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On an isoperimetric problem with a competing non-local term: quantitative results

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Abstract This paper provides a quantitative version of the recent result of Knüpfer and Muratov (Commun Pure Appl Math 66:1129–1162, 2013) concerning the solutions of an extension of the classical isoperimetric problem in which a non-local repulsive term involving Riesz potential is present. In that work, it was shown that in two space dimensions the minimizer of the considered problem is either a ball or does not exist, depending on whether or not the volume constraint lies in an explicit interval around zero, provided that the Riesz kernel decays sufficiently slowly. Here, we give an explicit estimate for the exponents of the Riesz kernel for which the result holds.

Keywords Geometric variational problems · Competing interactions · Global minimizers · Gamow's model

1 Introduction

This paper is concerned with a study of the following non-local extension of the classical isoperimetric problem: minimize the energy

$$E(F) = P(F) + V(F), \tag{1}$$

among all Lebesgue measurable sets $F \subset \mathbb{R}^n$, $n \geq 2$, subject to the constraint $|F| = m$, with some $m > 0$ fixed. Here, $P(F)$ is the perimeter of the set F in the sense of De Giorgi [3]:

$$P(F) := \sup \left\{ \int_F \nabla \cdot \phi \, \mathrm{d}x \, : \, \phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \quad |\phi| \leq 1 \right\}, \tag{2}$$

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and $V(F)$ describes a non-local repulsive interaction:

$$V(F) := \int_F \int_F \frac{1}{|x-y|^\alpha} \, dx \, dy, \tag{3}$$

for some $\alpha \in (0, n)$. Minimizers of this problem will be referred to as "minimizers of E with mass m" in the rest of the paper.

The variational problem above arises in a number of contexts of mathematical physics, most notably in the case of $\alpha = 1$ and $n = 3$, when it corresponds to the classical Gamow's liquid drop model of an atomic nucleus [5,6,16]. Note that the case $\alpha = 1$ is of particular importance, since it corresponds to the repulsive Coulombic interaction and is, therefore, also relevant to a wide variety of other physical situations, both in three- and two-space dimensions (see the discussion in [19] for further details). In particular, during their studies of a closely related Ohta-Kawasaki energy functional of diblock copolymer systems [21], Choksi and Peletier [11] asked if the minimizer of the above problem with $n = 3$ and $\alpha = 1$ is a ball whenever it exists (see also [10] for an overview of the problem and recent results). The answer to this question is not obvious at all, since the two terms in the energy in (1) are in direct competition with each other: while the first term tends to put the mass together, the second term favors spreading the mass apart as much as possible. As a consequence, a single ball is no longer a minimizer if the value of m becomes sufficiently large, since for large enough values of m it is advantageous to split the ball into two balls of equal volume and move the resulting smaller balls far apart. A far more difficult question, however, is whether the ball has the best *shape* among all competitors for the energy minimizers at a given $m > 0$.

A rather detailed study of the variational problem described above was recently performed by Knüpfer and Muratov [18,19] (see also [2,7,12,13,17,20,23,25] for some related recent work). Some basic existence and non-existence properties of the minimizers of the considered variational problem were established, together with the more detailed information about the shape of the minimizers in certain parameter regimes [18,19]. We summarize those findings under a simplifying assumption of $n \le 3$, corresponding to the spatial dimensionality of the problems of physical interest. In this case, it is known that a minimizer of E with mass m exists for all $m \le m_1$, for some $m_1 > 0$ depending on α and n. If, furthermore, $\alpha < 2$, then there exist $m_0 > 0$ and $m_2 > 0$ depending on α and n such that the minimizer is a ball whenever $m \le m_0$, and minimizers do not exist whenever $m > m_2$.

Clearly, if $\alpha < 2$ and $n \le 3$, we have $0 < m_0 \le m_1 \le m_2 < \infty$. However, it is not obvious whether one could choose $m_0 = m_1 = m_2$, indicating that minimizers are balls whenever they exist. A gap between the values of m_0 and m_1 would indicate existence of non-radial minimizers for certain values of m. Similarly, a gap between the values of m_1 and m_2 would indicate that the set of values of m for which minimizers exist is not a bounded interval around the origin. Both these possibilities would yield a negative answer to the question of Choksi and Peletier for the problem under consideration. Moreover, even if $m_0 = m_1 = m_2$, it is not yet clear that those values are equal to $m_{c1} > 0$, the value of m at which one ball of mass m has the same energy as two balls of mass $\frac{1}{2}m$ infinitely far apart, which is what one would expect if the splitting mechanism into two equal size balls was the dominant mechanism for reducing the energy at large masses.

Despite a general lack of understanding of the global structure of the energy minimizers in the considered problem, some partial results currently exist in the case of sufficiently slowly decaying kernels in (3) [7,18]. In [18], Knüpfer and Muratov, showed that when $n = 2$ there exists a universal constant $\alpha_0 > 0$ such that the minimizer of the considered problem is a ball whenever it exists, and one can choose $m_0 = m_1 = m_2$ for all $\alpha < \alpha_0$. This result was recently extended by Bonacini and Cristoferi [7] to higher dimensions, who also analyzed local minimality of balls and gave a rather precise characterization of minimizing sequences for $\alpha < \alpha_0$. At the same time, it was also shown in [18] that when $n = 2$ and $\alpha \leq \alpha_0$, one can, in fact, choose $m_0 = m_1 = m_2 = m_{c1}$, where the value of m_{c1} is explicitly given by

$$m_{c1}(\alpha) = \pi \left(\frac{\left(\sqrt{2} - 1\right) \Gamma\left(2 - \frac{\alpha}{2}\right) \Gamma\left(3 - \frac{\alpha}{2}\right)}{\pi \left(1 - 2^{\frac{\alpha-2}{2}}\right) \Gamma(2 - \alpha)} \right)^{\frac{2}{3-\alpha}}, \quad (4)$$

where $\Gamma(z)$ is the Euler Gamma function [1], confirming the global bifurcation picture suggested by Choksi and Peletier [10,11] in the case of $n = 2$ and α sufficiently small.

Whether such a picture remains valid for all $n \geq 2$ and all $\alpha \in (0, n)$ is still far from clear. In particular, as a starting point, it would be interesting to know if one could choose $\alpha_0 = 2$ for $n = 2$ in [18, Theorem 2.7]. At the very minimum, such a result would require a quantitative version of the analysis of [18]. The goal of this paper is to provide such an analysis. Here, is our main result, which gives the following quantitative version of this theorem.

