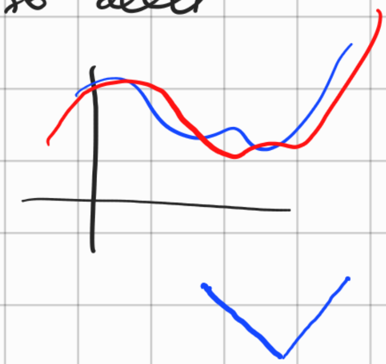
ANALISI MATEMATICA 2
LEZIONE 25 — 25.11.2025

Stone–Weierstrass

- i polinomi sono "densi" nello spazio delle funzioni continue (sui compatti)



PIÙ PRECISAMENTE:

Teo X sp. metrico compatto,

$C(X) = C^0(X,\mathbb{R}) = \{ f : X \to \mathbb{R} \text{ continue} \}$ (spazio vettoriale)

Sia $\mathcal{A} \subseteq C(X)$ tale che:

(algebra)
- (i) $f,g \in \mathcal{A}$, $\alpha,\beta \in \mathbb{R} \Rightarrow \alpha f + \beta g \in \mathcal{A}$
- (ii) $f,g \in \mathcal{A} \Rightarrow f \cdot g \in \mathcal{A}$

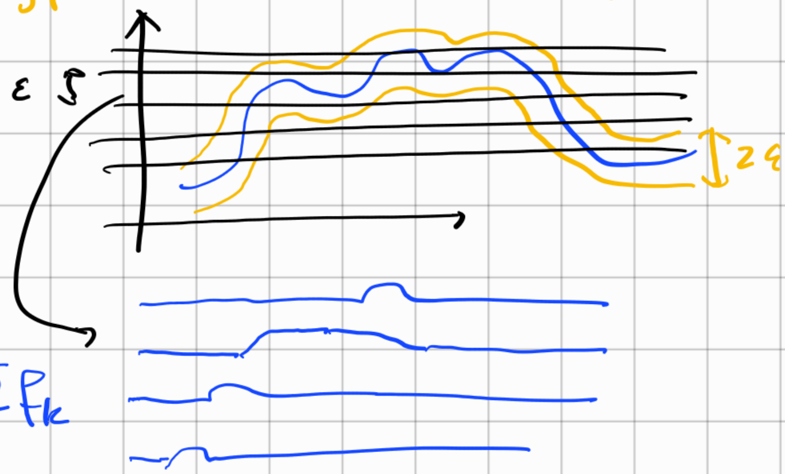
(contiene le costanti) (iii) $1 \in \mathcal{A}$

(separa i punti) (iv) $\forall x_1, x_2 \in X$, $x_1 \neq x_2$ $\exists f \in \mathcal{A}$ tale che $f(x_1) \neq f(x_2)$.

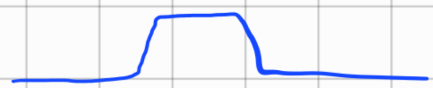
$\forall \varepsilon > 0$ $\forall f \in C(X)$ $\exists g \in \mathcal{A}$ t.c. $\| f - g \|_\infty < \varepsilon$.

($\mathcal{A}$ è denso in $C(X)$ rispetto alla distanza uniforme)

idea della dimostrazione:



$f = \sum f_k$



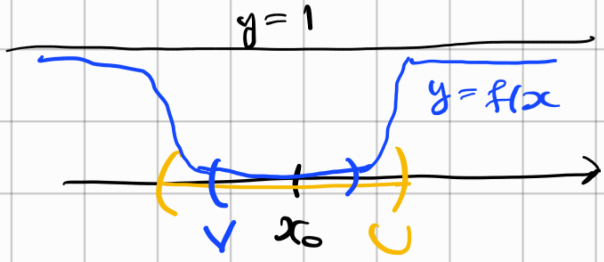
Vorrei poter approssimare funzioni di questo genere

Lemma Dato $x_0 \in X$, dato U intorno aperto di x_0 esiste V intorno di x_0, $V \subseteq U$ tale che $\forall \varepsilon > 0$ $\exists f \in \mathcal{A}$ tale che:

(1) $0 \le f(x) \le 1$

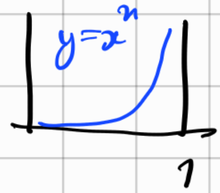
(2) $f(x) < \varepsilon \quad \forall x \in V$

