



A Quantitative Description of Skyrmions in Ultrathin Ferromagnetic Films and Rigidity of Degree ± 1 Harmonic Maps from \mathbb{R}^2 to \mathbb{S}^2

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Abstract

We characterize skyrmions in ultrathin ferromagnetic films as local minimizers of a reduced micromagnetic energy appropriate for quasi two-dimensional materials with perpendicular magnetic anisotropy and interfacial Dzyaloshinskii–Moriya interaction. The minimization is carried out in a suitable class of two-dimensional magnetization configurations that prevents the energy from going to negative infinity, while not imposing any restrictions on the spatial scale of the configuration. We first demonstrate the existence of minimizers for an explicit range of the model parameters when the energy is dominated by the exchange energy. We then investigate the conformal limit, in which only the exchange energy survives and identify the asymptotic profiles of the skyrmions as degree 1 harmonic maps from the plane to the sphere, together with their radii, angles and energies. A byproduct of our analysis is a quantitative rigidity result for degree ± 1 harmonic maps from the two-dimensional sphere to itself.

Contents

1. Introduction	220
1.1. Informal Discussion of Results	223
1.2. Outline of the Paper	225
2. Main Results	225
2.1. The Energy and the Admissible Class	225
2.2. Statement of the Results	227
2.3. Notation	232
3. An Explicit Representation of the Energy	233
4. Rigidity of Degree ± 1 Harmonic Maps	235
4.1. The Spectral Gap Property for the Linearized Problem	235
4.2. From Linear Stability to Rigidity	241
4.3. Proofs of Theorem 2.4, Lemma 2.5 and Corollary 2.6	246
5. Existence of Minimizers	252

5.1. First Lower Bounds	252
5.2. Upper Bounds via Minimization of a Reduced Energy	255
5.3. Existence of Minimizers via the Concentration Compactness Principle	265
6. The Conformal Limit	272
6.1. Convergence to Shrinking Belavin–Polyakov Profiles via Stability of the Reduced Energy $\mathcal{E}_{\sigma,\lambda;K}$	282
A. Appendix	284
A.1. Sobolev Spaces on the Sphere	284
A.2. The Topological Bound and Energy Minimizing Harmonic Maps of Degree 1	287
A.3. Integrals Involving Belavin–Polyakov Profiles	290
References	295

1. Introduction

A skyrmion is a topologically nontrivial field configuration that locally minimizes an energy functional of a nonlinear field theory. As topological solitons [61], they are localized, have finite energy and exhibit quasi-particle properties, including quantized topological charge, attractive or repulsive interactions between each other, etc.. Since its original formulation by Tony Skyrme in the early 1960s [76], the mathematical concept of skyrmion has spread over various branches of physics [73]. In condensed matter physics, a revival of the skyrmion topic was triggered by experimental observations of skyrmions in non-centrosymmetric bulk magnetic materials [67, 81] and ultrathin ferromagnets [13, 74] with distinct top and bottom interfaces [40]. These magnetic skyrmions consist of local swirls of spins that may exhibit nanometer size [74], room temperature thermal stability [13] and may be controlled via electric current [45] or electric field [42]. These properties are highly desirable for information technology applications, making magnetic skyrmions attractive for race track memory [79], spintronic logic [82], as well as stochastic [71] and neuromorphic computing [72].

At the level of the continuum, the starting point in the analysis of magnetic skyrmions in thin ferromagnetic films is the micromagnetic energy functional [11]

$$E(m) := E_{\text{ex}}(m) + E_{\text{a}}(m) + E_{\text{Z}}(m) + E_{\text{DMI}}(m) + E_{\text{s}}(m) \quad (1.1)$$

describing the energy (per unit of the film thickness) of a smooth map $m : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ that represents the normalized ($|m| = 1$) magnetization vector field in a ferromagnet. The terms in (1.1) are, in order of appearance: the exchange (also called the Dirichlet energy), the anisotropy, the Zeeman, the Dzyaloshinskii–Moriya interaction (DMI), and the stray field energies, respectively. The precise form of these terms is model-specific and will be spelled out for the particular situation we are interested in shortly. Coming back to the magnetization m , its topology may be characterized by the topological charge

$$\mathcal{N}(m) = \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot (\partial_1 m \times \partial_2 m) \, dx. \quad (1.2)$$

This integer-valued quantity corresponds to the Brouwer degree of a smooth map m which is constant sufficiently far away from the origin, up to the sign due to

a particular choice of an orientation of \mathbb{R}^2 . The topologically nontrivial localized magnetization configurations are, hence, characterized by a non-zero value of \mathcal{N} in (1.2). See Hoffman *et al.* [41] for a discussion of how to distinguish between skyrmions and antiskyrmions independently of the sign convention for \mathcal{N} .

In a two-dimensional model containing only the exchange energy $E_{\text{ex}}(m) = A_{\text{ex}} \int_{\mathbb{R}^2} |\nabla m|^2 dx$, where A_{ex} is the exchange stiffness, BELAVIN and POLYAKOV [6] predicted the existence of skyrmion-like solutions as energy minimizing configurations with constant energy and an explicit profile, which we refer to later as Belavin–Polyakov profiles. Note, however, that these solutions may not be considered proper skyrmions, since they exhibit dilation invariance and thus do not exhibit true particle-like properties. Furthermore, they are easily seen to cease to exist in the presence of an additional anisotropy term $E_a(m) = K_u \int_{\mathbb{R}^2} |m'|^2 dx$. Here K_u is the uniaxial anisotropy constant and $m' = (m_1, m_2)$ is the in-plane component of the magnetization vector $m = (m', m_3)$ [24]. Similarly, skyrmion solutions are destroyed in the presence of an out-of-plane applied magnetic field modeled by $E_Z(m) = -\mu_0 M_s \int_{\mathbb{R}^2} H(1 + m_3) dx$, where H is the magnetic field strength, M_s is the saturation magnetization, μ_0 is the permeability of vacuum, and we subtracted a constant to ensure that the Zeeman energy is finite when $m(x) \rightarrow -e_3$ sufficiently fast as $|x| \rightarrow \infty$. Therefore, additional energy terms are necessary to stabilize magnetic skyrmions.

Among the known stabilizing energies are higher order exchange [1, 44], DMI [11] and stray field [19, 47] terms. In particular, BOGDANOV and YABLONSKII [11] considered an additional DMI term of general form, which includes a bulk DMI term $E_{\text{DMI}}^{\text{bulk}}(m) = D_{\text{bulk}} \int_{\mathbb{R}^2} m \cdot (\nabla \times m) dx$, or an interfacial DMI term $E_{\text{DMI}}^{\text{surf}}(m) = D_{\text{surf}} \int_{\mathbb{R}^2} (m_3 \nabla \cdot m' - m' \cdot \nabla m_3) dx$, where D_{bulk} and D_{surf} are the bulk and the interfacial DMI strengths [40, 70], respectively, and showed that these terms may give rise to skyrmions. Their model accounts for the stray field in an infinite vortex-like magnetization configuration along the thickness direction [11]. This prediction was further verified numerically in the absence [10] and in the presence [9] of an applied out-of-plane magnetic field. Finally, the analysis of BÜTTNER *et al.* indicates that stray field energy alone (starting with an exact expression for the magnetostatic interaction energy of a thickness-independent magnetization configuration in a film) may be sufficient to stabilize magnetic skyrmions [19]. Notice that in all of the above studies it is assumed that skyrmion solutions possess radial symmetry.

Mathematically, the question of existence of skyrmions as topologically nontrivial energy minimizers was first systematically addressed (under no symmetry assumptions) by ESTEBAN [31–33] and by LIN and YANG [57, 58]. Specifically, for the energy of the form of (1.1) consisting of an exchange energy with an additional Skyrme-type higher order term, $E_{\text{ex}}(m) = A_{\text{ex}} \int_{\mathbb{R}^2} |\nabla m|^2 dx + A_s \int_{\mathbb{R}^2} |\partial_1 m \times \partial_2 m|^2 dx$, and a special form of an anisotropy/Zeeman term (see the remark in [28, p. 2]), $E_a(m) + E_Z(m) = K \int_{\mathbb{R}^2} |m + e_3|^4 dx$, existence of a skyrmion solution as a minimizer m of E with $\mathcal{N}(m) = \pm 1$ was proved in [53, 58]. Also, curvature of the underlying space has been explored as another possible mechanism for ensuring existence of skyrmions [51, 65].

Turning to DMI-stabilized skyrmions, in situations where the energy consists of exchange $E_{\text{ex}}(m) = A_{\text{ex}} \int_{\mathbb{R}^2} |\nabla m|^2 dx$, Zeeman $E_Z(m) = -\mu_0 M_s H \int_{\mathbb{R}^2} (1 +$

$m_3) dx$, and bulk DMI $E_{\text{DMI}}(m) = D_{\text{bulk}} \int_{\mathbb{R}^2} m \cdot (\nabla \times m) dx$ terms, existence of minimizers with non-zero topological charge for suitable values of the parameters was established by MELCHER [64], adapting an argument by BREZIS and CORON [16] for harmonic maps on bounded domains. Furthermore, Melcher demonstrated that the obtained minimizer is indeed a skyrmion, as the minimum of the energy is attained for $\mathcal{N} = 1$ (expressed using the sign conventions of the present paper). In the regime of dominating exchange energy E_{ex} , DÖRING and MELCHER [28] analyzed the compactness properties of these solutions and proved that they converge to a minimizer of E_{ex} of topological charge $\mathcal{N} = 1$ found by BELAVIN and POLYAKOV [6], which the lower order terms uniquely determine. However, as these limits do not decay sufficiently fast for the Zeeman energy to be finite, they had to choose a faster decaying version of the Zeeman energy $E_Z = K \int_{\mathbb{R}^2} |m + e_3|^p dx$ for $p \in (2, 4]$, which only corresponds to a physical model for $p = 4$, and even then only to the specific combination of anisotropy and Zeeman terms analyzed in [53, 58]. Furthermore, LI and MELCHER [54] proved that, for the above mentioned physical choices of E_{ex} , E_Z and E_{DMI} , axisymmetric skyrmions are stable also with respect to symmetry-breaking perturbations and are indeed local minimizers of the model considered by MELCHER [64]. For the same model, KOMINEAS et al. [49] formally established asymptotic formulas for the skyrmion radius and the energy by means of numerics and asymptotic matching. Finally, they also describe the skyrmion profile in a large radius regime on the basis of formal asymptotic analysis [50]. Existence of skyrmions with a uniaxial anisotropy term E_a rather than a Zeeman term has been shown by GRECO [37] in the context of cholesteric liquid crystals.

As one expects the minimizers of a perturbed exchange energy to be close to energy-minimizing harmonic maps (i.e., minimizers of the Dirichlet energy), it is natural to analyze the rigidity of these harmonic maps. The proper context for such an analysis is the theory of harmonic maps between manifolds, which is reviewed in papers by EELLS and LEMAIRE [29, 30], and by HÉLEIN and WOOD [39]. Here, we only discuss the immediately relevant results of the theory. First, the classification of harmonic maps from \mathbb{S}^2 to itself in the mathematical literature is independently due to LEMAIRE [52] and WOOD [80], see also [29, (11.5)]. Additionally, they observed that any harmonic map from \mathbb{S}^2 to \mathbb{S}^2 is also energy-minimizing in its homotopy class. It is worth noting that in the setting of maps from \mathbb{R}^2 to \mathbb{S}^2 the classification result was also formally obtained by BELAVIN and POLYAKOV [6]. Second, a linear version of our stability result, namely that the null-space of the Hessian only arise from minimality-preserving perturbations, is also well-known in the case of the identity map $\text{id} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ and has been established by SMITH [77, Example 2.13], as well as MAZET [62, Proposition 8]. A similar statement in the equivalent setting of harmonic maps from \mathbb{R}^2 to \mathbb{S}^2 has furthermore been recently proved by Chen, Liu and Wei for arbitrary degrees [21]. However, to the best of our knowledge the corresponding spectral gap estimate has only been obtained in a related setting by LI and MELCHER [54], as well as for the problem of H -bubbles by ISOBE [43], and by CHANILLO and MALCHIODI [20]. We also point out that, based on related stability considerations, DAVILA et al. [23] constructed solutions to the harmonic heat flow in which degree 1 harmonic maps bubble off at a specified time and at specified

blow-up locations. Furthermore, strict local minimality results closely related to our rigidity result, Theorem 2.4 below, have been given by LI and MELCHER [54]; Di FRATTA et al. [26]; and Di FRATTA et al. [25]. Also, in the more restrictive equivariant setting our rigidity result follows from GUSTAFSON et al. [38, Theorem 2.1]. Finally, very recently LUCKHAUS and ZEMAS [60] proved a quantitative stability result for conformal maps from \mathbb{S}^n to itself for all $n \geq 2$ under a Lipschitz assumption and closeness in H^1 to the identity.

1.1. Informal Discussion of Results

In this paper, we analyze the energy $E_{Q,\kappa,\delta}$, to be defined shortly in equation (2.9) below, which consists of the exchange, anisotropy, surface DMI energy, as well as the nonlocal stray field energy that is appropriate for thin films [48, 68, 69]. We remark that, while our methods are capable of dealing with a non-zero external field, we have chosen to consider the physically most basic case of vanishing external field. Note that in this case the energy is unbounded from below, which can be seen by considering large magnetic bubbles with topological charge $\mathcal{N} = 1$, see for example [7]. Therefore, an absolute minimizer with the desired topology does not exist, and instead we have to look for a local minimizer, which means that we need to identify a suitable constraint. We argue that in the present context one possible choice is given by [8]

$$\int_{\mathbb{R}^2} |\nabla m|^2 dx < 16\pi. \quad (1.3)$$

As the Dirichlet energy in the wall of a magnetic bubble scales with the radius, this bound clearly excludes such competitors. In contrast, to see why the condition (1.3) would yield skyrmion solutions, we turn to the classical Belavin–Polyakov bound relating the Dirichlet energy to the topological charge:

$$\int_{\mathbb{R}^2} |\nabla m|^2 dx \geq 8\pi |\mathcal{N}(m)|; \quad (1.4)$$

see the original paper by BELAVIN and POLYAKOV [6] or Lemma A.3 below for the proof in the present context. Together with the bound (1.3), it a priori excludes higher topological charges and only allows $\mathcal{N} = -1, 0, 1$. At the same time, we emphasize that, due to the scale invariance of the Dirichlet energy in two dimensions, the assumption (1.3) does not impose any constraints on the actual size of the skyrmion. Another minor point is that the energy cannot distinguish between m and $-m$ and thus only enforces $\lim_{|x| \rightarrow \infty} m(x) = e_3$ or $\lim_{|x| \rightarrow \infty} m(x) = -e_3$. For definiteness, we simply choose the latter in an averaged sense, which together with the assumption (1.3) defines our admissible class \mathcal{A} of magnetizations, see the definition in (2.15).

In Theorem 2.1, we prove that there exists an explicit constant $C > 0$ such that in the regime $0 < \frac{|\kappa| + \delta}{\sqrt{Q-1}} \leq C$ the energy $E_{Q,\kappa,\delta}$, defined in equation (2.9) below, does indeed admit minimizers over \mathcal{A} . In comparison to Melcher’s work [64], the main issues are, first, an a priori lack of control of the decay of out-of-plane component $m_3 + 1$ due to the absence of an external field, and second,

the presence of the nonlocal terms. To restore control of $m_3 + 1$, we combine the Gagliardo–Nirenberg–Sobolev inequality with a vectorial version of the Modica–Mortola argument. To handle the nonlocal terms, we mainly appeal to interpolation inequalities. The remaining argument closely follows the methods developed by BREZIS and CORON [16] and MELCHER [64], ruling out the vanishing and splitting alternatives of Lions’ concentration-compactness principle [59]. Vanishing, which heuristically is the collapse of skyrmions via shrinking, is ruled out by combining the topological bound (1.4) with a construction giving

$$\inf_{\mathcal{A}} E_{Q,\kappa,\delta} < 8\pi, \quad (1.5)$$

so that the scale-dependent contributions to the energy cannot go to zero. In the case of splitting, i.e., two configurations drifting infinitely far apart from each other, the combinatorics involved in the requirement $\mathcal{N} = 1$ and the two bounds in (1.4) and (1.3) imply that at least one of the two pieces has $\mathcal{N} = 1$. As the nonlocal interaction of two magnetic charges vanishes as they move infinitely far apart, the two pieces essentially do not interact so that the energy can be strictly lowered by discarding the piece with $\mathcal{N} \neq 1$. Thus splitting is excluded and the obtained compactness is sufficiently strong to prove existence of minimizers.

