A GLOBAL VARIATIONAL STRUCTURE AND PROPAGATION OF DISTURBANCES IN REACTION-DIFFUSION SYSTEMS OF GRADIENT TYPE

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Abstract. We identified a variational structure associated with traveling waves for systems of reaction-diffusion equations of gradient type with equal diffusion coefficients defined inside an infinite cylinder, with either Neumann or Dirichlet boundary conditions. We show that the traveling wave solutions that decay sufficiently rapidly exponentially at one end of the cylinder are critical points of certain functionals. We obtain a global upper bound on the speed of these solutions. We also show that for a wide class of solutions of the initial value problem an appropriately defined instantaneous propagation speed approaches a limit at long times. Furthermore, under certain assumptions on the shape of the solution, there exists a reference frame in which the solution of the initial value problem converges to the traveling wave solution with this speed at least on a sequence of times. In addition, for a class of solutions we establish bounds on the shape of the solution in the reference frame associated with its leading edge and determine accessible limiting traveling wave solutions. For this class of solutions we find the upper and lower bounds for the speed of the leading edge.

1. Introduction

In this paper, we study the following initial value problem

$$u_t = \Delta u + f(u), \ u(x,0) = u_0(x), \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^m$, $x \in \Sigma = \mathbb{R} \times \Omega$, $\Omega \subset \mathbb{R}^{n-1}$ is a bounded domain with smooth boundary, $f : \mathbb{R}^m \to \mathbb{R}^m$ is in C^∞, with either Neumann or Dirichlet boundary conditions

$$(n \cdot \nabla u)|_{\partial\Sigma} = 0, \text{ or } u|_{\partial\Sigma} = 0, \tag{1.2}$$

where n is the outward normal to $\partial\Sigma$. This problem describes a reaction-diffusion system with equal diffusion coefficients defined inside an n-dimensional cylinder Σ. We assume that $f(0) = 0$, so

$$u = 0 \tag{1.3}$$

is a trivial solution of Eqs. (1.1) and (1.2). Furthermore, we consider reaction-diffusion systems of *gradient type*. That is, there exists a function $V(u)$ such that

$$f = -\nabla_u V, \ V : \mathbb{R}^m \to \mathbb{R} \tag{1.4}$$

Note that for the scalar case ($m = 1$) the system is automatically of gradient type. As a special case, we also consider the problem in one space dimension. We will be interested in the *propagation of disturbances* in these systems. More precisely, we

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will study the solutions of this initial value problem which decay exponentially to zero at one or both ends of the cylinder.

This type of problems arises in a wide variety of applications. Specifically, the scalar case arises in the problems of combustion, chemical reactions, population dynamics, propagation of nerve impulses [1–4]. There, the scalar variable u may play the role of temperature, concentration of chemical species, the membrane voltage, etc. Specifically, the quintic Ginzburg-Landau equation

$$u_t = \Delta u + \mu u \pm u^3 - u^5, \tag{1.5}$$

where μ is a parameter, often arises as an amplitude equation that describes large-scale behavior of systems near a bifurcation point [1]. Other well-known examples include the Nagumo equation of a nerve axon, for which $f(u) = u(u-a)(1-u)$, with $0 < a < 1/2$, or the Arrhenius nonlinearity $f(u) = \mathrm{e}^{-\frac{a}{u}}(1-u)$, $a > 0$, for combustion problems [4, 5]. The vector case arises in the kinetics of phase transitions, in this case u may stand for the magnetization or polarization vector [6–8]. For example, if u_i are the three components of the magnetization vector in a ferromagnetic crystal with cubic symmetry near Curie temperature, and h_i are the components of the applied field, the kinetics of u may be described by the following Ginzburg-Landau equation:

$$\tau \frac{\partial u_i}{\partial t} = g\Delta u_i + h_i + a u_i - b u_i^3 - c u_i \sum_{i \neq j} u_j^2, \tag{1.6}$$

where τ, g, a, b, c are coefficients. Alternatively, in one spatial dimension this equation with $h = 0$, $\tau = g = a = b = c = 1$ and $m = 2$ (or, equivalently, u complex) is the Newell-Whitehead-Segel equation describing the Rayleigh-Bénard convection near the onset [1].

A ubiquitous feature of the systems of this kind is that they are capable of supporting traveling waves propagating with constant speed [1]. Numerical simulations of various examples show that at long times a localized initial condition generically evolves into these traveling waves. It is therefore important to understand the approach of the solutions of the initial value problem in Eq. (1.1) to the traveling wave solutions and wave selection [1].

These topics have been investigated by a great number of authors. In their pioneering work of 1937, Kolmogorov, Petrovsky, and Piskunov proved existence of traveling wave solutions in scalar reaction-diffusion equations with certain types of nonlinearities in one dimension [9]. Furthermore, they showed that an initial condition in the form of a step function converges in shape to a particular traveling wave solution at long times. Kanel' extended these results to a more general class of initial conditions in a different family of systems [10, 11]. Fife and McLeod studied bistable scalar reaction-diffusion systems in one dimension and proved existence and uniqueness of the traveling wave solutions in these systems [12]. They further proved that an initial condition that looks like a front converges asymptotically to a single traveling wave solution, while a localized initial condition converges to a pair of counter-propagating fronts. They also extended their results on existence of one-dimensional traveling waves to planar fronts in $\mathbb{R}^n$. Further existence results for scalar one-dimensional problem were obtained by Aronson and Weinberger for a class of nonlinearities with linearly unstable equilibrium state $u = 0$, and by Medvedev, Ono, and Holmes for the degenerate case [13, 14] (see also [15]). In another direction, Aronson and Weinberger used the comparison principle to investigate the speed of propagation of disturbances in scalar reaction-diffusion equations [13, 16]. They obtained asymptotic propagation speeds of certain classes of the initial conditions, while not addressing the question of convergence.

For higher-dimensional scalar equations, Gardner studied existence of the traveling wave solutions in the form of curved fronts on a strip [17]. Vega proved existence of the traveling wave solutions in cylinders with Dirichlet boundary conditions [18]. The most general existence results for traveling waves in cylinders were obtained by Berestycki, Larrouturou, Lions, and Nirenberg, who also considered the effect of advective terms [19, 20]. More recently, Volpert and Volpert proved existence and stability of multidimensional traveling waves for monotone parabolic systems of equations [21,22]. Heinze gave a proof of existence of the traveling wave solutions in rather general scalar systems with bistable and combustion-type nonlinearities in cylinders [23]. Convergence of the solutions of the initial value problem to traveling waves in scalar reaction-diffusion equations in cylinders was shown under rather general conditions by Roquejoffre, and for monotone systems in one dimension by Roquejoffre, Terman, and Volpert [24–26].

In addition, the problem of traveling wave selection was considered in a number of papers. Van Saarloos studied the problem of linear and nonlinear front selection in the scalar equations in one space dimension [27,28] (see also [29]). More recently, Brazhnik used the asymptotic theory of slightly curved fronts to construct geometrically the solutions in the form of the V-shaped fronts, and Bonnet and Hamel proved existence of such waves in a combustion model [30, 31]. Numerical methods for finding traveling wave solutions in cylinders were developed by Lord *et al.* [32]. This list of references is far from exhaustive, for example, see also [1, 20, 33–35].

Most of the works mentioned above rely heavily on the applications of the maximum principle and are, therefore, limited to scalar equations and monotone systems. A natural question that arises is whether there exists some unifying structure within the framework of Eqs. (1.1) – (1.4) that invariably leads to the formation of traveling wave solutions of certain types. In this paper, we show that a common feature in this situation is the existence of a general variational structure. It turns out that a large class of traveling wave solutions are critical points of certain functionals. The key feature of these functionals is the fact that, in contrast to Eq. (1.1), they are not invariant under translations of u. In fact, the functionals involve exponentially weighted Sobolev norms. This may limit somewhat the usefulness of these functionals. Nevertheless, a number of general conclusions can be made based on their certain special properties.

Let us give a brief summary of our paper here. In Sec. 2, we introduce a one-parameter family of functionals and show the variational structure of the problem in the reference frames moving with constant speeds. This leads to the need to work with the exponentially weighted Sobolev norms. In the context of Eq. (1.1), we introduce convenient families of functions that ensure that the solutions decay exponentially, faster than a prescribed rate, at one end of the cylinder Σ. In Sec. 2 we also derive a crucial inequality, which is an analogue to the Poincare inequality, in Lemma 2.5.

In Sec. 3, we identify a class of the traveling wave solutions that are critical points of the functionals in the family. We then establish a number of properties of these so-called *variational traveling waves*. Our main result in this section is the global upper bound on the speed of these traveling waves obtained in Theorem 3.7. We also establish non-existence of the variational traveling waves of certain shapes, including localized solitary waves, in Theorem 3.11. In addition, we discuss ways to obtain lower bounds on the speeds of such waves.

Then, in Sec. 4 we analyze propagation of *disturbances* in Eq. (1.1). We first establish monotonicity of the functionals as functions of time evaluated at the solutions of Eq. (1.1) in the reference frames moving with constant speeds. We show that no disturbance that decay sufficiently rapidly at one end of the cylinder can propagate with the speed greater than the global upper bound on the speed of the variational traveling waves established in Sec. 3, see Theorem 4.2. We then introduce the notion of a *wave-like solution*, which is a particular class of solutions of Eq. (1.1). These solutions behave like traveling waves in a certain sense. An important feature of these solutions is the fact that for each such solution it is possible to introduce a function $\bar{c}(t)$ with the meaning of the instantaneous propagation speed, which turns out to be a monotonic function of time. This leads to the conclusion, given in Theorem 4.7, that for wave-like solutions the speed $\bar{c}(t)$ approaches a limit at long times. Then, in Theorem 4.8 we formulate sufficient conditions for a wave-like solution to converge to a variational traveling wave at least on a sequence of times. One crucial assumption of this theorem is about the rate of exponential decay of the solution. Essentially, this theorem applies if the solution remains sufficiently localized. Here we also discuss ways to show convergence of the solutions to variational traveling waves for all times.

The second group of results of Sec. 4 has to do with a particular class of wave-like solutions of Eq. (1.1). These solutions may not be realized in all reaction-diffusion systems of gradient type. Let us point out that this class of solutions involves all solutions in systems, in which the equilibrium state $u = 0$ is linearly stable. In Theorem 4.11 we make conclusions about the motion of the *leading edge* of a solution in this class and its shape. Here we obtain the lower bound on the average speed of the leading edge. These results also allow to exclude certain kinds of the traveling wave solutions as potential asymptotic states.

Finally, in Sec. 5 we give a few concrete applications of the obtained results, and in Sec. 6 draw conclusions.

2. Variational formulation

Existence and uniqueness of solutions of Eq. (1.1) has been studied under very general assumptions in great detail (see, for example, [34,36]). For smooth $f(u)$ and continuous $u_0(x)$ global existence of solutions is guaranteed by uniform boundedness of $u(x,t)$. Furthermore, due to the smoothing action of Eq. (1.1), its solutions are smooth both in x and t for all $(x,t) \in \Sigma \times \mathbb{R}^+$. Moreover, in this situation all the derivatives of u are uniformly bounded for all $t > t_0$ with arbitrary $t_0 > 0$. The uniform boundedness of solutions of Eq. (1.1) is apriorily guaranteed by the existence of an invariant rectangle $\mathcal{R} = \{u : a_i \leq u_i \leq b_i\} \subset \mathbb{R}^m$, which has the property

$$\nu \cdot f(u)|_{\partial\mathcal{R}} < 0, \tag{2.1}$$

where ν is the outward normal to the boundary $\partial\mathcal{R}$ [36]. This means that the flow in $\mathbb{R}^m$ generated by the system of ordinary differential equations obtained from Eq. (1.1) by setting the spatial derivatives to zero is *into* $\mathcal{R}$. Note that for gradient systems studied in this paper the existence of such an invariant rectangle is guaranteed, if, for example, the function $V(u)$ (see Eq. (1.4)) is a monotonically increasing function of $|u_i|$, where u_i are the components of u, for large enough $|u_i|$, and hence has a global minimum. In the following, we will assume the existence of an invariant rectangle and will consider only the solutions that remain in $\mathcal{R}$.

