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Uniqueness of one-dimensional Néel wall profiles

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We study the domain wall structure in thin uniaxial ferromagnetic films in the presence of an in-plane applied external field in the direction normal to the easy axis. Using the reduced one-dimensional thin-film micromagnetic model, we analyse the critical points of the obtained non-local variational problem. We prove that the minimizer of the one-dimensional energy functional in the form of the Néel wall is the unique (up to translations) critical point of the energy among all monotone profiles with the same limiting behaviour at infinity. Thus, we establish uniqueness of the one-dimensional monotone Néel wall profile in the considered setting. We also obtain some uniform estimates for general one-dimensional domain wall profiles.

1. Introduction

Thin soft ferromagnetic films have been widely used as a data storage solution in modern computer technology [1–3]. It is well established that for sufficiently thin films, the magnetization vector of the material lies almost entirely in the film plane. Such ultra-thin ferromagnetic films often exhibit magnetization patterns consisting of domains in which the magnetization vector is nearly constant and is aligned along one of the directions of the easy axis of materials. Domains with different orientation of the magnetization are separated by thin transition layers called *domain walls* in which the magnetization vector rotates rapidly from one direction to another.

The study of the domain wall structure in ferromagnetic materials has attracted a lot of attention. One of the common domain wall types in ultrathin ferromagnetic

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films is the *Néel wall*. In this wall type, the magnetization vector exhibits an in-plane 180° rotation in the absence of an applied magnetic field. At present, the structure of the Néel wall is rather well understood. Within the framework of micromagnetic modelling, the overall physical picture has been summarized in books [2,4] (see also [5–8], etc.). Experimental observations of the one-dimensional Néel wall profiles can be found in [9–11]. Rigorous mathematical analysis of Néel wall is more recent, starting from the work of García–Cervera on the analysis of the associated one-dimensional variational problem [5,12]. Melcher studied one-dimensional energy minimizers in thin uniaxial films and obtained symmetry, monotonicity of the one-dimensional minimizing profile, as well as the logarithmic decay beyond the core region for very soft films [13]. Linearized stability of the one-dimensional Néel wall with respect to one-dimensional perturbations in a reduced thin film model was proved in [14]. Asymptotic stability of one-dimensional Néel walls with respect to large two-dimensional perturbations in a reduced two-dimensional thin film model was demonstrated in [15].

Recently, Chermisi & Muratov [16] studied the reduced one-dimensional energy in the presence of an applied in-plane magnetic field in the direction perpendicular to the easy axis. They expressed the magnetic energy in terms of the phase angle rather than the usual two-dimensional unit vector representation of the magnetization. They obtained uniqueness and strict monotonicity of the angle variable for the minimizing Néel wall structure. Moreover, they proved precise asymptotic behaviour of the minimizing Néel wall profiles at infinity.[1] The associated Euler–Lagrange equation in their setting is expressed as an ordinary differential equation for the phase angle with a non-local term present.

We note that while from the physical point of view the Néel walls are believed to be the energy-minimizing configurations of the magnetization connecting the two oppositely oriented domains in uniaxial films, it is natural to ask whether other, metastable Néel wall-type configurations connecting the two domains, are also possible. For example, in the presence of a transverse in-plane magnetic field, one can distinguish normal and reverse domain walls, which differ by the rotation sense of the magnetization [17]. Clearly, the reverse domain wall is not an energy minimizer, because the magnetization vector opposes the applied field in such a wall. Still, in view of the highly nonlinear and non-local character of the problem it is not *a priori* clear whether there could exist other one-dimensional domain wall profiles connecting the domains of opposing magnetization which are only local, but not global minimizers of the micromagnetic energy.

In this paper, we follow the variational setting introduced in [16] and consider the *critical points* of the associated energy functional which are monotone in the angle variable. We prove that any monotone critical point of the reduced one-dimensional energy is unique (up to translations) and, therefore, is the minimizer. Thus, we establish that monotone one-dimensional magnetization profiles that are not global energy minimizers do not exist, corroborating the expected physical picture. This also provides a better understanding of the results of the numerical solution of the considered problem and allows to conclude that the obtained one-dimensional profiles [7] indeed correspond to the Néel walls. In addition, we address the question of uniform regularity of the critical points of the one-dimensional energy and establish uniform bounds and, hence, decay of all the derivatives of such solutions at infinity. This result can be applied to other types of domain wall profiles of interest, such as those of the 360° walls [18,19].

The rest of this paper is organized as follows: in §2, we recall some basic facts about the micromagnetic energy and the reduced one-dimensional energy in the presence of an applied in-plane field oriented normally to the easy axis. The main results are stated at the end of §2. The proof of the uniqueness theorem is presented in §3, and the proof of the uniform estimates for the derivatives of domain wall solutions is given in §4. Finally, we briefly revisit the question of the decay of Néel walls at infinity in appendix A and present the proof of a technical lemma in appendix B.

[1] We point out that the proof of the asymptotic decay of the Néel wall profiles in [16] contained an error, which, however, does not affect the result. For the reader's convenience, we present the correction in appendix A.

2. Variational setting and statement of the main result

In this paper, we are interested in the analysis of magnetization configurations in thin uniaxial ferromagnetic films of large extent with in plane easy axis and applied in-plane field normal to the easy axis. The energy functional related to such a system, introduced by Landau and Lifschitz, can be written in CGS units as a combination of five terms:

$$E(\mathbf{M}) = \frac{A}{2|M_s|^2}\int_{\Omega} |\nabla \mathbf{M}|^2\,\mathrm{d}r + \frac{K}{2|M_s|^2}\int_{\Omega} \Phi(\mathbf{M})\,\mathrm{d}r - \int_{\Omega} \mathbf{H}_{\mathrm{ext}}\cdot \mathbf{M}\,\mathrm{d}r \\ + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla\cdot\mathbf{M}(\mathbf{r})\nabla\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}r\,\mathrm{d}r' + \frac{M_s^2}{2K}\int_{\Omega} |\mathbf{H}_{\mathrm{ext}}|^2\,\mathrm{d}r. \tag{2.1}$$

Here, $\Omega \subset \mathbb{R}^3$ is the domain occupied by the ferromagnetic material, $\mathbf{M}:\mathbb{R}^3 \to \mathbb{R}^3$ is the magnetization vector that satisfies $|\mathbf{M}| = M_s$ in Ω and $\mathbf{M} = 0$ outside Ω, the positive constants M_s, A and K are the material parameters denoting the saturation magnetization, exchange constant and the anisotropy constant, respectively, $\mathbf{H}_{\mathrm{ext}}$ is the applied external field, and $\Phi:\mathbb{R}^3 \to \mathbb{R}$ is a non-negative potential which vanishes at finitely many points. The divergence of $\mathbf{M}$ in the double integral is understood in the distributional sense. The five terms in (2.1) represent the exchange energy, the anisotropy energy, the Zeeman energy, the stray-field energy and an inessential constant term added for convenience.