The most important part of this paper is the description of the asymptotic behavior of the obtained minimizers in Theorem 2.2 for $0 < \frac{|\kappa|+\delta}{\sqrt{Q-1}} \ll 1$, corresponding to the regime dominated by the Dirichlet energy: the minimizers of $E_{Q,\kappa,\delta}$ approach the set of minimizers of the Dirichlet energy $\int_{\mathbb{R}^2} |\nabla m|^2 dx$, i.e., the set of Belavin–Polyakov profiles. Here, the challenge is to capture the fact that the skyrmion radius converges to zero in the limit of dominating exchange energy in order to compensate all Belavin–Polyakov profiles having infinite anisotropy energy. This intuition can be gained by making an ansatz-based minimization of suitably truncated Belavin–Polyakov profiles, which provides us with an upper bound for the minimal energy in the form of a finite-dimensional reduced energy depending only on the scale of truncation, the skyrmion radius and rotation angle [8]. Therefore, in order to find a matching lower bound and to conclude the proof, one has to quantitatively control closeness of the minimizers to the set of Belavin–Polyakov profiles. To this end, we prove a rigidity result for Belavin–Polyakov profiles, Theorem 2.4, estimating the Dirichlet distance of \dot{H}^1 maps of degree 1 to the set of Belavin–Polyakov profiles (see the definition in (2.38)) in terms of the Dirichlet excess $\int_{\mathbb{R}^2} |\nabla m|^2 dx - 8\pi$. Said excess can be directly linked to the scale of truncation, and the stability result allows us to prove that the lower order contributions to $E_{Q,\kappa,\delta}$ match the upper bound. A subtle issue here is the fact that the Belavin–Polyakov profile obtained in Theorem 2.4 does not necessarily approach $-e_3$ at infinity or even have a limit which is close to $-e_3$. This is related to the logarithmic failure of the critical Sobolev embedding $H^1 \not\hookrightarrow L^\infty$. Instead, we have to ensure the correct behavior at infinity by proving that otherwise the anisotropy energy is too large. The coercivity properties of the reduced energy finally allow to conclude the proof.

The proof of the rigidity result, Theorem 2.4, relies on first proving a corresponding linear estimate in the form of a spectral gap estimate for the Hessian at a Belavin–Polyakov profile. To this end, we diagonalize the Hessian using a

vector-valued version of spherical harmonics. The main difficulty is then to pass to the nonlinear estimate, specifically in the case where the Dirichlet excess is small. Existence of a Belavin–Polyakov profile that is close to the minimizer follows from known compactness properties of minimizing sequences in the harmonic map problem, and to remove the trivial degeneracies of the Hessian resulting from the invariances of the energy, we pick the closest Belavin–Polyakov profile in the \dot{H}^1 -topology. In order to then apply the spectral gap estimate, we have to justify that the Hessian gives a good description of the energy close to the minimizer. However, the higher order terms turn out to be radially weighted L^p -norms for which standard attempts at estimation fail logarithmically. Therefore, we have to find some problem-specific cancellations, for which we exploit the fact that the harmonic map problem is conformally invariant and that the Belavin–Polyakov profiles are conformal maps. This allows us to formulate the rigidity problem for maps from \mathbb{S}^2 to \mathbb{S}^2 , where the error terms turn into unweighted L^p -norms which are amenable to the Sobolev inequality. The required cancellation is then the fact that the average of the identity map over \mathbb{S}^2 vanishes. Finally, we obtain a Moser–Trudinger type inequality for maps in which the vanishing average assumption is replaced by closeness to the identity in the \dot{H}^1 -topology.

1.2. Outline of the Paper

In Sect. 2 we state and discuss our main results in detail. Section 3 is devoted to providing an explicit representation of the energy $E_{\sigma,\lambda}$ by continuously extending the nonlocal terms F_{vol} and F_{surf} . In Sect. 4 we give the proof of Theorem 2.4. The upper bound for the minimal energy and Theorem 2.1, the existence of skyrmions, can be found in Sect. 5. The proof of Theorem 2.2 is completed in Sect. 6. Finally, Appendix A collects an introduction to Sobolev spaces on the sphere, the proof of the topological lower bound along with a classification of its degree 1 extremizers, and a number of calculations involving Bessel functions necessary for calculating the energy of the ansatz. Within each subsection, we always first present all propositions, lemmas and corollaries, while their proofs can be found at the end of the subsection. Remarks concerning notation can be found at the end of Sect. 2.

2. Main Results

2.1. The Energy and the Admissible Class

In this paper, we consider the following model [69], based on a rigorous asymptotic expansion of the stray field energy given in [48]: For the quality factor $Q > 1$, non-dimensionalized film-thickness $\delta > 0$ and DMI-strength κ and on

$$\mathcal{D} := \{m \in C^\infty(\mathbb{R}^2; \mathbb{S}^2) : m + e_3 \text{ has compact support}\} \quad (2.1)$$

we choose, recalling that $m = (m', m_3)$,

$$E_{\text{ex}}(m) := \int_{\mathbb{R}^2} |\nabla m|^2 dx, \quad (2.2)$$

$$E_a(m) := Q \int_{\mathbb{R}^2} |m'|^2 dx, \tag{2.3}$$

$$E_Z(m) := 0, \tag{2.4}$$

$$E_{\text{DMI}}(m) := \kappa \int_{\mathbb{R}^2} (m_3 \nabla \cdot m' - m' \cdot \nabla m_3) dx, \tag{2.5}$$

$$E_s(m) := - \int_{\mathbb{R}^2} |m'|^2 dx + \delta (F_{\text{vol}}(m') - F_{\text{surf}}(m_3)), \tag{2.6}$$

where the normalized, nonlocal contributions $F_{\text{vol}}(m')$ and $F_{\text{surf}}(m_3)$ of the volume and surface charges, respectively, are defined via

$$F_{\text{vol}}(f) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot f(x) \nabla \cdot f(\tilde{x})}{|x - \tilde{x}|} d\tilde{x} dx, \tag{2.7}$$

$$F_{\text{surf}}(\tilde{f}) := \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\tilde{f}(x) - \tilde{f}(\tilde{x}))^2}{|x - \tilde{x}|^3} d\tilde{x} dx, \tag{2.8}$$

for $f \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$ and $\tilde{f} \in C^\infty(\mathbb{R}^2)$ such that there exists $c \in \mathbb{R}$ with $\tilde{f} + c$ having compact support. They can be interpreted as multiples of the squares of the $\dot{H}^{-\frac{1}{2}}$ -norm of $\nabla \cdot m'$ and the $\dot{H}^{\frac{1}{2}}$ -norm of m_3 , respectively, and an extension of these terms of sufficient generality for our purposes can be found in Sect. 3. In total, our functional may then be expressed as

$$E_{Q,\kappa,\delta}(m) := \int_{\mathbb{R}^2} (|\nabla m|^2 + (Q - 1)|m'|^2 - 2\kappa m' \cdot \nabla m_3) dx + \delta (F_{\text{vol}}(m') - F_{\text{surf}}(m_3)), \tag{2.9}$$

where we integrated by parts to simplify the DMI term.

In order to remove one of the parameters and make the mathematical structure of the energy explicit, we further rescale our functional (2.9). We first point out that the sign of κ is not essential: If we have $\kappa < 0$, then considering $\tilde{m}(x) := m(-x)$ gives

$$E_{Q,\kappa,\delta}(m) = E_{Q,-\kappa,\delta}(\tilde{m}). \tag{2.10}$$

and thus we may additionally suppose $\kappa \geq 0$. Furthermore, provided $\kappa + \delta > 0$ we use the rescaling

$$\bar{x} := \frac{Q - 1}{\kappa + \delta} x \text{ and } \bar{m}(\bar{x}) := m\left(\frac{\kappa + \delta}{Q - 1} \bar{x}\right) \tag{2.11}$$

in the energy (2.9), so that for

$$\sigma := \frac{\kappa + \delta}{\sqrt{Q - 1}} \text{ and } \lambda := \frac{\kappa}{\kappa + \delta}, \tag{2.12}$$

we finally obtain $E_{Q,\kappa,\delta}(m) = E_{\sigma,\lambda}(\bar{m})$, where

$$\begin{aligned}
 E_{\sigma,\lambda}(\bar{m}) &:= \int_{\mathbb{R}^2} |\nabla \bar{m}|^2 \, d\bar{x} \\
 &+ \sigma^2 \left(\int_{\mathbb{R}^2} |\bar{m}'|^2 \, d\bar{x} - 2\lambda \int_{\mathbb{R}^2} \bar{m}' \cdot \nabla \bar{m}_3 \, d\bar{x} + (1 - \lambda) (F_{\text{vol}}(\bar{m}') - F_{\text{surf}}(\bar{m}_3)) \right)
 \end{aligned}
 \tag{2.13}$$

for $\bar{m} \in \mathcal{D}$.

Of course, the assumption on regularity and decay at infinity encoded in \mathcal{D} is much too restrictive to allow for existence of minimizers. In view of the discussion in Sect. 1.1, it would be natural to consider instead the energy on the \mathbb{S}^2 -valued variant of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^2)$, which we define as a space of functions

$$\dot{H}^1(\mathbb{R}^2) := \left\{ u \in H_{\text{loc}}^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla u|^2 \, dx < \infty \right\},
 \tag{2.14}$$

equipped with the L^2 -norm of the gradient (note, however, some technical issues associated with such a critical space [4, Section 1.3]). Consequently, we aim to consider the energy $E_{\sigma,\lambda}$ on the set

$$\mathcal{A} := \left\{ m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2) : \int_{\mathbb{R}^2} |\nabla m|^2 \, dx < 16\pi, m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3), \mathcal{N}(m) = 1 \right\},
 \tag{2.15}$$

where the condition $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ is the appropriate way of prescribing $\lim_{|x| \rightarrow \infty} m(x) = -e_3$, see Lemma 5.1. Note that the definition of the topological charge, also referred to as the degree,

$$\mathcal{N}(m) := \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot (\partial_1 m \times \partial_2 m) \, dx,
 \tag{2.16}$$

is valid for all $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ and is consistent with equation (1.2) for smooth maps that are constant sufficiently far from the origin. However, due to the non-local terms, some care needs to be taken in extending the energy to \mathcal{A} . To avoid technicalities before the statement of results, we extend by relaxation, i.e., for $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ with $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ we set

$$E_{\sigma,\lambda}(m) := \inf \left\{ \liminf_{n \rightarrow \infty} E_{\sigma,\lambda}(m_n) : m_n \in \mathcal{D} \text{ for } n \in \mathbb{N} \text{ with } \lim_{n \rightarrow \infty} \|m_n - m\|_{H^1} = 0 \right\}.
 \tag{2.17}$$

Corollary 3.2 states that the representation (2.13) is still valid, provided the nonlocal terms F_{vol} and F_{surf} are interpreted appropriately.

2.2. Statement of the Results

We first establish that the energy $E_{\sigma,\lambda}$ admits minimizers over \mathcal{A} for all $\lambda \in [0, 1]$, provided σ is sufficiently small. In particular, we get existence of skyrmions even in the case $\lambda = 0$, which corresponds to no DMI being present. Our model therefore predicts skyrmions purely stabilized by the stray field. The

proof of Theorem 2.1 below closely follows the previous works by MELCHER [64] and DÖRING and MELCHER [28] and relies on the concentration compactness principle of LIONS [59]. The main new aspect is the inclusion of the nonlocal terms due to the stray field, which we deal with by standard interpolation inequalities.

Theorem 2.1. *Let $\sigma > 0$ and $\lambda \in [0, 1]$ be such that $\sigma^2(1 + \lambda)^2 \leq 2$. Then there exists $m_{\sigma,\lambda} \in \mathcal{A}$ such that*

$$E_{\sigma,\lambda}(m_{\sigma,\lambda}) = \inf_{\tilde{m} \in \mathcal{A}} E_{\sigma,\lambda}(\tilde{m}). \tag{2.18}$$

Note that throughout the rest of the paper we suppress λ in the index of $m_{\sigma,\lambda}$ for simplicity of notation.

We now turn to the heart of the paper, namely, the analysis of the limit $\sigma \rightarrow 0$ in which the Dirichlet energy dominates. As was already pointed out by DÖRING and MELCHER [28], in this limit one expects minimizers m_σ of $E_{\sigma,\lambda}$ to converge to minimizers of

$$F(m) := \int_{\mathbb{R}^2} |\nabla m|^2 \, dx \tag{2.19}$$

for $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ with $\mathcal{N}(m) = 1$, i.e., minimizing harmonic maps of degree 1. These have been identified by BELAVIN and POLYAKOV [6], see also BREZIS and CORON [17, Lemma A.1] or Lemma A.3 below, to be given by the previously mentioned Belavin–Polyakov profiles

$$\mathcal{B} := \left\{ S\Phi(\rho^{-1}(\bullet - x)) : S \in \text{SO}(3), \rho > 0, x \in \mathbb{R}^2 \right\}, \tag{2.20}$$

where Φ is a rotated variant of the stereographic projection with respect to the south pole

$$\Phi(x) := \left(-\frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right) \tag{2.21}$$

for $x \in \mathbb{R}^2$. One can moreover see that they achieve equality in the topological bound (1.4) in view of

$$\int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx = 8\pi \tag{2.22}$$

for all $\phi \in \mathcal{B}$. It is even known, see [29, (11.5)], that \mathcal{B} comprises all solutions $\phi : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ of the harmonic map equation

$$\Delta \phi + |\nabla \phi|^2 \phi = 0 \tag{2.23}$$

with $\mathcal{N}(\phi) = 1$, meaning all critical points of F of degree 1 are absolute minimizers.

The task then is to identify which Belavin–Polyakov profiles $\phi = S\Phi(\rho^{-1}(\bullet - x))$ for $S \in \text{SO}(3)$ and $\rho > 0$ are selected in the limit $\sigma \rightarrow 0$. By the requirement $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$, we can certainly expect to have $Se_3 = e_3$ in the limit, so that $S = S_\theta$ for some angle $\theta \in [-\pi, \pi)$ and

$$S_\theta := \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.24}$$

However, even for such Belavin–Polyakov profiles it holds that $\phi + e_3 \notin L^2(\mathbb{R}^2; \mathbb{R}^3)$ due to logarithmic divergence of the anisotropy term. Consequently, we expect minimizers to be truncated Belavin–Polyakov profiles which will shrink to keep the anisotropy energy finite in the limit $\sigma \rightarrow 0$ in the spirit of the construction by DÖRING and MELCHER [28, Lemma 3].

Indeed, careful minimization in a corresponding class of ansätze [8], see also Sect. 5.2, leads one to believe that the optimal skyrmion radius ρ_0 is given asymptotically by

$$\rho_0 \simeq \frac{\bar{g}(\lambda)}{16\pi} \frac{1}{|\log \sigma|}, \tag{2.25}$$

where the auxiliary function

$$\bar{g}(\lambda) := \begin{cases} \left(8 + \frac{\pi^2}{4}\right) \pi \lambda - \frac{\pi^3}{4} & \text{if } \lambda \geq \lambda_c, \\ \frac{128\lambda^2}{3\pi(1-\lambda)} + \frac{\pi^3}{8}(1-\lambda) & \text{else,} \end{cases} \tag{2.26}$$

in which the critical threshold λ_c is defined as

$$\lambda_c := \frac{3\pi^2}{32 + 3\pi^2}, \tag{2.27}$$

results from the balance of the DMI and stray field terms. The function $\bar{g}(\lambda)$ can straightforwardly be seen to be continuous and satisfy

$$\frac{1}{C} \leq \bar{g}(\lambda) \leq C \tag{2.28}$$

for a universal constant $C > 0$. Furthermore, the two optimal rotation angles $\theta_0^+ \in [0, \frac{\pi}{2}]$ and $\theta_0^- \in [-\frac{\pi}{2}, 0]$ are asymptotically

$$\theta_0^\pm := \begin{cases} 0 & \text{if } \lambda \geq \lambda_c, \\ \pm \arccos\left(\frac{32\lambda}{3\pi^2(1-\lambda)}\right) & \text{else.} \end{cases} \tag{2.29}$$

Here, the angle $\theta_0^\pm = 0$ corresponds to a Néel-type skyrmion profile present in the regime $\lambda \geq \lambda_c$ of DMI dominating over the stray field, while skyrmions purely stabilized by the stray field have Bloch-type profiles in view of $\theta_0^\pm = \pm \frac{\pi}{2}$ for $\lambda = 0$. The following convergence theorem confirms these expectations:

Theorem 2.2. *Let $\lambda \in [0, 1]$. Let m_σ be a minimizer of $E_{\sigma,\lambda}$ over \mathcal{A} . Then there exist $x_\sigma \in \mathbb{R}^2$, $\rho_\sigma > 0$ and $\theta_\sigma \in [-\pi, \pi)$ such that $m_\sigma - S_{\theta_\sigma} \Phi(\rho_\sigma^{-1}(\bullet - x_\sigma)) \rightarrow 0$ in $\mathring{H}^1(\mathbb{R}^2; \mathbb{R}^3)$ as $\sigma \rightarrow 0$, and*

$$\lim_{\sigma \rightarrow 0} |\log \sigma| \rho_\sigma = \frac{\bar{g}(\lambda)}{16\pi}, \quad \lim_{\sigma \rightarrow 0} |\theta_\sigma| = \theta_0^\pm, \tag{2.30}$$

as well as

$$\lim_{\sigma \rightarrow 0} \frac{|\log \sigma|^2}{\sigma^2 \log |\log \sigma|} \left| E_{\sigma,\lambda}(m_\sigma) - 8\pi + \frac{\sigma^2}{|\log \sigma|} \left(\frac{\bar{g}^2(\lambda)}{32\pi} - \frac{\bar{g}^2(\lambda)}{32\pi} \frac{\log |\log \sigma|}{|\log \sigma|} \right) \right| = 0. \tag{2.31}$$

Remark 2.3. For the convergences in Theorem 2.2, our methods also allow to provide the following non-optimal (with the exception of the estimate (2.35)) rates:

$$\int_{\mathbb{R}^2} \left| \nabla \left(m_\sigma(x) - S_{\theta_\sigma} \Phi(\rho_\sigma^{-1}(x - x_\sigma)) \right) \right|^2 dx \leq C\sigma^2, \tag{2.32}$$

$$\left| \log \sigma |\rho_\sigma - \frac{\bar{g}(\lambda)}{16\pi}| \right| \leq \frac{C}{|\log \sigma|}, \tag{2.33}$$

$$|\theta_\sigma - \theta_0^+|^4 + |\lambda - \lambda_c| |\theta_\sigma - \theta_0^+|^2 \leq \frac{C}{|\log \sigma|}, \tag{2.34}$$

as well as

$$\frac{1}{C} \frac{\sigma^2}{|\log \sigma|^2} \leq \int_{\mathbb{R}^2} |\nabla m_\sigma|^2 dx - 8\pi \leq C \frac{\sigma^2}{|\log \sigma|^2}, \tag{2.35}$$

and

$$\left| \frac{|\log \sigma|}{\sigma^2} (E_{\sigma,\lambda}(m_\sigma) - 8\pi) - \left(-\frac{\bar{g}^2(\lambda)}{32\pi} + \frac{\bar{g}^2(\lambda) \log |\log \sigma|}{32\pi |\log \sigma|} \right) \right| \leq \frac{C}{|\log \sigma|}, \tag{2.36}$$

for $C > 0$ universal and $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough and universal. In fact, our proof does establish all these rates except the one for the angles θ_σ , whose proof relies on some lengthy, but elementary estimates. Note that the loss in the rate of convergence of θ_σ to θ_0^\pm for the parameter $\lambda = \lambda_c$ coincides with Néel profiles becoming linearly unstable.

