{N-1}\Big(\bigcup_{i=1}^M \partial^* E_i\Big).$$

We define then

$$\mathfrak{J}(m) = \inf\big\{P(\mathcal{E}) : |\mathcal{E}| = m\big\}.$$

The isoperimetric problem for cluster consists then in looking for minimizers of the functional $\mathfrak{J}$. Each cluster $\mathcal{E}$ such that $P(\mathcal{E}) = \mathfrak{J}(|\mathcal{E}|)$ is called a *minimal cluster*. Our goal is to show the following result.

Theorem 1. *For every $m \in (\mathbb{R}^+)^M$ there exist minimal clusters of volume m.*

To prove the result, we start with some simple observations.

Lemma 2. *For every m, m', $m'' \in (\mathbb{R}^+)^M$ one has*

$$\mathfrak{J}(m' + m'') \leq \mathfrak{J}(m') + \mathfrak{J}(m''), \tag{1}$$

$$\mathfrak{J}(m) \geq \mathfrak{J}(|m|, 0, \dots, 0) = N\omega_N^{1/N}|m|^{\frac{N-1}{N}}, \tag{2}$$

where as usual $|m| = m_1 + m_2 + \dots + m_M$ is the norm of the vector m.

Proof. To prove the first inequality, for every $\varepsilon > 0$ we take two bounded clusters $\mathcal{E}'$ and $\mathcal{E}''$ with volumes $|\mathcal{E}'| = m'$ and $|\mathcal{E}''| = m''$ and such that $P(\mathcal{E}') < \mathfrak{J}(m') + \varepsilon$ and $P(\mathcal{E}'') < \mathfrak{J}(m'') + \varepsilon$. Up to a translation we can assume that the union of the sets E_i' does not intersect the union of the sets E_j''. Hence, define $\mathcal{E}$ the cluster defined by $E_i = E_i' \cup E_i''$ for every $1 \leq i \leq M$. By construction, we have that $|\mathcal{E}| = m' + m''$, thus

$$\mathfrak{J}(m' + m'') \leq P(\mathcal{E}) \leq P(\mathcal{E}') + P(\mathcal{E}'') < \mathfrak{J}(m') + \mathfrak{J}(m'') + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, (1) is proved.

To obtain (2), for any cluster $\mathcal{E}$ with volume $|\mathcal{E}| = m$ we just define F the set $F = \cup_{i=1}^M E_i$. The set F has volume $|m|$, and by construction $\partial^* F \subseteq \cup_{i=1}^M \partial^* E_i$, so

$$P(\mathcal{E}) \geq P(F) \geq \mathfrak{J}(|m|, 0, \dots, 0).$$

Taking the infimum over all possible clusters $\mathcal{E}$, we obtain (2). Notice that $N\omega_N^{1/N}|m|^{\frac{N-1}{N}}$ is the perimeter of a ball of volume m, which coincides with $\mathfrak{J}(|m|, 0, \dots, 0)$ by the isoperimetric inequality. $\square$

It is reasonable to guess that inequality (1) is strict unless either m' or m'' are the zero vector. We give then the following definition.

Definition 3. *Let $m \in (\mathbb{R}^+)^M$. We say that m is* irreducible *if for every m', m'' such that $m' + m'' = m$ and $\min\{|m'|, |m''|\} > 0$ one has*

$$\mathfrak{J}(m) < \mathfrak{J}(m') + \mathfrak{J}(m'')\,.$$

We are going to prove Theorem 1 in some steps. First of all, we will show that there exist minimal clusters for every irreducible volume; then, we will use this to deduce the existence of minimal cluster for every volume. In the end, we will notice that actually every volume is irreducible, see Remark 12.