In the case of extended monocrystalline thin films with an in-plane easy axis, we have $\Omega = \mathbb{R}^2 \times (0, d)$. Without loss of generality, we shall assume that the easy axis is in the $\mathbf{e}_2$-direction. Here, $\mathbf{e}_i$ is the unit vector in the ith coordinate direction. For moderately soft thin films, a *reduced thin film energy* has been derived [7,20,21], providing a significant simplification to the considered variational problem. For a better understanding of the parameter regime, we introduce the following quantities

$$l = \left(\frac{A}{4\pi M_s^2}\right)^{1/2}, \quad L = \left(\frac{A}{K}\right)^{1/2} \quad \text{and} \quad Q = \left(\frac{l}{L}\right)^2,$$

representing the exchange length, the Bloch wall thickness and the material quality factor, respectively. For *ultra-thin* and *soft* film, we have $d \lesssim l \lesssim L$, balanced as $Ld \sim l^2$. We can then introduce a dimensionless parameter

$$\nu = \frac{4\pi M_s^2 d}{KL} = \frac{Ld}{l^2} = \frac{d}{l\sqrt{Q}},$$

which measures the relative strength of the magnetostatic interaction. For the reduced thin film energy, we can write, after an appropriate non-dimensionalization [16]:

$$E(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla \mathbf{m}|^2 + (\mathbf{m}\cdot\mathbf{e}_1 - h)^2)\,\mathrm{d}r + \frac{\nu}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\cdot\mathbf{m}(\mathbf{r})\nabla\cdot\mathbf{m}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}r\,\mathrm{d}r',$$

where $\mathbf{m}:\mathbb{R}^2 \to \mathbb{S}^1$ is the unit magnetization vector in the film plane and h is the dimensionless applied magnetic field.

To study one-dimensional Néel wall profiles, we assume further that $\mathbf{m}$ depends only on $x = \mathbf{e}_1 \cdot \mathbf{r}$. Introducing the variable $\theta = \theta(x)$ that represents the angle between $\mathbf{m}$ and the easy axis $\mathbf{e}_2$ in the anticlockwise direction, we have

$$\mathbf{m}(x) = (-\sin\theta(x),\ \cos\theta(x)),$$

for every $x \in \mathbb{R}$. One can rewrite the energy of such a magnetization configuration per unit length of the wall in terms of θ as

$$E(\theta) = \frac{1}{2}\int_{\mathbb{R}} \left\{ \left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + (\sin\theta - h)^2 + \frac{\nu}{2}(\sin\theta - h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\theta - h) \right\} \mathrm{d}x. \tag{2.2}$$

Here, $(-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}$ represents the linear operator whose Fourier symbol is $|k|$ and can be understood as a bounded linear map from $H^1(\mathbb{R})$, modulo additive constants, to $L^2(\mathbb{R})$. Because two distinct global minima of the energy in (2.2) exist only if $|h| < 1$, we shall always assume that $0 \le h < 1$ in most of the paper.

Let $\eta_h \in C^\infty(\mathbb{R}, [0, \pi])$ be a fixed non-increasing function with

$$\eta_h(x) = \begin{cases} \theta_h & \text{if } x > 1, \\ \pi - \theta_h & \text{if } x < -1, \end{cases} \qquad \theta_h := \arcsin h,$$

and consider an admissible class

$$\mathcal{A} := \{\theta \in H^1_{\mathrm{loc}}(\mathbb{R}) : \theta - \eta_h \in H^1(\mathbb{R})\}.$$

Note that the definition of $\mathcal{A}$ is independent of the choice of η_h. The following result was obtained in [16] addressing the uniqueness, strict monotonicity, symmetry properties and decay of one-dimensional Néel walls.

Theorem 2.1 ([16]). *For every $\nu > 0$ and every $h \in [0, 1)$, there exists a minimizer of $E(\theta)$ in $\mathcal{A}$, which is unique (up to translations), strictly decreasing with the range equal to $(\theta_h, \pi - \theta_h)$ and is smooth. Moreover, if θ is a minimizer satisfying $\theta(0) = \pi/2$, then $\theta(x) = \pi - \theta(-x)$, and there exists a constant $c > 0$ such that $\lim_{x\to\infty} x^2(\theta(x) - \theta_h) = c$.*

The Euler–Lagrange equation associated with the functional in (2.2) is given by

$$-\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + (\sin\theta - h)\cos\theta + \frac{\nu}{2}\cos\theta\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\sin\theta = 0, \tag{2.3}$$

with the boundary conditions at infinity

$$\lim_{x\to+\infty}\theta(x) = \theta_h \quad \text{and} \quad \lim_{x\to-\infty}\theta(x) = \pi - \theta_h. \tag{2.4}$$

Our main result is the following uniqueness theorem.

Theorem 2.2. *For every $\nu > 0$ and every $h \in [0, 1)$, there exists a unique (up to translations) non-increasing smooth solution of (2.3) which satisfies the conditions at infinity in (2.4) and has bounded energy.*

Thus, the only possible monotone Néel wall profile is that of the minimizer of the energy in (2.2), whose existence and uniqueness was established in theorem 2.1. This confirms the long-standing physical intuition that the Néel wall profiles observed in ultrathin uniaxial ferromagnetic films minimize the one-dimensional micromagnetic energy among all such profiles.

We also obtain the following estimates for the general one-dimensional domain wall profiles. Here, by a one-dimensional domain wall profile, we mean a smooth solution of (2.3) connecting zeroes of $\sin\theta - h$ at $x = \pm\infty$. From the estimates in [14] or [16, section 5], we know that any solution θ of (2.3) with bounded energy is smooth, and it is easy to see that any solution of (2.3) with bounded energy should approach a zero of $\sin\theta - h$ at infinity. We note that the obtained estimates also apply to winding domain walls and, in particular, to 360° domain walls studied in [18,19].

Theorem 2.3. *There exist $C_i > 0$, $i = 1, 2, \ldots$, such that for any solution θ of (2.3) with $E(\theta) < \infty$, we have*

$$\sup_{x\in\mathbb{R}}\left(\left|\frac{\mathrm{d}^i\theta}{\mathrm{d}x^i}\right|\right) \le C_i,$$

where $C_i = C_i(\nu, h, E(\theta))$. Moreover, all the derivatives of θ vanish at infinity.

The main idea to prove the uniqueness result is as follows. Given any two monotone solutions θ_1 and θ_2 of (2.3) satisfying (2.4) and $\theta_1(0) = \theta_2(0) = \pi/2$, consider a suitable curve γ connecting θ_1

and θ_2. The curve γ is chosen in such a way that any $\theta^t \in \gamma$ satisfies $\sin\theta^t = t\sin\theta_1 + (1-t)\sin\theta_2$ for some $t \in [0,1]$. We then show that if $f(t) := E(\theta^t)$, then $f \in C^2([0,1])$ and $f''(t) > 0$ for any $t \in [0,1]$, which implies strict convexity of f. At the same time, because θ_i are solutions of (2.3), we must have $f'(t)|_{t=0,1} = 0$, which is impossible. A similar argument, using a hidden convexity of the considered energy functional, was used recently in [22] to prove uniqueness of solutions for a very different variational problem.