(3) $f(x) > 1-\varepsilon \quad \forall x \in X \setminus U$



dim

$\forall x \in X \setminus U \quad \exists g_x \in \mathcal{A}$ t.c. $g_x(x) \neq g_x(x_0)$



$h_x = g_x - g_x(x_0) \cdot 1 \in \mathcal{A} \qquad h_x(x) \neq h_x(x_0) = 0$

$p_x = \dfrac{h_x^2}{\|h_x\|_\infty^2} \in \mathcal{A} \quad p_x(x_0) = 0, \quad p_x(x) > 0,$

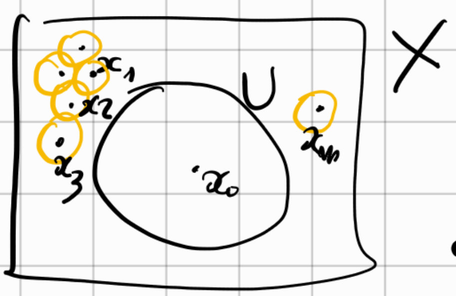
$0 \le p_x \le 1$

$U_x = \{ y \in X : p_x(y) > 0 \}$ è un intorno di x.

$X \setminus U$ è un compatto ricoperto da $\bigcup_{x \in X \setminus U} U_x$

(K compatto (metrico), $C \subseteq K$ è compatto $\Leftrightarrow$ C è chiuso)

Posso estrarre un sottoricoprimento finito: esistono $x_1, \dots, x_m \in X \setminus U$ t.c. $X \setminus U \subseteq U_{x_1} \cup \dots \cup U_{x_m}$



$p = \dfrac{p_{x_1} + \dots + p_{x_m}}{m} \in \mathcal{A}$

$0 \le p \le 1, \quad p(x_0) = 0, \quad p(x) > 0 \quad \forall x \in X \setminus U$

$X \setminus U$ è compatto $\Rightarrow$ (Weierstrass) esiste $\delta > 0$ tale che

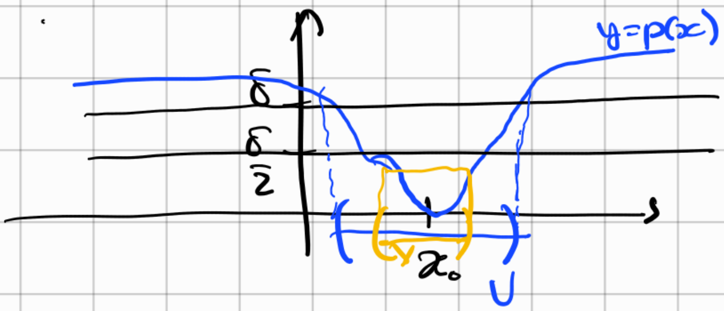
$$p(x) \geq \delta \quad \text{su } X \setminus U \qquad \left(\delta = \min_{X\setminus U} p\right)$$

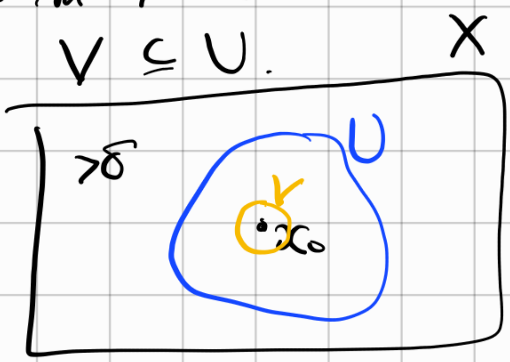
$$0 < \delta \leq 1.$$

$$V = \left\{x \in X : p(x) < \frac{\delta}{2}\right\} \qquad p(x_0) = 0$$

V è un intorno di x_0, $p(x) \geq \delta$ su $X \setminus U$

$\Rightarrow V \subseteq U$.





Sia $k \in \mathbb{N}$ t.c. $\frac{1}{\delta} < k \leq \frac{1}{\delta} + 1 = \frac{\delta + 1}{\delta} < \frac{2}{\delta}$

$$1 < k\delta < 2$$

$$q_n(x) = \left[1 - p^n(x)\right]^{k^n} \in \mathcal{A} \qquad 0 \leq q_n \leq 1 \qquad q_n(x_0) = 1.$$