Theorem 1.1 *Let $n = 2$, let $\alpha \leq 0.034$, and let m_{c1} be given by* (4)*. Then, minimizers of E with mass m exist if and only if $m \leq m_{c1}$, and every minimizer of E is a disk of radius $\sqrt{m/\pi}$.*

The proof of Theorem 1.1 mostly follows along the lines of [18], while keeping track of the constants appearing in all the estimates. We recall that the strategy in proving [18, Theorem 2.7] was to demonstrate, for all $m \leq M$ with $M > 0$ fixed, that for small enough $\alpha > 0$ depending only on M the minimizer, if it exists, must be a convex set which is only a small perturbation of the disk of radius $\sqrt{m/\pi}$ in the Hausdorff sense. This is achieved by combining the quantitative version of the isoperimetric inequality with suitable a priori upper bounds for the energy, together with a careful analysis of the rigidity of disks with respect to small perturbations. This result is then combined with a non-existence result for minimizers with $m > M$ for some $M > 0$, which is uniform in $\alpha \ll 1$. Inevitably, this strategy is guaranteed to work only for sufficiently small values of α. Yet, it is rather surprising that we were only able to prove our result for such a narrow range of values of α, despite our attempts to strive for the best constants in the analysis wherever possible. Perhaps this is an indication that the global bifurcation structure of the considered variational problem may be more complex, and further non-perturbative studies of the problem are needed. Let us also mention that since our methods rely in essential ways on the two-dimensional character of the problem (e.g., via bounds on the diameter of sets of finite perimeter by the perimeter or via the use of Bonnesen inequality), they cannot be readily extended to higher-dimensional problems. For the latter, however, one should be able to use the available quantitative results of the regularity theory for quasi-minimizers of the perimeter to obtain similar lower bounds for the value of α_0 in higher dimensions as well (see, e.g., [14]).

The rest of the paper is organized as follows. In Sect. 2, we summarize several facts about minimizers of E with mass m and state a few technical facts. In Sect. 3, we derive a tight upper bound on the minimal energy that scales linearly with m for large masses. In Sect. 4, we give a quantitative version of the non-existence result for large masses. In Sect. 5, we give a quantitative criterion about when minimizers are balls whenever they exist. Finally, in Sect. 6, we put all the obtained estimates together and prove the main Theorem. In this section, we also give a numerical estimate of the value of $\alpha_0 \approx 0.04273$ that is slightly better than our analytical estimate.

Throughout the paper, all the constants may depend implicitly on α and ε, a parameter related to m that appears in Sect. 5. These dependences will be suppressed whenever it does not cause ambiguity to simplify the notation. The algebraic computations were performed with the help of MATHEMATICA 8.0 software.

2 Preliminaries

We start by collecting some basic facts about the considered variational problem. Even if we stated the original problem in general spatial dimensionality, in two space dimensions the minimization problem is equivalent to minimizing E among open sets with a C^1 boundary. This is because of the basic regularity property that minimizers of E inherit from being quasi-minimizers of the perimeter (see, e.g., [4,24]). We have the following basic regularity result for the minimizers of E (in the rest of the paper, we always assume that $n = 2$).

Proposition 2.1 ([18], Proposition 2.1) *Let $m > 0$ and let Ω be a minimizer of E among all open sets with boundary of class C^1 and $|\Omega| = m$. Then*

(i) *$\partial\Omega$ is of class $C^{2,\beta}$, for some $\beta \in (0, 1)$, with the regularity constants depending only on m and α.*
(ii) *Ω is bounded, connected and contains at most finitely many holes.*
(iii) *$\partial\Omega$ satisfies the Euler–Lagrange equation*

$$\kappa(x) + 2v(x) - \mu = 0, \qquad v(x) := \int_\Omega \frac{1}{|x-y|^\alpha}\,\mathrm{d}y, \tag{5}$$

where $\kappa(x)$ is the curvature (positive if Ω is convex) at $x \in \partial\Omega$ and $\mu \in \mathbb{R}$.

Note that since Ω in Proposition 2.1 is connected, we also have the following elementary bound:

$$\mathrm{diam}(\Omega) \le \frac{1}{2}P(\Omega), \tag{6}$$

which will be repeatedly used throughout our paper.

Concerning the minimizers of E (in a wider class of sets of finite perimeter, also for any dimension $n \ge 2$ and any fixed $\alpha \in (0, n)$), we know that their existence is guaranteed for all sufficiently small values of m [19, Theorem 3.1]. For $n = 2$, existence of minimizers in the sense of Proposition 2.1 then follows, possibly after a redefinition of Ω on a set of Lebesgue measure zero [19, Proposition 2.1]. In the context of the present paper, however, we have the following quantitative improvement of the existence result in [18].

Proposition 2.2 *Suppose that the minimizer of E with mass m, whenever it exists, is a disk of radius $\sqrt{m/\pi}$ for all $m \le m_0$ with some $m_0 > m_{c1}$, where m_{c1} is defined in (4). Then*

(i) *The minimizer exists and is a disk for all $m \le m_{c1}$.*
(ii) *There is no minimizer for all $m_{c1} < m \le m_0$.*

Proof The proof follows from [18, Lemma 3.6 and Proposition 8.8]. Indeed, suppose that $m \le m_0$. If the minimizer of E with mass m exists, it is a disk by assumption of the proposition. However, by [18, Lemma 3.6] this is not possible if $m > m_{c1}$, yielding the second statement. To prove the first statement, suppose, by contradiction, that there is no minimizer and $m \le m_{c1}$. Then by [18, Proposition 8.8] and the assumption of the proposition, for any measurable set $F \subset \mathbb{R}^2$ with $|F| = m$ there is a set $\tilde{F} = \cup_{i=1}^N B_{R_i}(x_i)$, a union of finitely many disjoint disks, such that $|\tilde{F}| = m$ and $E(\tilde{F}) \le E(F)$. Then, again, repeatedly applying [18, Lemma 3.6], we have $E\left(B_{\sqrt{m/\pi}}(0)\right) < E(\tilde{F})$, indicating that $B_{\sqrt{m/\pi}}(0)$ is a minimizer of E with mass m, a contradiction. □

We will need several additional properties related to disks as test configurations. First, we give an explicit formula for the potential energy of a disk of radius R.

Lemma 2.3 ([18], Corollary 3.5) *We have*

$$E(B_R(0)) = 2\pi R + \frac{2\pi^2\Gamma(2-\alpha)}{\Gamma\left(2-\frac{\alpha}{2}\right)\Gamma\left(3-\frac{\alpha}{2}\right)} R^{4-\alpha}. \tag{7}$$

Next, we introduce the potential associated with the unit disk:

$$v^B(|x|) := \int_{B_1(0)} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y. \tag{8}$$

An explicit computation shows that [18, Lemma 3.8]

$$v^B(r) = \begin{cases} \left(\frac{\pi}{r^\alpha}\right) {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha}{2}; 2; \frac{1}{r^2}\right), & r \geq 1, \\ \left(\frac{2\pi}{2-\alpha}\right) {}_2F_1\left(\frac{\alpha-2}{2}, \frac{\alpha}{2}; 1; r^2\right), & r < 1. \end{cases} \tag{9}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [1]. We also have the following useful properties of v^B.