The uniform-bound theorem relies on the uniform estimate on the non-local term in (2.3). To obtain the estimate on the non-local term, we used local smoothness of the solutions, together with the integral representation of the non-local term and energy-type estimates for the first derivatives. Decay property of derivatives of solution at infinity follows directly once we get those uniform derivative bounds.

3. Uniqueness of the critical point

Assume that $\theta_1 \not\equiv \theta_2$ are two non-increasing solutions of (2.3) satisfying (2.4) and $E(\theta_i) < \infty$. By a suitable translation, we can ensure that $\theta_i(0) = \pi/2$. Let now

$$\theta^t(x) := \begin{cases} \arcsin(t\sin\theta_1 + (1-t)\sin\theta_2) & x \geq 0, \\ \pi - \arcsin(t\sin\theta_1 + (1-t)\sin\theta_2) & x < 0. \end{cases} \tag{3.1}$$

From the arguments of Chermisi & Muratov [16], we know that θ_i are smooth and $\mathrm{d}\theta_i/\mathrm{d}x < 0$ on $\mathbb{R}$. We first prove lemma 3.1 regarding differentiability of θ^t. We note that the latter is not obvious *a priori*, because the definition of θ^t involves the arcsine function, which is *not* differentiable when its argument equals $\pi/2$. This could potentially create problems near $x = 0$. In fact, the conclusions of this section would clearly be incorrect, if there were multiple points at which either θ_1 or θ_2 equals $\pi/2$. Indeed, uniqueness of solutions of (2.3) and (2.4) with finite energy is false in view of the translational symmetry of the problem. Therefore, the somewhat delicate estimates near $x = 0$ in the lemmas in the following are not merely technical, they are what enables the intuitive arguments of [14,16] to be used to establish uniqueness of the solutions that are translated so as to equal $\pi/2$ at $x = 0$.

In the following, the subscripts x and t denote the partial derivatives with respect to the corresponding variables.

Lemma 3.1. *For any $t \in [0,1]$, the function $\theta^t(x)$ is continuously differentiable with respect to $x \in \mathbb{R}$. For any $x \in \mathbb{R}$, $\theta^t_x(x)$ is twice continuously differentiable with respect to t on $[0,1]$, with the understanding of one-sided derivatives at the boundary. All derivatives $\theta^t_x(x)$, $\theta^t_{xt}(x)$ and $\theta^t_{xtt}(x)$ are continuous functions of x and t separately on $\mathbb{R} \times [0,1]$. Moreover, there exists a constant $K > 0$ depending only on θ_1 and θ_2 such that for all $x \in \mathbb{R}$*

$$\max\{|\theta^t_x(x)|, |\theta^t_{xt}(x)|, |\theta^t_{xtt}(x)|\} \leq K\left(\left|\frac{\mathrm{d}\theta_1}{\mathrm{d}x}(x)\right| + \left|\frac{\mathrm{d}\theta_2}{\mathrm{d}x}(x)\right|\right) \quad \textit{for all } t \in [0,1].$$

We present the proof of lemma 3.1, which is a rather tedious exercise in calculus, in appendix B. To proceed with the proof of our theorem 2.2, we first prove differentiability of $E(\theta^t)$.

Recall that

$$\begin{aligned} E(\theta^t) &= \frac{1}{2}\int_{\mathbb{R}} \left\{ |\theta^t_x|^2 + (\sin\theta^t - h)^2 + \frac{\nu}{2}(\sin\theta^t - h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\theta^t - h) \right\} \mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}} (|\theta^t_x|^2 + (t(\sin\theta_1 - h) + (1-t)(\sin\theta_2 - h))^2)\,\mathrm{d}x \\ &\quad + \frac{\nu}{4}t^2 \int_{\mathbb{R}} (\sin\theta_1 - h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\theta_1 - h)\,\mathrm{d}x \end{aligned}$$

$$+\frac{\nu}{2}t(1-t)\int_{\mathbb{R}}(\sin\theta_1-h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\theta_2-h)\,\mathrm{d}x$$
$$+\frac{\nu}{4}(1-t)^2\int_{\mathbb{R}}(\sin\theta_2-h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\theta_2-h)\,\mathrm{d}x. \tag{3.2}$$

We shall write $f(t)=E(\theta^t)$ for shorthand. Lemma 3.2 is a direct corollary of lemma 3.1.

Lemma 3.2. *We have $E(\theta^t)<\infty$ for all $t\in[0,1]$. Moreover, $f(t)$ is twice continuously differentiable, and $f''(t)>0$ on $[0,1]$.*

Proof. By lemma 3.1 and (3.2), we have

$$E(\theta^t)\le K^2E(\theta_1)+K^2E(\theta_2).$$

Therefore, $f(t)=E(\theta^t)$ is well defined. To ensure that $f(t)$ is sufficiently regular, observe that from (3.2), we can write

$$E(\theta^t)=\int_{\mathbb{R}}\frac{1}{2}|\theta_x^t|^2\mathrm{d}x+P_2(t),$$

where $P_2(t)$ is a quadratic polynomial in t with bounded coefficients depending on θ_1, θ_2. The question of differentiability of $f(t)$ thus reduces to that of

$$g(t)=\int_{\mathbb{R}}\frac{1}{2}|\theta_x^t|^2\mathrm{d}x.$$

By lemma 3.1, for all $x\in\mathbb{R}$, we have

$$|\theta_x^t(x)\theta_{xt}^t(x)|\le 2K^2(\theta_{1x}^2(x)+\theta_{2x}^2(x))\quad\text{for all } t\in[0,1], \tag{3.3}$$

and

$$|\theta_{xt}^t(x)|^2+|\theta_x^t(x)\theta_{xtt}^t(x)|\le 6K^2(\theta_{1x}^2(x)+\theta_{2x}^2(x))\quad\text{for all } t\in[0,1]. \tag{3.4}$$

Because $E(\theta_i)<\infty$ and $\theta_x^t(x),\theta_{xt}^t(x)$ and $\theta_{xtt}^t(x)$ are continuous in t on $[0,1]$ for each $x\in\mathbb{R}$, we conclude from the dominated convergence theorem and continuity of integral theorem that for $t\in[0,1]$

$$g'(t)=\int_{\mathbb{R}}\theta_x^t\theta_{xt}^t\,\mathrm{d}x\quad\text{and}\quad g''(t)=\int_{\mathbb{R}}(|\theta_{xt}^t|^2+\theta_x^t\theta_{xtt}^t)\,\mathrm{d}x,$$

and that $g'(t)$ and $g''(t)$ are both continuous on $[0,1]$. A direct computation then yields