Let us denote the coordinate along the axis of the cylinder Σ as z and the perpendicular coordinate as y, so $y \in \Omega$. Let us then fix a constant $c > 0$ and go to the reference frame traveling with constant speed c in the direction of positive z. In this frame let us define the coordinate ξ:

$$\xi = z - ct. \tag{2.2}$$

Then, Eq. (1.1) in this reference frame becomes

$$u_t = u_{\xi\xi} + \nabla_y^2 u + c u_\xi + f(u). \tag{2.3}$$

It turns out that this equation can be formally recast in the variational form:

$$u_t = -\mathrm{e}^{-c\xi} \frac{\delta \Phi_c}{\delta u}, \tag{2.4}$$

where

$$\Phi_c[u(z, y, t)] = \int_\Sigma \mathrm{e}^{cz} \left(\frac{1}{2} \nabla u^T \cdot \nabla u + V(u) \right) dz dy, \tag{2.5}$$

where we treat u and u^T as the m-component column and row vectors, respectively, and "$\cdot$" implies scalar product in $\mathbb{R}^n$. Sometimes for convenience we will denote $v^T v = v^2$ for $v \in \mathbb{R}^m$. We will choose the additive constant in V such that $V(0) = 0$. Thus, we have

$$V(0) = 0, \ \nabla_u V(0) = 0, \tag{2.6}$$

since by assumption $f(0) = 0$ as well.

Let us give a formal derivation of Eq. (2.4) here. For $u = u(\xi, y, t)$ satisfying Eq. (2.3) the first variation $\delta\Phi_c = \delta\Phi_c(u, \delta u)$ of Φ_c is

$$\begin{aligned} \delta\Phi_c &= \int_\Sigma \mathrm{e}^{c\xi} \left(u_\xi^T \delta u_\xi + \nabla_y u^T \cdot \delta \nabla_y u + \nabla_u^T V \delta u \right) d\xi dy \\ &= \left(\int_\Omega \mathrm{e}^{c\xi} u_\xi^T \delta u \ dy \right) \Bigg|_{\xi=-\infty}^{\xi=+\infty} + \int_{\partial\Sigma} \mathrm{e}^{c\xi} (n \cdot \nabla_y u^T) \delta u \ d\xi dy \\ &\quad - \int_\Sigma \mathrm{e}^{c\xi} \left(u_{\xi\xi} + \nabla_y^2 u + c u_\xi + f(u) \right)^T \delta u \ d\xi dy, \end{aligned} \tag{2.7}$$

where we performed integration by parts. The second integral in the second line of Eq. (2.7) vanishes because of the boundary conditions in Eq. (1.2). Assuming that δu decays sufficiently rapidly as $\xi \to \pm\infty$, the first integral in Eq. (2.7) is also zero, and we arrive at Eq. (2.4).

Note that the functional similar to Φ_c was identified by Fife and McLeod in their study of convergence of the solutions of scalar bistable reaction-diffusion equations to the traveling waves in [12] (see also [23, 24, 37]). However, only the situation in which c is the speed of the traveling wave was considered. We, on the other hand, will allow c to be an arbitrary constant and will therefore work with the *one-parameter family* of functionals Φ_c.

The variational structure represented by Eq. (2.4) is a fundamental property of Eq. (1.1). Note, however, that in the definition of Φ_c in Eq. (2.5) there is a factor e^{cz} which diverges as $z \to +\infty$. Therefore, the functional $\Phi_c[u]$ may not be well-defined for all solutions of Eq. (1.1), so one needs to specify the class of functions for which the integral in Eq. (2.5) converges. Clearly, convergence of this integral requires that $u(z, y, t)$ decays exponentially as $z \to +\infty$. The following class of functions will be appropriate for working with the functional Φ_c.

Definition 2.1. *Let $\mathcal{Q}_c(\Sigma)$ be the family of functions*

$$\mathcal{Q}_c(\Sigma) = \{u \in C^\infty(\Sigma) : \\ |D^\alpha u| < M_\alpha;\ e^{\lambda z}|D^\alpha u| < \infty,\ z \to +\infty\}, \tag{2.8}$$

where α is the derivative multi-index, $M = \{M_\alpha\}$, and M_α and $\lambda > \frac{c}{2}$ are some positive constants.

One can see that $\mathcal{Q}_c(\Sigma)$ determines a class of functions that decay exponentially together with all their derivatives at plus infinity, with the rate greater than $e^{-cz/2}$. Note the obvious inclusion $\mathcal{Q}_{c'}(\Sigma) \subset \mathcal{Q}_c(\Sigma)$ for $c' > c$, as well as the fact that if $u \in \mathcal{Q}_c(\Sigma)$, then $u \in \mathcal{Q}_{c+\epsilon}(\Sigma)$ for small enough $\epsilon > 0$. Also note that the members of $\mathcal{Q}_c(\Sigma)$ lie in the exponentially weighted Sobolev spaces $H^1_c(\Sigma)$ with the norm

$$||u||_{1,c} = ||u||_{L^2_c} + ||\nabla u||_{L^2_c}, \tag{2.9}$$

where

$$||u||^2_{L^2_c} = \int_\Sigma e^{cz} u^T u \, dzdy,\ ||\nabla u||^2_{L^2_c} = \int_\Sigma e^{cz} \nabla u^T \cdot \nabla u \, dzdy. \tag{2.10}$$

These spaces are intimately related with the functional Φ_c (see also [23, 39, 40]).

Definition 2.2. *We will say that $u \in \mathcal{Q}_c(\Sigma, \mathbb{R}^+)$, if $u \in C^\infty(\Sigma \times \mathbb{R}^+)$, $u(\cdot, t) \in \mathcal{Q}_c(\Sigma)$ and that for any $T > 0$ and α there exists a constant C_α such that $|D^\alpha u(\cdot, t)| < C_\alpha e^{-\lambda z}$, $\lambda > \frac{c}{2}$, for all $0 \le t \le T$.*

It is not difficult to see that the solutions of the initial value problem in Eq. (1.1) with the initial data in $\mathcal{Q}_c(\Sigma)$ should lie in $\mathcal{Q}_c(\Sigma, \mathbb{R}^+)$. Indeed, setting $u(z, y, t) = e^{\lambda^2 t - \lambda z} v(z - 2\lambda t, y, t)$, we obtain for v

$$v_t = \Delta v + g(v, x, t), \tag{2.11}$$

where $g = e^{\lambda z + \lambda^2 t} f(e^{-\lambda z - \lambda^2 t} v)$ is Lipschitz whenever u is in $\mathcal{R}$. So, by standard parabolic theory we obtain uniform bounds on v and its derivatives on arbitrary finite time intervals, and hence the exponential estimates for u. More generally, one should expect the solutions to be in $\mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ for appropriately bounded initial data for $t \ge t_0$ with arbitrary $t_0 > 0$. In particular, this should be the case for the initial data with compact support.

Since V is smooth and $V(0) = \nabla_u V(0) = 0$, for $u \in \mathcal{Q}_c(\Sigma)$ there exist constants μ_- and μ_+ such that

$$\mu_- = \min_{u \neq 0} \frac{2V(u)}{u^T u},\ \mu_+ = \max_{u \neq 0} \frac{2V(u)}{u^T u}. \tag{2.12}$$

From the existence of $\mu_\pm$ it is, in turn, clear that the functional Φ_c is well-defined for all $u \in \mathcal{Q}_c(\Sigma)$. Let us point out the following obvious property of the functional Φ_c that has to do with the translational symmetry of the problem

Lemma 2.3. *Let $u \in \mathcal{Q}_c(\Sigma)$. Then*

$$\Phi_c[u(z - R, y)] = e^{cR} \Phi_c[u(z, y)]. \tag{2.13}$$

This lemma also applies to the L^2_c-norm. Also, from this lemma follows

Corollary 2.4. *The sign of Φ_c is invariant with respect to translations.*

Before concluding this section, let us derive an important inequality that relates $||u||_{L^2_c}$ and $||u_z||_{L^2_c}$. This inequality turns out to be crucial for the entire analysis that follows.

Lemma 2.5. *Let $u \in \mathcal{Q}_c(\Sigma)$. Then*

$$\frac{c^2}{4} \int_\Sigma \mathrm{e}^{cz} u^T u \, dz dy \leq \int_\Sigma \mathrm{e}^{cz} u_z^T u_z \, dz dy. \tag{2.14}$$

Proof. By Schwartz inequality

$$\left(\int_\Sigma \mathrm{e}^{cz} u^T u_z dz dy \right)^2 \leq \int_\Sigma \mathrm{e}^{cz} u^T u \, dz dy \int_\Sigma \mathrm{e}^{cz} u_z^T u_z dz dy. \tag{2.15}$$

On the other hand, integrating by parts,

$$\int_\Sigma \mathrm{e}^{cz} u^T u_z dz dy = \left(\frac{1}{2} \int_\Omega \mathrm{e}^{cz} u^T u \, dy \right) \Bigg|_{z=-\infty}^{z=+\infty} - \frac{c}{2} \int_\Sigma \mathrm{e}^{cz} u^T u \, dz dy. \tag{2.16}$$

Since Ω is bounded and $\mathrm{e}^{cz} u^T u \to 0$ at $z = \pm\infty$, the surface term in the right-hand side of Eq. (2.16) vanishes. Then, squaring both sides of this equation and substituting it to Eq. (2.15), we obtain

$$\frac{c^2}{4} \left(\int_\Sigma u^T u \, dz dy \right)^2 \leq \int_\Sigma \mathrm{e}^{cz} u^T u \, dz dy \int_\Sigma \mathrm{e}^{cz} u_z^T u_z dz dy \tag{2.17}$$

Now, canceling a factor of $\int_\Sigma \mathrm{e}^{cz} u^T u \, dz dy$, we obtain Eq. (2.14). □

3. Traveling wave solutions

Now we are going to study the properties of the functional Φ_c and its critical points. Let $u(z, y, t) = \bar{u}(z - ct, y)$ be a traveling wave solution with speed c. In the reference frame moving with the wave (Eq. (2.2)) $\bar{u}$ satisfies

$$\bar{u}_{\xi\xi} + \nabla_y^2 \bar{u} + c \bar{u}_\xi + f(\bar{u}) = 0, \tag{3.1}$$

with the boundary conditions from Eq. (1.2). Note that existence of solutions of Eq. (3.1) was proved in a variety of contexts [9–14, 16–23] (see also discussion at the end of this section). It is not difficult to see by repeating the steps of the derivation of Eq. (2.7) that the following proposition holds.

Proposition 3.1. *Let $\bar{u} \in \mathcal{Q}_c(\Sigma)$ be a traveling wave solution with speed c. Then for any $v \in C_0^\infty(\Sigma)$ we have*

$$\delta \Phi_c(\bar{u}, v) = 0. \tag{3.2}$$

In other words, some traveling wave solutions are critical points of the functional Φ_c. Clearly, the converse is also true if $\bar{u} \in \mathcal{Q}_c(\Sigma)$ is a critical point of Φ_c. Note, however, that the important point in this Proposition is that $\bar{u}$ lies in $H_c^1(\Sigma)$. Therefore, not all the traveling wave solutions are critical point of Φ_c. In order to distinguish those traveling wave solutions which, in fact, are, it is convenient to introduce the following

Definition 3.2. *The solution $\bar{u}$ of Eq. (3.1) is called a* variational traveling wave*, if it has speed $c > 0$ and lies in $\mathcal{Q}_c(\Sigma)$.*

Note that more generally variational traveling waves can be defined as those solutions of Eq. (3.1) that lie in $H_c^1(\Sigma) \cap L^\infty(\Sigma)$ [39, 40]. In other words, only bounded traveling wave solutions that decay sufficiently rapidly at plus infinity can be variational traveling waves.