Lemma 2.4 *We have*

$$v^B(0) = \frac{2\pi}{2-\alpha}. \tag{10}$$

If also $\alpha < 1$, we have $v^B \in C^1([0,\infty))$ and

$$\left|\frac{\mathrm{d}v^B(1)}{\mathrm{d}r}\right| = \max_{r\geq 0}\left|\frac{\mathrm{d}v^B(r)}{\mathrm{d}r}\right| = \frac{\pi\alpha(2-\alpha)\Gamma(1-\alpha)}{2\Gamma^2(2-\frac{\alpha}{2})}. \tag{11}$$

Proof The formula in (10) follows from (9) by noting that ${}_2F_1\left(\frac{\alpha-2}{2}, \frac{\alpha}{2}; 1; 0\right) = 1$. Smoothness of v^B follows from [18, Lemma 3.8]. Finally, to obtain (11), we differentiate the formula in (9) twice with respect to r. We get for $r > 1$:

$$\frac{\mathrm{d}^2 v^B}{\mathrm{d}r^2} = \frac{1}{4}\pi\alpha r^{-\alpha-4}\left(4(\alpha+1)r^2\,{}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+2}{2}; 2; \frac{1}{r^2}\right)\right.$$
$$\left. +\alpha(\alpha+2)\,{}_2F_1\left(\frac{\alpha}{2}+1, \frac{\alpha}{2}+2; 3; \frac{1}{r^2}\right)\right), \tag{12}$$

and for $r < 1$:

$$\frac{\mathrm{d}^2 v^B}{\mathrm{d}r^2} = -\frac{1}{4}\pi\alpha\left(\alpha(\alpha+2)r^2\,{}_2F_1\left(\frac{\alpha}{2}+1, \frac{\alpha}{2}+2; 3; r^2\right) + 4\,{}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+2}{2}; 2; r^2\right)\right). \tag{13}$$

An inspection of these formulas shows that $v^B(r)$ is a concave function for $r < 1$ and a convex function for $r > 1$. Therefore, since $\mathrm{d}v^B(0)/\mathrm{d}r = 0$ and $\mathrm{d}v^B(\infty)/\mathrm{d}r = 0$, the derivative of $v^B(r)$ is negative for all $r > 0$ and reaches its absolute minimum at $r = 1$. The second equality in (11) again follows from [18, Lemma 3.8]. □

Finally, we introduce the isoperimetric deficit of a measurable set $F \subset \mathbb{R}^2$:

$$D(F) := \frac{P(F)}{\sqrt{4\pi|F|}} - 1. \tag{14}$$

The following quantitative version of the isoperimetric inequality due to Bonnesen [8] will be useful (see also [9,15,22]).

Lemma 2.5 ([8]) *Let $F \subset \mathbb{R}^2$ be a convex open set which is bounded. Then, there exists $x_0 \in \mathbb{R}^2$ and r_1, r_2 satisfying $0 < r_1 \leq r_2$ such that $B_{r_1}(x_0) \subseteq F \subseteq B_{r_2}(x_0)$ and*

$$\frac{(r_2 - r_1)^2}{|F|} \leq (2 + D(F))D(F). \tag{15}$$

Note that for $D(F) \ll 1$ the constant in the right-hand side of (15) is optimal [8,9,15,22].

3 An upper bound for the minimal energy

In this section, we derive an ansatz-based upper bound for the minimal energy scaling linearly with m for large m, which will be useful in a number of proofs. Our ansatz consists of n disks of equal mass, spaced arbitrarily far apart. We choose n as a function of m to optimize our bound.

For now, it is convenient to work in terms of $R := \sqrt{m/\pi}$. We shall switch back to m in the final step. Also, we shall focus on bounding the energy per unit area; this will then yield a corresponding energy bound in terms of m. We define the constant

$$V_0(\alpha) := V(B_1(0)) = \frac{2\pi^2\Gamma(2-\alpha)}{\Gamma(2-\frac{\alpha}{2})\Gamma(3-\frac{\alpha}{2})}, \tag{16}$$

which is just the potential energy of a unit ball. Then, $E_1(R) := 2\pi R + V_0(\alpha)R^{4-\alpha}$ denotes the energy of one disk of radius R by Lemma 2.3. We denote the infimum of the energy obtained by splitting the mass $m = \pi R^2$ into n disks of equal radius by $E_n(R) := nE_1(R/\sqrt{n})$. Here, we noted that the non-local interaction between these disks can be made arbitrarily small by translating the disks sufficiently far apart. Finally, we define the corresponding energy per unit area $\rho_n(R) := E_n(R)/(\pi R^2)$. For notational convenience, we let $\rho(R) := \rho_1(R) = E_1(R)/(\pi R^2)$. Note that

$$\rho_n(R) = nE_1(R/\sqrt{n})/(\pi R^2) = \rho(R/\sqrt{n}). \tag{17}$$

To find our upper bound, we characterize the envelope of the graphs of the sequence of functions which are appropriate dilations of $\rho(R)$, the energy per unit area of a single disk of radius R. Three of these functions are illustrated in Fig. 1. From this figure, one may suspect that the envelope can be determined completely by locating the intersections of the adjacent graphs. This is proved in the following sequence of lemmas, which are intended to deal with the elementary, but rather tedious algebra involved.

Lemma 3.1 *For $n = 1, 2, \ldots$, the equation $\rho_n(R) = \rho_{n+1}(R)$ has a unique positive solution given by*

$$R_{cn} := \left(\frac{2\pi\left(\sqrt{n+1}-\sqrt{n}\right)}{V_0\left(n^{\frac{\alpha}{2}-1}-(n+1)^{\frac{\alpha}{2}-1}\right)}\right)^{\frac{1}{3-\alpha}}.$$

Proof Since $R > 0$, we can solve the equivalent equation $E_n(R) = E_{n+1}(R)$. Solving

$$n\left[\frac{2\pi R}{\sqrt{n}} + V_0\left(\frac{R}{\sqrt{n}}\right)^{4-\alpha}\right] = (n+1)\left[\frac{2\pi R}{\sqrt{n+1}} + V_0\left(\frac{R}{\sqrt{n+1}}\right)^{4-\alpha}\right]$$

for R gives the result. □

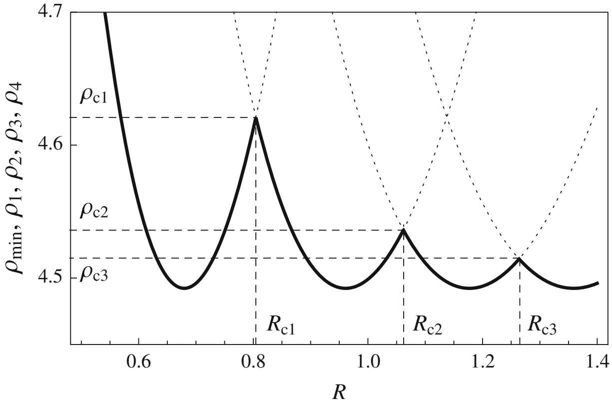

Fig. 1 The energy per unit area $\rho_n(R)$ for $n = 1, 2, 3$ (*dotted lines*), along with the intersection points (R_{cn}, ρ_{cn}) and the minimum $\rho_{\min}(R)$ (*solid line*) for $\alpha = 0.1$

Lemma 3.2 *Suppose the four functions* $f_1, f_2, g_1, g_2 \in C^1([0, 1])$ *satisfy*

1. $f_i(0) = g_i(1) = 0$ for $i = 1, 2$;
2. $0 < f_1'(x) < f_2'(x) \quad \forall x \in (0, 1)$;
3. $0 > g_1'(x) > g_2'(x) \quad \forall x \in (0, 1)$.

Then for $i = 1, 2$ *there exist unique* $x_i \in (0, 1)$ *such that* $f_i(x_i) = g_i(x_i) =: y_i$, *with* $y_1 < y_2$.