$$\frac{\mathrm{d}^2(E(\theta^t))}{\mathrm{d}t^2}=\int_{\mathbb{R}}(|\theta_{xt}^t|^2+\theta_x^t\theta_{xtt}^t)\,\mathrm{d}x+\int_{\mathbb{R}}(\sin\theta_1-\sin\theta_2)^2\,\mathrm{d}x$$
$$+\frac{\nu}{2}\int_{\mathbb{R}}(\sin\theta_1-h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\theta_1-h)\,\mathrm{d}x$$
$$+\frac{\nu}{2}\int_{\mathbb{R}}(\sin\theta_2-h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}(\sin\theta_2-h)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \left(\frac{\sin\theta_1 - \sin\theta_2}{\sqrt{(1-\sin^2\theta^t)^3}} \theta_x^t \sin 2\theta^t + \frac{\theta_{1x}\cos\theta_1 - \theta_{2x}\cos\theta_2}{\sqrt{1-\sin^2\theta^t}} \right)^2 \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \frac{(t\theta_{1x}\cos\theta_1 + (1-t)\theta_{2x}\cos\theta_2)^2 (\sin\theta_1 - \sin\theta_2)^2}{(1-\sin\theta^t)^2} \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} (\sin\theta_1 - \sin\theta_2)^2 \,\mathrm{d}x + \frac{\nu}{2}\int_{\mathbb{R}} (\sin\theta_1 - h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} (\sin\theta_1 - h)\,\mathrm{d}x \\
&\quad + \frac{\nu}{2}\int_{\mathbb{R}} (\sin\theta_2 - h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} (\sin\theta_2 - h)\,\mathrm{d}x \\
&> 0.
\end{aligned}
$$

■

Proof of theorem 2.2. Existence and smoothness of solutions follows from theorem 2.1 in [16]. We argue by contradiction and assume that $\theta_1 \not\equiv \theta_2$ are two monotone decreasing solutions of (2.3) satisfying (2.4), together with $E(\theta_i) < \infty$ and $\theta_i(0) = \pi/2$. Let θ^t be defined by (3.1) and let $f(t) = E(\theta^t)$. Differentiating (3.2) at $t = 0$, we get

$$
\begin{aligned}
f'(0) = \lim_{t\to 0^+} \int_{\mathbb{R}} \frac{|\theta_x^t|^2 - \theta_{2x}^2}{2t}\,\mathrm{d}x &+ \int_{\mathbb{R}} (\sin\theta_2 - h)(\sin\theta_1 - \sin\theta_2)\,\mathrm{d}x \\
&- \frac{\nu}{2}\int_{\mathbb{R}} (\sin\theta_2 - h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} (\sin\theta_2 - h)\,\mathrm{d}x \\
&+ \frac{\nu}{2}\int_{\mathbb{R}} (\sin\theta_1 - h)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} (\sin\theta_2 - h)\,\mathrm{d}x.
\end{aligned} \tag{3.5}
$$

By (3.3) and dominated convergence theorem, we have

$$
\begin{aligned}
\lim_{t\to 0^+} \int_{\mathbb{R}} \frac{|\theta_x^t|^2 - \theta_{2x}^2}{2t}\,\mathrm{d}x &= \int_{\mathbb{R}} \lim_{t\to 0^+} \frac{|\theta_x^t|^2 - \theta_{2x}^2}{2t}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \theta_{2x}\theta_{xt}^t|_{t=0}\mathrm{d}x = -\int_{\mathbb{R}} \theta_{2xx}\theta_t^t|_{t=0}\,\mathrm{d}x.
\end{aligned} \tag{3.6}
$$

Here, we used the fact that

$$
\theta_{xt}^t(x)|_{t=0} = \begin{cases} \dfrac{\theta_{1x}\cos\theta_1 - \theta_{2x}\cos\theta_2}{\cos\theta_2} + \dfrac{\sin\theta_2(\sin\theta_1 - \sin\theta_2)}{\cos^2\theta_2}\theta_{2x} & x \neq 0, \\ -\dfrac{\theta_{2x}^2(0) - \theta_{1x}^2(0)}{2\theta_{2x}(0)} & x = 0, \end{cases}
$$

and

$$
\theta_t^t(x)|_{t=0} = \begin{cases} \dfrac{\sin\theta_1 - \sin\theta_2}{\cos\theta_2} & x \neq 0, \\ 0 & x = 0, \end{cases} \tag{3.7}
$$

are both continuous on $\mathbb{R}$, which follows from (B 4), and the fact

$$
\lim_{x\to\pm\infty} \frac{\sin\theta_1 - \sin\theta_2}{\cos\theta_2} = 0.
$$

We conclude from (3.5)–(3.7) that

$$
f'(0) = \int_{\mathbb{R}} \left\{ -\frac{\mathrm{d}^2\theta_2}{\mathrm{d}x^2} + \cos\theta_2(\sin\theta_2 - h) + \frac{\nu}{2}\cos\theta_2\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\sin\theta_2 \right\} \frac{\mathrm{d}\theta^t}{\mathrm{d}t}(x)\bigg|_{t=0} \mathrm{d}x = 0. \tag{3.8}
$$

A similar argument gives

$$f'(1) = 0. \tag{3.9}$$

On the other hand, it follows from lemma 3.2 that

$$f(t) \in C^2[0,1] \quad \text{and} \quad f''(t) > 0 \ \text{ on } [0,1].$$

Therefore, one cannot have (3.8) and (3.9) to hold at the same time, a contradiction. ∎

Remark 3.3. Our proof of uniqueness works as long as $\theta(x)$ has range $(\theta_h, \pi - \theta_h)$, satisfies (2.4) and passes through $\pi/2$ only once.

4. Uniform bounds and decay of the derivatives

(a) Uniform bound for solutions with bounded energy

Let

$$u(x) = \sin\theta(x) - h, \quad v(x) = \left(-\frac{d^2}{dx^2}\right)^{1/2} \sin\theta = \left(-\frac{d^2}{dx^2}\right)^{1/2} u(x). \tag{4.1}$$

We first recall from the proof in §5, step 2, in [16] that any solution θ of (2.3) with bounded energy is smooth. We shall use this fact for the rest of the section.

Lemma 4.1. *Let $\nu > 0$, let $h \in \mathbb{R}$ and let θ be a solution of (2.3) such that $E(\theta) < \infty$. Then, there exists a constant $C = C(\nu, h, E(\theta)) > 0$ such that $|v(x)| \le C$ for all $x \in \mathbb{R}$.*

Proof. Using the identity (see, for example, formula (3.1) in [23])

$$\left(-\frac{d^2}{dx^2}\right)^{1/2} u(x) = \frac{1}{\pi}\ \text{p.v.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{(x-y)^2}\, dy,$$

for every $x \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where p.v. stands for the principal value of the integral, we can write

$$v(x) = \frac{1}{\pi}\ \text{p.v.} \int_{\mathbb{R}} \frac{\sin\theta(x) - \sin\theta(y)}{(x-y)^2}\, dy.$$

Given $\delta > 0$, we have

$$\begin{aligned} \pi v(x) &= \text{p.v.} \int_{\mathbb{R}} \frac{\sin\theta(x) - \sin\theta(y)}{(x-y)^2}\, dy \\ &= \int_{x+\delta}^{\infty} \frac{\sin\theta(x) - \sin\theta(y)}{(x-y)^2}\, dy + \int_{-\infty}^{x-\delta} \frac{\sin\theta(x) - \sin\theta(y)}{(x-y)^2}\, dy \\ &\quad + \text{p.v.} \int_{x-\delta}^{x+\delta} \frac{\sin\theta(x) - \sin\theta(y)}{(x-y)^2}\, dy. \end{aligned} \tag{4.2}$$