The fact whether $\bar{u}$ lies in $H_c^1(\Sigma)$ is determined by the linearization of Eq. (3.1) around zero (for examples, see Sec. 5). Suppose that $\bar{u} \sim v(y) \mathrm{e}^{-\lambda \xi}$, where v :

$\Omega \to \mathbb{R}^m$, as $\xi \to +\infty$. Then λ satisfies the following eigenvalue problem (with the appropriate boundary conditions) in Ω:

$$[-\nabla_y^2 + H(0)]v = \lambda(\lambda - c)v, \tag{3.3}$$

where H is the Hessian of V:

$$H(u) = (\nabla_u \otimes \nabla_u)V. \tag{3.4}$$

Let us denote the spectrum of the operator on the left-hand side of this equation as $\{\nu_k\}$. Since Ω is bounded, this spectrum is discrete. Also, since H is a symmetric matrix, all ν_k's are real. We need to look at the relationship between this spectrum and $\lambda = \lambda_k$ for each k (see also [18, 20]). This relationship is given by the following quadratic equation:

$$\lambda_k^2 - c\lambda_k - \nu_k = 0. \tag{3.5}$$

We are only interested in the real positive solutions of this equation. Indeed, Re λ has to be positive in order for the solution to decay at $\xi = +\infty$. If λ_k is complex, which corresponds to $\nu_k < -\frac{c^2}{4}$, then Re $\lambda_k = \frac{c}{2}$, so $\bar{u}$ only lies in $H^1_{c'}(\Sigma)$ with $c' < c$. This is also true for a double real root, when $\nu_k = -\frac{c^2}{4}$. Note that when $\nu_k < 0$, the traveling wave solutions whose speed $c = 2\sqrt{-\nu_k}$ are sometimes referred to as "marginal fronts" [1, 27].

On the other hand, for real solutions of Eq. (3.5), which satisfy $-\frac{c^2}{4} < \nu_k < 0$ there are two solutions $\lambda_k^\pm$ for each k, such that $\lambda_k^- < \frac{c}{2}$ and $\lambda_k^+ > \frac{c}{2}$. Therefore, in this case $\bar{u}$ can lie in $H^1_c(\Sigma)$ only if $\lambda = \lambda_k^+$. Note that in this situation there may exist continuous families of traveling wave solutions corresponding to λ_k^- (sometimes called "linear fronts"), and solutions corresponding to λ_k^+ with some particular speeds (sometimes called "nonlinear fronts") [1, 28]. As can be seen from the arguments above, only the latter can be the variational traveling waves. Note, however, that these solutions play an important role for propagation of disturbances in Eq. (1.1) (see below). Finally, for $\nu_k \geq 0$ we have $\lambda_k^- \leq 0$ and $\lambda_k^+ > c > \frac{c}{2}$, so the traveling wave solution $\bar{u} \in H^1_c(\Sigma)$ always. Note that this is the case when all the eigenvalues of the matrix $H(0)$ are positive, meaning that $u = 0$ is a locally stable equilibrium point.

The variational formulation described above allows to make a number of general conclusions about the variational traveling waves. We will need a few properties of the functional Φ_c that have to do with these waves.

Proposition 3.3. *Let $\bar{u}$ be a variational traveling wave with speed c. Then*

$$\Phi_c[\bar{u}] = 0. \tag{3.6}$$

Proof. In order for $\bar{u}$ to be a critical point of the functional, Φ_c should not change with infinitesimal translations of $\bar{u}$. By Lemma 2.3, this can only be the case, if $\Phi_c[\bar{u}] = 0$. □

Another interesting property of the variational traveling waves has to do with the rate of change of $\Phi_{c'}[\bar{u}]$ considered as a function of c'. Let us first prove the following lemma.

Lemma 3.4. *Let $u \in \mathcal{Q}_c(\Sigma)$ and $0 < c' < c''$, where $c'' > c$ is sufficiently close to c. Then $\Phi_{c'}[u]$ is a smooth function of c'.*

Proof. From the definition of Φ_c, we have

$$\frac{d^k \Phi_{c'}[u]}{dc'^k} = \int_\Sigma \xi^k e^{c'\xi} \left(\frac{1}{2} \nabla u^T \cdot \nabla u + V(u) \right) d\xi dy. \tag{3.7}$$

Let us look at the first term in Eq. (3.7). For any k there exists a constant K_k such that $|\xi|^k e^{c'\xi} \leq K_k e^{c''\xi}$, $c'' > c'$, for all $\xi \geq 0$. Therefore

$$\left| \int_\Sigma \xi^k e^{c'\xi} \nabla u^T \cdot \nabla u d\xi dy \right|$$

$$\leq M|\Omega| \int_{-\infty}^0 |\xi|^k e^{c'\xi} d\xi + K_k \int_\Omega \int_0^{+\infty} e^{c''\xi} \nabla u^T \cdot \nabla u d\xi dy < \infty. \tag{3.8}$$

With the help of Eq. (2.12), a similar estimate can be obtained for the second term in Eq. (3.7). □

Let us now show the following property of Φ_c evaluated at a variational traveling wave.

Proposition 3.5. *Let $\bar{u}$ be a variational traveling wave with speed c. Then $\Phi_{c'}[\bar{u}]$ is a strictly increasing function of c' in a small neighborhood of $c' = c$.*

Proof. Integrating Eq. (3.7) with $k = 1$ by parts, we obtain

$$\frac{d\Phi_{c'}[\bar{u}]}{dc'} = \left(\frac{1}{c'} \int_\Omega \xi e^{c'\xi} \left(\frac{1}{2} \nabla \bar{u}^T \cdot \nabla \bar{u} + V(\bar{u}) \right) dy \right) \Bigg|_{\xi=-\infty}^{\xi=+\infty}$$

$$-\frac{1}{c'} \Phi_{c'}[\bar{u}] - \frac{1}{c'} \int_\Sigma \xi e^{c'\xi} \left(\bar{u}_{\xi\xi}^T \bar{u}_\xi + \nabla_y \bar{u}_\xi^T \cdot \nabla_y \bar{u} - \bar{u}_\xi^T f(\bar{u}) \right) d\xi dy$$

$$= -\frac{1}{c'} \Phi_{c'}[\bar{u}] - \frac{1}{c'} \int_{\partial\Sigma} \xi e^{c'\xi} \bar{u}_\xi^T (n \cdot \nabla_y \bar{u}) d\xi dy$$

$$+ \frac{1}{c'} \int_\Sigma \xi e^{c'\xi} \bar{u}_\xi^T \left(-\bar{u}_{\xi\xi} + \nabla_y^2 \bar{u} + f(\bar{u}) \right) d\xi dy$$

$$= -\frac{1}{c'} \Phi_{c'}[\bar{u}] + \frac{1}{c'} \int_\Sigma \xi e^{c'\xi} \bar{u}_\xi^T \left(-\bar{u}_{\xi\xi} + \nabla_y^2 \bar{u} + f(\bar{u}) \right) d\xi dy. \tag{3.9}$$

In arriving at the last line in Eq. (3.9) we used the fact that the integrals over Ω at $\xi = \pm\infty$ vanish due to decay of u, and the integrals over $\partial\Sigma$ vanish because of the boundary conditions. Now, using Eqs. (3.1) and (3.6) and evaluating the derivative at $c' = c$, we obtain

$$\left. \frac{d\Phi_{c'}[\bar{u}]}{dc'} \right|_{c'=c} = -\frac{1}{c} \int_\Sigma \xi e^{c\xi} \bar{u}_\xi^T (2\bar{u}_{\xi\xi} + c\bar{u}_\xi) d\xi dy$$

$$= \left(-\frac{1}{c} \int_\Omega \xi e^{c\xi} \bar{u}_\xi^T \bar{u}_\xi dy \right) \Bigg|_{\xi=-\infty}^{\xi=+\infty} + \frac{1}{c} \int_\Sigma e^{c\xi} \bar{u}_\xi^T \bar{u}_\xi d\xi dy$$

$$= \frac{1}{c} \int_\Sigma e^{c\xi} \bar{u}_\xi^T \bar{u}_\xi d\xi dy > 0. \tag{3.10}$$

Since, by Lemma 3.4, $d\Phi_{c'}[\bar{u}]/dc'$ is a continuous function of c' in a small neighborhood of c, the functional $\Phi_{c'}[\bar{u}]$ is a strictly increasing function of c' for fixed $u = \bar{u}$, in that neighborhood. □

Note that from Propositions 3.5 and 3.3 follows that the functional Φ'_c evaluated on the variational traveling wave *changes sign* from negative to positive at $c' = c$ when the value of c' is increased, so c is a simple zero of $\Phi_{c'}[\bar{u}]$.

We will need the following important lemma.

Lemma 3.6. *Let $u \in \mathcal{Q}_c(\Sigma)$ with $c > c_{\max}$, where*

$$c_{\max} = \begin{cases} 2\sqrt{-\mu_-}, & \mu_- < 0, \\ 0, & \mu_- \geq 0. \end{cases} \tag{3.11}$$

and μ_- is defined in Eq. (2.12). Then there exists a constant $K > 0$ such that

$$\Phi_c[u] \geq K \int_\Sigma e^{cz} \nabla u^T \cdot \nabla u dz dy \geq 0. \tag{3.12}$$

Proof. For $\mu_- \geq 0$ the statement is obvious. So, suppose that $\mu_- < 0$. According to Eq. (2.12) and Lemma 2.5, we have

$$\begin{aligned} \Phi_c[u] &\geq \frac{1}{2} \int_\Sigma e^{cz} \left(\nabla u^T \cdot \nabla u + \mu_- u^T u \right) dz dy \\ &\geq \frac{1}{2} \left(1 + \frac{4\mu_-}{c^2} \right) \int_\Sigma e^{cz} \nabla u^T \cdot \nabla u dz dy \geq 0, \end{aligned} \tag{3.13}$$

which has the form of Eq. (3.12). □

Note that sharper estimates are possible with more information about the shape of the domain Ω for Dirichlet boundary conditions.

With the use of this lemma, we are ready to obtain one of the main results of this paper that gives an upper bound for the speed of the variational traveling waves.

Theorem 3.7. *Let $\bar{u}$ be a variational traveling wave with speed c. Then, $c \leq c_{\max}$, where $c_{\max}$ is given by Eq. (3.11).*

Proof. To see this, let us assume that there exists a variational traveling wave $\bar{u} \in \mathcal{Q}_c(\Sigma)$ with speed $c > c_{\max}$. Then, by Proposition 3.3, $\Phi_c[\bar{u}] = 0$. But by Lemma 3.6 this is impossible for any non-zero u. □

Corollary 3.8. *If $V(u) \geq 0$ for all $u \in \mathcal{R}$, the variational traveling waves in $\mathcal{R}$ do not exist.*

In other words, the variational traveling waves can exist only if $V(u) < 0$ somewhere in $\mathcal{R}$. This is a generalization of the well-known condition for the scalar reaction diffusion equations in one dimension [34].