Proof Existence of unique x_i for $i = 1, 2$ follows from applying the intermediate value theorem to $h_i := f_i - g_i$. To prove $y_1 < y_2$, first consider $f_1(x)$ and $g_2(x)$. These also have a unique intersection point; call it (x_3, y_3). Define $h_3 := f_1 - g_2$. Then, $h_3(0) = -g_2(0) = h_2(0)$, and $0 < h_3' = f_1' - g_2' < f_2' - g_2' = h_2'$. So for $x \in (0, 1)$, h_2 and h_3 are both increasing and $h_3 < h_2$. So x_3, the root of h_3, satisfies $x_3 > x_2$. Since g_2 is decreasing, we have $g_2(x_3) < g_2(x_2)$, which implies that $y_3 < y_2$. Similarly, by comparing the intersection of f_1 and g_2 to that of f_1 and g_1, we can show that $y_1 < y_3$. Combining with the above, we get $y_1 < y_2$. □

Lemma 3.3 *Let* f_1, g_1 *and* y_1 *be as in Lemma 3.2, and suppose* $g_3(x) = g_1(x + a)$, *where* $0 < a < 1$. *Then, the unique solution* $x_4 \in (0, 1)$ *of* $f_1(x_4) = g_3(x_4) =: y_4$ *satisfies* $y_4 < y_1$.

Proof Analogous to that of Lemma 3.2. □

Lemma 3.4 *Suppose* $f_1 \in C^2(\mathbb{R}^+)$ *satisfies* $f_1'' > 0$ *and attains its minimum at* x_0. *Let* $f_2(x) := f_1(x/a)$, *where* $a > 1$. *Then*

(i) f_2 *satisfies* $f_2'' > 0$ *and attains its minimum at* ax_0.
(ii) *There is a unique* $x_1 \in (x_0, ax_0)$ *such that* $f_1(x_1) = f_2(x_1)$.
(iii) *For all* $x < x_1$, *we have* $f_1(x) < f_2(x)$ *and for all* $x > x_1$ *we have* $f_1(x) > f_2(x)$.
(iv) *Define a new function* $f_3(x) := f_2(x + (a - 1)x_0)$ *which is* f_2 *shifted so its minimum coincides with that of* f_1. *Then,* $|f_3'| \le |f_1'|$, *with equality only when* $x = x_0$.

Proof (i) is obvious. To prove (ii), we use the fact that $f_1(x_0) = f_2(ax_0)$ and $f_1' > 0$, $f_2' < 0$ for all $x \in (x_0, ax_0)$. By a slightly modified version of Lemma 3.2, we know that there is a unique intersection point in (x_0, ax_0). Also, we see that now (iii) certainly holds in $[x_0, ax_0]$.

The possibility of intersection *outside* of this interval will be ruled out when we prove the rest of (iii).

To prove $f_2(x) > f_1(x)$ for $x < x_0$, define the map $\sigma : (x, y) \mapsto (ax, y)$. Then, σ maps the graph of f_1 on $(0, x_0/a)$ onto the graph of f_2 on $(0, x_0)$. If $x \in (0, x_0/a)$, then σ maps $(x, f_1(x))$ to a point above the graph of f_1, because f_1 decreases on $(0, x_0)$. Similarly, since $f_1(x)$ increases for $x > x_0$, the image under σ of its graph on this interval lies below its own graph. But this image is the graph of $f_2(x)$ for $x > ax_0$. So $f_2(x) < f_1(x)$ for all $x > ax_0$. This completes the proof of (iii).

Finally, in part (iv), $f_1'(x_0) = f_3'(x_0) = 0$. Also, $f_3'(x) < 0$ for $x < x_0$ and $f_3'(x) > 0$ for $x > x_0$. Then, by the chain rule

$$f_3'(x) = f_2'(x + (a-1)x_0) = \frac{1}{a} f_1'\left(\frac{x + (a-1)x_0}{a}\right) = \frac{1}{a} f_1'\left(x_0 + \frac{x - x_0}{a}\right). \tag{18}$$

Suppose $x > x_0$. Then

$$x_0 < x_0 + \frac{x - x_0}{a} < x.$$

Since $f_1'(x)$ is positive and increasing for $x > x_0$, by (18) we get

$$0 < f_3'(x) < f_1'\left(x_0 + \frac{x - x_0}{a}\right) < f_1'(x).$$

Now suppose $x < x_0$. Then

$$x < x_0 + \frac{x - x_0}{a} < x_0.$$

Since $f_1'(x)$ is negative and increasing for $x < x_0$, we get analogously

$$f_1'(x) < f_1'\left(x_0 + \frac{x - x_0}{a}\right) < f_3'(x) < 0.$$

Thus, part (iv) is proved. □

Lemma 3.5 *For $n = 1, 2, \ldots,$ and $\alpha \leq 1$, $\rho_n(R)$ has a positive second derivative for all $R > 0$ and attains the unique minimum at*

$$R_n := \sqrt{n}\left(\frac{2\pi}{V_0(2-\alpha)}\right)^{\frac{1}{3-\alpha}}.$$

Proof First, we prove the $n = 1$ case by differentiating ρ twice:

$$\rho(R) = \frac{2}{R} + \frac{V_0}{\pi} R^{2-\alpha}$$
$$\rho'(R) = -\frac{2}{R^2} + \frac{V_0}{\pi}(2-\alpha) R^{1-\alpha}$$
$$\rho''(R) = \frac{4}{R^3} + \frac{V_0}{\pi R^\alpha}(2-\alpha)(1-\alpha).$$

The second derivative is clearly positive for all $R > 0$. Hence $\rho(R)$ attains the unique minimum at $R = R_1$, where

$$R_1 = \left(\frac{2\pi}{V_0(2-\alpha)}\right)^{\frac{1}{3-\alpha}}.$$

In the case of general $n > 1$, (17) and part (i) of Lemma 3.4 yield the result. □

Lemma 3.6 *For $R > 0$, let $\rho_{min}(R) := \min_{n\in\mathbb{N}} \rho_n(R)$. If we partition $\mathbb{R}^+$ into disjoint intervals*

$$I_1 := (0, R_{c1}],$$
$$I_n := (R_{c(n-1)}, R_{cn}], \qquad n = 2, 3, \ldots,$$

then

$$R \in I_n \implies \rho_{min}(R) = \rho_n(R).$$

Proof First, by part (ii) of Lemma 3.4, we have that R_{cn} lie between the successive minima of ρ_n, so they increase in n. By part (iii) of Lemma 3.4, we have

$$R < R_{cn} \implies \rho_n(R) < \rho_{n+1}(R) \tag{19}$$
$$R > R_{cn} \implies \rho_n(R) > \rho_{n+1}(R). \tag{20}$$

Suppose $R \le R_{cn}, n \ge 1$. Then, $R \le R_{cn} < R_{c(n+1)} < \cdots$, and repeatedly using (19) gives:

$$\rho_n(R) \le \rho_{n+1}(R) \le \cdots .$$

The result of the Lemma for $R \in I_1$ then follows immediately. Otherwise, suppose $R > R_{c(n-1)}$, where $n > 1$. Then, $R > R_{c(n-1)} > R_{c(n-2)} > \cdots$, so (20) gives

$$\rho_n(R) < \rho_{n-1}(R) < \cdots < \rho_1(R).$$

Thus, $R \in (R_{c(n-1)}, R_{cn}]$ implies that $\rho_n(R) \le \rho_k(R)$ for every $k \in \mathbb{N}$, yielding the claim. □