The first two terms are bounded by $2/\delta$ after direct integration. Because θ is smooth, it follows from Taylor expansion that the third term on the right-hand side of (4.2) can be bounded by

$$\left| \text{p.v.} \int_{x-\delta}^{x+\delta} \frac{\sin\theta(x) - \sin\theta(y)}{(x-y)^2}\, dy \right| \le \max_{[x-\delta, x+\delta]} |u_{xx}|\delta. \tag{4.3}$$

Furthermore, we have

$$\begin{aligned} \max_{[x-\delta, x+\delta]} |u_{xx}| &= \max_{[x-\delta, x+\delta]} \left| \theta_{xx}\cos\theta - \theta_x^2 \sin\theta \right| \\ &\le \max_{[x-\delta, x+\delta]} |\theta_{xx}| + \max_{[x-\delta, x+\delta]} |\theta_x^2|. \end{aligned} \tag{4.4}$$

To estimate the first term on the right-hand side of (4.4), we use (2.3) to obtain

$$\max_{[x-\delta, x+\delta]} |\theta_{xx}| \le 1 + |h| + \frac{\nu}{2} \max_{[x-\delta, x+\delta]} |v(x)|. \tag{4.5}$$

To obtain a bound on θ_x, we observe that because $E(\theta) < \infty$, there exists a sequence $\{x_n\} \to -\infty$ such that $\theta_x(x_n) \to 0$. Therefore, multiplying (2.3) by θ_x and integrating from x_n to x, we get

$$\frac{1}{2}\theta_x^2(x)\bigg|_{x_n}^{x} = \frac{1}{2}u^2(x)\bigg|_{x_n}^{x} + \frac{\nu}{2}\int_{x_n}^{x} v(y)\,\mathrm{d}u(y). \tag{4.6}$$

Because

$$\int_{x_n}^{x} v^2\,\mathrm{d}x \le \int_{\mathbb{R}} v^2\,\mathrm{d}x = \int_{\mathbb{R}} u_x^2\,\mathrm{d}x \le \int_{\mathbb{R}} \theta_x^2\,\mathrm{d}x \le 2E(\theta),$$

we can bound the integral in the second term on the right-hand side of (4.6) as follows

$$\left|\int_{x_n}^{x} v(y)\,\mathrm{d}u(y)\right| \le \left(\int_{x_n}^{x} v^2\,\mathrm{d}x\right)^{1/2}\left(\int_{x_n}^{x} u_y^2\mathrm{d}y\right)^{1/2} \le 2E(\theta). \tag{4.7}$$

Furthermore, because $|u(x)+h| \le 1$, we get

$$\theta_x^2(x) - \theta_x^2(x_n) \le (1+|h|)^2 + 2\nu E(\theta).$$

Finally, sending $n \to \infty$, we obtain

$$|\theta_x(x)| \le \sqrt{(1+|h|)^2 + 2\nu E(\theta)} \quad \text{for any } x \in \mathbb{R}. \tag{4.8}$$

From (4.2)–(4.5) and (4.8), we thus conclude

$$\pi \max_{\mathbb{R}} |v| \le \frac{2}{\delta} + \left(1 + |h| + \frac{\nu}{2}\max_{\mathbb{R}}|v| + 2\nu E(\theta) + (1+|h|)^2\right)\delta.$$

Choosing $\delta = \pi/\nu$, we get

$$\max_{\mathbb{R}} |v| \le \frac{4\nu}{\pi^2} + \frac{2}{\nu}(1 + |h| + (1+|h|)^2) + 4E(\theta).$$

∎

Corollary 4.2. *There exists $C_i = C_i(\nu, h, E(\theta)) > 0$ $(i = 1, 2, \ldots,)$ such that, given any solution θ of (2.3) with $E(\theta) < \infty$, we have*

$$\sup_{x\in\mathbb{R}}\left(\left|\frac{\mathrm{d}^i\theta}{\mathrm{d}x^i}(x)\right|\right) \le C_i.$$

Proof. The estimate for θ_x, θ_{xx} follows directly from (4.5), (4.8) and lemma 4.1. To estimate θ_{xxx}, differentiate (2.3). We have

$$\theta_{xxx} = \theta_x \cos^2\theta - \theta_x \sin\theta(\sin\theta - h) - \nu\theta_x \sin\theta + v_x \cos\theta.$$

It then follows that

$$|\theta_{xxx}| \le C + |v_x|.$$

Because

$$\pi\, v_x(x) = \mathrm{p.v.}\int \frac{u_x(x) - u_y(y)}{(x-y)^2}\,\mathrm{d}y,$$

using Taylor expansion, we get

$$\pi\,|v_x| \le \max|u_{xxx}|\delta + \frac{C}{\delta}.$$

On the other hand,

$$u_{xxx} = \theta_{xxx} - \theta_x^3\cos\theta - 3\theta_x\theta_{xx}\sin\theta.$$

We can then follow a similar argument as in lemma 4.1 to get a bound on $|v_x|$ and, thus, a bound on $|\theta_{xxx}|$. Differentiating repeatedly, we obtain similar estimates for all derivatives. ∎

Because any solution of (2.3) with bounded energy is in $W^{k,2}(\mathbb{R})$ for any $k \in \mathbb{N}$, as a direct corollary of our bound on the derivatives of θ, we conclude that any solution of (2.3) with bounded energy must have all its derivatives vanish at infinity.

Corollary 4.3. *If θ is a solution of (2.3) with bounded energy, then all the derivatives of θ vanish at infinity.*

The proof of theorem 2.2 is now complete, combining corollaries 4.2 and 4.3.

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Appendix A. Decay of Néel walls

Here, we revisit the question of the asymptotic decay of Néel wall solutions, whose existence and uniqueness is guaranteed by theorem 2.1. Let θ be the unique minimizer of E in $\mathcal{A}$ satisfying $\theta(0) = \pi/2$, and introduce

$$\rho(x) := \begin{cases} \theta(x), & x \geq 0, \\ \pi - \theta(x), & x < 0. \end{cases} \tag{A 1}$$

Note that $\rho - \theta_h \in H^1(\mathbb{R})$, $\sin\rho = \sin\theta$, $\cos\rho = \operatorname{sgn}(x)\cos\theta$ and $\rho(x)$ is a smooth even function of x, except at $x = 0$, where ρ_x undergoes a jump discontinuity.