Note, however, that the class of the variational traveling waves may be rather restrictive, since in general the solutions of Eq. (3.1) with speed c may not necessarily lie in $H_c^1(\Sigma)$ with the *same* c. Still, if these solutions decay exponentially at plus infinity, it is possible to modify the variational formulation in such a way that these solutions will be captured in that formulation. The main issue here is in the translational invariance of the problem and the fact that Φ_c is *not* translationaly invariant. To circumvent it, we need to eliminate translations from the consideration. Therefore, instead of looking for the critical points of the functional Φ_c, we may look for the critical points of the functional $\Phi_{c'}$, where c' is another positive constant, with respect to changes of the *shape* of the solution. That is, in finding the critical points, we will not allow the variation δu to contain the translational mode. Then, we have the following

Proposition 3.9. *Let $\bar{u} \in \mathcal{Q}_{c'}(\Sigma)$ be a traveling wave solution with speed c. Then $\bar{u}$ must satisfy*

$$\delta\Phi_{c'}(\bar{u}, v) = 0 \tag{3.14}$$

for any $v \in C_0^\infty(\Sigma)$ subject to

$$\int_\Sigma e^{c'\xi} \bar{u}_\xi^T v \, d\xi dy = 0. \tag{3.15}$$

Proof. This can be easily seen from the calculation similar to the one in Eq. (2.7) and using the fact that one can now add an arbitrary multiple of u_ξ to the bracket in the right-hand of this equation. □

Of course, in this case Eq. (3.6) is no longer valid. Instead, we get

Proposition 3.10. *Let $\bar{u} \in \mathcal{Q}_{c'}(\Sigma)$ be a traveling wave solution with speed c. Then*

$$c = c' - \frac{c' \Phi_{c'}[\bar{u}]}{\int_\Sigma e^{c'\xi} \bar{u}_\xi^T \bar{u}_\xi d\xi dy}. \tag{3.16}$$

Proof. To see this, let us multiply Eq. (3.1) by $e^{c'\xi}\bar{u}_\xi^T$ from the left and integrate over Σ. Integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_\Sigma e^{c'\xi} \left(\bar{u}_\xi^T \bar{u}_{\xi\xi} + \bar{u}_\xi^T \nabla_y^2 \bar{u} + c \bar{u}_\xi^T \bar{u}_\xi + \bar{u}_\xi^T f(\bar{u}) \right) d\xi dy \\
&= \left(\frac{1}{2} \int_\Omega e^{c'\xi} \bar{u}_\xi^T \bar{u}_\xi dy \right) \Big|_{\xi=-\infty}^{\xi=+\infty} - \left(\int_\Omega e^{c'\xi} V(\bar{u}) dy \right) \Big|_{\xi=-\infty}^{\xi=+\infty} \\
&\qquad + \int_{\partial\Sigma} e^{c'\xi} \bar{u}_\xi^T (n \cdot \nabla_y \bar{u}) d\xi dy \\
&\quad + \int_\Sigma e^{c'\xi} \left(\frac{1}{2}(2c - c') \bar{u}_\xi^T \bar{u}_\xi + c' V(\bar{u}) - \nabla_y \bar{u}_\xi^T \cdot \nabla_y \bar{u} \right) d\xi dy \\
&= - \left(\frac{1}{2} \int_\Omega e^{c'\xi} \nabla_y \bar{u}^T \cdot \nabla_y \bar{u} \; d\xi dy \right) \Big|_{\xi=-\infty}^{\xi=+\infty} \\
&\quad + \int_\Sigma e^{c'\xi} \left(\frac{1}{2}(2c - c') \bar{u}_\xi^T \bar{u}_\xi + \frac{c'}{2} \nabla_y \bar{u}^T \cdot \nabla_y \bar{u} + c' V(\bar{u}) \right) d\xi dy \\
&= c' \Phi_{c'}[\bar{u}] + (c - c') \int_\Sigma e^{c'\xi} \bar{u}_\xi^T \bar{u}_\xi d\xi dy,
\end{aligned} \tag{3.17}$$

which is equivalent to Eq. (3.16). In writing these equations, we used the fact that the surface terms in the integration by parts all vanish. ☐

This modified variational formulation allows to capture all the traveling wave solutions that decay exponentially as $\xi \to +\infty$ by choosing sufficiently small values of c'. Of course, one recovers Eq. (3.6) for $c = c'$.

Proposition 3.10 allows to exclude a large class of shapes for potential traveling wave solutions.

Theorem 3.11. *There are no traveling wave solutions $\bar{u} \in \mathcal{Q}_{c'}(\Sigma)$, with some $c' > 0$, moving with speed $c > 0$, which lie in $H^1(\Sigma)$.*

Proof. The proof is obtained by taking the limit $c' \to 0$ in Eq. (3.16). Then, the right-hand side of this equation goes to zero, leading to contradiction. ☐

Essentially, this Theorem shows that there are no traveling wave solutions which are localized.

A natural question that arises is whether the critical points of Φ_c can actually be *minimizers*. Clearly, variational traveling waves must be local minimizers of the functional in order to be linearly stable. The answer to this question is positive. In fact, under very general assumptions it is possible to use direct methods of calculus of variations to prove existence of variational traveling waves which are the minimizers of Φ_c (see also [23,40]). This proof will be presented elsewhere [39]. Let us comment here that it is easy to see from Proposition 3.10 that the nontrivial global minimizers of Φ_c may exist only for a unique value of $c = c^*$. Indeed, if there exist two minimizers $\bar{u}_1$ and $\bar{u}_2$ for $c = c_1$ and $c = c_2$, lying in $\mathcal{Q}_{c_{1,2}}(\Sigma)$, respectively, with $c_1 < c_2$, then by Eq. (3.16) we have $\Phi_{c_1}[\bar{u}_2] < 0$, which is justified since $\bar{u}_2 \in \mathcal{Q}_{c_1}(\Sigma)$ also. But this contradicts Proposition 3.3 and the fact that $\bar{u}_1$ is the global minimizer, so the value of c^* is unique.

4. Propagation of disturbances

Now we are going to turn to studying the properties of the functional Φ_c evaluated on the solutions of Eq. (1.1). We will assume that the solution $u = u(z, y, t)$ of Eq. (1.1) lies in $\mathcal{Q}_c(\Sigma, \mathbb{R}^+)$, so the functional $\Phi_c[u]$ is well-defined for all times. The following property of Φ_c has already been anticipated in Eq. (2.4) and is a crucial property of the dynamics.

Proposition 4.1. *Let $u \in \mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ be a solution of Eq. (2.3). Then, $\Phi_c[u(\cdot, t)]$ is a non-increasing function of time.*

Proof. Let us look at the rate of change of $\Phi_c[u(\cdot, t)]$, where u satisfies Eq. (2.3). Integrating by parts and using Eq. (2.3), we obtain

$$\begin{aligned}
\frac{d\Phi_c[u(\cdot,t)]}{dt} &= \int_\Sigma \mathrm{e}^{c\xi} \left(u_\xi^T u_{\xi t} + \nabla_y u^T \cdot \nabla_y u_t - f^T(u) u_t \right) d\xi dy \\
&= \left(\int_\Omega \mathrm{e}^{c\xi} u_\xi^T u_t d\xi dy \right) \Bigg|_{\xi=-\infty}^{\xi=+\infty} + \int_{\partial\Sigma} \mathrm{e}^{c\xi} u_t (n \cdot \nabla_y u) d\xi dy \\
&\quad - \int_\Sigma \mathrm{e}^{c\xi} \left(u_{\xi\xi} + c u_\xi + \nabla_y^2 u + f(u) \right)^T u_t d\xi dy \\
&= - \int_\Sigma \mathrm{e}^{c\xi} u_t^T u_t d\xi dy \leq 0.
\end{aligned} \tag{4.1}$$

This calculation is justified by the fact that for $u \in \mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ the integrals in Eq. (4.1) converge uniformly on any finite interval of t. As usual, in arriving at this expression we took into account that the surface integrals are equal to zero. This means that in the reference frame moving with speed c with respect to the cylinder $\Phi_c[u]$ is a non-increasing function of time. □

The existence of a monotonically decreasing functional $\Phi_c[u]$ for the dynamics in the reference frame moving with speed c strongly suggest that under very general assumptions the asymptotic solutions at long times should be the variational traveling waves. Proving this statement in general is a fundamental challenge (for recent results, see [25, 37]). Fife and McLeod used a functional similar to Φ_c to prove convergence of the solutions of the initial value problem to the traveling wave solution in the case of the bistable scalar reaction-diffusion equations in one dimension [12]. This, however, requires boundedness of the functional in the reference frame moving with speed c, where c is the speed of the traveling wave. For bistable scalar reaction-diffusion equations this can be done by using comparison arguments [12,13]. On the other hand, because of the exponentially decaying weight at $\xi \to -\infty$ in the right-hand side of Eq. (4.1) one also needs to make sure that the solution $u(\xi, y, t)$ actually converges to the traveling wave solution and not to zero. This can also be done by using comparison arguments in the case of bistable scalar reaction-diffusion equations [12].

In the general case, however, it is not clear under what conditions there exists such a speed c of the reference frame in which Φ_c is uniformly bounded from below in time with the solution remaining bounded away from zero on a fixed finite subdomain of Σ. A simple example of a system in which such a speed does not exist is the Fisher equation in one dimension, for which $f(u) = u(1 - u)$ and the initial condition in the form of a unit step. As was shown by Kolmogorov, Petrovsky and Piskunov, while the asymptotic speed of propagation of the solution of this equation is $c^* = 2$, the solution goes to zero in the reference frame moving with speed $c = c^*$ [9]. They also pointed out that at long times the solution converges to the traveling wave solution in *shape* and not uniformly in time.

Here, instead of the conventional approach, we are going to use certain special properties of the functional Φ_c that have to do with its sign to make some general conclusions about the behavior of the solutions of Eq. (1.1) at long times. In the following, we we will only consider solutions that decay sufficiently rapidly at plus infinity. More precisely, we will look at $u = u(z, y, t) \in \mathcal{Q}_{c_{\max}}(\Sigma, \mathbb{R}^+)$, where $c_{\max}$ is given by Eq. (3.11). In the previous section we showed that there are no variational traveling wave solutions for $c > c_{\max}$. Let us now show that solutions $u \in \mathcal{Q}_{c_{\max}}(\Sigma, \mathbb{R}^+)$ of Eq. (1.1) cannot propagate faster than the speed $c_{\max}$ in the limit $t \to \infty$.

Theorem 4.2. *Let $u \in \mathcal{Q}_{c_{\max}}(\Sigma, \mathbb{R}^+)$, where $c_{\max}$ is given by Eq. (3.11), be a solution of Eq. (1.1). Then u converges to zero in $H^1_{c_{\max}}(\Sigma)$ in the reference frame moving with any speed $c > c_{\max}$.*

Proof. Let us pick c' such that $c_{\max} < c' < c$ and $u \in \mathcal{Q}_{c'}(\Sigma)$. From Proposition 4.1 and Lemma 2.3 we have

$$\Phi_{c'}[u_0(z, y)] \geq \Phi_{c'}[u(z + c't, y, t)] = \mathrm{e}^{c'(c-c')t}\Phi_{c'}[u(z + ct, y, t)]. \tag{4.2}$$

This means that $\Phi_{c'}[u(z+ct, y, t)] \to 0$ as $t \to \infty$. But, by Lemma 3.6, this implies $\int_\Sigma \mathrm{e}^{c'(z-ct)}\nabla u^T \cdot \nabla u dz dy \to 0$ as $t \to \infty$, so taking into account Lemma 2.5, we see that $u \to 0$ in $H^1_{c'}(\Sigma)$ in the reference frame moving with speed c. Now, since for any $c'' < c'$

$$\int_\Sigma \mathrm{e}^{c''z}\nabla u^T \cdot \nabla u \, dydz =$$

$$\int_{-\infty}^{-R}\int_\Omega \mathrm{e}^{c''z}\nabla u^T \cdot \nabla u \; dydz + \int_{-R}^{\infty}\int_\Omega \mathrm{e}^{c''z}\nabla u^T \cdot \nabla u \, dydz$$

$$\leq M\mathrm{e}^{-c''R} + \mathrm{e}^{-c''R}\int_{-R}^{\infty}\int_\Omega \mathrm{e}^{c''(z+R)}\nabla u^T \cdot \nabla u \, dydz$$

$$\leq M\mathrm{e}^{-c''R} + \mathrm{e}^{(c'-c'')R}\int_{-R}^{\infty}\int_\Omega \mathrm{e}^{c'z}\nabla u^T \cdot \nabla u \, dydz \to M\mathrm{e}^{-c''R} \tag{4.3}$$

as $t \to \infty$, and R is arbitrary, u converges to zero in $H^1_{c_{\max}}(\Sigma)$ as well. □

Note that since ∇u is uniformly bounded, this implies that $u \to 0$ in $C(\Sigma^+)$, where Σ^+ is an arbitrary subdomain of Σ which is bounded at minus infinity.