Proposition 4. *Let $m \in (\mathbb{R}^+)^M$ be an irreducible volume. Then, there exist minimal clusters of volume m.*

Proof. Let $\{\mathcal{E}^n\}$, with $n \in \mathbb{N}$, be an optimal sequence of clusters of volume m, that is,

$$|\mathcal{E}^n| = m \quad \forall\, n \in \mathbb{N}\,, \qquad\qquad P(\mathcal{E}^N) \xrightarrow[n\to\infty]{} \mathfrak{J}(m)\,.$$

Since m is irreducible, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\mathfrak{J}(m) < \mathfrak{J}(m') + \mathfrak{J}(m'') - \delta \quad \forall\, m',\, m'' \,:\, m' + m'' = m\,, \quad \min\{|m'|,\, |m''|\} > \varepsilon\,, \tag{3}$$

as one readily obtains thanks to the observation that $\mathfrak{J} : (\mathbb{R}^+)^M \to \mathbb{R}$ is continuous. Let us then fix $\varepsilon > 0$, and let $n \in \mathbb{N}$ be any number such that

$$P(\mathcal{E}^n) < \mathfrak{J}(m) + \frac{\delta}{3}\,. \tag{4}$$

Calling for brevity $F^n = \cup_{i=1}^M E_i^n$, up to a translation we can assume that

$$\big|F^n \cap x_i > 0\big| = \frac{|m|}{2} \quad \forall\, 1 \le i \le N\,. \tag{5}$$

Let then $t \in \mathbb{R}$ be such that

$$\mathscr{H}^{N-1}\big(F^n \cap \{x_1 = t\}\big) < \frac{\delta}{3}\,. \tag{6}$$

Call now $\mathcal{E}'$ and $\mathcal{E}''$ the two clusters obtained by intersecting $\mathcal{E}^n$ with the two half-spaces $\{x_1 > t\}$ and $\{x_1 < t\}$, that is, for every $1 \le j \le M$ we set

$$E_j' = E_j \cap \{x_1 < t\}\,, \qquad\qquad E_j'' = E_j \cap \{x_1 > t\}\,.$$

We have then by (6) and (4)

$$\mathfrak{J}(|\mathcal{E}'|) + \mathfrak{J}(|\mathcal{E}''|) \le P(\mathcal{E}') + P(\mathcal{E}'') \le P(\mathcal{E}^n) + 2\mathscr{H}^{N-1}(F^n \cap \{x_1 = t\}) \le P(\mathcal{E}^n) + \frac{2}{3}\,\delta < \mathfrak{J}(m) + \delta\,,$$

and by (3) we deduce

$$\min\{|\mathcal{E}'|,\, |\mathcal{E}''|\} \le \varepsilon\,. \tag{7}$$

By Fubini Theorem, we can find $t_1 < 0 < t_2$ such that (6) is satisfied both with $t = t_1$ and $t = t_2$, and such that

$$\max\{|t_1|,\, |t_2|\} < 2\,\frac{|m|}{\delta}\,.$$

As a consequence, keeping in mind (5), (7) ensures that

$$\big|F^n\cap\{x_1>2|m|/\delta\}\big|\leq\big|F^n\cap\{x_1>t_2\}\big|\geq\varepsilon\,,$$

and similarly

$$\big|F^n\cap\{x_1<-2|m|/\delta\}\big|\leq\big|F^n\cap\{x_1<t_1\}\big|\geq\varepsilon\,,$$

Repeating the same argument in the other $N-1$ directions, we finally deduce that

$$\left|F^n\cap\left[-2\,\frac{|m|}{\delta},2\,\frac{|m|}{\delta}\right]\right|^N\geq|m|-2N\varepsilon \tag{8}$$

for every n for which (4) is true.

Notice now that, for every $1\leq j\leq M$, the characteristic functions $\chi_{E_j^n}$ are bounded in $BV(\mathbb{R}^N)$. Hence, up to a subsequence, we can assume that for every $1\leq j\leq M$ one has $\chi_{E_j^n}\xrightarrow[BV_{\rm loc}]{*}f_j$ when $n\to\infty$. Since the convergence is locally strong in L^1, each function f_j is in fact a characteristic function, say the characteristic function of a set $\overline{E}_j$. We immediately deduce that the sets $\overline{E}_j$ are essentially disjoint, hence $\overline{\mathcal{E}}=\{\overline{E}_1,\overline{E}_2,\,\dots\,,\overline{E}_M\}$ is a cluster. By lower semicontinuity of the perimeter, together with the simple observation that for any cluster $\mathcal{E}$ one has

$$P(\mathcal{E})=\frac{1}{2}\,\left(\sum_{j=1}^M P(E_i)+P\big(\mathbb{R}^N\setminus\cup_{j=1}^M E_j\big)\right),$$