Proceeding as in step 4 of the proof of theorem 2.1 in [16], we observe that ρ satisfies distributionally

$$L(\rho(x) - \theta_h) = f(x) + a\delta(x), \tag{A 2}$$

where

$$L := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\nu\cos^2\theta_h\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} + \cos^2\theta_h \tag{A 3}$$

is a one-to-one linear map from $\mathcal{S}'(\mathbb{R})$ to itself, which is also a bounded operator from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$, $f \in L^2(\mathbb{R})$ is defined in equation (66) of Chermisi & Muratov [16], $a = 2|\theta'(0)| > 0$ and $\delta(x)$ is the Dirac delta function. The last term in the right-hand side of (A 2) was inadvertently omitted in [16]. Nevertheless, its presence does not affect the rest of the proof. Namely, we invert the operator L with the help of the fundamental solution G (see [16, lemma A1] for an explicit definition and properties of G). In particular, we have

$$\rho(x) = \theta_h + 2aG(x) + \int_{\mathbb{R}} G(x-y)f(y)\,\mathrm{d}y, \tag{A 4}$$

for every $x \in \mathbb{R}$. The presence of the $2aG$ term in the right-hand side of (A 4) leaves all the remaining estimates unchanged, in view of the fact that $0 < G(x) \leq C/|x|^2$, for some $C > 0$.

Appendix B. Proof of lemma 3.1

Here, we present the proof of lemma 3.1.

Proof. By our assumption, when $x \neq 0$, we have

$$0 < t\sin\theta_1 + (1-t)\sin\theta_2 < 1$$

for any $t \in [0,1]$. Because $\arcsin(u)$ is differentiable for $u < 1$, chain rule applies when taking derivative of $\theta^t(x)$ with respect to x at $x \neq 0$ for any $t \in [0,1]$. From the assumption on θ_i and the definition of θ^t, we have

$$\cos\theta^t(x) = \mathrm{sgn}(x)\sqrt{1-\sin^2\theta^t(x)}.$$

Direct calculation then gives

$$\begin{aligned}
\sin\theta^t &= t\sin\theta_1 + (1-t)\sin\theta_2,\\
(\sin\theta^t)_x &= t\theta_{1x}\cos\theta_1 + (1-t)\theta_{2x}\cos\theta_2,\\
\theta^t_x &= \frac{t\theta_{1x}\cos\theta_1 + (1-t)\theta_{2x}\cos\theta_2}{\cos\theta^t}, \quad x\neq 0.
\end{aligned} \tag{B 1}$$

Observe that when $x\neq 0$, the function $1/\cos\theta^t$ is differentiable with respect to t for any $t\in[0,1]$. Differentiating (B 1) with respect to t, we get for $x\neq 0$

$$\theta^t_{xt} = \frac{\theta_{1x}\cos\theta_1 - \theta_{2x}\cos\theta_2}{\cos\theta^t} + \frac{\sin\theta^t(\sin\theta_1-\sin\theta_2)(t\theta_{1x}\cos\theta_1 + (1-t)\theta_{2x}\cos\theta_2)}{\cos^3\theta^t} \tag{B 2}$$

and

$$\begin{aligned}
\theta^t_{xtt} &= 3\frac{(\sin\theta_1-\sin\theta_2)^2\sin^2\theta^t(t\theta_{1x}\cos\theta_1 + (1-t)\theta_{2x}\cos\theta_2)}{\cos^5\theta^t}\\
&\quad + 2\frac{\sin\theta^t(\sin\theta_1-\sin\theta_2)(\theta_{1x}\cos\theta_1 - \theta_{2x}\cos\theta_2)}{\cos^3\theta^t}\\
&\quad + \frac{(\sin\theta_1-\sin\theta_2)^2(t\theta_{1x}\cos\theta_1 + (1-t)\theta_{2x}\cos\theta_2)}{\cos^3\theta^t}.
\end{aligned} \tag{B 3}$$

From (B 1) to (B 3), continuity of θ^t_x, θ^t_{xt} and θ^t_{xtt} with respect to x for all $x\neq 0$ follows. For $x=0$, we calculate the derivatives of θ^t via the definition as follows. By assumption, we have $0<\theta^t(x)<\pi/2$ when $x>0$ and $\pi/2<\theta^t(x)<\pi$ when $x<0$. From this, we obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos\theta^t(x)}{x} &= \lim_{x\to 0}\frac{|\cos\theta^t(x)|}{|x|} = \lim_{x\to 0}\frac{\sqrt{1-\sin^2\theta^t(x)}}{|x|}\\
&= \lim_{x\to 0}\sqrt{t\left(\frac{1-\sin\theta_1(x)}{x^2}\right) + (1-t)\left(\frac{1-\sin\theta_2(x)}{x^2}\right)}\\
&\quad \times \lim_{x\to 0}\sqrt{t(1+\sin\theta_1(x)) + (1-t)(1+\sin\theta_2(x))}\\
&= \sqrt{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)},
\end{aligned}$$

The last step in the limit above follows from applying L'Hospital's rule in

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\sin\theta_i(x)}{x^2} &= -\lim_{x\to 0}\frac{\theta_{ix}(x)\cos\theta_i(x)}{2x}\\
&= \lim_{x\to 0}\frac{\theta^2_{ix}(x)\sin\theta_i(x) - \theta_{ixx}(x)\cos\theta_i(x)}{2} = \frac{1}{2}\theta^2_{ix}(0).
\end{aligned} \tag{B 4}$$

We calculate the derivative of $\theta^t(x)$ with respect to x at $x=0$ as follows

$$\begin{aligned}
\theta^t_x(0) &= \lim_{x\to 0}\frac{\theta^t(x)-\theta^t(0)}{x}\\
&= \lim_{x\to 0}\frac{\theta^t(x)-\theta^t(0)}{\sin(\theta^t(x)-\theta^t(0))}\times\frac{\sin(\theta^t(x)-\theta^t(0))}{x}\\
&= \lim_{x\to 0}\frac{\sin\theta^t(x)\cos\theta^t(0) - \sin\theta^t(0)\cos\theta^t(x)}{x}\\
&= -\lim_{x\to 0}\frac{\cos\theta^t(x)}{x} = -\sqrt{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)}.
\end{aligned} \tag{B 5}$$

Moreover, by (B 4), we have

$$\lim_{x\to 0}\frac{\cos\theta_i(x)}{\cos\theta^t(x)} = \lim_{x\to 0}\sqrt{\left(\frac{1-\sin\theta_i(x)}{x^2}\right)(1+\sin\theta_i(x))} \times \lim_{x\to 0}\frac{1}{\sqrt{t\left(\frac{1-\sin\theta_1(x)}{x^2}\right)+(1-t)\left(\frac{1-\sin\theta_2(x)}{x^2}\right)}}$$

$$\times \lim_{x\to 0}\frac{1}{\sqrt{t(1+\sin\theta_1(x))+(1-t)(1+\sin\theta_2(x))}}$$

$$= -\frac{\theta_{ix}(0)}{\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}}. \tag{B 6}$$

where in the last step we used the fact that $\theta_{ix}(0)<0$. It then follows from (B 6) that

$$\lim_{x\to 0}\theta_x^t(x) = \lim_{x\to 0}\frac{(t\theta_{1x}(x)\cos\theta_1(x)+(1-t)\theta_{2x}(x)\cos\theta_2(x))}{\cos\theta^t(x)}$$

$$= -\frac{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}{\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}} = -\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}, \tag{B 7}$$

Equations (B 5) and (B 7) imply that $\theta_x^t(x)$ is continuous at $x=0$ for any $t\in[0,1]$. Continuity of θ_x^t with respect to t is obvious from (B 1) and (B 5).