Thus, we have established the upper bound on the speed of propagation for disturbances lying in $\mathcal{Q}_{c_{\max}}(\Sigma, \mathbb{R}^+)$. This means that the initial disturbance u_0 that decays sufficiently rapidly at plus infinity will not propagate faster than the speed $c_{\max}$. Notice that the obtained upper bound applies to the initial data with compact support.

Let us now study the question of the speed of propagation of disturbances in more detail. It turns out that the scalar product generated by the L^2_c norm provides a natural way to introduce an *instantaneous* propagation speed for the solution $u = u(z, y, t)$ of Eq. (1.1). Indeed, for a fixed constant $c > 0$ and a solution $u \in \mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ of Eq. (1.1), let us define the speed $v_c(t)$ as the projection of u_t on the appropriately normalized translational mode in L^2_c. That is, let

$$v_c(t) = -\frac{\int_\Sigma \mathrm{e}^{cz}u_t^T u_z dz dy}{\int_\Sigma \mathrm{e}^{cz}u_z^T u_z dz dy}. \tag{4.4}$$

It turns out that $v_c(t)$ can be explicitly calculated in terms of Φ_c.

Proposition 4.3. *Let $u \in \mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ be a solution of Eq. (1.1). Then*

$$v_c(t) = c - \frac{c\Phi_c[u]}{\int_\Sigma e^{cz} u_z^T u_z dz dy}. \tag{4.5}$$

Proof. This result can be obtained by going through exactly the same steps as in Eq. (3.17). □

Naturally, we recover Eq. (3.16) when $u = \bar{u}$ is a traveling wave solution.

Let us take a closer look at Eq. (4.5). This equation indicates that u propagates faster or slower than c depending on the *sign* of the functional Φ_c. By Corollary 2.4, the sign of the functional is invariant with respect to translations along the z-axis. Therefore, we can get the information about it by studying the problem in the reference frame moving with the speed c and then applying it to Eq. (4.5) which is written in the original reference frame.

Our first conclusion is that, since for $c \geq c_{\max}$ and $u \in \mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ the functional $\Phi_c[u] \geq 0$, we will have $v_c(t) \leq c_{\max}$. So, the propagation speed in the sense of Eq. (4.4) of a solution which decays sufficiently rapidly at plus infinity is bounded by $c_{\max}$. This is in agreement with the convergence of the solutions to zero on finite subdomains in the reference frames moving with speeds $c > c_{\max}$ obtained above. Secondly, let us introduce the following definition.

Definition 4.4. *Let $u \in \mathcal{Q}_{c_{\max}}(\Sigma, \mathbb{R}^+)$ be a solution of Eq. (1.1). We will call u* wave-like*, if there exists a constant $c = c_{\min} > 0$ such that $\Phi_c[u(\cdot, t_0)] < 0$ for some $t_0 \geq 0$.*

Observe that any solution of Eq. (1.1) that tends to a constant for which $V < 0$ at minus infinity is a wave-like solution. Indeed, for sufficiently small c the main contribution to Φ_c will be from the tail of the solution at minus infinity, which for this type of a solution is negative. So, it will always be possible to find such $c > 0$ that $\Phi_c[u] < 0$. Naturally, wave-like solutions exist only if $V(u) < 0$ for some u.

Since in the reference frame moving with speed c the functional $\Phi_c[u(\xi, y, t)]$ is monotonically decreasing with time, we will have $\Phi_c[u(\xi, y, t)] < 0$ for all $t \geq t_0$ in that reference frame. But, since the sign of $\Phi_c[u]$ is invariant with respect to translations, $\Phi_c[u(z, y, t)] < 0$ for all $t \geq t_0$ as well. This means that for $t \geq t_0$ we will have $v_c(t) \geq c_{\min}$. Thus, these arguments allow to determine a range of propagation speeds for the solution u in the sense of Eq. (4.4). In essence, Eq. (4.5) provides an integral comparison principle for the propagation speeds of the solutions of Eq. (1.1). Its advantage is in the fact that one only needs to evaluate the sign of $\Phi_c[u]$ at different values of c instead of constructing sub or supersolutions.

On the other hand, the propagation speed $v_c(t)$ suffers from the ambiguity introduced by its c-dependence. This dependence is in fact non-trivial, since the value of c determines in which space $H_c^1(\Sigma)$ the solution may lie. On the other hand, if u decays sufficiently rapidly at plus infinity, $v_c(t)$ will be well-defined for sufficiently large c, so there is a question of the choice of c in Eq. (4.4). It turns out that for these solutions it is possible to introduce another kind of the propagation speed which does not suffer from this ambiguity and has a few interesting properties that can be exploited in the analysis.

Indeed, let us assume that $u = u(z, y, t)$ is a wave-like solution. Since $\Phi_c[u(\cdot, t)]$ changes sign from positive to negative only once, if ever, as a function of time, we will have $\Phi_{c_{\min}}[u(\cdot, t)] < 0$ for all $t \geq t_0$. On the other hand, from the *a priori* estimate of Eq. (3.12) we have $\Phi_{c_{\max}}[u] \geq 0$ for all t. Since $\Phi_c[u]$ is a continuous function of c, it must become zero somewhere in the interval $c_{\min} \leq c \leq c_{\max}$ for each t. Therefore, the following definition may be introduced:

Definition 4.5. *Let u be a wave-like solution of Eq. (1.1). Then define the speed $\bar{c}(t)$:*

$$\bar{c}(t) = \sup S(t), \;\; S(t) = \{c : c_{\min} \leq c \leq c_{\max}, \;\; \Phi_c[u(\cdot, t)] < 0\} \tag{4.6}$$

for all $t \geq t_0$.

Clearly, by continuity of $\Phi_c[u]$ as a function of c

$$\Phi_{\bar{c}(t)}[u(\cdot, t)] = 0 \tag{4.7}$$

identically. Furthermore, $\Phi_c[u(\cdot, t)]$ as a function of c changes sign at $c = \bar{c}(t)$ and $\Phi_c[u] \geq 0$ for all $\bar{c}(t) \leq c \leq c_{\max}$. Comparing this with Eqs. (4.4) and (4.5), we can see that $\bar{c}(t) = v_{\bar{c}(t)}(t)$, that is, $\bar{c}$ is the speed of the reference frame in which $\int e^{\bar{c}\xi} u_t^T u_\xi d\xi dy = 0$ at time t.

Another important property of $\bar{c}(t)$ is the following.

Proposition 4.6. *The function $\bar{c}(t)$ is a non-decreasing function of t.*

Proof. By Proposition 4.1, once $\Phi_c[u(\cdot, t)] < 0$, it will remain like this for all time. Therefore, the set $S(t)$ cannot shrink. Since $\bar{c}(t) = \sup S(t)$, the function $\bar{c}(t)$ cannot decrease. □

But, observe that $\bar{c}(t)$ is bounded from above. This leads to the following striking observation.

Theorem 4.7. *There exists a limit*

$$c^* = \lim_{t \to \infty} \bar{c}(t), \tag{4.8}$$

where $c_{\min} \leq c^* \leq c_{\max}$.

While the argument above establishes existence of a limiting propagation speed c^*, it does not actually say what this speed is. It would be natural to suppose that this should be the speed of the traveling wave solution to which the solution u of Eq. (1.1) converges. This, however, would require to prove that u indeed converges to a traveling wave in a certain sense. Unfortunately, the variational structure alone does not seem to give such a proof in our problem. One might get encouraged by the fact that since $c^* = \sup \bar{c}(t)$, we have $\Phi_{c^*}[u] \geq 0$ for all t in the reference frame moving with speed c^*, so $d\Phi_{c^*}[u(\cdot, t)]/dt \to 0$ on a sequence $\{t_n\}$ in that reference frame. This, however, does not yet mean convergence of u to a traveling wave solution (on arbitrary compact subdomains of Σ) in that reference frame because the possibility of u converging to a trivial solution (for which $f(u) = 0$) is not excluded.

Furthermore, since convergence of the solutions of Eq. (1.1) to the traveling wave solution may not be uniform in any reference frame, one needs to make more precise what is meant by such convergence [9]. As Kolmogorov, Petrovsky, and Piskunov pointed out, one needs to look at the convergence in *shape*, that is, eliminate simple translations along the z-axis. So, the question may be asked in the following way: for a given sequence of times $\{t_n\}$ is there a suitable sequence of *shifts* $\{R_n\}$, such that

$$\lim_{n \to \infty} u(z + R_n, y, t_n) = \bar{u}(z, y), \tag{4.9}$$

where $\bar{u}$ is a traveling wave solution, in some norm? In the following, we will show that the answer to this question is positive at least on a sequence of times under certain assumptions on the shape of the solution.

Theorem 4.8. *Let u be a wave-like solution of Eq. (1.1). Assume that for each t there exists a shift $R(t)$ such that $u(z+R(t),y,t)$ is uniformly bounded in $H^1_c(\Sigma)$ with some $c > c^*$, and $f(u(z+R(t),y,t))$ is bounded away from zero at a fixed point in Σ. Then, there exists a sequence of times $\{t_n\}$ and shifts $\{R_n\}$ such that $u(z+R_n,y,t_n) \to \bar{u}$ as $t_n \to \infty$, where $\bar{u}$ is a variational traveling wave with speed $c = c^*$, in $C^2(\Sigma_0)$, where Σ_0 is an arbitrary compact subdomain of Σ.*

Proof. Let us differentiate Eq. (4.7) with respect to t. We get

$$\begin{aligned} 0 = \frac{d\Phi_{\bar{c}(t)}[u(\cdot,t)]}{dt} &= \left.\frac{\partial \Phi_c[u]}{\partial c}\right|_{c=\bar{c}(t)} \frac{d\bar{c}}{dt} \\ &+ \int_\Sigma \mathrm{e}^{\bar{c}(t)z}\left(u_z^T u_{zt} + \nabla_y u^T \cdot \nabla_y u_t - f^T(u) u_t\right) dz dy \\ &= \left.\frac{\partial \Phi_c[u]}{\partial c}\right|_{c=\bar{c}(t)} \frac{d\bar{c}}{dt} + \left.\left(\int_\Omega \mathrm{e}^{\bar{c}(t)z} u_z^T u_t dy\right)\right|_{z=-\infty}^{z=+\infty} \\ &+ \int_{\partial\Sigma} \mathrm{e}^{\bar{c}(t)z} u_t^T (n\cdot\nabla_y u) dz dy \\ &- \int_\Sigma \mathrm{e}^{\bar{c}(t)z}\left(u_{zz} + \bar{c}(t) u_z + \nabla_y^2 u + f(u)\right)^T u_t dz dy \\ &= \left.\frac{\partial \Phi_c[u]}{\partial c}\right|_{c=\bar{c}(t)} \frac{d\bar{c}}{dt} - \int_\Sigma \mathrm{e}^{\bar{c}(t)z}(u_t^T u_t + \bar{c}(t) u_z^T u_t) dz dy, \end{aligned} \tag{4.10}$$

whenever $d\bar{c}/dt$ exists. Here $\partial\Phi_c/\partial c$ is as in Eq. (3.7), we performed integration by parts and took into account that the surface terms are all equal to zero. Now, taking into account that $\Phi_{\bar{c}(t)}[u] = 0$ and so $\int_\Sigma \mathrm{e}^{\bar{c}(t)z} u_t^T u_z dz dy = -\bar{c}(t)\int_\Sigma \mathrm{e}^{\bar{c}(t)z} u_z^T u_z dz dy$ (see Eq. (4.4) and (4.5)), we obtain

$$\left.\frac{\partial \Phi_c[u]}{\partial c}\right|_{c=\bar{c}(t)} \frac{d\bar{c}}{dt} = \int_\Sigma \mathrm{e}^{\bar{c}(t)z}\left(u_t + \bar{c}(t) u_z\right)^2 dz dy. \tag{4.11}$$

This equation is invariant with respect to constant shifts of u along the z-axis. Indeed, according to Eq. (3.7) with $k=1$ and Eq. (4.7), we have

$$\left.\frac{\partial \Phi_c[u(z-R,y,t)]}{\partial c}\right|_{c=\bar{c}(t)} = \mathrm{e}^{\bar{c}(t)R} \left.\frac{\partial \Phi_c[u(z,y,t)]}{\partial c}\right|_{c=\bar{c}(t)}, \tag{4.12}$$

so a finite shift by R produces a factor of $\mathrm{e}^{\bar{c}(t)R}$ in both sides of Eq. (4.11). Thus, this equation only depends on the shape of the solution.

