LEZIONE 10.01.25

$\alpha \in \mathbb{R}$ $\qquad x \to 0$ Polinomio di Tylor?

$f(x) = (1+x)^\alpha$

$f'(x) = \alpha(1+x)^{\alpha-1}$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$ $\qquad$ Il prodotto di T. di ordine n centrato in $x_0 = 0$

$\vdots$

$f^{(k)}(x) = \overbrace{\alpha(\alpha-1)\dots(\alpha-k+1)}^{k \text{ fattori}}(1+x)^{\alpha-k}$

$P(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{k!}x^2 + \dots + \binom{\alpha}{n}x^n$

$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$

Esempio:

$\alpha = -1$ $\qquad f(x) = (1+x)^{-1} = \frac{1}{1+x}$

$\binom{-1}{k} = \frac{(-1)(-1-1)\cdot(-1-2)\cdots(-1-k+1)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k$

$P(x) = 1 - x + x^2 - x^3 \dots + (-1)^n x^n$

E' sensato!

$\sum_{k=0}^{\infty}(x)^k = \frac{1}{1+x}$

Altro esempio (caso $\alpha = \frac{1}{2}$):

$\alpha = \frac{1}{2}$ $\qquad f(x) = (1+x)^{\frac{1}{2}} = \sqrt{1+x}$

$\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!} = \frac{\overbrace{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdots -\frac{2k+3}{2}}^{k \text{ fattori}}}{k!} = \frac{(-1)^{k-1}(2k-3)!!}{2^k k!} = \frac{(-1)^{k-1}(2k-3)!!}{(2k)!!}$

$P(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots + (-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}x^n$

Esercizio: Scrivere i primi termini dello sviluppo di: $\sqrt{\cos x} = \sqrt{1 - \frac{x^2}{2} + o(x^2)} = \dots$

SVILUPPI DI TAYLOR DELLE FUNZIONI ELEMENTARI

$P(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

- $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}$
- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!}$
- $\sin x = x - \frac{x^3}{3!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $\ln(1+x)$ $\qquad$ P. Taylor?

$x \neq ?$

$e^{1+x} = e \cdot e^x$

$\cos(1+x) = \cos x - \sin 1 \cdot \sin x$

oss.

Polinomio $\qquad$ Ordine

$f \to P \qquad n$

$f' \to P' \qquad n-1$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \rightsquigarrow P'(x) = 1 - x + x^2 + \dots$

$x^\alpha \longrightarrow \alpha x^{\alpha-1}$ $\qquad P(x) = \cancel{f(0)} + x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1}\frac{x^n}{n}$ $\quad (f(0) = 0)$

$\frac{x^{\alpha+1}}{\alpha+1} \longrightarrow x^\alpha$

- $f(x) = \mathrm{arctg}(x)$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = \underbrace{1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n}} + o(x^{2n})$

$\Downarrow$

E' il P. di Taylor di $f'(x)$ di

ordine $2n$ centrato in 0

$P(x) = \cancel{f(0)} + x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^n\frac{x^{2n+1}}{2n+1}$ $\quad (f(0) = 0)$

- $f(x) = \arcsin(x)$

$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 - \frac{3}{8}x^4 + \dots + \frac{(-1)^{n+1}(2n-1)!!}{(2n)!!}x^{2n} + o(x^{2n})$

$\binom{-1/2}{k} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-k+1\right)}{k!} = \frac{(-1)^k\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2k-1}{2}}{k!} = \frac{(-1)^k(2k-1)!!}{(2k)!!}$

$P(x) = x + \frac{x^2}{6} - \frac{3x^5}{40} + \dots + \frac{(-1)^{n+1}}{2n+1}\frac{(2n-1)!!}{(2n)!!} \cdot x^{2n+1}$

- $f(x) = \arccos(x) = \pi - \arcsin(x)$
- $f(x) = \mathrm{tg}(x)$

per $x \to 0$ $\qquad \mathrm{tg}(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5)$

$t = \mathrm{tg}(x) \xrightarrow{t=0} 0$ $\qquad \frac{2}{3!}$

$t' = 1 + t^2 \longrightarrow 1$

$t'' = 2t(1+t^2) \longrightarrow 0$

$t''' = 2(1+t^2)^2 + 4t^2(1+t^2) \longrightarrow 2$

$\vdots$

Teorema (formula di Taylor con resto di Lagrange)

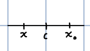
Sia f... $\qquad$ Sia P pol. di Taylor di f centrato in x_0 di ordine n

$f(x) = P(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$

$\forall x \neq x_0$

$\exists c \in (x_0, x)$

$c = c(x)$



Dimostrazione (per induzione su m)

Devo mostrare che:

$\forall x \in [\quad \exists c \in (x_0, x) \quad \frac{f(x) - P(x)}{(x - x_0)^{m+1}} = \frac{f^{(m+1)}(c)}{m}$
$x \neq x_0$

Per $m = 0 \quad \exists c \in (x_0, x)$ t.c. $\frac{f(x) - P(x)}{x - x_0} = \frac{f'(c)}{(m+1)!}$

È il teorema di Lagrange applicato a $f - P$ perché $f(x_0) - P(x_0) = 0$

- Passo induttivo: $(m \Rightarrow m+1)$ $\quad \exists c \in (x_0, x) \quad \exists d \in (x_0, c)$

$$\frac{f(x) - P(x)}{(x - x_0)^{m+2}} = \frac{f(x) - P(x) - f(x_0) - P(x_0)}{(x - x_0)^{m+2} - (x_0 - x_0)^{m+2}} \overset{\text{Cauchy}}{=} \frac{f'(c) - P'(c)}{(m+2)(c - x_0)^{m+1}} \overset{\text{Ipotesi induttiva}}{=} \frac{1}{(m+2)} \frac{(f')^{(m+1)}(d)}{(m+1)!} = \frac{f^{(m+2)}(d)}{(m+2)!}$$

$\square$

Esercizio

$\boxed{x \leq 1}$

$$\sum_{k=1}^{\infty} \frac{(-x)^{k-1}}{k} = \ln(1 + x)$$