Bernoulli $\left[(1+t)^n \geq 1 + nt,\ t \geq -1\right]$

$\forall x \in V \quad k \cdot p(x) \leq k\frac{\delta}{2} < 1 \quad \Rightarrow \quad q_n(x) \geq 1 - k^n p^n(x)$

$$\geq 1 - \left(k\frac{\delta}{2}\right)^n \to 1$$

cioè $q_n \rightrightarrows 1$ su V

mentre se $x \in X \setminus U \qquad kp \geq k\delta > 1$

$$q_n(x) = \left[1 - p^n(x)\right]^{k^n} \leq \frac{1 + k^n p^n(x)}{k^n p^n(x)} \left[1 - p^n(x)\right]^{k^n}$$

Bernoulli
$$\leq \frac{\left(1 + p^n(x)\right)^{k^n}}{k^n p^n(x)} \left[1 - p^n(x)\right]^{k^n}$$

$$= \frac{[1-p^{2n}(x)]^{k^n}}{k^n p^n(x)} \le \frac{1}{k^n p^n(x)} \le \frac{1}{(k\delta)^n} \longrightarrow 0$$

dunque $q_n \rightrightarrows 0$ su $X \setminus U$

Prendendo $n \gg 1$ $\quad q_n < \varepsilon$ su $X \setminus U$

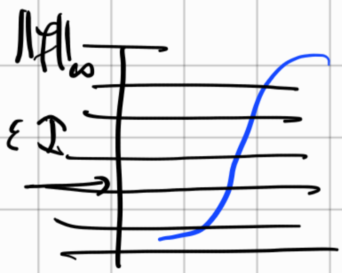
e $q_n > 1-\varepsilon$ su V

$f = 1 - q_n$ soddisfa le proprietà richieste $\square$

<u>dim</u> (Teo)

Sia $f \in C(X)$, sia $\varepsilon > 0$.

Pur di prendere $f + \|f\|_\infty$ posso supporre $f \geq 0$. Supponiamo $\varepsilon < \frac{1}{3}$.



Prendiamo n $\quad (n-1)\varepsilon \geq \|f\|_\infty$

Per $j = 0, \dots, n$ definiamo:

$$A_j = \{x \in X : \quad f(x) \leq (j - \tfrac{1}{3})\varepsilon\}$$

$$B_j = \{x \in X : \quad f(x) \geq (j + \tfrac{1}{3})\varepsilon\}$$

$A_j \cap B_j = \emptyset$, A_j, B_j chiusi

$$\emptyset \subseteq A_0 \subseteq A_1 \subseteq \dots \subseteq A_n = X$$

$$B_0 \supseteq B_1 \supseteq \dots \supseteq B_n = \emptyset$$

(lemma) (2) (PROSSIMA LEZIONE) $\Rightarrow$ $\exists\ g_j \in \mathcal{A}$ t.c. $0 \leq g_j \leq 1$ t.c.

$$g_j < \frac{\varepsilon}{n} \text{ su } A_j \quad e \quad g_j > 1-\frac{\varepsilon}{n} \text{ su } B_j$$

$$g = \varepsilon \sum_{i=0}^{n} g_i \in \mathcal{A} \ : \quad \forall x \in X \quad \underline{\exists j} \quad x \in A_j \setminus A_{j-1}$$

$\Downarrow$ definizione di A_j

$i > j \qquad g_i < \varepsilon/n$

$$\left(j - \frac{4}{3}\right)\varepsilon < f(x) \leq \left(j - \frac{1}{3}\right)\varepsilon$$

$\forall i \leq j-2 \quad x \in B_i \Rightarrow g_i(x) > 1 - \frac{\varepsilon}{n}$

$$\begin{aligned} g(x) &= \varepsilon \sum_{i=0}^{j-1} g_i(x) + \varepsilon \sum_{i=j}^{n} g_i(x) \\ &\leq j\varepsilon + \varepsilon (n-j+1)\frac{\varepsilon}{n} \\ &\leq j\varepsilon + \varepsilon^2 < \left(j+\frac{1}{3}\right)\varepsilon \end{aligned}$$

$$\begin{aligned} g(x) &\geq \varepsilon \sum_{i=0}^{j-2} g_i(x) \geq (j-1)\varepsilon\left(1-\frac{\varepsilon}{n}\right) \\ &= (j-1)\varepsilon - \frac{j-1}{n}\varepsilon^2 > \dots > \left(j-\frac{4}{3}\right)\varepsilon \end{aligned}$$

$$|f(x) - g(x)| \leq \left(j+\frac{1}{3}\right)\varepsilon - \left(j-\frac{4}{3}\right)\varepsilon < 2\varepsilon$$

$\square$