Since $\bar{c}(t)$ is bounded from above, there exists a sequence of times $\{t_n\}$ going to infinity such that $d\bar{c}(t_n)/dt \to 0$. Indeed, by Lebesgue Theorem, $\bar{c}(t)$ is differentiable almost everywhere, and for any $T>0$ we have $\int_{t_0}^{t_0+T} \frac{d\bar{c}}{dt} dt \le c^*$. Applying the Mean Value Theorem to this integral, we obtain that $0 \le \inf_{t_0 \le t \le t_0+T} \frac{d\bar{c}}{dt} \le \frac{c^*}{T}$, which can be arbitrarily small for large T, so the sequence $\{t_n\}$ exists.

Then, there exists a subsequence $\{t'_n\}$ such that $u(z+R(t'_n),y,t'_n) \to \bar{u}(z,y)$ in $C^2(\Sigma_0)$, where $\bar{u}$ is a traveling wave solution with speed c^*. The proof follows from the fact that from uniform boundedness of $u(z+R(t),y,t)$ in $H^1_c(\Sigma)$ with $c > c^* \ge \bar{c}(t)$ follows uniform boundedness of $\left.\frac{\partial\Phi[u(z+R(t),y,t)]}{\partial c}\right|_{c=\bar{c}(t)}$ (see Eq. (3.8) and the arguments there). This means that the right-hand side of Eq. (4.11) goes to zero on a sequence of $u(z+R(t_n),y,t_n)$. Since $u(\cdot,t) \in \mathcal{Q}_c(\Sigma)$ are equicontinuous together with all its derivatives, and Σ_0 is compact, by Arzela-Ascoli Theorem there exists a subsequence $\{t'_n\}$ such that $u(z+R(t'_n),y,t'_n) \to \bar{u}(z,y)$ in $C^2(\Sigma_0)$. Since also $\bar{c}(t) \to c^*$, $\bar{u}$ satisfies Eq. (3.1) with $c = c^*$. By assumption, the trivial solutions are excluded from the possible choices of $\bar{u}$, so this completes the proof. □

Corollary 4.9. *Suppose there exists a wave-like solution of Eq. (1.1) that satisfies the assumptions of Theorem 4.8. Then there exists a variational traveling wave solution with speed* $c_{\min} \leq c \leq c_{\max}$.

Note that if $\bar{u}(z + R(t), y, t) \in H_c^1(\Sigma)$ with $c > c^*$, u can only converge to a variational traveling wave. So, the question here is to find some *a priori* bounds on the rate of decay of u and ∇u at plus infinity in a suitable reference frame. Let us also comment on the way to show whether $u \to \bar{u}$ *asymptotically* as $t \to \infty$. Theorem 4.8 allows to establish that c^* is the speed of the traveling wave solution. Suppose that for any $u \in \mathcal{Q}_c(\Sigma, \mathbb{R}^+)$ there exists $R = R(t)$ that minimizes

$$d(R,t) = \frac{1}{2} \int_\Sigma e^{\bar{c}(t)z} (u(z,y,t) - \bar{u}(z-R,y))^2 dz dy \tag{4.13}$$

with respect to R for any t. Then, convergence of the shape of the solution to the traveling wave in the reference frame characterized by the shift $R(t)$ may be shown by establishing that $e^{-\bar{c}(t)R(t)} d(R(t), t) \to 0$ as $t \to \infty$.

Let us see what sorts of assumptions on Φ_c this would require. Introducing $v = u - \bar{u}$, where $\bar{u} = \bar{u}(z - R(t), y)$, expanding Φ_c around $\bar{u}$, and using Eqs. (4.7) and (3.16), we get

$$\begin{aligned} 0 = \frac{\bar{c}(t) - c^*}{\bar{c}(t)} \int_\Sigma e^{\bar{c}(t)z} \bar{u}_z^T \bar{u}_z dz dy + (c^* - \bar{c}(t)) \int_\Sigma e^{\bar{c}(t)z} \bar{u}_z^T v dz dy \\ + \frac{1}{2} \int_\Sigma e^{\bar{c}(t)z} (\nabla v^T \cdot \nabla v + v^T H(\tilde{u}) v) dz dy, \end{aligned} \tag{4.14}$$

where H is given by Eq. (3.4), for some $\tilde{u}(z, y, t)$. In fact, the second integral in the right-hand side of this equation is zero. Indeed, by construction $0 = \frac{\partial}{\partial R} d(R,t)|_{R=R(t)} = \int_\Sigma e^{\bar{c}(t)z} \bar{u}_z^T v dz dy$. If now we assume a version of strong positivity of the last integral in Eq. (4.14):

$$\int_\Sigma e^{cz} \left(\nabla v^T \cdot \nabla v + v^T H(\tilde{u}) v \right) dz dy \geq K \int_\Sigma e^{cz} v^T v dz dy \tag{4.15}$$

for sufficiently small $e^{-cR(t)} ||v||_{L_c^2}^2$ in some interval of $c < c^*$, we can prove that $e^{-\bar{c}(t)R(t)} d(R(t), t) \to 0$ as $t \to \infty$. Indeed, the first integral in Eq. (4.14) behaves like $(c^* - \bar{c}(t)) e^{\bar{c}(t)R(t)}$, so multiplying both sides of Eq. (4.14) by $e^{-\bar{c}(t)R(t)}$ and taking into account that $\bar{c}(t) \to c^*$ as $t \to \infty$, we obtain the result.

A number of other general conclusions can be made for the asymptotic behavior of the wave-like solutions, when the nonlinearity has certain properties. Suppose, for example, that $H(0)$ is a positive-definite matrix, so zero is a locally stable equilibrium point. Clearly, in this situation some of the initial conditions may decay to zero. This, however, cannot happen under certain assumptions if u is a wave-like solution.

Proposition 4.10. *Let u be a wave-like solution of Eq. (1.1) and $c^* > 2\sqrt{-\nu_0}$, where ν_0 is the lowest eigenvalue of Eq. (3.3), if $\nu_0 < 0$. Then there exists a constant $b > 0$ such that*

$$\max_{(z,y) \in \Sigma} u^2(z,y,t) \geq b \tag{4.16}$$

for all t.

Proof. We only need to show this for $t \geq t_0$. So, suppose the opposite is true. We have

$$\int_\Omega \left(\nabla_y u^T \cdot \nabla_y u + u^T H(0) u\right) dy \geq \nu_0 \int_\Omega u^T u dy, \tag{4.17}$$

and V is smooth with $V(0) = \nabla_u V(0) = 0$. So, for any $\epsilon > 0$ it is possible to find $b > 0$ such that $V(u) \geq \frac{1}{2} u^T (H(0) - \epsilon) u$ for all $u^2 < b$. Therefore

$$\begin{aligned} &\Phi_c[u(\cdot,t)] \\ &\geq \frac{1}{2} \int_\Sigma \mathrm{e}^{cz} \left(u_z^T u_z + \epsilon \nabla_y u^T \cdot \nabla_y u + ((1-\epsilon)\nu_0 - \epsilon)\, u^T u + \epsilon u^T H(0) u\right) dz dy \\ &\geq \frac{\epsilon}{2} \int_\Sigma \mathrm{e}^{cz} \nabla u^T \cdot \nabla u dz dy \\ &+ \frac{1}{2}\left((1-\epsilon)\left(\frac{c^2}{4} + \nu_0\right) - \epsilon + \epsilon \min H(0)\right) \int_\Sigma \mathrm{e}^{cz} u^T u dz dy. \end{aligned} \tag{4.18}$$

Now, for sufficiently small ϵ there exists $t \geq t_0$ such that the last bracket in the integral on the right-hand side of Eq. (4.18) is positive for $c = \bar{c}(t)$. Therefore, $\Phi_{\bar{c}(t)}[u(\cdot,t)] > 0$, which contradicts Eq. (4.7). □

Under the same assumptions, it is also possible to show that the solutions remain uniformly bounded in $H_c^1(\Sigma)$ with any $c < c^*$, in a certain special reference frame.

Theorem 4.11. *Let u be a wave-like solution of Eq. (1.1) and $c^* > 2\sqrt{-\nu_0}$, where ν_0 is the lowest eigenvalue of Eq. (3.3), if $\nu_0 < 0$. Then*

1. *There exists a shift $R(t)$ such that $u^2(R(t), y, t) = b$ at some y, and $u^2(z, y, t) < b$ for all $z > R(t)$ for sufficiently small $b > 0$ and all t.*
2. *$u(z + R(t), y, t)$ is uniformly bounded in $H_c^1(\Sigma)$ for any $c < c^*$.*
3. *$R(t) \geq ct + R_0$ with some R_0 for any $c < c^*$ for all t.*

Proof. By Proposition 4.10, there exists a constant $b > 0$ such that $u^2(\cdot, t) \geq b$ somewhere in Σ. Since $u(\cdot,t) \to 0$ as $z \to +\infty$, the set of all points in Σ at which $u^2 \geq b$ is bounded at plus infinity. Therefore, for each t there exists the least upper bound of that set, call it $R(t)$. Since u is continuous, there exists y such that $u^2(R(t), y, t) = b$ and $u^2(\cdot, t) < b$ for all $z > R(t)$. So, $R(t)$ is the function sought.

Let us now show that $u(z + R(t), y, t)$ remains uniformly bounded in $H_c^1(\Sigma)$ for arbitrary $c < c^*$. We only need to show that for those values of c that are sufficiently close to c^*, this would then imply uniform boundedness in $H_{c'}^1(\Sigma)$ with any $0 < c' < c$. Following the argument of Eq. (4.18), for any $\epsilon > 0$ it is possible to choose such $b > 0$ that $V(u) \geq \frac{1}{2} u^T (H(0) - \epsilon) u$ for all $u^2 < b$. Then, using this value of b to define $R(t)$ and repeating the arguments of Eq. (4.18), we get

$$\begin{aligned} &\Phi_c[u(z + R(t), y, t)] = \mathrm{e}^{-cR(t)} \Phi_c[u(\cdot, t)] \\ &\geq \frac{\epsilon}{2} \int_\Sigma \mathrm{e}^{c(z - R(t))} \nabla u^T \cdot \nabla u dz dy + V_{\min} \int_\Omega \int_{-\infty}^{R(t)} \mathrm{e}^{c(z-R(t))} dz dy \\ &+ \frac{1}{2}\left((1-\epsilon)\left(\frac{c^2}{4} + \nu_0\right) - \epsilon + \epsilon \min H(0)\right) \int_\Omega \int_{R(t)}^{+\infty} \mathrm{e}^{c(z-R(t))} u^T u dz dy, \end{aligned} \tag{4.19}$$

where $V_{\min} = \min\limits_{u \in \mathcal{Q}_c(\Sigma)} V(u)$, and we used Lemma 2.3. Let us choose c sufficiently close to c^*, so $c_{\min} < c < c^*$. Then, there exists a time T such that $\Phi_c[u(\cdot,t)] < 0$ for all $t > T$. Let us choose ϵ so small that the last integral in Eq. (4.19) is positive.