we deduce that

$$P(\overline{\mathcal{E}})\leq\liminf P(\mathcal{E}^n)=\mathfrak{J}(m)\,.$$

Of course, the proof will be concluded as soon as we check that $|\overline{\mathcal{E}}|=m$. In fact, by lower semicontinuity it is clear that $|\overline{\mathcal{E}}|\leq m$ (that is, for every $1\leq j\leq M$ one has $|\overline{E}_j|\leq m_j$), hence it suffices to check that $|F|=|m|$, being $F=\cup_{j=1}^M\overline{E}_j$. Since the convergence of the characteristic functions is strong in $L^1(D)$, being $D=[-2|m|/\delta,2|m|/\delta]^N$, thanks to (8) we know that

$$|F|\geq|m|-2N\varepsilon\,.$$

And finally, since $\varepsilon>0$ was arbitrary, the proof is concluded. □

We can also show that every minimal cluster is bounded.

Proposition 5. *Let $\mathcal{E}$ be a minimal cluster. Then $\mathcal{E}$ is bounded.*

In order to proof this proposition, the following technical result is needed. The result is true in a wide generality, and its proof in the case of minimal clusters is not particularly difficult.

Lemma 6. *Let $\mathcal{E}$ be any cluster. There exist three positive constants R, $\bar\varepsilon$, $C>0$ such that the following holds. Let some constants $-\bar\varepsilon<\varepsilon_j<\bar\varepsilon$ for $1\leq j\leq M$ be given. Then, there exists another cluster $\mathcal{E}'$ such that $E_j\setminus B_R=E'_j\setminus B_R$, that is, the two clusters differ only inside the ball of radius R centered at the origin, and moreover*

$$|E'_j|=|E_j|+\varepsilon_j\quad\forall\,1\leq j\leq M\,,\qquad\qquad P(\mathcal{E}')\leq P(\mathcal{E})+C\bigg(\sum_{i=1}^M|\varepsilon_i|\bigg)\,. \tag{9}$$

Proof (of Proposition 5). Let us call again $F = \cup_{i=1}^M E_i$, and let $R, \bar\varepsilon, C > 0$ be given by Lemma 6. For every $t > R$, let us call

$$\varphi(t) = \big|F \setminus B_t\big|\,,$$

where B_t is the ball with radius t centered at the origin. For every $1 \leq j \leq M$, let us call $E_j^{\rm ext} = E_j \setminus B_t$, and $\varepsilon_j = |E_j^{\rm ext}|$, so that the cluster $\mathcal{E}^{\rm ext}$ satisfies $|\mathcal{E}^{\rm ext}| = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)$. By construction, $\sum \varepsilon_j = \varphi(t)$. If t is big enough, then $\varphi(t) < \bar\varepsilon$. As a consequence, applying Lemma 6 we can find a cluster $\mathcal{E}'$ such that (9) holds. For every $1 \leq j \leq M$, let us then call $E_j'' = E_j' \cap B_t$. By construction, $\mathcal{E}''$ is a cluster satisfying $|\mathcal{E}''| = |\mathcal{E}|$. Being $\mathcal{E}$ a minimal cluster, also by (9) we get then

$$P(\mathcal{E}) \leq P(\mathcal{E}'') \leq P(\mathcal{E}) + C\varphi(t) - P(\mathcal{E}^{\rm ext}) + \mathscr{H}^{N-1}(F \cap \partial B_t)\,. \tag{10}$$

Let us now notice that $\mathscr{H}^{N-1}(F \cap \partial B_t) = -\varphi'(t)$. Moreover, by (2) we have

$$P(\mathcal{E}^{\rm ext}) \geq J(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M) \geq J(\varphi(t), 0, \ldots, 0) = N\omega_N^{1/N}\varphi(t)^{\frac{N-1}{N}}\,.$$

From the estimate (10) we deduce then

$$-\varphi'(t) \geq N\omega_N^{1/N}\varphi(t)^{\frac{N-1}{N}} - C\varphi(t)\,.$$

SInce φ is a decreasing positive function which converges to 0 when $t \to \infty$, for t large enough we deduce

$$-\varphi'(t) \geq \frac{N\omega_N^{1/N}}{2}\,\varphi(t)^{\frac{N-1}{N}}\,,$$

and since $\frac{N-1}{N} < 1$ we further obtain that φ vanishes in finite time, that is, there exists some $\bar t > 0$ such that $\varphi(\bar t) = 0$. This precisely means that $\mathcal{E}$ is bounded. □