While not strictly speaking adhering to a Γ -convergence framework, the proof of Theorem 2.2 is very much in the spirit of Γ -equivalence [15] in that we compare the sequence of energies at minimizers to a sequence of finite-dimensional reduced energies. This simplification allows us to explicitly compute approximate minimizers and even analyze their stability properties. As is usual in the theory of Γ -convergence, the comparison is done via upper bounds obtained by construction and ansatz-free lower bounds. The constructions have already been alluded to above. The main ingredient for the lower bounds is the following Theorem 2.4, a quantitative stability estimate for degree 1 harmonic maps from \mathbb{R}^2 to \mathbb{S}^2 , i.e., for the maps in the set \mathcal{B} , see definition (2.20). Once we know that the minimizers are close to Belavin–Polyakov profiles, we use this information to estimate the remaining lower order terms in the energy.

To state the theorem, we first introduce the family of all \dot{H}^1 -maps from \mathbb{R}^2 to \mathbb{S}^2 of degree 1 :

$$\mathcal{C} := \left\{ \tilde{m} \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2) : \mathcal{N}(\tilde{m}) = 1 \right\}. \tag{2.37}$$

We next introduce a notion of distance between elements in this family and Belavin–Polyakov profiles, which we term the *Dirichlet distance*:

$$D(m; \mathcal{B}) := \inf_{\tilde{\phi} \in \mathcal{B}} \left(\int_{\mathbb{R}^2} |\nabla (m - \tilde{\phi})|^2 dx \right)^{\frac{1}{2}}. \tag{2.38}$$

With these definitions we have the following theorem:

Theorem 2.4. *For every $m \in \mathcal{C}$ there exists $\phi \in \mathcal{B}$ that achieves the infimum in the Dirichlet distance $D(m; \mathcal{B})$. Furthermore, there exists a universal constant $\eta > 0$ such that*

$$\eta D^2(m; \mathcal{B}) \leq F(m) - 8\pi. \tag{2.39}$$

Notice that this result in the more restrictive equivariant setting is contained in [38, Theorem 2.1].

Well understood compactness properties of minimizing sequences for the Dirichlet energy [56] ensure the existence of a Belavin–Polyakov profile ϕ that is close to an almost minimizer m of the Dirichlet energy but do not provide us with a rate of closeness. To overcome this issue, we pass to the corresponding linearized problem, which can easily be solved using a suitable vectorial version of spherical harmonics, see Proposition 4.2 below. However, naive attempts at explicitly estimating the error terms arising in the linearization procedure tend to break down due to the logarithmic failure of the critical Sobolev embedding $H^1 \not\hookrightarrow L^\infty$ in two dimensions. Therefore, the main conceptual issue is to find additional cancellations resulting from the structure of the problem.

The relevant structure, it turns out, is the fact that the harmonic map problem is conformally invariant, and that all Belavin–Polyakov profiles are conformal maps. This allows us to reformulate the problem as stability of the identity map $\text{id} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, denoted from now on as $\text{id}_{\mathbb{S}^2}$, by considering $\tilde{m} := m \circ \phi^{-1}$. Nonlinear terms can then be estimated using the standard Sobolev embedding on the sphere, and the required cancellation is that the identity map on the sphere has average zero. This idea leads us to the following estimates, which when expressed on \mathbb{R}^2 also provides topologies in which m itself converges to ϕ :

Lemma 2.5. *There exists a universal constant $\tilde{\eta} > 0$ such that the following holds: Let $p \in [1, \infty)$. Then there exists a constant $C_p > 0$ such that if $m \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ satisfies $\int_{\mathbb{S}^2} |\nabla(m - \text{id}_{\mathbb{S}^2})|^2 d\mathcal{H}^2 \leq \tilde{\eta}$, then we have the estimate*

$$\left(\int_{\mathbb{S}^2} |m - \text{id}_{\mathbb{S}^2}|^p d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{S}^2} |\nabla(m - \text{id}_{\mathbb{S}^2})|^2 d\mathcal{H}^2 \right)^{\frac{1}{2}}. \tag{2.40}$$

Furthermore, there exists a universal constant $C > 0$ such that the Moser–Trudinger type inequality

$$\int_{\mathbb{S}^2} e^{\frac{2\pi}{3} \frac{|m - \text{id}_{\mathbb{S}^2}|^2}{\|\nabla(m - \text{id}_{\mathbb{S}^2})\|_2^2}} d\mathcal{H}^2 \leq C \tag{2.41}$$

holds.

We furthermore point out that Theorem 2.4 implies a corresponding statement for degree one harmonic maps on \mathbb{S}^2 , i.e., for minimizers of

$$F_{\mathbb{S}^2}(\tilde{m}) := \int_{\mathbb{S}^2} |\nabla \tilde{m}|^2 d\mathcal{H}^2 \tag{2.42}$$

over

$$C_{\mathbb{S}^2} := \left\{ \tilde{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2) : \mathcal{N}_{\mathbb{S}^2}(\tilde{m}) = 1 \right\}, \tag{2.43}$$

where

$$\mathcal{N}_{\mathbb{S}^2}(\tilde{m}) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \det(\nabla \tilde{m}) \, d\mathcal{H}^2 \tag{2.44}$$

denotes the degree for maps from $\tilde{m} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, see for example BREZIS and NIRENBERG [18] or Sect. A.1 in the appendix for details. Recalling the definition (2.21) of Φ , it can be seen that the minimizers are given by the set of Möbius transformations

$$\mathcal{B}_{\mathbb{S}^2} := \left\{ \phi \circ \Phi^{-1} : \phi \in \mathcal{B} \right\} \tag{2.45}$$

$$= \left\{ \Phi \circ f \circ \Phi^{-1} : f(z) := \frac{az + b}{cz + d} \text{ for } a, b, c, d \in \mathbb{C} \text{ with } ad - bc \neq 0 \right\}, \tag{2.46}$$

where points $x \in \mathbb{R}^2$ in the plane are identified with the points $z \in \mathbb{C}$ in the complex plane. Indeed, this follows from the conformal invariance of the harmonic map problem, see Lemma A.2. The second equality is a classical fact we will prove in Lemma A.3 for the convenience of the reader. A similar nonlinear stability statement for degree -1 maps is a simple result of the identity $\mathcal{N}_{\mathbb{S}^2}(-\tilde{m}) = -\mathcal{N}_{\mathbb{S}^2}(\tilde{m})$ for $\tilde{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$.

Corollary 2.6. *For $\tilde{m} \in C_{\mathbb{S}^2}$ we have*

$$\eta \min_{\tilde{\phi} \in \mathcal{B}_{\mathbb{S}^2}} \int_{\mathbb{R}^2} |\nabla(\tilde{m} - \tilde{\phi})|^2 \, dx \leq F_{\mathbb{S}^2}(\tilde{m}) - 8\pi, \tag{2.47}$$

where $\eta > 0$ is the universal constant of Theorem 2.4. Furthermore, for $\tilde{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ with $\mathcal{N}_{\mathbb{S}^2}(\tilde{m}) = -1$ we have the corresponding statement

$$\eta \min_{\tilde{\phi} \in (-\mathcal{B}_{\mathbb{S}^2})} \int_{\mathbb{R}^2} |\nabla(\tilde{m} - \tilde{\phi})|^2 \, dx \leq F_{\mathbb{S}^2}(\tilde{m}) - 8\pi. \tag{2.48}$$

Notice that our result is stronger than the one in [60] for $n = 2$ in that it does not require the assumption that the map \tilde{m} be Lipschitz and close in $H^1(\mathbb{S}^2; \mathbb{R}^3)$ to \mathcal{B} .

2.3. Notation

Throughout the paper, the symbols C and η denote universal, positive constants that may change from inequality to inequality, and where we think of C as large and η as small. Whenever we use O -notation, the involved constants are understood to be universal. For matrices $A \in \mathbb{R}^{n \times m}$ for $n, m \in \mathbb{N}$, we use the Frobenius norm $|A| := \sqrt{\text{tr}(A^T A)}$.

3. An Explicit Representation of the Energy

Here, we extend the functionals F_{vol} and F_{surf} to a sufficiently big space of functions to ensure that our energy $E_{\sigma,\lambda}$ has a practical representation on \mathcal{A} , and that F_{vol} and F_{surf} are defined for the relevant components of the stereographic projection Φ . The main tool to obtain the relevant estimates will be the Fourier transform, for which we use the convention

$$\mathcal{F}f(k) := \int_{\mathbb{R}^2} e^{-ik \cdot x} f(x) \, dx \tag{3.1}$$

for $f \in L^1(\mathbb{R}^2)$ and which we extend to functions $f \in L^p(\mathbb{R}^2)$ for $1 < p \leq 2$ in the usual way, see [55, Section 5.4 and 5.6].

The situation for F_{surf} is straightforward: As it is obviously non-negative, we can simply use the definition (2.8) for all $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ with $F_{\text{surf}}(f) < \infty$. The Fourier space representation of $F_{\text{surf}}(f)$ obtained from (3.9) for $f \in H^1(\mathbb{R}^2)$ below will nevertheless be helpful to prove estimates and to compute the surface charge contribution of a Belavin–Polyakov profile. Furthermore, we define

$$F_{\text{surf}}(f, g) := \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(f(x) - f(\tilde{x}))(g(x) - g(\tilde{x}))}{|x - \tilde{x}|^3} \, d\tilde{x} \, dx \tag{3.2}$$

whenever $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are measurable with $F_{\text{surf}}(f) < \infty$ and $F_{\text{surf}}(g) < \infty$.

Turning to the volume charges, for measurable functions $\tilde{f}, \tilde{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\nabla \cdot \tilde{f} \in L^2(\mathbb{R}^2)$ and $\nabla \cdot \tilde{g} \in L^2(\mathbb{R}^2)$ we define

$$F_{\text{vol}}(\tilde{f}) := \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\mathcal{F}(\nabla \cdot \tilde{f})|^2}{|k|} \frac{dk}{(2\pi)^2}, \tag{3.3}$$

$$F_{\text{vol}}(\tilde{f}, \tilde{g}) := \frac{1}{2} \int_{\mathbb{R}^2} \frac{\mathcal{F}(\nabla \cdot \tilde{f}) \overline{\mathcal{F}(\nabla \cdot \tilde{g})}}{|k|} \frac{dk}{(2\pi)^2}, \tag{3.4}$$

the latter of which requires $F_{\text{vol}}(\tilde{f}) < \infty$ and $F_{\text{vol}}(\tilde{g}) < \infty$.

The following, standard lemma ensures this is indeed an extension of the original definition (2.7) and provides a number of interpolation inequalities for both F_{surf} and F_{vol} we will use throughout the paper:

Lemma 3.1. *For maps $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that there exist $c, d \in \mathbb{R}$ with $f + c, g + d \in H^1(\mathbb{R}^2)$ and $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\nabla \cdot \tilde{f} \in L^2(\mathbb{R}^2)$ we have $F_{\text{surf}}(f, g) \in \mathbb{R}$ and*

$$F_{\text{surf}}(f) \geq 0, \tag{3.5}$$

$$F_{\text{vol}}(\tilde{f}) \geq 0, \tag{3.6}$$

$$|F_{\text{surf}}(f, g)| \leq \frac{1}{2} \|f + c\|_2 \|\nabla g\|_2. \tag{3.7}$$

If, for $p \in (1, \infty)$, we additionally have $\tilde{f}, \tilde{g} \in L^p(\mathbb{R}^2; \mathbb{R}^2) \cap \dot{W}^{1,p'}(\mathbb{R}^2; \mathbb{R}^2)$ with $\nabla \cdot \tilde{g} \in L^2(\mathbb{R}^2)$ and $p' = p/(p-1)$, then we have $F_{\text{vol}}(\tilde{f}, \tilde{g}) \in \mathbb{R}$ with the estimate

$$F_{\text{vol}}(\tilde{f}, \tilde{g}) \leq C_p \|\tilde{f}\|_p \|\nabla \tilde{g}\|_{p'}. \tag{3.8}$$

Finally, we have the representation

$$F_{\text{surf}}(f, g) = \frac{1}{2} \int_{\mathbb{R}^2} |k| \mathcal{F}(f + c) \overline{\mathcal{F}(g + d)} \frac{dk}{(2\pi)^2}, \tag{3.9}$$

and for $\tilde{f}, \tilde{g} \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$ we also have

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \tilde{f}(x) \nabla \cdot \tilde{g}(\tilde{x})}{|x - \tilde{x}|} d\tilde{x} dx = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\mathcal{F}(\nabla \cdot \tilde{f}) \overline{\mathcal{F}(\nabla \cdot \tilde{g})}}{|k|} \frac{dk}{(2\pi)^2}. \tag{3.10}$$

In particular, the definition (3.3) extends that in (2.7).

With these extensions, we prove that the representation (2.13) of $E_{\sigma,\lambda}$ is still valid. Notice that the density result below is a variant of [63, Lemma 4.1] (see also SCHOEN and UHLENBECK [75]).

Corollary 3.2. For $\sigma > 0, \lambda > 0$ and $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ with $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ there exists a sequence $m_n \in \mathcal{D}$ with $\lim_{n \rightarrow \infty} \|m_n - m\|_{H^1} = 0$, and we have

$$E_{\sigma,\lambda}(m) = \int_{\mathbb{R}^2} |\nabla m|^2 dx + \sigma^2 \left(\int_{\mathbb{R}^2} |m'|^2 dx - 2\lambda \int_{\mathbb{R}^2} m' \cdot \nabla m_3 dx + (1 - \lambda) (F_{\text{vol}}(m') - F_{\text{surf}}(m_3)) \right). \tag{3.11}$$

Proof of Lemma 3.1. We first deal with the surface term. The estimate (3.5) is trivial. The Fourier representation (3.9) follows immediately from [55, Theorem 7.12, identity (4)]. The estimate (3.7) is then a straightforward consequence of the Cauchy–Schwarz inequality and Plancherel’s theorem, [55, Theorem 5.3].

Next, we turn to the volume terms. Again, non-negativity (3.6) is a trivial consequence of the definition (3.3). For $\tilde{f}, \tilde{g} \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$, the equality (3.10) is a result of [55, Theorem 5.2, identity (2)].