Then from Eq. (4.19) for $t > T$ we obtain

$$(4.20)\quad 0 > \Phi_c[u(z+R(t),y,t)] \geq \frac{\epsilon}{2}\int_\Sigma e^{c(z-R(t))}\nabla u^T \cdot \nabla u dz dy + \frac{V_{\min}A}{c},$$

where $A = \int_\Omega dy$. This means that

$$(4.21)\qquad ||\nabla u(z+R(t),y,t)||^2_{L^2_c} \leq -\frac{2V_{\min}A}{\epsilon c},$$

so, by Lemma 2.5, $||u(z+R(t),y,t)||_{1,c}$ is uniformly bounded for all $t > T$. But since $u \in \mathcal{Q}_{c_{\max}}(\Sigma,\mathbb{R}^+)$, this means that $u(z+R(t),y,t)$ is uniformly bounded in $H^1_c(\Sigma)$ for all t as well.

Now, by Proposition 4.1, Lemma 2.3 and Eq. (4.20), for $t > T$ we have

$$\begin{aligned} e^{-c(R(t)-ct)}\Phi_c[u(z+cT,y,T)] &\geq e^{-c(R(t)-ct)}\Phi_c[u(z+ct,y,t)] \\ &= \Phi_c[u(z+R(t),y,t)] \geq \frac{V_{\min}A}{c}. \end{aligned} \qquad (4.22)$$

Dividing this inequality by a negative number $\Phi_c[u(z+cT,y,T)]$ and taking the logarithm of both sides, we obtain

$$(4.23)\qquad R(t) \geq ct + \ln\frac{c\Phi_c[u(z+cT,y,T)]}{V_{\min}A},$$

which is the last statement of the Theorem. □

The quantity $R(t)$ represents the position of the *leading edge* of a wave-like solution. Thus, under the assumptions of Theorem 4.11 the leading edge will propagate faster than the speed $c^* - \epsilon$ with ϵ arbitrarily small, establishing c^* as a lower bound on the average propagation speed of the leading edge of the solution. Note that Theorem 4.2 establishes the upper bound for the speed of propagation of the leading edge.

Also, under the assumptions of Theorem 4.11 the shape of the solution remains sufficiently well-behaved: $u(z+R(t),y,t) \in H^1_{c^*-\epsilon}(\Sigma)$. This fact provides a selection criterion for the solutions in that reference frame. In particular, it allows to exclude all the non-variational traveling wave solutions as potential candidates for the asymptotic state in this reference frame. Note that this, however, is not yet sufficient for applying Theorem 4.8. For this theorem to be valid, we would have to have $u(z+R(t),y,t) \in H^1_{c^*+\epsilon}(\Sigma)$ with some ϵ.

Let us point out that when $\nu_0 < 0$, the speed $c = 2\sqrt{-\nu_0}$ is the smallest propagation speed for a traveling wave solution that behaves asymptotically like $v_0(y)e^{-\lambda z}$, where v_0 is the eigenfunction of the operator in Eq. (3.3) corresponding to ν_0, as $z \to +\infty$. In this situation it is possible to have families of traveling wave solutions with speeds greater or equal than $2\sqrt{-\nu_0}$ [1]. Moreover, for scalar equations the solution with the speed $c = 2\sqrt{-\nu_0}$ may often be the selected solution at long times [9, 27, 28]. However, the assumptions of Theorem 4.11 do not allow that. In the language of Ref. [28], these conditions exclude the possibility of linear selection mechanism.

5. Some applications

As a simple example, consider the subcritical version of Eq. (1.5) in one space dimension:

$$(5.1)\qquad u_t = u_{xx} + \mu u + u^3 - u^5.$$

For this equation, certain exact traveling wave solutions were found in [28] (see also [1, 29]). When $\mu > -\frac{1}{4}$, this equation has three equilibrium points. For

$-\frac{3}{16} < \mu < 0$ there is a unique traveling wave solution with positive speed $c = c^*$, where

$$c^* = \frac{2\sqrt{1+4\mu} - 1}{\sqrt{3}}, \tag{5.2}$$

connecting zero with a positive equilibrium point [12]. According to the discussion above, this is a variational traveling wave. When $\mu = 0$, there exists one positive traveling wave solution with speed $c^* = 1/\sqrt{3}$ that decays exponentially for $z \to +\infty$ and a continuous family of positive solutions with $c > c^*$ and algebraic decay [14,15]. Out of these, only the former is a variational traveling wave. For $0 < \mu < \frac{3}{4}$, a family of exponentially decaying solutions with $c \geq 2\sqrt{\mu}$ and slower decay, and a unique positive solution with $c = c^* > 2\sqrt{\mu}$, where c^* is given by Eq. (5.2), and faster decay exist [28,29]. Only the latter is the variational traveling wave. Finally, for $\mu \geq \frac{3}{4}$ a family of exponentially decaying solutions for $c \geq 2\sqrt{\mu}$ exists. There is also a family of solutions with oscillatory behavior at infinity for $0 < c < 2\sqrt{\mu}$. None of these solutions is a variational traveling wave.

A simple calculation for Eq. (5.1) shows that the upper bound for the speed of the variational traveling wave is

$$c_{\max} = \frac{1}{2}\sqrt{16\mu + 3}. \tag{5.3}$$

As it should, this equation is in agreement with Eq. (5.2). It also provides the value of $\mu = -\frac{3}{16}$ at which the variational traveling waves cease to exist. Note that the upper bound in Eq. (5.3) is typically very close to the exact value in Eq. (5.2). For example, numerically we have $c^* = 1.423$, while $c_{\max} = 1.658$, for $\mu = \frac{1}{2}$, within $\sim 15\%$ of each other.

Note that it can be shown that the variational traveling wave solutions of Eq. (5.1) are in fact minimizers of Φ_c at $c = c^*$ [40]. This then allows to obtain lower bounds for the value of c^*. For example, consider Eq. (5.1) with $\mu = \frac{1}{2}$. Let u_m be the positive equilibrium solution of this equation, then consider a trial function

$$\bar{u}_a(x) = \begin{cases} u_m, & x < 0, \\ u_m e^{-ax}, & x \geq 0, \end{cases} \tag{5.4}$$

where a is an adjustable parameter. Let us substitute $\bar{u}_a$ into Φ_c, minimize the functional with respect to a and then find the largest value of $c = c_{\min}$ for which $\Phi_c[\bar{u}_a] \leq 0$. A straightforward numerical calculation then gives $c_{\min} = 1.415$. This is below the exact value by just $\sim 0.5\%$!

Let us now discuss the results of Sec. 4 applied to Eq. (5.1). For simplicity, consider the case of $\mu = 0$. Theorem 4.11 applies to any initial condition that decays sufficiently rapidly at plus infinity and approaches a nonzero limit whose absolute value is less than $\sqrt{\frac{3}{2}}$ at minus infinity. For example, an initial condition $u_0 = 1$ if $x < 0$ and $u_0 = e^{-x}\cos x$ if $x \geq 0$ will generate a wave-like solution with $\bar{c}(t) \geq 0.2138$. The value of b for this solution can be chosen to be, say, 0.01. Then, according to Theorem 4.11, the first point on the right at which $u^2 = 0.01$ will move with an average speed greater than $c_{\min} = 0.2138$. Better lower bounds can be obtained from estimates of $\bar{c}(t)$ obtained by the numerical solution of Eq. (5.1) on finite time intervals, which can be done with any reasonable precision. Furthermore, the only accessible traveling wave solutions for this initial value problem will be the two variational traveling waves (which differ by the sign) with speed $c = 1/\sqrt{3}$, since they are the only variational traveling wave solutions of Eq. (5.1) at $\mu = 0$.

Let us now demonstrate that Theorem 4.8 can be applied to a class of monotone initial conditions connecting zero and one, and decay sufficiently rapidly at plus infinity. To do that, we need to show that in the reference frame associated with the leading edge of the solution the functions u and u_x decay with a sufficiently fast exponential rate. This can be done using phase plane arguments. Indeed, for $u_x < 0$ it is possible to rewrite Eq. (5.1) in terms of $v = -u_x$ as a function of t and u. After a change of variables, we obtain

$$v_t = v^2 \frac{\partial}{\partial u}\left(\frac{\partial v}{\partial u} + \frac{u^3 - u^5}{v}\right), \quad v(0) = v(1) = 0, \tag{5.5}$$

with $v(u,t) > 0$ for all $0 < u < 1$. This degenerate parabolic equation obeys the comparison principle (see, for example, [41]). To construct suitable lower solutions, we use the phase plane trajectories of the stable manifold of

$$u_{xx} + \alpha u_x + u^3 - u^5 = 0, \tag{5.6}$$

approaching zero from above as $x \to +\infty$, with $\alpha < 1/\sqrt{3}$. Clearly, these trajectories will intersect the $v = 0$ axis at some $u = u_0 < 1$. Therefore, since $v(u_0) > 0$, the obtained trajectories $v = \underline{v}_\alpha(u)$ are subsolutions on the interval $0 \le u \le u_0$ [41]. Similarly, by choosing $\beta > 1/\sqrt{3}$ (instead of α in Eq. (5.6)), we will obtain the supersolution $\bar{v}_\beta(u)$ for $0 \le u \le 1$, since obviously $\bar{v}_\beta(1) > 0$.

Since $\underline{v}_\alpha(u)$ and $\bar{v}_\beta(u)$ are obtained from the stable manifold of Eq. (5.6), we will have

$$\underline{v}_\alpha(u) \ge (\alpha - \epsilon)u, \;\; \bar{v}_\beta \le (\beta + \epsilon)u, \;\; 0 \le u \le \delta, \tag{5.7}$$

for some sufficiently small $\delta > 0$, given an arbitrary $\epsilon > 0$. Now, choosing the initial data for Eq. (5.1) which lie in the phase plane between $\underline{v}_\alpha$ and $\bar{v}_\beta$, we are guaranteed that $(\alpha - \epsilon)u \le v(u,t) \le (\beta + \epsilon)u$ as long as $u \le \delta$. This, in turn implies for solutions of Eq. (5.1) that

$$u(x,t) \le \delta\, \mathrm{e}^{-(\alpha-\epsilon)(x - R(t))}, \tag{5.8}$$

$$|u_x(x,t)| \le \delta(\beta + \epsilon)\mathrm{e}^{-(\alpha-\epsilon)(x-R(t))}, \tag{5.9}$$

with $R(t)$ defined by $u(R(t),t) = \delta$. Indeed, recalling the definition of $v(u)$ and using the first inequality in Eq. (5.7), we have

$$u_x \le -(\alpha - \epsilon)u, \;\; \text{for all } x > R(t). \tag{5.10}$$

Dividing by u, integrating from $R(t)$ to $x > R(t)$, and taking into account the definition of $R(t)$, we obtain Eq. (5.8). Also, recalling the second inequality in Eq. (5.7), we obtain Eq. (5.9). Then, taking into account that both u and u_x are uniformly bounded, we conclude that $u(x + R(t), t)$ is uniformly bounded in $H^1_c(\mathbb{R})$ with $c = 2\alpha - 3\epsilon$.