Let us now concentrate ourselves on the question whether some given $m \in (\mathbb{R}^+)^M$ is irreducible or not. In positive case, Proposition 4 ensures that there is a minimal cluster of volume m, hence Theorem 1 is proved. In negative case, instead, there are some nonzero vectors m', m'' such that $m' + m'' = m$ and $\mathfrak{J}(m) = \mathfrak{J}(m') + \mathfrak{J}(m'')$. If both m' and m'' are irreducible, then there are a minimal cluster of volume m' and another minimal cluster of volume m''. Since we can assume the two clusters to have empty intersection, because they are both bounded by Proposition 5, their union is a minimal cluster of volume m, hence Theorem 1 is again proved. As a consequence, in some sense we only have to avoid that this argument has to be repeated infinitely many times. Let us be more precise.

Definition 7. *Let $m^1, m^2, \ldots, m^K$ be vectors in $(\mathbb{R}^+)^M$ such that $|m^1| \geq |m^2| \geq \cdots \geq |m^K|$, and let $m = \sum_{\ell=1}^K m^\ell$. We say that $(m^1, m^2, \ldots, m^K)$ is a* reduction of *m if*

$$\mathfrak{J}(m) = \sum_{\ell=1}^K \mathfrak{J}(m^\ell)\,.$$

Moreover, we say that the reduction $(m^1, m^2, \ldots, m^K)$ is balanced *if $K = 1$ or*

$$|m^{K-1}| \geq \frac{|m^1|}{2}\,.$$

Lemma 8. *Let $m \in (\mathbb{R}^+)^M$ be a vector. If $(m^1, m^2, \dots, m^K)$ is a reduction of m, then there exists also a balanced reduction $(\widetilde{m}^1, \widetilde{m}^2, \dots, \widetilde{m}^H)$ of m with $\widetilde{m}^1 = m^1$.*

Proof. If $K < 3$ or $K \geq 3$ and $|m^{K-1}| \geq |m^1|/2$, then the original reduction is already balanced. Otherwise, let us define a reduction $(\widehat{m}^1, \widehat{m}^2, \dots, \widehat{m}^{K-1})$ as follows. The $K-1$ vectors $\widehat{m}^\ell$ coincide with the $K-2$ vectors m^ℓ for $1 \leq \ell \leq K-2$ together with the vector $m^{K-1} + m^K$, and they are numbered so that $|\widehat{m}^1| \geq |\widehat{m}^2| \geq \cdots \geq |\widehat{m}^{K-1}|$. We have that $\widehat{m}^1 = m^1$ since by construction

$$|m^{K-1}| + |m^K| \leq 2|m^{K-1}| < |m^1|\,.$$

Moreover, by (1) we have

$$\mathfrak{J}(m) = \sum_{\ell=1}^{K} \mathfrak{J}(m^\ell) \geq \sum_{\ell=1}^{K-2} \mathfrak{J}(m^\ell) + J(m^{K-1} + m^K) = \sum_{\ell=1}^{K-1} \mathfrak{J}(\widehat{m}^\ell) \geq \mathfrak{J}(m)\,,$$

so $(\widehat{m}^1, \widehat{m}^2, \dots, \widehat{m}^{K-1})$ is actually a reduction. With an obvious recursion, after at most $K-2$ steps we obtain a balanced reduction. □

Lemma 9. *For every reduction $(m^1, m^2, \dots, m^K)$ of any vector $m \in (\mathbb{R}^+)^M$ one has*

$$|m^1| \geq \frac{1}{M}\,|m|\,.$$

Proof. A possible cluster of volume m is given by M disjoint balls of volumes $m_1, m_2, \dots, m_M$. Hence,

$$\mathfrak{J}(m) \leq N\omega_N^{1/N}\Big(m_1^{\frac{N-1}{N}} + m_2^{\frac{N-1}{N}} + \cdots + m_M^{\frac{N-1}{N}}\Big)\,.$$