By a density argument, it is sufficient to prove the interpolation result (3.8) still under the assumption $\tilde{f}, \tilde{g} \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$. To this end, we define a vectorial variant of the Riesz transform

$$T \tilde{f} := \mathcal{F}^{-1} \left(i \frac{k}{|k|} \cdot \mathcal{F} \tilde{f} \right). \tag{3.12}$$

By the standard fact that $\mathcal{F}(\nabla \cdot \tilde{f})(k) = ik \cdot \mathcal{F} \tilde{f}(k)$ for a.e. $k \in \mathbb{R}^2$ and Plancherel’s identity, we have

$$\frac{1}{2} \int_{\mathbb{R}^2} \frac{\mathcal{F}(\nabla \cdot \tilde{f}) \overline{\mathcal{F}(\nabla \cdot \tilde{g})}}{|k|} \frac{dk}{(2\pi)^2} = \frac{1}{2} \int_{\mathbb{R}^2} T(\tilde{f}) \nabla \cdot \tilde{g} dx. \tag{3.13}$$

By the Mihlin–Hörmander multiplier theorem [36, Theorem 6.2.7], T extends to a bounded operator from $L^p(\mathbb{R}^2; \mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for all $p \in (1, \infty)$. As a result, Hölder’s inequality implies the desired inequality (3.8). \square

Proof of Corollary 3.2. By the density result [63, Lemma 4.1] we may choose a sequence $m_n \in C^\infty(\mathbb{R}^2; \mathbb{S}^2)$ with $m_n + e_3 \in L^2$ such that $\lim_{n \rightarrow \infty} \|m_n - m\|_{H^1} = 0$. The proof of [28, Lemma 8] implies that we may furthermore take $m_n + e_3$ to have compact support for all $n \in \mathbb{N}$, so that we have $m_n \in \mathcal{A}$. The local terms are obviously continuous in the H^1 -topology. Continuity of F_{vol} and F_{surf} is ensured by Lemma 3.1. \square

4. Rigidity of Degree ± 1 Harmonic Maps

The goal of this section is to prove Theorem 2.4, the quantitative stability statement for Belavin–Polyakov profiles with respect to the Dirichlet energy $F(m)$. As explained in Sect. 2.2, it will be helpful at times to think of maps $\tilde{m} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ by setting $\tilde{m} := m \circ \phi^{-1}$ for some appropriately chosen $\phi \in \mathcal{B}$. The maps $\phi \in \mathcal{B}$ have the nice property of being *conformal*, see [27, Chapter 4, Definition 3]. As such, the above re-parametrization leaves the harmonic map problem invariant, see Lemma A.2, and we gain compactness of the underlying sets, as well as a greater conceptual clarity in some of our arguments.

The definitions of gradients, Laplace operators and Sobolev spaces on the sphere can be found in Sect. A.1. In particular, we use the same symbol for the Euclidean and Riemannian versions of gradients and Laplace operators as it is always clear from context which one is meant.

4.1. The Spectral Gap Property for the Linearized Problem

This subsection is devoted to the solution of the linear problem corresponding to Theorem 2.4 and Corollary 2.6, i.e., we establish the sharp spectral gap property for the Hessian of F , or equivalently $F_{\mathbb{S}^2}$, at minimizers. The notions and arguments needed are fairly standard. Here we provide the proof for the convenience of the reader.

Given a map $m \in \mathcal{C}$ that is close to $\phi \in \mathcal{B}$ in $\dot{H}^1(\mathbb{R}^2; \mathbb{R}^3)$, by Lemma A.2 we also have that $m \circ \phi^{-1}$ is close to $\text{id}_{\mathbb{S}^2}$ in $H^1(\mathbb{S}^2; \mathbb{R}^3)$. Therefore, we only have to compute the Hessian at the identity map $\text{id}_{\mathbb{S}^2} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. The corresponding Hessian on \mathbb{S}^2 and in local coordinates given by $\phi \in \mathcal{B}$ is, respectively

$$\mathfrak{H}(\zeta, \xi) := \int_{\mathbb{S}^2} (\nabla \zeta : \nabla \xi - 2\zeta \cdot \xi) \, d\mathcal{H}^2, \tag{4.1}$$

$$\mathfrak{H}_\phi(\zeta_\phi, \xi_\phi) := \int_{\mathbb{R}^2} \left(\nabla \zeta_\phi : \nabla \xi_\phi - \zeta_\phi \cdot \xi_\phi |\nabla \phi|^2 \right) \, dx, \tag{4.2}$$

see [62,77], defined for tangent vector fields $\zeta, \xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$, see equation (A.5) for the definition of this space, and

$$\zeta_\phi, \xi_\phi \in H_w^1(\mathbb{R}^2; T_\phi \mathbb{S}^2)$$

$$:= \left\{ \tilde{\xi}_\phi \in H_w^1(\mathbb{R}^2; \mathbb{R}^3) : \tilde{\xi}_\phi(x) \cdot \phi(x) = 0 \text{ for almost all } x \in \mathbb{R}^2 \right\}. \quad (4.3)$$

Here, we introduced a vector-valued variant $H_w^1(\mathbb{R}^2; \mathbb{R}^3)$ of the weighted Sobolev space

$$H_w^1(\mathbb{R}^2) := \left\{ u \in H_{loc}^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \frac{|u|^2}{1 + |x|^4} \right) dx < \infty \right\}, \quad (4.4)$$

arising from $H^1(\mathbb{S}^2; \mathbb{R}^3)$ under parametrization by $\phi \in \mathcal{B}$ due to Lemma A.2, and the pullback $T_\phi \mathbb{S}^2 := \bigcup_{x \in \mathbb{R}^2} \{\phi(x)\} \times T_{\phi(x)} \mathbb{S}^2$ of the tangent bundle $T\mathbb{S}^2$ of the sphere. In particular, note that $\zeta_\phi \cdot \xi_\phi |\nabla \phi|^2$ is integrable. As $\text{id}_{\mathbb{S}^2}$ is a minimizer of $F_{\mathbb{S}^2}$, the Hessians are non-negative bilinear forms in the sense that for all $\xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$ and $\xi_\phi \in H_w^1(\mathbb{R}^2; T_\phi \mathbb{S}^2)$ we have

$$\mathfrak{H}(\xi, \xi) \geq 0, \quad (4.5)$$

$$\mathfrak{H}_\phi(\xi_\phi, \xi_\phi) \geq 0. \quad (4.6)$$

In view of identity (4.1), the inequality (4.5) can be interpreted as a Poincaré type inequality on the space $H^1(\mathbb{S}^2; T\mathbb{S}^2)$ that does not rely on subtracting averages.

The next step is to identify the null space of the Hessian:

$$J := \left\{ \zeta \in H^1(\mathbb{S}^2; T\mathbb{S}^2) : \mathfrak{H}(\zeta, \zeta) = 0 \right\}. \quad (4.7)$$

It is well known that $\zeta \in J$ is equivalent to ζ solving the so-called *Jacobi equation*

$$L(\zeta)(y) := -\Delta \zeta(y) - 2\zeta(y) - 2(\nabla y : \nabla \zeta(y))y = 0 \quad (4.8)$$

for all $y \in \mathbb{S}^2$, where the Laplace–Beltrami operator is taken component-wise. We call solutions to the Jacobi equation *Jacobi fields*. In local coordinates given by $\phi \in \mathcal{B}$, i.e., for $\zeta_\phi := \zeta \circ \phi^{-1}$, this equation is

$$L_\phi(\zeta_\phi) := -\Delta \zeta_\phi - |\nabla \phi|^2 \zeta_\phi - 2(\nabla \phi : \nabla \zeta_\phi) \phi = 0. \quad (4.9)$$

For our purposes we only need to rigorously ensure that $\zeta \in J$ solves a weak version of the equation in local coordinates under the (a posteriori unnecessary) assumption that $\zeta \in J$ is smooth, which we will do in Lemma 4.1 below for the convenience of the reader.

Lemma 4.1. *Let $\zeta \in J$ be smooth and $\phi \in \mathcal{B}$. Then for all $\xi \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3)$ the function $\zeta_\phi := \zeta \circ \phi$ satisfies*

$$\int_{\mathbb{R}^2} \left(\nabla \zeta_\phi : \nabla \xi - \zeta_\phi \cdot \xi |\nabla \phi|^2 - 2\phi \cdot \xi \nabla \phi : \nabla \zeta_\phi \right) dx = 0. \quad (4.10)$$

Using the characterization of the Hessian \mathfrak{H} as the second derivative of $F_{\mathbb{S}^2}$ at $\text{id}_{\mathbb{S}^2}$, we can readily find Jacobi fields: If for $\varepsilon > 0$ and $t \in (-\varepsilon, \varepsilon)$, the function u_t is a smooth curve of minimizers of $F_{\mathbb{S}^2}$ with $u_0 = \text{id}_{\mathbb{S}^2}$, then $\frac{d}{dt}|_{t=0} u_t \in J$. To use this idea, we recall the representation

$$u_t = \Phi \circ f_t \circ \Phi^{-1} \quad (4.11)$$

for $f_t(z) = \frac{a_t z + b_t}{c_t z + 1}$ and $a_t, b_t, c_t \in \mathbb{C}$ with $a_t - b_t c_t \neq 0$ and $a_0 = 1, b_0 = c_0 = 0$, see equation (2.46). Differentiating in t , we see that $j \in J$, where by the chain rule we have

$$j(y) := \left(\nabla \Phi \circ \Phi^{-1}(y) \right) g \circ \Phi^{-1}(y) \quad y \in \mathbb{S}^2, \tag{4.12}$$

for the complex polynomial $g(z) := -\frac{d}{dt}|_{t=0} c_t z^2 + \frac{d}{dt}|_{t=0} a_t z + \frac{d}{dt}|_{t=0} b_t$. In particular, we know that $\dim J \geq 6$.

In the next Proposition (4.2), we prove that all Jacobi fields arise in such a manner and we compute the spectral gap. To this end, we use to the notion of vector spherical harmonics [35, Chapter 5.2], as they turn out to diagonalize \mathfrak{H} . They are related to the spherical harmonics $Y_{n,j} : \mathbb{S}^2 \rightarrow \mathbb{R}$ for $n \geq 0$ and $j = -n, \dots, n$, which are eigenfunctions of the Laplace–Beltrami operator Δ with eigenvalues $-n(n+1)$. Here, we take them to be normalized such that they form a real-valued, orthonormal system for $L^2(\mathbb{S}^2)$. Their definition is well-known and we do not need their explicit expressions in the following (an interested reader may refer to [35, Chapter 3.4]). The vector spherical harmonics, see [35, equation (5.36)] are defined for $y \in \mathbb{S}^2$ as

$$\mathcal{Y}_{0,0}^{(1)}(y) := \frac{1}{\sqrt{4\pi}} y, \tag{4.13}$$

and for $n \geq 1$ and $j = -n, \dots, n$ as

$$\mathcal{Y}_{n,j}^{(1)}(y) := Y_{n,j}(y) y, \tag{4.14}$$

$$\mathcal{Y}_{n,j}^{(2)}(y) := \frac{1}{\sqrt{n(n+1)}} \nabla Y_{n,j}(y), \tag{4.15}$$

$$\mathcal{Y}_{n,j}^{(3)}(y) := \frac{1}{\sqrt{n(n+1)}} y \times \nabla Y_{n,j}(y). \tag{4.16}$$

Similarly to their scalar counterparts, they are eigenfunctions with eigenvalues $-n(n+1)$ for a suitably defined vectorial Laplace–Beltrami operator [35, Theorem 5.28 and Definition 5.26]: For $\xi \in C^2(\mathbb{S}^2; \mathbb{R}^3)$, using the projections π_t and π_n onto tangential and normal components, we set

$$\Delta_v \xi := \pi_n(\Delta + 2)(\pi_n \xi) + \pi_t \Delta(\pi_t \xi), \tag{4.17}$$

where Δ is to be understood as the component-wise Laplace–Beltrami operator. Furthermore, they form an orthonormal system for $L^2(\mathbb{S}^2; \mathbb{R}^3)$, see [35, Theorem 5.9].

Turning to tangential vector fields, we note that by the above results the set $\{\mathcal{Y}_{n,j}^{(2)}, \mathcal{Y}_{n,j}^{(3)} : n \geq 1, j = -n, \dots, n\}$ of tangential vector spherical harmonics forms an orthonormal system for

$$L^2(\mathbb{S}^2; T\mathbb{S}^2) := \left\{ \xi \in L^2(\mathbb{R}^2; \mathbb{R}^3) : \xi(y) \cdot y = 0 \text{ for almost all } y \in \mathbb{S}^2 \right\}. \tag{4.18}$$

Additionally, we can use the fact that the vector spherical harmonics are eigenfunctions of Δ_v to integrate by parts, see equation (A.7) below, to obtain

$$\int_{\mathbb{S}^2} \nabla \mathcal{Y}_{n,j}^{(k)} : \nabla \mathcal{Y}_{p,i}^{(o)} d\mathcal{H}^2 = n(n+1)\delta_{n,p}\delta_{j,i}\delta_{k,o} \tag{4.19}$$

for $n, p \geq 1, j = -n, \dots, n, i = -p, \dots, p$ and $k, o \in \{2, 3\}$, where the δ symbols denote the corresponding Kronecker deltas.

With this information, we are finally able to characterize both the space of Jacobi functions J and the spectral gap of the Hessian \mathfrak{H} with respect to the \mathring{H}^1 -scalar product. To this end, we define the space of tangent vector fields which are \mathring{H}^1 -orthogonal to the space J of Jacobi fields

$$\mathbb{H}^1 := \left\{ \xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2) : \int_{\mathbb{S}^2} (\nabla \xi : \nabla \zeta) d\mathcal{H}^2 = 0 \text{ for all } \zeta \in J \right\}. \tag{4.20}$$

The choice of the \mathring{H}^1 -scalar product is motivated by Theorem 2.4 requiring us to estimate the \mathring{H}^1 -distance of any given $m \in \mathcal{C}$ to \mathcal{B} .

Proposition 4.2. *We have $J = \text{span} \{ \mathcal{Y}_{1,j}^{(2)}, \mathcal{Y}_{1,j}^{(3)}; j = -1, 0, 1 \}$. In particular, all Jacobi fields are smooth and it holds that $\dim J = 6$. Furthermore, we have the spectral gap property*

$$\mathfrak{H}(\xi, \xi) \geq \frac{2}{3} \int_{\mathbb{S}^2} |\nabla \xi|^2 d\mathcal{H}^2 \tag{4.21}$$

for all $\xi \in \mathbb{H}^1$. Finally, the L^2 -orthogonal projection $\pi_J : L^2(\mathbb{S}^2; T\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2; T\mathbb{S}^2)$ onto J is well-defined and orthogonal with respect to the inner product in $\mathring{H}^1(\mathbb{S}^2)$.

Thus all Jacobi fields arise from variations of the form (4.11).

Having presented all statements of this subsection, we provide their proofs below.

Proof of Lemma 4.1. *Step 1: We have $\zeta \in J$ if and only if the condition*

$$\int_{\mathbb{S}^2} (\nabla \zeta : \nabla \xi - 2\zeta \cdot \xi) d\mathcal{H}^2 = 0 \tag{4.22}$$

holds for all $\xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$.

Let $\zeta \in J$, meaning we have $\zeta \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$ with $\mathfrak{H}(\zeta, \zeta) = 0$. As \mathfrak{H} is a non-negative bilinear form, the Cauchy-Schwarz inequality implies for all $\xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$ that

$$0 \leq |\mathfrak{H}(\zeta, \xi)| \leq \mathfrak{H}^{\frac{1}{2}}(\zeta, \zeta)\mathfrak{H}^{\frac{1}{2}}(\xi, \xi) = 0, \tag{4.23}$$

which yields (4.22). Furthermore, by choosing $\xi = \zeta$ we get that (4.22) is equivalent to $\mathfrak{H}(\zeta, \zeta) = 0$.

Step 2: Prove equation (4.10).