Lemma 3.7 *For $\alpha \le 1$, the sequence $\rho_{cn} := \rho_n(R_{cn})$ is decreasing in n.*

Proof Let $d_n := R_{c(n+1)} - R_{cn} = R_{c1}(\sqrt{n+1} - \sqrt{n})$ be the distance between the successive minima described in Lemma 3.5. Clearly (d_n) is decreasing. Now consider the graphs of functions $\rho_n(R)$ and $\rho_{n+1}(R)$, whose unique intersection point is (R_{cn}, ρ_{cn}). Shift these horizontally, so that their minima are at $R = 0$ and $R = d_n$, and call the new functions whose graphs these are as f_2 and g_2, respectively. Note that the ρ value of the intersection of these new functions is still ρ_{cn}. Next, consider the graphs of $\rho_{n+1}(R)$ and $\rho_{n+2}(R)$, whose intersection point is $(R_{c(n+1)}, \rho_{c(n+1)})$. As before, slide these so their respective minima are at $R = 0$ and $R = d_n$, and call the new functions whose graphs these are as f_1, g_1. By (17) and part (iv) of Lemma 3.4, f_1, f_2, g_1, g_2 satisfy, up to translations and dilations, the hypotheses of Lemma 3.2. Thus, the intersection of the graphs of f_1 and g_1 lies below that of f_2 and g_2.

Let g_3 be g_1 shifted to the left so that its minimum is now at d_{n+1} rather than d_n. By Lemma 3.3, the intersection point of the graphs of f_1 and g_3 is still below that of f_2 and g_2. But the ρ value of this intersection is $\rho_{c(n+1)}$. Thus, $\rho_{c(n+1)} < \rho_{cn}$ for every $n \in \mathbb{N}$. □

Lemma 3.8 *For $\alpha \le 1$ and $R \ge R_{c1}$, we have $\rho_{min}(R) \le \rho_{c1}$.*

Proof Suppose $R \ge R_{c1}$. Since $\rho_{\min}(R)$ is convex between successive R_{cn}, it lies below the piecewise linear function connecting the (R_{cn}, ρ_{cn}) points. By Lemma 3.7, this function decreases. □

We are now able to prove the main result of this section.

Proposition 3.9 (Upper Bound on Minimal Energy) *If $\alpha \le 1$ and Ω is a minimizer of E with mass $m \ge m_{c1}$, where m_{c1} is defined in (4), then*

$$E(\Omega) \le m\rho_{c1} = \frac{m}{m_{c1}} E\left(B_{\sqrt{m_{c1}/\pi}}(0)\right).$$

Proof In view of Lemma 3.8, testing E with a union of $n \geq 1$ disks of mass m/n sufficiently far apart and choosing n optimally, we get configurations whose energy can be made arbitrarily close to $m\rho_{c1}$. The desired inequality then follows. □

4 Non-existence of minimizers

The upper bound for the minimum of E obtained in the preceding section allows us to find a condition guaranteeing non-existence of minimizers as in [18]. Here, however, we will further refine those estimates to ensure that the threshold value of m for non-existence approaches $m_{c1}(0) \approx 2.051$ as $\alpha \to 0$, which is sharp.

We introduce an auxiliary function

$$\rho_0(R) := \frac{2}{R} + \frac{2^\alpha \pi^{1-\alpha}}{\rho_{c1}^\alpha} R^{2-2\alpha}, \tag{21}$$

where, as in Proposition 3.9, $\rho_{c1} = E_1(R_{c1})/(\pi R_{c1}^2)$ and $R_{c1} = \sqrt{m_{c1}/\pi}$. Then, we have the following result concerning the roots of the equation $\rho_0(R) = \rho_{c1}$ that exceed R_{c1}.

Lemma 4.1 *For every $\alpha \leq \frac{1}{2}$, there exists a unique value of $R_0 \geq R_{c1}$ such that $\rho_0(R_0) = \rho_{c1}$. Moreover, if $R > R_{c1}$ then $\rho_0(R) > \rho_{c1}$ if and only if $R > R_0$.*

Proof First of all, observe that by (6), we have

$$\begin{aligned} \pi R_{c1}^2 \rho_{c1} &= E(B_{R_{c1}}(0)) = P(B_{R_{c1}}(0)) + V(B_{R_{c1}}(0)) \\ &\geq P(B_{R_{c1}}(0)) + \frac{\pi^2 R_{c1}^4}{\left(\frac{1}{2} P(B_{R_{c1}}(0))\right)^\alpha}. \end{aligned}$$

Therefore, $P(B_{R_{c1}}(0)) < \pi R_{c1}^2 \rho_{c1}$ and

$$\pi R_{c1}^2 \rho_{c1} > 2\pi R_{c1} + \frac{2^\alpha \pi^{2-\alpha}}{\rho_{c1}^\alpha} R_{c1}^{4-2\alpha} = \pi R_{c1}^2 \rho_0(R_{c1}).$$

Thus $\rho_0(R_{c1}) < \rho_{c1}$. On the other hand, since $\rho_0(R)$ is continuous and $\rho_0(R) \to +\infty$ as $R \to +\infty$, there exists $R_0 \geq R_{c1}$ such that $\rho_0(R_0) = \rho_{c1}$. Moreover, since

$$\rho_0''(R) = \frac{4}{R^3} + \frac{2^{1+\alpha}\pi^{1-\alpha}}{\rho_{c1}^\alpha R^{2\alpha}}(1-\alpha)(1-2\alpha) > 0 \qquad \forall R > 0,$$

the function $\rho_0(R)$ is strictly convex and, hence, the value of R_0 is unique. Finally, the last statement follows from the fact that $\rho_0(R) - \rho_{c1}$ changes sign from negative to positive as R increases. □

We now state the non-existence result.

Proposition 4.2 (Non-existence of Minimizers) *Let $\alpha \leq \frac{1}{2}$ and let $m_2 = \pi R_0^2$, where R_0 is as in Lemma 4.1. Then, there is no minimizer of E with mass m for any $m > m_2$.*

Proof Suppose, to the contrary, that a minimizer Ω exists and $m > m_2$. By Proposition 3.9 and (6), we have

$$m\rho_{c1} \geq E(\Omega) = P(\Omega) + V(\Omega) \geq P(\Omega) + \frac{m^2}{\left(\frac{1}{2}P(\Omega)\right)^\alpha}.$$

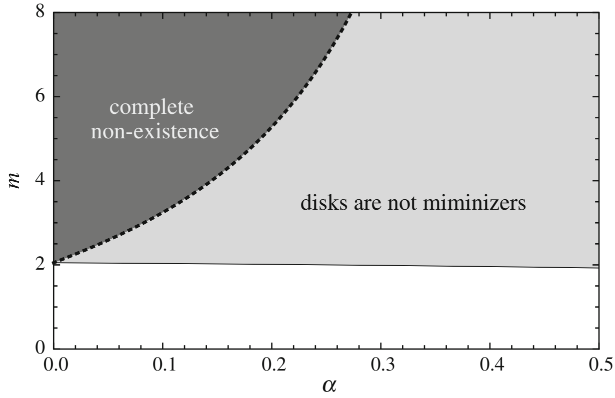

Fig. 2 Regions of guaranteed non-existence of minimizers. The *dotted line* shows the plot of $m_2(\alpha)$ from Proposition 4.2 and the *solid line* shows the plot of $m_{c1}(\alpha)$ from (4), both obtained numerically. *Dark shaded* area shows the region where minimizers of E with mass m do not exist. *Light-shaded* area shows the region in which disks cannot be minimizers