On the other hand, for every reduction $(m^1, m^2, \dots, m^K)$ of m by (2) we have

$$\mathfrak{J}(m) = \sum_{\ell=1}^{K} \mathfrak{J}(m^\ell) \geq N\omega_N^{1/N} \sum_{\ell=1}^{K} |m^\ell|^{\frac{N-1}{N}} \geq \frac{N\omega_N^{1/N}}{|m^1|^{1/N}} \sum_{\ell=1}^{K} |m^\ell| = \frac{N\omega_N^{1/N}|m|}{|m^1|^{1/N}}\,.$$

Putting together the last two estimates we get

$$|m^1| \geq |m|\,\frac{|m|^{N-1}}{\Big(m_1^{\frac{N-1}{N}} + m_2^{\frac{N-1}{N}} + \cdots + m_M^{\frac{N-1}{N}}\Big)^N}\,,$$

which concludes the proof since by concavity we have

$$\frac{\Big(m_1^{\frac{N-1}{N}} + m_2^{\frac{N-1}{N}} + \cdots + m_M^{\frac{N-1}{N}}\Big)^N}{|m|^{N-1}} \leq M\,.$$

□

Corollary 10. *A balanced reduction $(m^1, m^2, \dots, m^K)$ of any vector $m \in (\mathbb{R}^+)^M$ has at most length $K = 2M - 1$.*

Lemma 11. *Let $m \in (\mathbb{R}^+)^M$ be a given vector. Then, there is a reduction $(m^1, m^2, \dots, m^K)$ of m such that m^1 is irreducible.*

Proof. Let us define

$$\eta = \inf\left\{|m^1| :\ (m^1, m^2, \dots, m^K) \text{ is a reduction of } m\right\},$$

and notice that $\eta \geq |m|/M$ by Lemma 9. We aim to prove that η is in fact a minimum.

To do so, by Lemma 8 we can take a sequence of balanced reductions for which $|m^1|$ converges to η. Keeping in mind Corollary 10, and calling $K = 2M - 1$, this means that there are reductions $(m^{1,n}, m^{2,n}, \dots, m^{K,n})$ of m with $n \in \mathbb{N}$ such that $|m^{1,n}| \to \eta$ for $n \to \infty$. By compactness and by the continuity of $\mathfrak{J}$, we immediately deduce that there is a reduction $(\overline{m}^1, \overline{m}^2, \dots, \overline{m}^K)$ of m for which $|\overline{m}^1| = \eta$.

To complete the proof, we have to show that $\overline{m}^1$ is necessarily irreducible. More precisely, since the order or vectors with the same norm is not fixed, we aim to show that there is some irreducible $\overline{m}^\ell$ with $|\overline{m}^\ell| = \eta$. In fact, if none of the vectors $\overline{m}^\ell$ with $|\overline{m}^\ell| = \eta$ is irreducible, substituting each of them with a non-trivial reduction provides another reduction of m having all vectors of norm strictly smaller than η, against the definition of η. □

We are finally in position to prove our main result.

Proof (of Theorem 1). Let us fix the vector $m \in (\mathbb{R}^+)^M$. By Lemma 11 and Lemma 9, we can find a sequence of irreducible vectors $m^n \in (\mathbb{R}^+)^M$ such that

$$m = \sum_{n\in\mathbb{N}} m^n, \qquad \mathfrak{J}(m) = \sum_{n\in\mathbb{N}} \mathfrak{J}(m^n). \tag{11}$$

By Proposition 4, we find a minimal cluster for each volume m^n, that is, a cluster $\mathcal{E}^n$ with $|\mathcal{E}^n| = m^n$ and $P(\mathcal{E}^n) = \mathfrak{J}(m^n)$. Since all the clusters $\mathcal{E}^n$ are bounded by Proposition 5, up to translations we can assume them to be pairwise disjoint. And finally, calling $\mathcal{E}$ the cluster obtained by the union of the clusters $\mathcal{E}^n$, by (11) we have that $\mathcal{E}$ is a cluster of volume m and with perimeter $\mathfrak{J}(m)$, hence a minimal cluster for mass m. The proof is then concluded. □

Remark 12. *One Theorem 1 has been proved, it is actually not hard to observe that every volume is irreducible. In fact, otherwise we could find a minimal cluster which is not connected. And in turn, it is easy to see how to lower the perimeter for such a cluster: the idea is to bring two different connected components very close to each other, so that there are two small pieces of the boundaries close to each other and more or less parallel. And then, it is enough to "glue these two pieces together", strictly lowering the total perimeter.*