If $\xi \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3) \subset H^1_{\text{w}}(\mathbb{R}^2; \mathbb{R}^3)$ satisfies $\xi(x) \cdot \phi(x) = 0$ for almost all $x \in \mathbb{R}^2$, then the statement immediately follows from (4.22) and Lemma A.2. Consequently, it is sufficient to consider $\xi \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3)$ with $\xi = (\xi \cdot \phi)\phi$. In this case, by virtue of $\zeta_\phi \cdot \xi = (\xi \cdot \phi)(\zeta_\phi \cdot \phi) = 0$ almost everywhere we can write

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(\nabla \zeta_\phi : \nabla \xi - \zeta_\phi \cdot \xi |\nabla \phi|^2 - 2(\nabla \phi : \nabla \zeta_\phi)(\phi \cdot \xi) \right) dx \\ &= \int_{\mathbb{R}^2} \left(\nabla \zeta_\phi : \nabla[(\phi \cdot \xi)\phi] - 2(\nabla \phi : \nabla \zeta_\phi)(\phi \cdot \xi) \right) dx \\ &= \left(\sum_{i=1}^2 \sum_{j=1}^3 \int_{\mathbb{R}^2} \phi_j \partial_i (\xi \cdot \phi) \partial_i \zeta_{\phi,j} dx \right) - \int_{\mathbb{R}^2} (\nabla \phi : \nabla \zeta_\phi)(\phi \cdot \xi) dx. \end{aligned} \tag{4.24}$$

For $i = 1, 2$ using the identity $0 = \partial_i(\zeta_\phi \cdot \phi) = \sum_{j=1}^3 (\zeta_{\phi,j} \partial_i \phi_j + \phi_j \partial_i \zeta_{\phi,j})$ we obtain

$$\sum_{i=1}^2 \sum_{j=1}^3 \int_{\mathbb{R}^2} \phi_j \partial_i (\xi \cdot \phi) \partial_i \zeta_{\phi,j} dx = - \sum_{i=1}^2 \sum_{j=1}^3 \int_{\mathbb{R}^2} \zeta_{\phi,j} \partial_i (\xi \cdot \phi) \partial_i \phi_j dx. \tag{4.25}$$

As ξ, ϕ and ζ_ϕ are bounded by assumption and $|\nabla \phi|$ decays quadratically at infinity, we can integrate by parts on the right-hand side of equation (4.25) without incurring additional boundary terms at infinity. Thus we get

$$- \sum_{i=1}^2 \sum_{j=1}^3 \int_{\mathbb{R}^2} \zeta_{\phi,j} \partial_i (\xi \cdot \phi) \partial_i \phi_j dx = \int_{\mathbb{R}^2} (\xi \cdot \phi) (\zeta_\phi \cdot \Delta \phi + \nabla \zeta_\phi : \nabla \phi) dx. \tag{4.26}$$

The first term drops out due to ϕ solving the harmonic map equation $\Delta \phi + |\nabla \phi|^2 \phi = 0$ and ζ_ϕ being a tangential field. The remaining term cancels with the other term on the right-hand side of equation (4.24), yielding (4.10). \square

Proof of Proposition 4.2. As the tangential vector spherical harmonics form an orthonormal basis for $L^2(\mathbb{S}^2; T\mathbb{S}^2)$, for each $\xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$ we have the Plancherel identity

$$\int_{\mathbb{S}^2} |\xi|^2 d\mathcal{H}^2 = \sum_{n \geq 1} \sum_{j=-n}^n \sum_{k=2,3} \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right)^2. \tag{4.27}$$

For $N \in \mathbb{N}$ let

$$H_N := \text{span} \left\{ \mathcal{Y}_{n,j}^{(k)} : 0 \leq n \leq N; j = -n, \dots, n; k = 2, 3 \right\}. \tag{4.28}$$

In view of inequality (4.5) the expression $\int_{\mathbb{S}^2} \nabla \zeta : \nabla \xi d\mathcal{H}^2$ defines a scalar product on $H^1(\mathbb{S}^2; T\mathbb{S}^2)$, which we call the \dot{H}^1 -scalar product. Let π_N be the \dot{H}^1 -orthogonal

projection onto H_N and let $\xi \in H^1(\mathbb{S}^2; T\mathbb{S}^2)$. Then for $1 \leq n \leq N, j = -n, \dots, n$ and $k = 2, 3$, we can integrate by parts, see identity (A.7), and use the fact that $\mathcal{Y}_{n,j}^{(k)}$ is an eigenvector of Δ_v with eigenvalue $-n(n+1)$ to get

$$0 = \int_{\mathbb{S}^2} \nabla(\xi - \pi_N \xi) : \nabla \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 = n(n+1) \int_{\mathbb{S}^2} (\xi - \pi_N \xi) \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2. \tag{4.29}$$

As a result, we have

$$\int_{\mathbb{S}^2} \pi_N \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 = \int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \tag{4.30}$$

for all $1 \leq n \leq N, j = -n, \dots, n$ and $k = 2, 3$, so that we get

$$\pi_N \xi = \sum_{n=1}^N \sum_{j=-n}^n \sum_{k=2,3} \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right) \mathcal{Y}_{n,j}^{(k)}. \tag{4.31}$$

Therefore, from identity (4.19) we obtain

$$\begin{aligned} \sum_{n=1}^N \sum_{j=-n}^n \sum_{k=2,3} n(n+1) \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right)^2 &= \int_{\mathbb{S}^2} |\nabla \pi_N \xi|^2 d\mathcal{H}^2 \\ &\leq \int_{\mathbb{S}^2} |\nabla \xi|^2 d\mathcal{H}^2. \end{aligned} \tag{4.32}$$

In the limit $N \rightarrow \infty$, we consequently deduce

$$\sum_{n \geq 1} \sum_{j=-n}^n \sum_{k=2,3} n(n+1) \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right)^2 \leq \int_{\mathbb{S}^2} |\nabla \xi|^2 d\mathcal{H}^2. \tag{4.33}$$

By the identities (4.31) and (4.27) we have $\pi_N \xi \rightarrow \xi$ in $L^2(\mathbb{S}^2; T\mathbb{S}^2)$, which implies that $\nabla \pi_N \xi \rightharpoonup \nabla \xi$ in $L^2(\mathbb{S}^2; \mathbb{R}^9)$. As a result, from the equality in (4.32) and lower semicontinuity of the $L^2(\mathbb{S}^2; \mathbb{R}^9)$ norm we get

$$\begin{aligned} \sum_{n \geq 1} \sum_{j=-n}^n \sum_{k=2,3} n(n+1) \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right)^2 &= \lim_{N \rightarrow \infty} \int_{\mathbb{S}^2} |\nabla \pi_N \xi|^2 d\mathcal{H}^2 \\ &\geq \int_{\mathbb{S}^2} |\nabla \xi|^2 d\mathcal{H}^2. \end{aligned} \tag{4.34}$$

Combining the two inequalities (4.33) and (4.34), we obtain

$$\sum_{n \geq 1} \sum_{j=-n}^n \sum_{k=2,3} n(n+1) \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right)^2 = \int_{\mathbb{S}^2} |\nabla \xi|^2 d\mathcal{H}^2. \tag{4.35}$$

By the two equalities (4.27) and (4.35) we obtain the representation

$$\mathfrak{H}(\xi, \xi) = \sum_{n \geq 1} \sum_{j=-n}^n \sum_{k=2,3} (n(n+1) - 2) \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right)^2, \tag{4.36}$$

from which the representation $J = H_1$ immediately follows by virtue of $n(n + 1) \geq 6$ for $n \geq 2$. To deduce the spectral gap property, note that by the same token we have the sharp estimate $n(n + 1) - 2 \geq \frac{2}{3}n(n + 1)$. If $\xi \in \mathbb{H}^1$ we therefore have

$$\mathfrak{H}(\xi, \xi) \geq \frac{2}{3} \sum_{n \geq 1} \sum_{j=-n}^n \sum_{k=2,3} n(n + 1) \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{n,j}^{(k)} d\mathcal{H}^2 \right)^2 = \frac{2}{3} \int_{\mathbb{S}^2} |\nabla \xi|^2 d\mathcal{H}^2, \tag{4.37}$$

which concludes the proof of estimate (4.21).

Finally, in view of the fact that the tangential vector spherical harmonics are an orthonormal system for $L^2(\mathbb{S}^2; T\mathbb{S}^2)$ we see that

$$\pi_J(\xi) := \sum_{j=-1,0,1} \left[\left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{1,j}^{(2)} d\mathcal{H}^2 \right) \mathcal{Y}_{1,j}^{(2)} + \left(\int_{\mathbb{S}^2} \xi \cdot \mathcal{Y}_{1,j}^{(3)} d\mathcal{H}^2 \right) \mathcal{Y}_{1,j}^{(3)} \right] \tag{4.38}$$

is the L^2 -orthogonal projection onto J , which by identity (4.31) coincides with the \mathring{H}^1 -orthogonal projection. \square

4.2. From Linear Stability to Rigidity

In order to make use of the spectral gap property of Proposition 4.2, we first have to find a degree one harmonic map to which to apply it. It turns out that it is advantageous to take $\phi \in \mathcal{B}$ minimizing the Dirichlet distance $D(m; \mathcal{B})$, see definition (2.38), between \mathcal{B} and $m \in \mathcal{C}$, which is possible due to the following Lemma 4.3. As in its proof it is more convenient to deal with Belavin–Polyakov profiles rather than Möbius transformations, we formulate it in the \mathbb{R}^2 -setting.

Lemma 4.3. *For any $m \in \mathcal{C}$ there exists $\phi \in \mathcal{B}$ such that*

$$D(m; \mathcal{B}) = \left(\int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 dx \right)^{\frac{1}{2}}. \tag{4.39}$$

With this statements, we are in a position to prove a local version of Theorem 2.4 by projecting $m - \phi$ onto a vector field tangent to ϕ and using Lemma 2.5 to control the resulting higher order terms.

Lemma 4.4. *Let $\tilde{\eta} > 0$ be as in Lemma 2.5. For $m \in \mathcal{C}$ with $D^2(m; \mathcal{B}) < \tilde{\eta}$ we have*

$$\left(\frac{2}{3} - \frac{2}{3}C_4^2 D(m; \mathcal{B}) - \frac{19}{12}C_4^4 D^2(m; \mathcal{B}) \right) D^2(m; \mathcal{B}) \leq F(m) - 8\pi, \tag{4.40}$$

where C_4 is the constant from Lemma 2.5.

Proof of Lemma 4.3. Towards a contradiction, we assume that $\inf_{\phi \in \mathcal{B}} \int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 dx$ is not attained. Throughout the proof, we ignore the relabeling of subsequences without further comment.

Step 1: If the infimum is not attained, then $\int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 dx > \int_{\mathbb{R}^2} |\nabla m|^2 dx + 8\pi$ for all $\phi \in \mathcal{B}$.

For $n \in \mathbb{N}$, let $R_n \in \text{SO}(3)$, $0 < \rho_n < \infty$ and $x_n \in \mathbb{R}^2$ be such that $\phi_n := R_n \Phi(\rho_n^{-1}(\bullet - x_n)) \in \mathcal{B}$ satisfies

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla(m - \phi_n)|^2 dx = \inf_{\phi \in \mathcal{B}} \int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 dx. \tag{4.41}$$

As $\text{SO}(3)$ is compact, there exists a subsequence and $R \in \text{SO}(3)$ such that $\lim_{n \rightarrow \infty} R_n = R$. By direct computation, we have uniformly for all $\tilde{\phi} \in \mathcal{B}$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \left| \nabla(R_n \tilde{\phi}) - \nabla(R \tilde{\phi}) \right|^2 dx = 0. \tag{4.42}$$

We may thus suppose that $R_n = R$ for all $n \in \mathbb{N}$. Due to the fact that there does not exist an optimal approximating Belavin–Polyakov profile we have $\lim_{n \rightarrow \infty} \rho_n = 0$, $\lim_{n \rightarrow \infty} \rho_n = \infty$, or $\lim_{n \rightarrow \infty} x_n = \infty$.

Let us first deal with the case $\lim_{n \rightarrow \infty} \rho_n = 0$, which implies $\nabla \phi_n \rightharpoonup 0$ in L^2 . Consequently, by expanding the square we get

$$\inf_{\phi \in \mathcal{B}} \int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla(m - \phi_n)|^2 dx = \int_{\mathbb{R}^2} |\nabla m|^2 dx + 8\pi. \tag{4.43}$$

As the infimum is not achieved, we obtain

$$\int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 dx > \int_{\mathbb{R}^2} |\nabla m|^2 dx + 8\pi \tag{4.44}$$

for all $\phi \in \mathcal{B}$.

In the case $\lim_{n \rightarrow \infty} \rho_n = \infty$, we rescale $m_n := m(\rho_n x + x_n)$ and observe $\nabla m_n \rightharpoonup 0$ in L^2 . Similarly as in the previous case we thus get for all $\phi \in \mathcal{B}$:

$$\int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 dx > \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla(m_n - R\Phi)|^2 dx = \int_{\mathbb{R}^2} |\nabla m|^2 dx + 8\pi. \tag{4.45}$$

In dealing with the case $\lim_{n \rightarrow \infty} x_n = \infty$ we may consequently assume that $\lim_{n \rightarrow \infty} \rho_n = \rho \in (0, \infty)$. Once again we then get $\nabla \phi_n \rightharpoonup 0$ in L^2 , and we conclude as in the first case.

Step 2: Derive the contradiction.

Expanding the square in the result of Step 1 yields

$$\int_{\mathbb{R}^2} \nabla m : \nabla \phi dx < 0 \tag{4.46}$$

for every $\phi \in \mathcal{B}$. Now, for $x \in \mathbb{R}^2$ we define the four Belavin–Polyakov profiles:

$$\phi_{+,+}(x) := (\Phi_1(x), \Phi_2(x), \Phi_3(x)), \tag{4.47}$$

$$\phi_{-,+}(x) := (\Phi_2(x), \Phi_1(x), -\Phi_3(x)), \tag{4.48}$$

$$\phi_{+,-}(x) := (-\Phi_1(x), -\Phi_2(x), \Phi_3(x)), \tag{4.49}$$

$$\phi_{-,-}(x) := (-\Phi_2(x), -\Phi_1(x), -\Phi_3(x)). \tag{4.50}$$

It is straightforward to see that

$$\int_{\mathbb{R}^2} \nabla m_3 \cdot \nabla \phi_{+,+;3} \, dx = - \int_{\mathbb{R}^2} \nabla m_3 \cdot \nabla \phi_{-,-;3} \, dx, \tag{4.51}$$

$$\int_{\mathbb{R}^2} \nabla m_3 \cdot \nabla \phi_{+,-;3} \, dx = - \int_{\mathbb{R}^2} \nabla m_3 \cdot \nabla \phi_{-,+;3} \, dx, \tag{4.52}$$

$$\int_{\mathbb{R}^2} \nabla m' : \nabla \phi'_{\pm,+} \, dx = - \int_{\mathbb{R}^2} \nabla m' : \nabla \phi'_{\pm,-} \, dx. \tag{4.53}$$

Therefore, by (4.46) and (4.51)–(4.53) we get

$$\begin{aligned} 0 &> \int_{\mathbb{R}^2} \nabla m : (\nabla \phi_{+,+} + \nabla \phi_{+,-} + \nabla \phi_{-,+} + \nabla \phi_{-,-}) \, dx \\ &= \int_{\mathbb{R}^2} \nabla m_3 \cdot (\nabla \phi_{+,+;3} + \nabla \phi_{+,-;3} + \nabla \phi_{-,+;3} + \nabla \phi_{-,-;3}) \, dx \\ &\quad + \int_{\mathbb{R}^2} \nabla m' : (\nabla \phi'_{+,+} + \nabla \phi'_{+,-} + \nabla \phi'_{-,+} + \nabla \phi'_{-,-}) \, dx = 0, \end{aligned} \tag{4.54}$$

which is a contradiction. \square

Proof of Lemma 4.4. Lemma 4.3 ensures the existence of $\phi \in \mathcal{B}$ such that

$$\int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 \, dx = D^2(m; \mathcal{B}). \tag{4.55}$$

As ϕ arises from Φ purely by invariances of the energy, we may without loss of generality suppose $\phi = \Phi$ by re-defining m . Throughout the proof, we abbreviate $\tilde{J} := \{\xi \circ \Phi : \xi \in J\}$.

Step 1: We decompose $m - \Phi$ into a vector field parallel to Φ , a Jacobi field and a tangent vector field normal to Jacobi fields. Furthermore, we state a few preliminary estimates and identities.