Therefore, we get $P(\Omega) \leq m\rho_{c1}$ and, hence, with the help of the isoperimetric inequality we obtain

$$\rho_{c1} \geq \sqrt{\frac{4\pi}{m}} + \frac{2^\alpha m^{1-\alpha}}{\rho_{c1}^\alpha} = \rho_0\left(\sqrt{m/\pi}\right),$$

which contradicts Lemma 4.1. □

We note that the value of $m_2 = m_2(\alpha)$ in Proposition 4.2 satisfies

$$\lim_{\alpha\to 0} m_2(\alpha) = m_{c1}(0).$$

This follows from the fact that the statement of Lemma 4.1 remains valid up to $\alpha = 0$ and that $\lim_{\alpha\to 0}\rho_0(R) = \rho(R)$ for every $R > 0$. Therefore, since $\rho(R)$ is strictly increasing when $R > R_{c1}$, we have that $R_0 = R_{c1}$ in this case. The numerically computed dependence of m_2 on α in Proposition 4.2, alongside with $m_{c1}(\alpha)$ from (4), is presented in Fig. 2. This figure also shows regions in the (α, m)-plane in which minimizers of different kinds are guaranteed to fail to exist.

5 Shape of minimizers

We now investigate under which conditions the unique, up to translations, minimizer of E with mass m is a disk. This is to be expected in the regime of sufficiently small values of m depending on α [18]. Here, we quantify this statement by finding a mass $m_0 = m_0(\alpha)$ below which the minimizer, if it exists, is a disk, and such that $m_0(\alpha)$ diverges as $\alpha \to 0$.

Let Ω be a minimizer of E with mass m. In this section it is convenient to introduce a rescaling $\Omega_\varepsilon := \Omega\sqrt{\pi/m}$ which ensures that $|\Omega_\varepsilon| = |B_1(0)| = \pi$. Here

$$\varepsilon := \left(\frac{m}{\pi}\right)^{\frac{3-\alpha}{2}}, \tag{22}$$

is the new parameter, whose smallness implies smallness of m. In terms of Ω_ε, we have $\sqrt{\pi/m}\,E(\Omega) = E_\varepsilon(\Omega_\varepsilon)$, where [18]

$$E_\varepsilon(\Omega_\varepsilon) := P(\Omega_\varepsilon) + \varepsilon V(\Omega_\varepsilon), \tag{23}$$

and Ω_ε is a minimizer of E_ε among all open bounded sets with boundary of class C^1 and area equal to π. Throughout the rest of this section, Ω_ε always stands for a minimizer of E_ε.

We wish to estimate the range of values of $\varepsilon > 0$ for which Ω_ε must be a unit disk. We proceed via a sequence of lemmas.

Lemma 5.1 (Bound on isoperimetric deficit) *If $D(\Omega_\varepsilon)$ is the isoperimetric deficit of Ω_ε defined in* (14)*, then*

$$D(\Omega_\varepsilon) < C_0, \qquad C_0(\alpha,\varepsilon) := \frac{\varepsilon}{2\pi}\left(V_0(\alpha) - \frac{\pi^{2-\alpha}}{\left(1+\frac{\varepsilon}{2\pi}V_0(\alpha)\right)^\alpha}\right) > 0.$$

Proof Testing the energy with a unit ball and using Lemma 2.3, we obtain

$$P(\Omega_\varepsilon) + \varepsilon V(\Omega_\varepsilon) = E(\Omega_\varepsilon) \le E(B_1(0)) = 2\pi + \varepsilon V_0.$$

In particular, $P(\Omega_\varepsilon) \le 2\pi + \varepsilon V_0$ and, therefore, by (6) we have

$$V(\Omega_\varepsilon) \ge \frac{2^\alpha \pi^2}{P^\alpha(\Omega_\varepsilon)} \ge \frac{\pi^{2-\alpha}}{\left(1+\frac{\varepsilon}{2\pi}V_0\right)^\alpha}.$$

Combining the two inequalities above gives the result. □

Remark 5.2 Observe that $C_0(\alpha,\varepsilon)$ is a monotonically increasing function of ε for α fixed, and that $C_0(\alpha,\varepsilon) \to 0$ as $\alpha \to 0$ with ε fixed. In particular, in view of Fig. 2 minimizers of E_ε, if they exist, are small perturbations of unit disks for $\alpha \ll 1$.

Lemma 5.3 (Bounds on potential) *Let*

$$v(x) := \int_{\Omega_\varepsilon} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y \tag{24}$$

be the potential associated with Ω_ε. Then, we have

$$C_1 < v(x) < C_2 \qquad \forall x \in \overline{\Omega}_\varepsilon,$$

where

$$C_1(\alpha,\varepsilon) := \frac{\pi^{1-\alpha}}{(1+C_0(\alpha,\varepsilon))^\alpha} \qquad \textit{and} \qquad C_2(\alpha) := \frac{2\pi}{2-\alpha}.$$

Proof For any $x \in \overline{\Omega}_\varepsilon$, let v^B be as in (8). Then

$$\begin{aligned} v^B(0) - v(x) &= \int_{B_1(x)} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y - \int_{\Omega_\varepsilon} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y \\ &= \int_{B_1(x)\setminus\Omega_\varepsilon} \frac{1}{|x-y|^\alpha}\,d\mathrm{d}y - \int_{\Omega_\varepsilon\setminus B_1(x)} \frac{1}{|x-y|^\alpha}\,d\mathrm{d}y \\ &> |B_1(x)\setminus\Omega_\varepsilon| - |\Omega_\varepsilon\setminus B_1(x)| = 0, \end{aligned}$$

since $|\Omega| = |B_1(x)|$. Therefore, $v(x)$ is bounded from above by $v^B(0)$, whose value is given by (10).

On the other hand, with the help of Lemma 5.1 and (6), we obtain

$$v(x) \ge \frac{2^\alpha \pi}{P^\alpha(\Omega_\varepsilon)} = \frac{\pi^{1-\alpha}}{(1+D(\Omega_\varepsilon))^\alpha} > C_1,$$

which yields the lower bound. □

Lemma 5.4 (Convexity) *There exists a unique* $\varepsilon = \varepsilon_0(\alpha) > 0$ *which solves*

$$\frac{1}{1 + C_0(\alpha, \varepsilon)} + 2\varepsilon(C_1(\alpha, \varepsilon) - C_2(\alpha)) = 0. \tag{25}$$

Furthermore, if $\varepsilon < \varepsilon_0$ *then* Ω_ε *is strictly convex.*

Proof First, in view of Remark 5.2 observe that since $C_2 > C_1$ and since $C_1(\alpha, \varepsilon)$ is decreasing as a function of ε, the left-hand side of (25) is a monotonically decreasing continuous function of ε. Therefore, existence of a unique solution of (25) is guaranteed by the fact that its left-hand side approaches unity as $\varepsilon \to 0$, while tending to $-\infty$ when $\varepsilon \to +\infty$.