Clearly, α and ϵ can be chosen in such a way that $c > c_{\max}$, where $c_{\max} = \sqrt{3}/2$ (see Eq. (5.3)). Since, in turn, $\lim_{t\to\infty} \bar{c}(t) \le c_{\max}$, for this class of initial conditions the conclusion of Theorem 4.8 holds. For example, this is the case for the initial conditions $u_0 = 1$ if $x < 0$ and $u_0 = \mathrm{e}^{-ax}$ if $x \ge 0$, with $a > \sqrt{3}/4$, which is guaranteed to lie between $\underline{v}_\alpha$ (recall that $\underline{v}_\alpha \sim \alpha u$ in the limit $u \to 0$, and that $f(u) > 0$ implies that $\underline{v}_\alpha(u)$ is convex down) and $\bar{v}_\beta$ (with large enough β) in the phase plane. Note that local stability of the variational traveling wave then implies global convergence of the solution to the traveling wave solution as $t \to \infty$ (see [25]). In addition, these arguments with, say, $a = \frac{1}{2}$ give existence of a variational traveling wave with $0.5220 < c < 0.8660$, which agrees with the known exact solution.

Let us point out that the above analysis is for illustrative purposes only, we are not aiming at complete generality here. Various generalizations of the above mentioned arguments are possible. For example, these results can be extended to the initial data with faster than exponential decay (including data with compact support), or to the case $0 < \mu < \frac{3}{4}$. One can also come up with heuristic forms of subsolutions. For example, one can choose $\underline{v}_\alpha(u) = \alpha\sqrt{1 - \frac{u^2}{u_0^2}}$, with $\alpha = 0.44$ and $u_0 = 0.7$, the fact that this is a subsolution for Eq. (5.5) is verified directly. We shall not dwell on these issues here any further.

On the other hand, it is more difficult to apply our results to nonlinearities of Fisher type. Indeed, as a simple example, consider the Fisher equation in one dimension, so $f(u) = u(1-u)$. It is easy to see that in this case $c_{\max} = 2$. However, it is known that positive traveling wave solutions for this equation exist only for $c \geq 2$ and have slow exponential decay, which make them lie *outside* of spaces $H_c^1(\mathbb{R})$. Clearly, Theorem 4.11 cannot be applied since in view of the value of $c_{\max}$ we must have $c^* \leq 2$, which contradicts the assumption of the Theorem. Similarly, Theorem 4.8 cannot hold, since Corollary 4.9 contradicts non-existence of variational traveling waves for the Fisher equation. Of course, Theorem 4.7 holds for all equations of the considered type, including the Fisher equation.

Nevertheless, the techniques developed in this paper allow to make a number of conclusions about the behavior of solutions of the Fisher equation and the like. Once again, consider the Fisher equation in one dimension. We can use the properties of $\mathcal{Q}_c(\mathbb{R})$ to establish global stability of the traveling wave solutions with speed $c > 2$ with respect to perturbations decaying faster than the wave at plus infinity. Indeed, let $v = u - \bar{u}$, where $\bar{u}$ is the traveling wave solution with speed $c > 2$, and we consider positive solutions: $u > 0$. Then the Fisher equation can be rewritten as

$$v_t = v_{xx} + cv_x + v - 2\bar{u}v - v^2. \tag{5.11}$$

Let us multiply this equation by $e^{c'x}v$ and integrate over space. After a number of integrations by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}} e^{c'x} v v_t dx &= \int_{\mathbb{R}} e^{c'x}\left(-v_x^2 - \frac{(c-c')c'}{2}v^2 + v^2 - 2\bar{u}v^2 - v^3\right)dx \\ &\leq \int_{\mathbb{R}} e^{c'x}\left(\frac{c'^2}{4} - \frac{cc'}{2} + 1 - 2\bar{u} - v\right)v^2 dx,\end{aligned} \tag{5.12}$$

where in arriving at the last inequality we used Eq. (2.14). Observe that in view of positivity of u, we have $-\bar{u} - v < 0$, so the last two terms in the bracket in Eq. (5.12) are negative. Therefore, if we choose $c' = c - \sqrt{c^2 - 4} + \epsilon$, with sufficiently small $\epsilon > 0$, we get

$$\frac{d}{dt}\int_{\mathbb{R}} e^{c'x} v^2 dx \leq -\alpha \int_{\mathbb{R}} e^{c'x} v^2 dx, \tag{5.13}$$

with some constant $\alpha > 0$, and we immediately conclude that the traveling wave solution $\bar{u}$ is exponentially stable with respect to arbitrary perturbations in $\mathcal{Q}_{c'}(\Sigma)$, as long as $u > 0$. Note that the traveling wave solutions with speed $c > 2$ behave asymptotically like $\bar{u}(x) \sim e^{-\lambda_- x}$, where $\lambda_- = \frac{1}{2}\left(c - \sqrt{c^2 - 4}\right)$, as $x \to +\infty$ [9,13]. So, the above mentioned requirement amounts to the fact that v must decay with a faster exponential rate than $\bar{u}$ for large x. This also implies that the propagation speed and the asymptotic profile of the solutions of the Fisher equation is determined solely by the asymptotic behavior of the initial data for sufficiently slow exponential decay rates. This generalizes the well-known result of McKean [42]. Note that the argument above can be trivially extended to cylinders with Neumann boundary conditions, or to convex down nonlinearities $f(u)$.

6. Conclusions

In conclusion, we have identified a variational structure of the gradient reaction-diffusion systems with equal diffusion coefficients in cylinders. This variational structure is intimately related with the traveling wave solutions. The obtained functionals are well-defined for functions that decay sufficiently rapidly exponentially at one end of the cylinder.

We found that a certain class of the traveling wave solutions which we call variational traveling waves are critical points of these functionals. For systems with linearly stable equilibrium solution $u = 0$ all traveling wave solutions are variational traveling waves. In the opposite case the variational traveling waves are special in the sense that the exponential decay rate of $|u|$ ahead of the wave for these waves corresponds to the *larger* value of λ that satisfies Eq. (3.5) (see Sec. 3).

Our main result about the variational traveling waves is the global upper bound for their speed (Theorem 3.7). Note that this upper bound is only determined by the nonlinearity f (plus, perhaps, the information about the spectrum of the Laplace operator in Ω) and can therefore be readily calculated for a given system. Our method also allows to estimate the lower bounds of the speeds of the variational traveling waves. Note that different kinds of other variational estimates can also be obtained for the speeds of the traveling waves in reaction-diffusion systems [35, 38].

Perhaps the most interesting results associated with the variational structure of the problem have to do with the solutions of the initial value problem. The variational formulation allows to study the evolution of disturbances, that is, the solutions of Eq. (1.1) that decay exponentially at one end of the cylinder. Our analysis allows to make predictions about the *propagation* of the disturbances that decay sufficiently rapidly as $z \to +\infty$.

For such solutions, the obtained functionals are monotonically decreasing with time in the reference frames moving with constant speeds. Our main observation here is that the *sign* of the functionals is translationaly invariant, so one can make certain predictions about the propagation of disturbances based on the information about the sign. It turns out that this allows one to introduce a definition of the instantaneous propagation speed for a wide class of solutions which we call wave-like. A striking property of this propagation speed is the fact that it approaches a limit as $t \to \infty$ (Theorem 4.7), thus suggesting that these kinds of disturbances propagate asymptotically *linearly* with time. In turn, this definition can be related to other definitions of the speed of the traveling wave, such as the average speed of the leading edge of the solution, for example (see Sec. 4). In this case under certain verifiable assumptions one can obtain bounds on such speeds. Essentially, the variational structure provides an integral comparison principle for the solutions of the initial value problem with the results similar to those obtained using the maximum principle (see, for example, [13, 16, 34]). The advantage of our method, however, is that it works with very general classes of initial conditions in arbitrary dimensions with multicomponent variable u and does not require construction of the upper and lower solutions.

One of our main results about the propagation of disturbances has to do with convergence of wave-like solutions to variational traveling waves (Theorem 4.8). What we found is that if the profile of the disturbance remains sufficiently compact

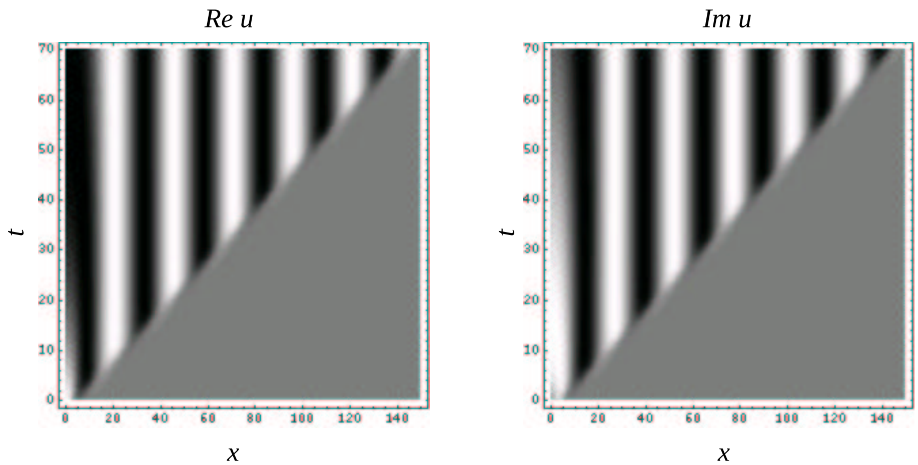

FIGURE 1. Development of complex spatiotemporal dynamics in the Newell-Whitehead-Segel equation. The shades of gray go from black to white when $Re\, u$ and $Im\, u$ go from –1 to 1. See text for details. Only half of the solution is shown.

in a certain reference frame, the solution of the initial value problem will converge to a variational traveling wave at least on a sequence of times. In other words, this will happen if the distribution of u does not spread out, or, more precisely, when the front of the disturbance remains sufficiently steep for all times. So, for this one needs some apriory bounds on the behavior of the solution in the front of the disturbance. For example, in the case of certain scalar reaction-diffusion equations in one dimension one can obtain such bounds by phase plane analysis and using the comparison principle for certain types of initial conditions (see Sec. 5). Note, however, that this result does not work with all gradient reaction-diffusion systems (like the Fisher equation).

A general question related to this is whether the traveling wave solutions are the only possible asymptotic states for the gradient reaction-diffusion systems with equal diffusion coefficients. For scalar reaction-diffusion equations the answer to this question is positive in a large number of situations [9–12, 24, 25]. Yet, it is not clear whether this can be extended to more general classes of initial conditions or multicomponent systems (for monotone systems, see [26]). To illustrate the nontriviality of this point, let us give a numerical example of a situation in which the asymptotic state is *not* a traveling wave. Consider the Newell-Whitehead-Segel equation (see Sec. 1). We solved numerically the initial value problem with $u_0(x) = \cosh^{-2}\left(\frac{x}{4}\right)\exp\left(\frac{ix}{2}\tanh\frac{x}{2}\right)$. The result is presented in Fig. 1. It is clear from this figure that complex spatiotemporal dynamics develop at long times. At the same time, the leading edge of the solution propagates linearly with time, in agreement with Theorem 4.7.

On the other hand, Theorem 4.11 gives uniform bounds in $H^1_c(\Sigma)$ for the solutions in the reference frame associated with the leading edge and, therefore, allows to *exclude* certain classes of solutions as potential asymptotic states. In particular, it allows to exclude all non-variational traveling waves and therefore provides a selection criterion. Note that this selection criterion partially verifies the nonlinear selection hypothesis introduced in Ref. [28].

Finally, let us mention a number of possible generalizations of our results. It can be easily seen that our results remain valid when the function f explicitly depends on the transversal coordinate y. Also, the variational formulation can be modified to include a constant mean flow in the transverse directions, as well as nonlinear sources at the boundaries. In addition, the variational formulation can be extended to Eq. (1.1) with an extra term αu_{tt}, where α is an arbitrary positive constant, added to its left-hand side. The latter can be particularly useful even for scalar problems in one dimension since the maximum principle does not apply to these equations.

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