For $\xi \in H_w^1(\mathbb{R}^2; T_\Phi \mathbb{S}^2)$, let $\pi_{\tilde{J}}(\xi) := \pi_J(\xi \circ \Phi^{-1}) \circ \Phi$, where π_J is defined in (4.38), which makes sense in view of Lemma A.2. We decompose $\zeta := m - \Phi$ pointwise into the three parts:

$$\zeta_{\parallel} := (\zeta \cdot \Phi) \Phi = -\frac{1}{2} |m - \Phi|^2 \Phi, \tag{4.56}$$

$$\zeta_{\tilde{J}} := \pi_{\tilde{J}}(\zeta - \zeta_{\parallel}), \tag{4.57}$$

$$\zeta^* := \zeta - \zeta_{\parallel} - \zeta_{\tilde{J}}, \tag{4.58}$$

where we noted that $\zeta - \zeta_{\parallel} \in \mathring{H}^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3) \subset H_w^1(\mathbb{R}^2; \mathbb{R}^3)$. Since by Proposition 4.2 the map π_J is both an $L^2(\mathbb{S}^2; T\mathbb{S}^2)$ -orthogonal and an $\mathring{H}^1(\mathbb{S}^2; T\mathbb{S}^2)$ -orthogonal projection, Lemma A.2 implies that

$$\int_{\mathbb{R}^2} \zeta_{\tilde{J}} \cdot \zeta^* |\nabla \Phi|^2 \, dx = 0, \tag{4.59}$$

$$\int_{\mathbb{R}^2} \nabla \zeta_{\bar{j}} : \nabla \zeta^* \, dx = 0. \tag{4.60}$$

Lemma 2.5, which we may apply due to our smallness assumption $D^2(m; \mathcal{B}) < \tilde{\eta}$, together with Lemma A.2 tells us that

$$\int_{\mathbb{R}^2} |\zeta_{\parallel}|^2 |\nabla \Phi|^2 \, dx = \frac{1}{4} \int_{\mathbb{R}^2} |m - \Phi|^4 |\nabla \Phi|^2 \, dx \leq \frac{C_4^4}{4} D^4(m; \mathcal{B}). \tag{4.61}$$

Step 2: We claim that

$$\int_{\mathbb{R}^2} |\zeta|^2 |\nabla \Phi|^2 \, dx \leq \int_{\mathbb{R}^2} |\zeta^*|^2 |\nabla \Phi|^2 \, dx + \frac{5}{4} C_4^4 D^4(m; \mathcal{B}). \tag{4.62}$$

Indeed, by construction we have $(\zeta_{\bar{j}} + \zeta^*) \cdot \zeta_{\parallel} = 0$ almost everywhere. Therefore, by (4.59) we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} |\zeta|^2 |\nabla \Phi|^2 \, dx \\ &= \int_{\mathbb{R}^2} \left(|\zeta_{\parallel}|^2 + 2\zeta_{\parallel} \cdot (\zeta_{\bar{j}} + \zeta^*) + |\zeta_{\bar{j}}|^2 + 2\zeta_{\bar{j}} \cdot \zeta^* + |\zeta^*|^2 \right) |\nabla \Phi|^2 \, dx \\ &= \int_{\mathbb{R}^2} \left(|\zeta_{\parallel}|^2 + |\zeta_{\bar{j}}|^2 + |\zeta^*|^2 \right) |\nabla \Phi|^2 \, dx. \end{aligned} \tag{4.63}$$

The ζ_{\parallel} -term in (4.63) is controlled by (4.61), so that we only have to estimate the $\zeta_{\bar{j}}$ -term. Furthermore, since $\zeta_{\bar{j}}$ is a Jacobi field, Proposition 4.2 implies that we can find $\varepsilon > 0$ and a smooth map $\phi : (-\varepsilon, \varepsilon) \rightarrow \mathcal{B}$ such that $\phi(0) = \Phi$ and $\partial_t \phi(t)|_{t=0} = \zeta_{\bar{j}}$. Differentiating the expression $\int_{\mathbb{R}^2} |\nabla(m - \phi(t))|^2 \, dx$ in t and using the fact that $t = 0$ is its minimum, we obtain

$$\int_{\mathbb{R}^2} \nabla \zeta : \nabla \zeta_{\bar{j}} \, dx = 0. \tag{4.64}$$

Thus we have together with $\zeta_{\bar{j}} = \zeta - \zeta_{\parallel} - \zeta^*$ and the identity (4.60) that

$$\int_{\mathbb{R}^2} |\nabla \zeta_{\bar{j}}|^2 \, dx = \int_{\mathbb{R}^2} \nabla(\zeta - \zeta_{\parallel} - \zeta^*) : \nabla \zeta_{\bar{j}} \, dx = - \int_{\mathbb{R}^2} \nabla \zeta_{\parallel} : \nabla \zeta_{\bar{j}} \, dx. \tag{4.65}$$

By Proposition 4.2, $\zeta_{\bar{j}}$ is smooth, and we may use Lemma 4.1, the fact that $\zeta_{\parallel} \cdot \zeta_{\bar{j}} = 0$ almost everywhere, as well as (4.56) to obtain

$$\begin{aligned} - \int_{\mathbb{R}^2} \nabla \zeta_{\parallel} : \nabla \zeta_{\bar{j}} \, dx &= - \int_{\mathbb{R}^2} 2(\zeta_{\parallel} \cdot \Phi)(\nabla \Phi : \nabla \zeta_{\bar{j}}) \, dx \\ &= \int_{\mathbb{R}^2} |m - \Phi|^2 (\nabla \Phi : \nabla \zeta_{\bar{j}}) \, dx. \end{aligned} \tag{4.66}$$

The two identities (4.65) and (4.66) allow us to obtain from the Cauchy–Schwarz inequality and the estimate (4.61) that

$$\int_{\mathbb{R}^2} |\nabla \zeta_j|^2 dx \leq C_4^4 D^4(m; \mathcal{B}). \tag{4.67}$$

This and the Poincaré type inequality (4.6) furthermore implies

$$\int_{\mathbb{R}^2} |\zeta_j|^2 |\nabla \Phi|^2 dx \leq C_4^4 D^4(m; \mathcal{B}), \tag{4.68}$$

which together with (4.61) yields the claim.

Step 3: We also claim that

$$\int_{\mathbb{R}^2} |\nabla \zeta|^2 dx = \int_{\mathbb{R}^2} \left(|\nabla \zeta_{\parallel}|^2 + 2\nabla \zeta_{\parallel} : \nabla(\zeta - \zeta_{\parallel}) + |\nabla \zeta_j|^2 + |\nabla \zeta^*|^2 \right) dx \tag{4.69}$$

and

$$\int_{\mathbb{R}^2} 2\nabla \zeta_{\parallel} : \nabla(\zeta - \zeta_{\parallel}) dx + \int_{\mathbb{R}^2} |\nabla \zeta_{\parallel}|^2 dx \geq -2C_4^2 D^3(m; \mathcal{B}) - C_4^4 D^4(m; \mathcal{B}). \tag{4.70}$$

The first equality follows directly from (4.60). With the help of the identities $\partial_k[\Phi \cdot (\zeta - \zeta_{\parallel})] = 0$ and $\partial_k \zeta_{\parallel, l} = -\frac{1}{2}|m - \Phi|^2 \partial_k \Phi_l - \frac{1}{2}\Phi_l \partial_k |m - \Phi|^2$ a.e. for $k = 1, 2$ and $l = 1, 2, 3$ obtained from (4.56), the second term in the right-hand side of (4.69) is

$$\begin{aligned} & \int_{\mathbb{R}^2} 2\nabla \zeta_{\parallel} : \nabla(\zeta - \zeta_{\parallel}) dx \\ &= \int_{\mathbb{R}^2} \sum_{k=1}^2 \sum_{l=1}^3 \left(-|\zeta|^2 \partial_k \Phi_l \partial_k (\zeta - \zeta_{\parallel})_l - \Phi_l \partial_k |\zeta|^2 \partial_k (\zeta - \zeta_{\parallel})_l \right) dx \\ &= \int_{\mathbb{R}^2} \sum_{k=1}^2 \sum_{l=1}^3 \left((\zeta - \zeta_{\parallel})_l \partial_k |\zeta|^2 \partial_k \Phi_l - |\zeta|^2 \partial_k \Phi_l \partial_k (\zeta - \zeta_{\parallel})_l \right) dx. \end{aligned} \tag{4.71}$$

As $|\nabla \Phi(x)| = O(|x|^{-2})$ for $|x| \rightarrow \infty$, we may integrate by parts in the first term to get

$$\begin{aligned} & \int_{\mathbb{R}^2} \sum_{k=1}^2 \sum_{l=1}^3 (\zeta - \zeta_{\parallel})_l \partial_k |\zeta|^2 \partial_k \Phi_l dx \\ &= - \int_{\mathbb{R}^2} \left(|\zeta|^2 (\zeta - \zeta_{\parallel}) \cdot \Delta \Phi + \sum_{k=1}^2 \sum_{l=1}^3 |\zeta|^2 \partial_k (\zeta - \zeta_{\parallel})_l \partial_k \Phi_l \right) dx. \end{aligned} \tag{4.72}$$

The first term drops out by the harmonic map equation $\Delta \Phi + |\nabla \Phi|^2 \Phi = 0$ and the fact that $\Phi \cdot (\zeta - \zeta_{\parallel}) = 0$ almost everywhere, while the second one combines with the second term on the right-hand side of identity (4.71) to give

$$\int_{\mathbb{R}^2} 2\nabla \zeta_{\parallel} : \nabla(\zeta - \zeta_{\parallel}) dx = - \int_{\mathbb{R}^2} 2|\zeta|^2 \nabla \Phi : \nabla \zeta dx + \int_{\mathbb{R}^2} 2|\zeta|^2 \nabla \Phi : \nabla \zeta_{\parallel} dx. \tag{4.73}$$

The Cauchy–Schwarz inequality and the equality in (2.38) applied to the first term on the right-hand side, as well as Young’s inequality applied to the second term imply

$$\begin{aligned}
 - \int_{\mathbb{R}^2} 2\nabla\zeta_{\parallel} : \nabla(\zeta - \zeta_{\parallel}) \, dx &\leq 2 \left(\int_{\mathbb{R}^2} |\zeta|^4 |\nabla\Phi|^2 \, dx \right)^{\frac{1}{2}} D(m; \mathcal{B}) \\
 &\quad + \int_{\mathbb{R}^2} |\zeta|^4 |\nabla\Phi|^2 \, dx + \int_{\mathbb{R}^2} |\nabla\zeta_{\parallel}|^2 \, dx,
 \end{aligned}
 \tag{4.74}$$

By estimate (4.61), this gives the second part of the claim.

Step 4: Conclusion.

We now use Steps 2 and 3 to decompose the conclusion of Lemma A.4 in terms of ζ^* , ζ_{\parallel} and $\zeta_{\bar{j}}$, taking care to not estimate $\int_{\mathbb{R}^2} |\nabla\zeta^*|^2 \, dx$ in the identity (4.69) and only to estimate one third of the remaining terms, by which we obtain

$$\begin{aligned}
 F(m) - 8\pi &= \int_{\mathbb{R}^2} |\nabla(m - \Phi)|^2 - (m - \Phi)^2 |\nabla\Phi|^2 \, dx \\
 &\geq \mathfrak{H}_{\Phi}(\zeta^*, \zeta^*) + \frac{2}{3} \int_{\mathbb{R}^2} \left(|\nabla\zeta_{\parallel}|^2 + 2\nabla\zeta_{\parallel} : \nabla(\zeta - \zeta_{\parallel}) + |\nabla\zeta_{\bar{j}}|^2 \right) \, dx \\
 &\quad - \frac{2}{3} C_4^2 D^3(m; \mathcal{B}) - \frac{19}{12} C_4^4 D^4(m; \mathcal{B}).
 \end{aligned}
 \tag{4.75}$$

As we have $\zeta^* \circ \Phi^{-1} \in \mathbb{H}^1$ by construction, see definition (4.58), the spectral gap proved in Proposition 4.2 along with Lemma A.2 gives

$$\begin{aligned}
 \mathfrak{H}_{\Phi}(\zeta^*, \zeta^*) &= \mathfrak{H}(\zeta^* \circ \Phi^{-1}, \zeta^* \circ \Phi^{-1}) \geq \frac{2}{3} \int_{\mathbb{S}^2} \left| \nabla \left(\zeta^* \circ \Phi^{-1} \right) \right|^2 \, d\mathcal{H}^2 \\
 &= \frac{2}{3} \int_{\mathbb{R}^2} |\nabla\zeta^*|^2 \, dx.
 \end{aligned}
 \tag{4.76}$$

As a result, the estimate (4.75) and Step 3 imply

$$F(m) - 8\pi \geq \frac{2}{3} \int_{\mathbb{R}^2} |\nabla(m - \Phi)|^2 \, dx - \frac{2}{3} C_4^2 D^3(m; \mathcal{B}) - \frac{19}{12} C_4^4 D^4(m; \mathcal{B}),
 \tag{4.77}$$

concluding the proof. \square

4.3. Proofs of Theorem 2.4, Lemma 2.5 and Corollary 2.6

Having collected all the necessary intermediate statements in the two previous subsections, we now proceed with proving our main results, starting with Theorem 2.4.

Proof of Theorem 2.4. *Step 1:* For all $\alpha > 0$ there exists $\beta > 0$ such that for all $m \in \mathcal{C}$ with $F(m) - 8\pi \leq \beta$ we have $D(m; \mathcal{B}) < \alpha$.

Towards a contradiction, we assume that there exists $\alpha > 0$ and a sequence of $m_n \in \mathcal{C}$ for $n \in \mathbb{N}$ such that $F(m_n) - 8\pi \leq \frac{1}{n}$ and $D(m_n; \mathcal{B}) \geq \alpha$. Then, for $r > 0$ we introduce the Lévy concentration function $Q_n(r) := \sup_{x \in \mathbb{R}^2} \mu_n(B_r(x))$ associated with the measure μ_n such that $d\mu_n := |\nabla m_n|^2 dx$. Observe that $Q_n(r)$ is a non-decreasing continuous function of r and satisfies $\lim_{r \rightarrow 0} Q_n(r) = 0$ and $\lim_{r \rightarrow \infty} Q_n(r) = F(m_n) \geq 8\pi$. Using translation and scale invariance of F , we can thus assume the sequence (m_n) to satisfy

$$\int_{B_1(0)} |\nabla m_n|^2 dx = Q_n(1) = 4\pi \tag{4.78}$$

for all $n \in \mathbb{N}$. Lemma A.2 implies that the sequence of $\tilde{m}_n(y) := m_n \circ \Phi^{-1}(y)$ for $y \in \mathbb{S}^2$ and $n \in \mathbb{N}$ is a minimizing sequence for $F_{\mathbb{S}^2}$. Consequently, [56, Theorem 1''] implies that there exists a harmonic map $\tilde{m} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ and a defect measure ν on \mathbb{S}^2 supported on an at most countable set such that $\tilde{m}_n \rightharpoonup \tilde{m}$ in $H^1(\mathbb{S}^2; \mathbb{R}^3)$ and $|\nabla \tilde{m}_n|^2 dy \xrightarrow{*} |\nabla \tilde{m}|^2 dy + \nu$ as Radon measures. Furthermore, [56, Theorem 5.8] implies that for $\nu \neq 0$ there exist $P \in \mathbb{N}$ points $\{y_p\}_{p=1}^P \in \mathbb{S}^2$ such that $\nu = \sum_{p=1}^P 8\pi N_p \delta_{y_p}$ for some $N_p \in \mathbb{N}$.

If we had $\nu \neq 0$, then on account of $\lim_{n \rightarrow \infty} F_{\mathbb{S}^2}(\tilde{m}_n) = 8\pi$ there can be at most a single defect $y_1 \in \mathbb{S}^2$ such that $\nu = 8\pi \delta_{y_1}$, and we must have $\nabla \tilde{m} = 0$ almost everywhere. We must consequently have $y_1 \in \Phi(B_1(0))$, which, however, implies

$$\lim_{n \rightarrow \infty} \int_{B_1(\Phi^{-1}(y_1))} |\nabla m_n|^2 dx = 8\pi, \tag{4.79}$$

contradicting the second equality in (4.78).

Therefore, we must have $\nu = 0$, in which case the convergence $|\nabla \tilde{m}_n|^2 dy \xrightarrow{*} |\nabla \tilde{m}|^2 dy$ gives

$$\int_{\mathbb{S}^2} |\nabla \tilde{m}|^2 dy = 8\pi, \tag{4.80}$$

which, in turn, implies that $m_n \rightarrow m$ in $\dot{H}^1(\mathbb{R}^2; \mathbb{R}^3)$. However, this contradicts the assumption $D(m_n; \mathcal{B}) \geq \alpha$ for all $n \in \mathbb{N}$.

Step 2: Conclusion.

By Step 1 we can choose $\beta > 0$ such that for all $m \in \mathcal{C}$ with $F(m) - 8\pi \leq \beta$ we have $D^2(m; \mathcal{B}) \leq \tilde{\eta}$, where $\tilde{\eta}$ is as in Lemma 2.5. If we additionally choose $\beta > 0$ small enough, Lemma 4.4 thus implies that there exists $\hat{\eta} > 0$ such that

$$\hat{\eta} D^2(m; \mathcal{B}) \leq F(m) - 8\pi \tag{4.81}$$

for all $m \in \mathcal{C}$ with $F(m) - 8\pi \leq \beta$. For $m \in \mathcal{C}$ with $F(m) - 8\pi \geq \beta$ we use Lemma A.4 together with $\int_{\mathbb{R}^2} (m - \phi)^2 |\nabla \phi|^2 dx \leq 32\pi$ for all $\phi \in \mathcal{B}$ to get

$$D^2(m; \mathcal{B}) \leq F(m) + 24\pi \leq \left(1 + \frac{32\pi}{\beta}\right) (F(m) - 8\pi). \tag{4.82}$$

Given $m \in \mathcal{C}$, existence of ϕ such that

$$\int_{\mathbb{R}^2} |\nabla (m - \phi)|^2 dx = D^2(m; \mathcal{B}) \tag{4.83}$$

was proved in Lemma 4.3. Thus the theorem holds for $\eta := \min \left\{ \hat{\eta}, \left(1 + \frac{32\pi}{\beta} \right)^{-1} \right\}$. □

Proof of Lemma 2.5. By Jensen’s inequality, it is sufficient to prove the estimate for $p \geq 2$. The Sobolev inequality applied to the map $u(y) := m(y) - y - \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) d\mathcal{H}^2(\tilde{y})$ for $y \in \mathbb{S}^2$ implies (see, for example, [5, Theorem 4])

$$\begin{aligned} & \left(\int_{\mathbb{S}^2} \left| m(y) - y - \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) d\mathcal{H}^2(\tilde{y}) \right|^p d\mathcal{H}^2(y) \right)^{\frac{2}{p}} \\ & \leq \frac{p-2}{2} \int_{\mathbb{S}^2} |\nabla (m(y) - y)|^2 d\mathcal{H}^2(y) \\ & \quad + \int_{\mathbb{S}^2} \left| m(y) - y - \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) d\mathcal{H}^2(\tilde{y}) \right|^2 d\mathcal{H}^2(y). \end{aligned} \tag{4.84}$$

The sharp Poincaré type inequality, following from the first nontrivial eigenvalue of the negative Laplace–Beltrami operator $-\Delta$ on \mathbb{S}^2 , see for example [35, Theorem 3.67], implies

$$\begin{aligned} & \int_{\mathbb{S}^2} \left| m(y) - y - \int_{\mathbb{S}^2} (m(\tilde{z}) - \tilde{y}) d\mathcal{H}^2(\tilde{y}) \right|^2 d\mathcal{H}^2(y) \\ & \leq \frac{1}{2} \int_{\mathbb{S}^2} |\nabla (m(y) - y)|^2 d\mathcal{H}^2(y), \end{aligned} \tag{4.85}$$

so that we obtain the Sobolev–Poincaré inequality

$$\begin{aligned} & \left(\int_{\mathbb{S}^2} \left| m(y) - y - \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) d\mathcal{H}^2(\tilde{y}) \right|^p d\mathcal{H}^2(y) \right)^{\frac{2}{p}} \\ & \leq \frac{p-1}{2} \int_{\mathbb{S}^2} |\nabla (m(y) - y)|^2 d\mathcal{H}^2(y). \end{aligned} \tag{4.86}$$

As the right-hand side of this estimate is part of the desired Sobolev inequality, we only have to control the average. To this end, for the moment we only consider the case that m is smooth. By symmetry of \mathbb{S}^2 , we obviously have

$$\int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) d\mathcal{H}^2(\tilde{y}) = \int_{\mathbb{S}^2} m(\tilde{y}) d\mathcal{H}^2(\tilde{y}). \tag{4.87}$$

With the goal of finding a similar cancellation for the remaining term, we work towards writing it on the image of m .