By Proposition 2.1 (after rescaling), the Euler–Lagrange equation for $\partial\Omega_\varepsilon$ at $x \in \partial\Omega_\varepsilon$ is

$$\kappa(x) + 2\varepsilon v(x) - \mu = 0, \tag{26}$$

where κ is curvature (positive if Ω_ε is convex) and $\mu \in \mathbb{R}$ is the Lagrange multiplier. Integrating (26) over the outer portion $\partial\Omega_\varepsilon^0$ of the boundary $\partial\Omega_\varepsilon$ with respect to arclength and using Lemmas 5.1 and 5.3 yields

$$\mu = \frac{2\pi}{|\partial\Omega_\varepsilon^0|} + 2\varepsilon\bar{v} \geq \frac{2\pi}{P(\Omega_\varepsilon)} + 2\varepsilon\bar{v} > \frac{1}{1 + C_0} + 2\varepsilon C_1,$$

where $\bar{v}$ is the average of v over $\partial\Omega_\varepsilon^0$. Then by (26) and Lemma 5.3, we have

$$\kappa(x) = \mu - 2\varepsilon v(x) > \frac{1}{1 + C_0} + 2\varepsilon(C_1 - C_2),$$

which is positive under the assumption of the Lemma. □

Lemma 5.5 (Confinement to an annulus) *If* Ω_ε *is convex, there exist* $x_0 \in \mathbb{R}^2$ *and* $\delta \geq 0$ *such that*

$$B_{1-\delta}(x_0) \subseteq \Omega_\varepsilon \subseteq B_{1+\delta}(x_0)$$

and

$$\delta \leq \sqrt{\pi D(\Omega_\varepsilon)(D(\Omega_\varepsilon) + 2)}, \tag{27}$$

with the convention that $B_{1-\delta}(x_0) = \varnothing$ *if* $\delta \geq 1$.

Proof By Lemma 2.5, there exist $x_0 \in \mathbb{R}^2$ and $0 < r_1 \leq 1 \leq r_2$ such that $B_{r_1}(x_0) \subseteq \Omega_\varepsilon \subseteq B_{r_2}(x_0)$ and $r_2 - r_1 \leq \sqrt{\pi D(\Omega_\varepsilon)(D(\Omega_\varepsilon) + 2)}$. Hence $1 - r_1 \leq \sqrt{\pi D(\Omega_\varepsilon)(D(\Omega_\varepsilon) + 2)}$ and $r_2 - 1 \leq \sqrt{\pi D(\Omega_\varepsilon)(D(\Omega_\varepsilon) + 2)}$, and the result follows. □

We are now ready to prove a quantitative criterion which guarantees that the minimizer of E_ε, if it exists and is convex, is a unit disk for a given value of ε. The proof follows the ideas in the proof of [18, Proposition 7.5] in a quantitative way.

Proposition 5.6 (Minimizers are disks) *There exists a unique* $\varepsilon = \varepsilon_1(\alpha) > 0$ *solving*

$$\varepsilon C_3(\alpha, \varepsilon)\Big[\varepsilon C_3(\alpha, \varepsilon)C_0(\alpha, \varepsilon)\Big(C_0(\alpha, \varepsilon) + 2\Big) + 2\Big] = 1, \tag{28}$$

where

$$C_3(\alpha, \varepsilon) := \frac{\pi^2\alpha(2 - \alpha)\Gamma(1 - \alpha)}{2\Gamma^2(2 - \frac{\alpha}{2})}\left(1 + \frac{2}{3}\sqrt{\pi C_0(\alpha, \varepsilon)(C_0(\alpha, \varepsilon) + 2)}\right). \tag{29}$$

Furthermore, if $\varepsilon < \varepsilon_1$ *and* Ω_ε *is convex, then* Ω_ε *is a unit disk.*

Proof In view of Remark 5.2, the left-hand side of (29) increases monotonically from zero to infinity as ε runs from zero to infinity. Hence there is a unique solution to (28).

Now, testing E_ε with $B_1(x_0)$, where x_0 is as in Lemma 5.5, gives

$$P(\Omega_\varepsilon) + \varepsilon V(\Omega_\varepsilon) = E_\varepsilon(\Omega_\varepsilon) \leq E_\varepsilon(B_1(x_0)) = 2\pi + \varepsilon V_0,$$

which is equivalent to

$$D(\Omega_\varepsilon) \leq \frac{\varepsilon}{2\pi}(V_0 - V(\Omega_\varepsilon)) =: \frac{\varepsilon}{2\pi}\Delta V. \tag{30}$$

Combining (27) and (30), then gives

$$\delta^2 \leq \frac{\varepsilon}{2}\Delta V \left(\frac{\varepsilon}{2\pi}\Delta V + 2\right). \tag{31}$$

On the other hand, arguing as in [18, Eqs. (7.14) and (7.15)] and applying Lemma 2.4, we then find that

$$\begin{aligned} \Delta V &\leq 2\int_{B_1(x_0)\Delta\Omega_\varepsilon} |v^B(|x-x_0|) - v^B(1)|\, d\mathrm{d}x \\ &\leq 2\left|\frac{\mathrm{d}v^B(1)}{\mathrm{d}r}\right| \int_{B_{1+\delta}(x_0)\setminus B_1(x_0)} (|x-x_0|-1)\,\mathrm{d}x \\ &= 4\pi \left|\frac{\mathrm{d}v^B(1)}{\mathrm{d}r}\right| \int_0^\delta t(1+t)\mathrm{d}t \\ &= 2\pi \left|\frac{\mathrm{d}v^B(1)}{\mathrm{d}r}\right| \left(1+\frac{2}{3}\delta\right)\delta^2, \end{aligned} \tag{32}$$

where to arrive at the second line in (32) we reflected all the points of the set $B_1\setminus\Omega_\varepsilon$ with respect to $\partial B_1(x_0)$. Furthermore, since in view of Lemmas 5.1 and 5.5, we have

$$\delta \leq \sqrt{\pi C_0(C_0+2)}, \tag{33}$$

from (32) and (11) we get

$$\Delta V \leq 2\delta^2 C_3. \tag{34}$$

Therefore, substituting the inequality in (34) back to (31) and then using (33) again yields that either $\delta = 0$ or

$$\varepsilon C_3(\alpha,\varepsilon)\Big[\varepsilon C_3(\alpha,\varepsilon)C_0(\alpha,\varepsilon)\Big(C_0(\alpha,\varepsilon)+2\Big)+2\Big] \geq 1. \tag{35}$$

Since the latter is impossible by our assumption, the rest of the proposition is proved. □

The dependences of $m(\varepsilon_0)$ (thin solid line) and $m(\varepsilon_1)$ on α (dotted/solid line) computed numerically are presented in Fig. 3. These curves, together with the curve $m_{c1}(\alpha)$ (dashed/solid line), separate the parameters into several regions (see the caption for an explanation). Specifically, the region below the thick solid line indicates the parameters for which the minimizer of E with mass m exists and is a disk, while the region above the solid line and below the dotted line is where no minimizers exist. Our numerical results indicate that one can choose $m_0(\alpha) = m(\varepsilon_1(\alpha))$, using Proposition 5.6 and Eq. (22). Then for any mass $m \in (0, m_0(\alpha))$, the minimizer, provided it exists, is a disk.