Next, we evaluate θ_{xt}^t at $x=0$. Recall that $\theta_{ix}(0)<0$ and differentiate (B 5) with respect to t. We get

$$\theta_{xt}^t(0) = \frac{\theta_{2x}^2(0)-\theta_{1x}^2(0)}{2\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}}. \tag{B 8}$$

On the other hand, (B 4) yields

$$\lim_{x\to 0}\frac{\sin\theta_1(x)-\sin\theta_2(x)}{1-\sin^2\theta^t(x)} = \lim \frac{(\sin\theta_1(x)-1)/x^2+(1-\sin\theta_2(x))/x^2}{t((1-\sin\theta_1(x))/x^2)+(1-t)(1-\sin\theta_2(x))/x^2}$$

$$\times \lim_{x\to 0}\frac{1}{t(1+\sin\theta_1(x))+(1-t)(1+\sin\theta_2(x))}$$

$$= \frac{-\frac{1}{2}\theta_{1x}^2(0)+\frac{1}{2}\theta_{2x}^2(0)}{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)} \tag{B 9}$$

Using (B 6) and (B 9), we evaluate the limit of θ_{xt}^t at $x=0$ as

$$\lim_{x\to 0}\theta_{xt}^t(x) = \lim_{x\to 0}\frac{\theta_{1x}(x)\cos\theta_1(x)-\theta_{2x}(x)\cos\theta_2(x)}{\cos\theta^t(x)}$$

$$+\lim_{x\to 0}\frac{\sin\theta^t(x)(\sin\theta_1(x)-\sin\theta_2(x))(t\theta_{1x}(x)\cos\theta_1(x)+(1-t)\theta_{2x}(x)\cos\theta_2(x))}{\cos^3\theta^t(x)}$$

$$= \frac{-\theta_{1x}^2(0)+\theta_{2x}^2(0)}{\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}} + \lim_{x\to 0}\frac{\sin\theta_1(x)-\sin\theta_2(x)}{1-\sin^2\theta^t(x)}$$

$$\times \frac{t\theta_{1x}(x)\cos\theta_1(x)+(1-t)\theta_{2x}(x)\cos\theta_2(x)}{\cos\theta^t(x)}$$

$$= \frac{-\theta_{1x}^2(0)+\theta_{2x}^2(0)}{\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}} - \frac{-(1/2)\theta_{1x}^2(0)+(1/2)\theta_{2x}^2(0)}{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)} \times \frac{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}{\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}}$$

$$= \frac{1}{2}\frac{-\theta_{1x}^2(0)+\theta_{2x}^2(0)}{\sqrt{t\theta_{1x}^2(0)+(1-t)\theta_{2x}^2(0)}}. \tag{B 10}$$

We conclude from (B 8) and (B 10) that $\theta_{xt}^t(x)$ is continuous at $x=0$ and continuity of $\theta_{xt}^t(x)$ with respect to t follows from (B 2) and (B 8). Lastly, recall that $\theta_{ix}(0)<0$ and differentiate (B 8) with respect to t. This yields

$$\theta^t_{xtt}(0) = \frac{1}{4}\frac{(\theta^2_{2x}(0) - \theta^2_{1x}(0))^2}{\left(\sqrt{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)}\right)^3}. \tag{B 11}$$

To derive continuity of $\theta^t_{xtt}(x)$ at $x = 0$, we calculate the limit of $\theta^t_{xtt}(x)$ at $x = 0$. By (B 6) and (B 9),

$$\begin{aligned}
\lim_{x\to 0}\theta^t_{xtt}(x) &= \lim_{x\to 0}\left\{3\frac{(\sin\theta_1(x) - \sin\theta_2(x))^2\sin^2\theta^t(x)(t\theta_{1x}(x)\cos\theta_1(x) + (1-t)\theta_{2x}(x)\cos\theta_2(x))}{\cos^5\theta^t(x)}\right.\\
&\quad + 2\frac{\sin\theta^t(x)(\sin\theta_1(x) - \sin\theta_2(x))(\theta_{1x}(x)\cos\theta_1(x) - \theta_{2x}(x)\cos\theta_2(x))}{\cos^3\theta^t(x)}\\
&\quad \left. + \frac{(\sin\theta_1(x) - \sin\theta_2(x))^2(t\theta_{1x}(x)\cos\theta_1(x) + (1-t)\theta_{2x}(x)\cos\theta_2(x))}{\cos^3\theta^t(x)}\right\}\\
&= -3\left(\frac{-(1/2)\theta^2_{1x}(0) + (1/2)\theta^2_{2x}(0)}{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)}\right)^2 \times \sqrt{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)}\\
&\quad + 2\frac{-(1/2)\theta^2_{1x}(0) + (1/2)\theta^2_{2x}(0)}{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)}\,\frac{-\theta^2_{1x}(0) + \theta^2_{2x}(0)}{\sqrt{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)}}\\
&= \frac{1}{4}\frac{(\theta^2_{2x}(0) - \theta^2_{1x}(0))^2}{(\sqrt{t\theta^2_{1x}(0) + (1-t)\theta^2_{2x}(0)})^3}.
\end{aligned} \tag{B 12}$$

Continuity of $\theta^t_{xtt}(x)$ at $x = 0$ follows from (B 11) and (B 12). Continuity of $\theta^t_{xtt}(x)$ with respect to t variable follows from (B 3) and (B 11).

Finally, we derive the bounds on those derivatives. When $x = 0$, it follows directly from (B 5), (B 8) and (B 11) that

$$|\theta^t_x(0)| \le |\theta_{1x}(0)| + |\theta_{2x}(0)|, \tag{B 13}$$

$$|\theta^t_{xt}(0)| \le \frac{1}{2}\max\left(\left|\frac{\theta_{1x}(0)}{\theta_{2x}(0)}\right|, \left|\frac{\theta_{2x}(0)}{\theta_{1x}(0)}\right|\right)(|\theta_{1x}(0)| + |\theta_{2x}(0)|) \tag{B 14}$$

and

$$|\theta^t_{xtt}(0)| \le \frac{1}{4}\max\left(\left|\frac{\theta_{1x}(0)}{\theta_{2x}(0)}\right|^3, \left|\frac{\theta_{2x}(0)}{\theta_{1x}(0)}\right|^3\right)(|\theta_{1x}(0)| + |\theta_{2x}(0)|). \tag{B 15}$$