Setting $\tilde{m} := m \circ \Phi$ and using Lemma A.2, we have

$$\int_{\mathbb{S}^2} m d\mathcal{H}^2 = \frac{1}{8\pi} \int_{\mathbb{R}^2} \tilde{m} |\nabla \Phi|^2 dx. \tag{4.88}$$

The identity $|\nabla\Phi|^2 - |\nabla\tilde{m}|^2 = 2\nabla\Phi : \nabla(\Phi - \tilde{m}) + |\nabla(\Phi - \tilde{m})|^2$ and Cauchy–Schwarz inequality imply

$$\begin{aligned} & \left| \frac{1}{8\pi} \int_{\mathbb{R}^2} \tilde{m} \left(|\nabla\tilde{m}|^2 - |\nabla\Phi|^2 \right) dx \right| \\ & \leq \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 dx \right)^{\frac{1}{2}} + \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 dx. \end{aligned} \quad (4.89)$$

With the goal of further rewriting $\frac{1}{8\pi} \int_{\mathbb{R}^2} \tilde{m} |\nabla\tilde{m}|^2 dx$, we use the estimate (A.23) to get

$$|\nabla\tilde{m}|^2 \geq 2|\tilde{m} \cdot (\partial_1\tilde{m} \times \partial_2\tilde{m})|, \quad (4.90)$$

and thus we obtain

$$\begin{aligned} & \left| \frac{1}{8\pi} \int_{\mathbb{R}^2} \tilde{m} \left(2|\tilde{m} \cdot (\partial_1\tilde{m} \times \partial_2\tilde{m})| - |\nabla\tilde{m}|^2 \right) dx \right| \\ & \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} \left(|\nabla\tilde{m}|^2 - 2|\tilde{m} \cdot (\partial_1\tilde{m} \times \partial_2\tilde{m})| \right) dx \\ & \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} \left(|\nabla\tilde{m}|^2 - 2\tilde{m} \cdot (\partial_1\tilde{m} \times \partial_2\tilde{m}) \right) dx \\ & = \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla\tilde{m}|^2 dx - \mathcal{N}(\tilde{m}). \end{aligned} \quad (4.91)$$

By choosing $\tilde{\eta} > 0$ small enough, we get $\mathcal{N}(\tilde{m}) = 1$ due to \dot{H}^1 -continuity of \mathcal{N} , so that the above turns into

$$\left| \frac{1}{8\pi} \int_{\mathbb{R}^2} \tilde{m} \left(2|\tilde{m} \cdot (\partial_1\tilde{m} \times \partial_2\tilde{m})| - |\nabla\tilde{m}|^2 \right) dx \right| \leq \frac{1}{8\pi} \left(\int_{\mathbb{R}^2} |\nabla\tilde{m}|^2 dx - 8\pi \right). \quad (4.92)$$

By Lemma A.4, we get

$$\frac{1}{8\pi} \left(\int_{\mathbb{R}^2} |\nabla\tilde{m}|^2 dx - 8\pi \right) \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 dx, \quad (4.93)$$

so that the above gives

$$\left| \frac{1}{8\pi} \int_{\mathbb{R}^2} \tilde{m} \left(2|\tilde{m} \cdot (\partial_1\tilde{m} \times \partial_2\tilde{m})| - |\nabla\tilde{m}|^2 \right) dx \right| \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 dx. \quad (4.94)$$

We now aim to use the area formula [3, Theorem 2.71] to rewrite the first term on the left-hand side as an integral over the image of \tilde{m} . As we have $\tilde{m} \cdot \partial_1\tilde{m} = \tilde{m} \cdot \partial_2\tilde{m} = 0$ everywhere, the two vectors \tilde{m} and $\partial_1\tilde{m} \times \partial_2\tilde{m}$ are parallel. Therefore, we have

$$\begin{aligned} |\tilde{m} \cdot (\partial_1\tilde{m} \times \partial_2\tilde{m})|^2 &= |\partial_1\tilde{m} \times \partial_2\tilde{m}|^2 = \partial_1\tilde{m} \cdot (\partial_2\tilde{m} \times (\partial_1\tilde{m} \times \partial_2\tilde{m})) \\ &= |\partial_1\tilde{m}|^2 |\partial_2\tilde{m}|^2 - (\partial_1\tilde{m} \cdot \partial_2\tilde{m})^2, \end{aligned} \quad (4.95)$$

so that $|\tilde{m} \cdot (\partial_1 \tilde{m} \times \partial_2 \tilde{m})|$ is the modulus of the Jacobian of \tilde{m} . Consequently, the area formula gives

$$\begin{aligned} \frac{1}{4\pi} \int_{\mathbb{R}^2} \tilde{m} |\tilde{m} \cdot (\partial_1 \tilde{m} \times \partial_2 \tilde{m})| \, dx &= \int_{\mathbb{S}^2} z \mathcal{H}^0(\{m^{-1}(z)\}) \, d\mathcal{H}^2(z) \\ &= \int_{\mathbb{S}^2} z \mathcal{H}^0(\{m^{-1}(z)\}) \, d\mathcal{H}^2(z). \end{aligned} \tag{4.96}$$

For all $z \in \mathbb{S}^2$ there exists at least one $y \in \mathbb{S}^2$ such that $m(y) = z$ since m has non-zero degree, see [18, Property 1]. On account of \mathcal{H}^0 being the counting measure, this means that we have $\mathcal{H}^0(\{m^{-1}(z)\}) \geq 1$ for almost all $z \in \mathbb{S}^2$. By symmetry of the sphere we get

$$\begin{aligned} \left| \int_{\mathbb{S}^2} z \mathcal{H}^0(\{m^{-1}(z)\}) \, d\mathcal{H}^2(z) \right| &= \left| \int_{\mathbb{S}^2} z \left(\mathcal{H}^0(\{m^{-1}(z)\}) - 1 \right) \, d\mathcal{H}^2(z) \right| \\ &\leq \int_{\mathbb{S}^2} \left(\mathcal{H}^0(\{m^{-1}(z)\}) - 1 \right) \, d\mathcal{H}^2(z). \end{aligned} \tag{4.97}$$

Going back to \mathbb{R}^2 and exploiting that averaging leaves constant functions invariant, by (4.90) we obtain

$$\begin{aligned} \int_{\mathbb{S}^2} \left(\mathcal{H}^0(\{m^{-1}(z)\}) - 1 \right) \, d\mathcal{H}^2(z) &= \frac{1}{4\pi} \int_{\mathbb{R}^2} |\tilde{m} \cdot (\partial_1 \tilde{m} \times \partial_2 \tilde{m})| \, dx - 1 \\ &\leq \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla \tilde{m}|^2 \, dx - 1. \end{aligned} \tag{4.98}$$

Straightforwardly concatenating the estimates (4.96), (4.97), (4.98) and (4.93), we then obtain

$$\left| \frac{1}{4\pi} \int_{\mathbb{R}^2} \tilde{m} |\tilde{m} \cdot (\partial_1 \tilde{m} \times \partial_2 \tilde{m})| \, dx \right| \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 \, dx. \tag{4.99}$$

Adding the estimates (4.89), (4.94), and (4.99) we get

$$\begin{aligned} \left| \frac{1}{8\pi} \int_{\mathbb{R}^2} \tilde{m} |\nabla \Phi|^2 \, dx \right| &\leq \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 \, dx \right)^{\frac{1}{2}} \\ &\quad + \frac{3}{8\pi} \int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 \, dx. \end{aligned} \tag{4.100}$$

Concatenating the identities (4.87) and (4.88), and again applying Lemma A.2, we thus obtain

$$\begin{aligned} \left| \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) \, d\mathcal{H}^2(\tilde{y}) \right| &\leq \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{S}^2} |\nabla(m(y) - y)|^2 \, d\mathcal{H}^2(y) \right)^{\frac{1}{2}} \\ &\quad + \frac{3}{8\pi} \int_{\mathbb{S}^2} |\nabla(m(y) - y)|^2 \, d\mathcal{H}^2(y). \end{aligned} \tag{4.101}$$

Under the assumption $\int_{\mathbb{R}^2} |\nabla(\tilde{m} - \Phi)|^2 dx \leq \tilde{\eta}$ for $\tilde{\eta} > 0$ small enough, we then get

$$\left| \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) d\mathcal{H}^2(\tilde{y}) \right| \leq \frac{1}{\sqrt{\pi}} \left(\int_{\mathbb{S}^2} |\nabla(m(y) - y)|^2 dy \right)^{\frac{1}{2}}, \tag{4.102}$$

which by a density result of SCHOEN and UHLENBECK [75] even holds for all $m \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ with $\int_{\mathbb{S}^2} |\nabla m|^2 dx - 8\pi \leq \tilde{\eta}$. Together with inequality (4.86), this proves the desired Sobolev inequality.

In order to obtain exponential integrability, we exploit the Moser–Trudinger inequality, [66, Theorem 2], which says

$$\int_{\mathbb{S}^2} e^{4\pi|u|^2} d\mathcal{H}^2 \leq C \tag{4.103}$$

for a universal constant $C > 0$ and $u : \mathbb{S}^2 \rightarrow \mathbb{R}$ satisfying $\int_{\mathbb{S}^2} |\nabla u|^2 d\mathcal{H}^2 \leq 1$ and $\int_{\mathbb{S}^2} u d\mathcal{H}^2 = 0$. To this end, for $i = 1, 2, 3$ and $y \in \mathbb{S}^2$ we define, assuming without loss of generality that $m \neq \text{id}_{\mathbb{S}^2}$:

$$\begin{aligned} u_i(y) &:= \left(\int_{\mathbb{S}^2} |\nabla(m(\tilde{y}) - \tilde{y})|^2 d\mathcal{H}^2(\tilde{y}) \right)^{-\frac{1}{2}} \\ &\quad \times \left((m(y) - y) - \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y}) d\mathcal{H}^2(\tilde{y}) \right)_i. \end{aligned} \tag{4.104}$$

By Hölder’s inequality and the Moser–Trudinger inequality we have

$$\int_{\mathbb{S}^2} e^{\frac{4\pi}{3}|u|^2} d\mathcal{H}^2 = \int_{\mathbb{S}^2} \prod_{i=1}^3 e^{\frac{4\pi}{3}|u_i|^2} d\mathcal{H}^2 \leq \prod_{i=1}^3 \left(\int_{\mathbb{S}^2} e^{4\pi|u_i|^2} d\mathcal{H}^2 \right)^{1/3} \leq C. \tag{4.105}$$

At the same time, as a result of the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for $a, b \in \mathbb{R}$, we have for all $y \in \mathbb{S}^2$ that

$$\begin{aligned} 2|u_i(y)|^2 &\geq \left(\int_{\mathbb{S}^2} |\nabla(m(y) - y)|^2 d\mathcal{H}^2(y) \right)^{-1} \\ &\quad \times \left(|m_i(y) - y_i|^2 - 2 \left| \int_{\mathbb{S}^2} (m(\tilde{y}) - \tilde{y})_i d\mathcal{H}^2(\tilde{y}) \right|^2 \right), \end{aligned} \tag{4.106}$$

which by the estimate (4.102) gives

$$2|u_i(y)|^2 \geq \frac{|m_i(y) - y_i|^2}{\|\nabla(m - \text{id}_{\mathbb{S}^2})\|_2^2} - C. \tag{4.107}$$

The two estimates (4.106) and (4.107) together imply

$$\int_{\mathbb{S}^2} e^{\frac{2\pi}{3} \frac{|m(y) - y|^2}{\|\nabla(m - \text{id}_{\mathbb{S}^2})\|_2^2}} d\mathcal{H}^2(y) \leq C, \tag{4.108}$$

concluding the proof. \square

Proof of Corollary 2.6. The statement for degree 1 maps immediately follows from Theorem 2.4 and Lemma A.2. The estimate for degree -1 maps is a simple consequence of the fact that $m \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ has degree -1 if and only if $-m$ has degree 1. \square

5. Existence of Minimizers

The goal of this section is to show that minimizers of the energy $E_{\sigma,\lambda}$ over \mathcal{A} exist in an appropriate range of parameters by proving Theorem 2.1.

5.1. First Lower Bounds

Here, we describe the basic coercivity properties of $E_{\sigma,\lambda}$. To do so, we ensure that exchange and anisotropy energies being finite forces either $m - e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ or $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$, justifying our choice of the latter as an assumption in the definition of \mathcal{A} . Recall that $m = (m', m_3)$, where $m' = (m_1, m_2)$ is the in-plane component of the magnetization vector.

Lemma 5.1. *Let $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ be such that $m' \in L^2(\mathbb{R}^2; \mathbb{R}^2)$. Then we have $m - e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ or $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ with the estimate*

$$\min \left\{ \int_{\mathbb{R}^2} |m_3 - 1|^2 dx, \int_{\mathbb{R}^2} |m_3 + 1|^2 dx \right\} \leq \frac{1}{4\pi} \int_{\mathbb{R}^2} |\nabla m|^2 dx \int_{\mathbb{R}^2} (1 - m_3^2) dx. \tag{5.1}$$

In particular, for $m \in \mathcal{A}$ we have

$$\int_{\mathbb{R}^2} |m_3 + 1|^2 dx \leq 4 \int_{\mathbb{R}^2} (1 - m_3^2) dx. \tag{5.2}$$

Using this estimate, we are in a position to prove that $E_{\sigma,\lambda}$ is bounded from below and controls both the Dirichlet and the anisotropy energies.

Lemma 5.2. *Let $\sigma > 0$ and $\lambda \in [0, 1]$. Let $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ with $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ and $\int_{\mathbb{R}^2} |\nabla m|^2 dx < 16\pi$. Then $E_{\sigma,\lambda}(m) < \infty$, and we have the lower bounds*

$$\left(1 - \sigma^2 \frac{(1 + \lambda)^2}{4} \right) \int_{\mathbb{R}^2} |\nabla m|^2 dx \leq E_{\sigma,\lambda}(m), \tag{5.3}$$

$$\left(1 - \sigma^2 \frac{(1 + \lambda)^2}{2} \right) \int_{\mathbb{R}^2} |\nabla m|^2 dx + \frac{\sigma^2}{2} \int_{\mathbb{R}^2} |m'|^2 dx \leq E_{\sigma,\lambda}(m). \tag{5.4}$$

In particular, in the regime $\sigma(1 + \lambda) \leq 2$ we have $E_{\sigma,\lambda}(m) \geq 0$ for all $m \in \mathcal{A}$.

Proof of Lemma 5.1. *Step 1:* We have either $m_3 - 1 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ or $m_3 + 1 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$.