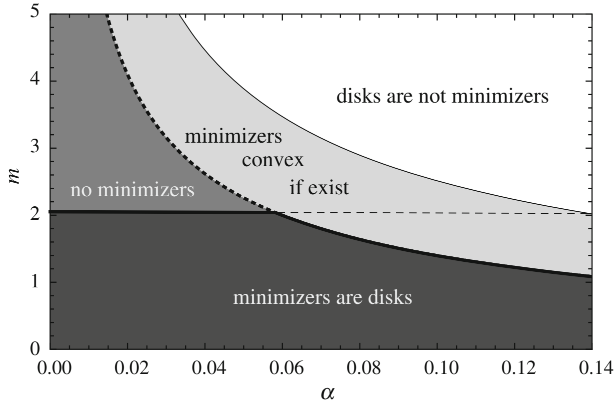

Fig. 3 Regions of guaranteed convexity and existence of minimizers. The *thin solid line* shows the plot of $m(\varepsilon_0(\alpha))$ from Lemma 5.4 and Eq. (22), the *dotted line* shows the plot of $m(\varepsilon_1(\alpha))$ from Proposition 5.6 and Eq. (22), the *dashed line* shows the plot of $m_{c1}(\alpha)$, and the *thick solid line* encloses the region in which minimizers exist and are disks. *Light gray* area shows the region where minimizers are convex, if they exist. *Medium gray* area shows the region where there are no minimizers. *Dark gray* area shows the region in which minimizers exist and are disks

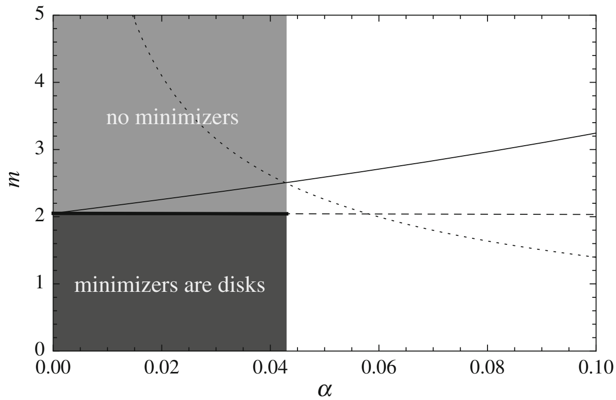

Fig. 4 Summary of the numerical results. Above the solid line no minimizers exist; below the *dotted line* minimizers, if they exist, are disks; above the *dashed/thick solid line* (*lighter gray* region) disks are not minimizers; below the *solid line* (*dark gray* region) minimizers are disks

6 Proof of the main Theorem

Figure 4 summarizes our results obtained numerically from evaluating the different criteria of existence and non-existence obtained in the preceding sections. From this figure, one can see that the curve $m_2(\alpha)$, above which non-existence of minimizers holds, intersects the curve $m_0(\alpha)$, below which minimizers must be disks, at $\alpha = \alpha_0 \approx 0.04273$. This indicates that the statement of Theorem 1.1 should hold below this value of α. In the rest of this section, we give an analytical proof of this fact with a slightly reduced value of α_0. The only difficulty at this point is that the dependences of m_0, m_2 and m_{c1} on α are given by extremely complicated algebraic formulas and, therefore, their qualitative behavior (e.g., monotonicity) is not easy to establish. Instead, we simply estimate those functions explicitly for $\alpha \in (0, \alpha_0]$, using the known behavior of the Gamma function and other functions appearing in the estimates. Note that our analytical estimates below are rather ad hoc and are not intended to be completely optimal. We believe that $\alpha_0 = 0.0427$, which comes from our numerical results, should give essentially the best constant with our approach. Proving this fact would be an extremely tedious exercise in calculus, which we decided not to pursue.

Proof of Theorem 1.1 Since $\Gamma(z)$ is a monotonically increasing function of z for $z \geq 1.966$ and a monotonically decreasing function of z for $0 < z \leq 1$, for all $0 < \alpha \leq 0.034$ we can bound its values that appear in our estimates as follows:

$$\begin{aligned} 0.986 \leq \Gamma(2-\alpha) &\leq 1, \\ 0.992 \leq \Gamma\left(2-\frac{\alpha}{2}\right) &\leq 1, \\ 1.968 \leq \Gamma\left(3-\frac{\alpha}{2}\right) &\leq 2, \\ 1 \leq \Gamma(1-\alpha) &\leq 1.021. \end{aligned}$$

Then from (4), we find that $2.007 \leq m_{c1}(\alpha) \leq 2.087$.

Next, define (here and everywhere below the constants are as in the previous sections, with the parametric dependences always indicated)

$$F_1(\alpha, \varepsilon) := \varepsilon C_3(\alpha, \varepsilon)\Big[\varepsilon C_3(\alpha, \varepsilon) C_0(\alpha, \varepsilon)\Big(C_0(\alpha, \varepsilon) + 2\Big) + 2\Big] - 1.$$

Assume $\varepsilon \leq 0.846$ and $0 < \alpha \leq 0.034$. Using the bounds above, we then get

$$\begin{aligned} C_0(\alpha, \varepsilon) &\leq 0.121, \\ C_3(\alpha, \varepsilon) &\leq 0.557, \\ F_1(\alpha, \varepsilon) &< 0. \end{aligned}$$

By Proposition 5.6, it then follows that $\varepsilon_1(\alpha) > 0.846$, and, hence, $m(\varepsilon_1) > 2.806$, for $0 < \alpha \leq 0.034$.

Similarly, we can define

$$F_2(\alpha, \varepsilon) := \frac{1}{1 + C_0(\alpha, \varepsilon)} - 2\varepsilon(C_2(\alpha) - C_1(\alpha, \varepsilon)).$$

We find that if $\varepsilon \leq 0.846$ and $0 < \alpha \leq 0.034$, then

$$\begin{aligned} C_1(\alpha, \varepsilon) &\geq 3.009, \\ C_2(\alpha) &\leq 3.196, \\ F_2(\alpha, \varepsilon) &\geq 0.575 > 0. \end{aligned}$$

Therefore, by Lemma 5.4 we have $\varepsilon_0(\alpha) > 0.846$ and, hence, $m(\varepsilon_0) > 2.806$ for $0 < \alpha \leq 0.034$.

Finally, we wish to obtain an upper bound for $m_2(\alpha)$. Define

$$F_3(\alpha, R) := \rho_0(\alpha, R) - \rho_{c1}(\alpha),$$

and recall that

$$\rho_{c1}(\alpha) = \frac{2}{R_{c1}(\alpha)} + \frac{V_0(\alpha)}{\pi} R_{c1}^{2-\alpha}(\alpha).$$

Then for $0 < \alpha \leq 0.034$ and $R = 0.945$, we have

$$\begin{aligned} 0.799 \leq R_{c1}(\alpha) &\leq 0.815 \\ \rho_{c1}(\alpha) &\leq 4.656, \\ \rho_0(\alpha, R) &\geq 4.677, \\ F_3(\alpha, R) &\geq 0.021 > 0. \end{aligned}$$

Thus, by Proposition 4.2 and the arguments in the proof of Lemma 4.1 we have $R_0(\alpha) < 0.945$ and, hence, $m_2(\alpha) < 2.806$ for $0 < \alpha \leq 0.034$.

In conclusion, $m_2(\alpha) < \min(m(\varepsilon_0), m(\varepsilon_1))$ for every $\alpha \in (0, 0.034]$, which proves the result. □

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