To obtain the bound for $x \neq 0$, we write

$$\begin{aligned}
\sqrt{1 - (t\sin\theta_1 + (1-t)\sin\theta_2)^2} &\ge \sqrt{t(1-\sin\theta_1) + (1-t)(1-\sin\theta_2)}\\
&\ge t\sqrt{1-\sin\theta_1} + (1-t)\sqrt{1-\sin\theta_2},
\end{aligned}$$

where the last inequality follows from the concavity of the function $F(s) = \sqrt{s}$. It then follows that for all $0 \le t \le 1$ and $x \neq 0$, we have

$$\begin{aligned}
\left|\frac{(t\theta_{1x}\cos\theta_1 + (1-t)\theta_{2x}\cos\theta_2)}{\cos\theta^t}\right| &\le |\theta_{1x}|\frac{\sqrt{2(1-\sin\theta_1)}}{t\sqrt{1-\sin\theta_1} + (1-t)\sqrt{1-\sin\theta_2}}\\
&\quad + |\theta_{2x}|\frac{\sqrt{2(1-\sin\theta_2)}}{t\sqrt{1-\sin\theta_1} + (1-t)\sqrt{1-\sin\theta_2}}\\
&\le \sqrt{2}N(x)(|\theta_{1x}| + |\theta_{2x}|),
\end{aligned} \tag{B 16}$$

where

$$N(x) = \max\left(\frac{\sqrt{1-\sin\theta_1(x)}}{\sqrt{1-\sin\theta_2(x)}}, \frac{\sqrt{1-\sin\theta_2(x)}}{\sqrt{1-\sin\theta_1(x)}}\right).$$

Similarly, we have

$$\begin{aligned}
&\left|\frac{(\theta_{1x}\cos\theta_1-\theta_{2x}\cos\theta_2)}{\cos\theta^t}\right|\\
&\quad\le|\theta_{1x}|\frac{\sqrt{2(1-\sin\theta_1)}}{t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2}}\\
&\qquad+|\theta_{2x}|\frac{\sqrt{2(1-\sin\theta_2)}}{t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2}}\\
&\quad\le\sqrt{2}N(x)(|\theta_{1x}|+|\theta_{2x}|),
\end{aligned}\tag{B 17}$$

$$\begin{aligned}
&\left|\frac{(\sin\theta_1-\sin\theta_2)(t\theta_{1x}\cos\theta_1+(1-t)\theta_{2x}\cos\theta_2)}{\cos^3\theta^t}\right|\\
&\quad\le|\theta_{1x}|\frac{(\sqrt{1-\sin\theta_1})^3+\sqrt{1-\sin\theta_1}(1-\sin\theta_2)}{(t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2})^3}\\
&\qquad+|\theta_{2x}|\frac{\sqrt{1-\sin\theta_2}(1-\sin\theta_1)+(\sqrt{1-\sin\theta_2})^3}{(t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2})^3}\\
&\quad\le(N^3(x)+N(x))(|\theta_{1x}|+|\theta_{2x}|),
\end{aligned}\tag{B 18}$$

$$\begin{aligned}
&\left|\frac{(\sin\theta_1-\sin\theta_2)(\theta_{1x}\cos\theta_1-\theta_{2x}\cos\theta_2)}{\cos^3\theta^t}\right|\\
&\quad\le|\theta_{1x}|\frac{(\sqrt{1-\sin\theta_1})^3+\sqrt{1-\sin\theta_1}(1-\sin\theta_2)}{(t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2})^3}\\
&\qquad+|\theta_{2x}|\frac{\sqrt{1-\sin\theta_2}(1-\sin\theta_1)+(\sqrt{1-\sin\theta_2})^3}{(t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2})^3}\\
&\quad\le(N^3(x)+N(x))(|\theta_{1x}|+|\theta_{2x}|)
\end{aligned}\tag{B 19}$$

and

$$\begin{aligned}
&\frac{(\sin\theta_1-\sin\theta_2)^2(t\theta_{1x}\cos\theta_1+(1-t)\theta_{2x}\cos\theta_2)}{\cos^5\theta^t}\\
&\quad\le|\theta_{1x}|\frac{(\sqrt{1-\sin\theta_1})^5+\sqrt{1-\sin\theta_1}(1-\sin\theta_2)^2}{(t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2})^5}\\
&\qquad+|\theta_{2x}|\frac{\sqrt{1-\sin\theta_2}(1-\sin\theta_1)+(\sqrt{1-\sin\theta_2})^3}{(t\sqrt{1-\sin\theta_1}+(1-t)\sqrt{1-\sin\theta_2})^3}\\
&\quad\le(N^5(x)+N(x))(|\theta_{1x}|+|\theta_{2x}|).
\end{aligned}\tag{B 20}$$

By (B 4), there exists $\delta_0>0$ such that for $i=1,2$, we have

$$\frac{1}{4}\theta_{ix}^2(0)<\frac{1-\sin\theta_i(x)}{x^2}<\frac{3}{4}\theta_{ix}^2(0)\quad\text{when } |x|<\delta_0.$$

Therefore, for $|x|<\delta_0$, we obtain

$$\max\left\{\frac{\sqrt{1-\sin\theta_1(x)}}{\sqrt{1-\sin\theta_2(x)}},\frac{\sqrt{1-\sin\theta_2(x)}}{\sqrt{1-\sin\theta_1(x)}}\right\}<3\max\left\{\frac{\theta_{1x}(0)}{\theta_{2x}(0)},\frac{\theta_{2x}(0)}{\theta_{1x}(0)}\right\},\tag{B 21}$$

and for $|x|\ge\delta_0$, we get

$$\begin{aligned}
&\max\left\{\frac{\sqrt{1-\sin\theta_1(x)}}{\sqrt{1-\sin\theta_2(x)}},\frac{\sqrt{1-\sin\theta_2(x)}}{\sqrt{1-\sin\theta_1(x)}}\right\}\\
&\le\max\left\{\frac{1-h}{1-\sin\theta_1(\delta_0)},\frac{1-h}{1-\sin\theta_1(-\delta_0)},\frac{1-h}{1-\sin\theta_2(\delta_0)},\frac{1-h}{1-\sin\theta_2(-\delta_0)}\right\}.\\
&=L_{\delta_0}.
\end{aligned}\tag{B 22}$$

Let

$$M = \max\left\{1, L_{\delta_0}, \left(\frac{\theta_{1x}(0)}{\theta_{2x}(0)}\right)^3, \left(\frac{\theta_{2x}(0)}{\theta_{1x}(0)}\right)^3, 3\frac{\theta_{1x}(0)}{\theta_{2x}(0)}, 3\frac{\theta_{2x}(0)}{\theta_{1x}(0)}\right\}.$$

Then, equations (B 1)–(B 3), (B 13)–(B 15), (B 16)–(B 20), (B 21) and (B 22) imply

$$|\theta_x^t(x)| \le M(|\theta_{1x}(x)| + |\theta_{2x}(x)|),$$

$$|\theta_{xt}^t(x)| \le (M^3 + M)(|\theta_{1x}(x)| + |\theta_{2x}(x)|),$$

$$|\theta_{xtt}^t(x)| \le (M^5 + M)(|\theta_{1x}(x)| + |\theta_{2x}(x)|),$$

for all $x \in \mathbb{R}$. The conclusion then follows by taking $K = M^5 + M$. ■

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