For almost all $x \in \mathbb{R}^2$ with $m_3(x) < 1$ we have the inequality $\frac{|\nabla m_3|^2}{1-m_3^2} \leq |\nabla m|^2$, see for example [69, (3.17)] for the argument, which together with the fact that $|\nabla m_3|(x) = 0$ for almost all $x \in \mathbb{R}^2$ with $|m_3(x)| = 1$ and together with Hölder’s inequality gives

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla m_3| \, dx &= \int_{\{m_3 < 1\}} |\nabla m_3| \, dx \\ &\leq \left(\int_{\mathbb{R}^2} |\nabla m|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} (1 - m_3^2) \, dx \right)^{\frac{1}{2}} < \infty. \end{aligned} \tag{5.5}$$

The co-area formula, see for example [3, Theorem 3.40], reads

$$\int_{\mathbb{R}^2} |\nabla m_3| \, dx = \int_{-1}^1 P(\{m_3 > t\}) \, dt. \tag{5.6}$$

Therefore, there exists $t \in (-\frac{1}{2}, \frac{1}{2})$ such that

$$P(\{m_3 > t\}) \leq \int_{\mathbb{R}^2} |\nabla m_3| \, dx < \infty. \tag{5.7}$$

The isoperimetric inequality [3, Theorem 3.46] then implies

$$\min \{ |\{m_3 > t\}|, |\{m_3 \leq t\}| \} \leq \frac{1}{4\pi} P^2(\{m_3 > t\}) < \infty. \tag{5.8}$$

In the following we only deal with the case $|\{m_3 \leq t\}| < \infty$, as the other case can be treated similarly. Using $0 \leq 1 - m_3(x) \leq 2$ for all $x \in \mathbb{R}^2$, and $1 + m_3(x) > 1 + t \geq \frac{1}{2}$ for all $x \in \{m_3 > t\}$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |m_3 - 1|^2 \, dx &\leq 2 \int_{\{m_3 > t\}} (1 - m_3) \, dx + 4|\{m_3 \leq t\}| \\ &\leq 4 \int_{\{m_3 > t\}} (1 + m_3)(1 - m_3) \, dx + 4|\{m_3 \leq t\}| \\ &\leq 4 \int_{\mathbb{R}^2} (1 - m_3^2) \, dx + 4|\{m_3 \leq t\}| < \infty. \end{aligned} \tag{5.9}$$

Step 2: Prove the quantitative estimates.

By [34, Theorem 2] the optimal Gagliardo–Nirenberg–Sobolev inequality is

$$\|u\|_2 \leq \frac{1}{2\sqrt{\pi}} \|Du\|_1 \tag{5.10}$$

for any $u \in L^r(\mathbb{R}^2)$ with $r \in [1, \infty)$. Combining this inequality with estimate (5.5) yields

$$\begin{aligned} \min \left\{ \int_{\mathbb{R}^2} |m_3 - 1|^2 \, dx, \int_{\mathbb{R}^2} |m_3 + 1|^2 \, dx \right\} &\leq \frac{1}{4\pi} \left(\int_{\mathbb{R}^2} |\nabla m_3| \, dx \right)^2 \\ &\leq \frac{1}{4\pi} \int_{\mathbb{R}^2} |\nabla m|^2 \, dx \int_{\mathbb{R}^2} (1 - m_3^2) \, dx, \end{aligned} \tag{5.11}$$

which concludes the proof of estimate (5.1). Finally, inequality (5.2) for $m \in \mathcal{A}$ follows from $m + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ and $\int_{\mathbb{R}^2} |\nabla m|^2 \, dx < 16\pi$. \square

Proof of Lemma 5.2. By Lemma 3.1, we have $0 \leq F_{\text{vol}}(m') < \infty$. Furthermore, the estimate (3.7) gives

$$F_{\text{surf}}(m_3) \leq \frac{1}{2} \|m_3 + 1\|_2 \|\nabla m_3\|_2. \tag{5.12}$$

Lemma 5.1 and the assumption $\|\nabla m\|_2 < 4\sqrt{\pi}$ for all $m \in \mathcal{A}$ then imply

$$F_{\text{surf}}(m_3) \leq \frac{1}{4\sqrt{\pi}} \|m'\|_2 \|\nabla m\|_2^2 \leq \|m'\|_2 \|\nabla m\|_2. \tag{5.13}$$

To handle the DMI term, note that the Cauchy–Schwarz inequality gives

$$2 \int_{\mathbb{R}^2} m' \cdot \nabla m_3 \, dx \leq 2 \|m'\|_2 \|\nabla m_3\|_2. \tag{5.14}$$

Combining these insights and applying Young’s inequality we obtain

$$\begin{aligned} E_{\sigma,\lambda}(m) &\geq \int_{\mathbb{R}^2} (|\nabla m|^2 + \sigma^2 |m'|^2) \, dx - \sigma^2 (1 + \lambda) \|m'\|_2 \|\nabla m\|_2 \\ &\geq \left(1 - \sigma^2 \frac{(1 + \lambda)^2}{4} \right) \int_{\mathbb{R}^2} |\nabla m|^2 \, dx \end{aligned} \tag{5.15}$$

and

$$E_{\sigma,\lambda}(m) \geq \int_{\mathbb{R}^2} \left(\left(1 - \sigma^2 \frac{(1 + \lambda)^2}{2} \right) |\nabla m|^2 + \frac{\sigma^2}{2} |m'|^2 \right) \, dx, \tag{5.16}$$

which gives the desired statements. \square

5.2. Upper Bounds via Minimization of a Reduced Energy

We now turn to defining a simplified energy that reduces the minimization to finding the best Belavin–Polyakov profile taking the correct value at infinity. As these profiles have logarithmically divergent anisotropy energy, a truncation is necessary to make sense of the energy.

Let

$$f(r) := \frac{2r}{1+r^2} \tag{5.17}$$

be the in-plane modulus of the Néel-type Belavin–Polyakov profile

$$\Phi(x) = \left(-\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right). \tag{5.18}$$

We consider the truncation at scale $L > 1$ defined as

$$\Phi_L(x) := \left(-f_L(|x|) \frac{x}{|x|}, \text{sign}(1-|x|) \sqrt{1-f_L^2(|x|)} \right), \tag{5.19}$$

where

$$f_L(r) := \begin{cases} f(r) & \text{if } r \leq L^{\frac{1}{2}}, \\ f\left(L^{\frac{1}{2}}\right) \frac{K_1(rL^{-1})}{K_1\left(L^{-\frac{1}{2}}\right)} & \text{if } r > L^{\frac{1}{2}}. \end{cases} \tag{5.20}$$

Here, K_1 is the modified Bessel function of the second kind of order 1, for a more detailed discussion see Sect. A.3. The ansätze are then given by

$$\phi_{\rho,\theta,L}(x) := S_\theta \Phi_L(\rho^{-1}x) \tag{5.21}$$

for $\rho > 0, \theta \in [-\pi, \pi), L > 1$ and where S_θ is given by (2.24). For convenience, we also define $\phi_{\rho,\theta,\infty}(x) := S_\theta \Phi(\rho^{-1}x)$.

For later computations, it turns out to be convenient to not quite use the variables ρ and L in the definition of the reduced energy, but to make the substitutions

$$\tilde{\rho} = |\log \sigma| \rho, \quad \tilde{L} = \frac{L}{2\sqrt{\pi}}. \tag{5.22}$$

We furthermore divide the energy by $\frac{\sigma^2}{|\log \sigma|}$. As we will show below, for $\sigma > 0, \lambda \in [0, 1]$ and a constant $K > 0$ the rescaled energy of $\phi_{\rho,\theta,L}$ is then given to the leading order in $\sigma \ll 1$ by

$$\mathcal{E}_{\sigma,\lambda;K}(\tilde{\rho}, \theta, \tilde{L}) := |\log \sigma| (\sigma \tilde{L})^{-2} + \frac{4\pi \log(K\tilde{L}^2)}{|\log \sigma|} \tilde{\rho}^2 - g(\lambda, \theta) \tilde{\rho}, \tag{5.23}$$

on the domain

$$V_\sigma := \left\{ (\tilde{\rho}, \theta, \tilde{L}) : \tilde{\rho} > 0, \theta \in [-\pi, \pi), \tilde{L} \geq \frac{1}{4\sigma\sqrt{\pi}} \right\}, \tag{5.24}$$

where

$$g(\lambda, \theta) := 8\pi\lambda \cos \theta + \frac{\pi^3}{8}(1 - \lambda) \left(1 - 3 \cos^2 \theta\right). \tag{5.25}$$

The first term on the right-hand side of definition (5.23) represents the Dirichlet excess. The constant $4\sqrt{\pi}$ in the bound for \tilde{L} in the definition of V_σ is the result of the a priori estimate (6.5) for the Dirichlet excess arising in the proof of the lower bound in Sect. 6. The second term captures the logarithmic blowup of the anisotropy energy as the profile approaches a Belavin–Polyakov profile. Finally, the third term combines the contributions of the DMI and stray field terms.

The details of the truncation (5.20) will only enter through the constant $K > 0$, with our construction giving $K = K^*$, where

$$K^* := \frac{16\pi}{e^{2(1+\gamma)}}, \tag{5.26}$$

in which $\gamma \approx 0.5772$ is the Euler–Mascheroni constant. As this constant matches the one obtained in Lemma 6.4 below, we do expect our ansatz to be optimal. However, we will lose a constant factor of $\frac{2}{3} - \varepsilon$ for $\varepsilon > 0$ sufficiently small in the application of the rigidity Theorem 2.4, so that we get different energies $\mathcal{E}_{\sigma,\lambda;K^*}$ and $\mathcal{E}_{\sigma,\lambda;(\frac{3}{2}+\varepsilon)^{-1}K^*}$ appearing in the upper and lower bounds for $\min E_{\sigma,\lambda}$. Nevertheless, we will see that the stability properties of the two reduced energies are strong enough to prove Theorem 2.2.

The next lemma contains an estimate comparing $E_{\sigma,\lambda}(\phi_{\rho,\theta,L})$ with $\mathcal{E}_{\sigma,\lambda;K}(\tilde{\rho}, \theta, \tilde{L})$. The rate $\sigma^{\frac{1}{2}}|\log \sigma|$ is likely not optimal, but it is sufficient for our argument. Additionally, we keep track of a number of identities which will be useful later.

Lemma 5.3. *There exist universal constants $C > 0$ and $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$ and for all $\lambda \in [0, 1]$ we have the following: For $\rho \in (0, 1]$, $\theta \in [-\pi, \pi)$ and $L \geq \frac{1}{2\sigma}$ we have $\phi_{\rho,\theta,L} \in \mathcal{A}$ and $(|\log \sigma| \rho, \theta, \frac{L}{2\sqrt{\pi}}) \in V_\sigma$. Furthermore, it holds that*

$$\left| \frac{|\log \sigma|}{\sigma^2} (E_{\sigma,\lambda}(\phi_{\rho,\theta,L}) - 8\pi) - \mathcal{E}_{\sigma,\lambda;K^*} \left(|\log \sigma| \rho, \theta, \frac{L}{2\sqrt{\pi}} \right) \right| \leq C \sigma^{\frac{1}{4}} |\log \sigma|, \tag{5.27}$$

where K^* is defined in Eq. (5.26). Additionally, for any $\rho > 0$ we have

$$\int_{\mathbb{R}^2} |\nabla \phi_{\rho,\theta,L}|^2 dx - 8\pi \leq \frac{4\pi}{L^2} + \frac{C \log^2 L}{L^3}, \tag{5.28}$$

$$\int_{\mathbb{R}^2} |\nabla (\phi_{\rho,\theta,\infty} - \phi_{\rho,\theta,L})|^2 dx \leq CL^{-2}, \tag{5.29}$$

$$\int_{\mathbb{R}^2} |\phi'_{\rho,\theta,L}|^2 dx \leq 4\pi\rho^2 \log \left(\frac{4L^2}{e^{2(1+\gamma)}} \right) + \frac{C\rho^2 \log^2 L}{L}, \tag{5.30}$$

$$\int_{\mathbb{R}^2} |\phi_{\rho,\theta,\infty;3} - \phi_{\rho,\theta,L;3}|^2 dx \leq \frac{C\rho^2}{L}, \tag{5.31}$$

$$\int_{\mathbb{R}^2} 2\phi'_{\rho,\theta,\infty} \cdot \nabla \phi_{\rho,\theta,\infty;3} \, dx = 8\pi\rho \cos \theta, \tag{5.32}$$

$$\int_{\mathbb{R}^2} 2\phi'_{\rho,\theta,L} \cdot \nabla \phi_{\rho,\theta,L;3} \, dx = 8\pi\rho \cos \theta + O\left(\rho L^{-\frac{1}{2}}\right), \tag{5.33}$$

$$F_{\text{vol}}\left(\phi'_{\rho,\theta,L}\right) - F_{\text{surf}}\left(\phi_{\rho,\theta,L;3}\right) = \left(\frac{3\pi^3}{8} \cos^2 \theta - \frac{\pi^3}{8}\right)\rho + O\left(\rho L^{-\frac{1}{4}}\right). \tag{5.34}$$

Having thus established the correspondence between $E_{\sigma,\lambda}$ and $\mathcal{E}_{\sigma,\lambda;K}$, we can carry it out explicitly for the first part of the following statement. The stability properties of $\mathcal{E}_{\sigma,\lambda;K}$ are collected in the second part, which will yield convergence of the skyrmion radius and angle in Sect. 6.

Proposition 5.4. *There exists a universal constant $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$, $\lambda \in [0, 1]$ and $K \in [\frac{1}{2}K^*, 2K^*]$ we have the following:*

(i) *The function $\mathcal{E}_{\sigma,\lambda;K}$ has at most two global minimizers $(\rho_0, \theta_0^\pm, L_0)$ over V_σ and no further critical points in V_σ . Recalling the definitions (2.26) and (2.27), the minimizers are given by*

$$\rho_0 = \frac{\bar{g}(\lambda)}{16\pi} + O\left(\frac{|\log |\log \sigma||}{|\log \sigma|}\right), \tag{5.35}$$

$$\theta_0^\pm = \begin{cases} 0 & \text{if } \lambda \geq \lambda_c, \\ \pm \arccos\left(\frac{32\lambda}{3\pi^2(1-\lambda)}\right) & \text{else,} \end{cases} \tag{5.36}$$

$$L_0 = \left(\frac{8\sqrt{\pi}}{\bar{g}(\lambda)} + O\left(\frac{|\log |\log \sigma||}{|\log \sigma|}\right)\right) \frac{|\log \sigma|}{\sigma}. \tag{5.37}$$

Furthermore, we have

$$\begin{aligned} \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K} &= -\frac{\bar{g}^2(\lambda)}{32\pi} + \frac{\bar{g}^2(\lambda) \log |\log \sigma|}{32\pi |\log \sigma|} - \frac{\bar{g}^2(\lambda) \log\left(\frac{\bar{g}^2(\lambda)}{64\pi eK}\right)}{64\pi |\log \sigma|} \\ &\quad + O\left(\frac{\log^2 |\log \sigma|}{|\log \sigma|^2}\right) \end{aligned} \tag{5.38}$$

and

$$g(\lambda, \theta_0^\pm) = \bar{g}(\lambda). \tag{5.39}$$

(ii) *Let $(\rho_\sigma, \theta_\sigma, L_\sigma) \in V_\sigma$ be such that*

$$\mathcal{E}_{\sigma,\lambda;K}(\rho_\sigma, \theta_\sigma, L_\sigma) \leq \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K} + \frac{\bar{g}^2(\lambda)}{64\pi |\log \sigma|}. \tag{5.40}$$

$$\lim_{\sigma \rightarrow 0} \rho_\sigma = \frac{\bar{g}(\lambda)}{16\pi}, \quad \lim_{\sigma \rightarrow 0} |\theta_\sigma| = \theta_0^+, \tag{5.41}$$

and there exists a universal constant $C > 0$ such that

$$\frac{1}{C} \frac{|\log \sigma|}{\sigma} \leq L_\sigma \leq C \frac{|\log \sigma|}{\sigma}. \tag{5.42}$$

Remark 5.5. We point out that it is possible to show the rates

$$\left(\rho_\sigma - \frac{\bar{g}(\lambda)}{16\pi}\right)^2 \leq \frac{C}{|\log \sigma|}, \quad \left||\theta_\sigma| - \theta_0^+\right|^4 + |\lambda - \lambda_c| \left||\theta_\sigma| - \theta_0^+\right|^2 \leq \frac{C}{|\log \sigma|} \tag{5.43}$$

for some $C > 0$ universal in the setting of the second part of Proposition 5.4. However, as we do not attempt to capture the sharp rates and as the proof is a somewhat lengthy calculus exercise, we will not reproduce it here.

Finally, we present two corollaries to these bounds. The first simply translates the minimal energy for $\mathcal{E}_{\sigma,\lambda;K}$ into an upper bound for the minimal value of $E_{\sigma,\lambda}$.

Corollary 5.6. *There exist universal constants $\sigma_0 > 0$ and $C > 0$ such that for all $\sigma \in (0, \sigma_0)$ and $\lambda \in [0, 1]$ we have*

$$\frac{|\log \sigma|}{\sigma^2} \left(\inf_{\mathcal{A}} E_{\sigma,\lambda} - 8\pi\right) \leq \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K^*} + C\sigma^{\frac{1}{4}} |\log \sigma|. \tag{5.44}$$

While Corollary 5.6 is concerned with asymptotically precise minimization, the existence of minimizers relies on an upper bound by 8π for general $\sigma > 0$ (see also [64]).

Lemma 5.7. *For $\sigma > 0$ and $\lambda \in [0, 1]$ we have*

$$\inf_{\mathcal{A}} E_{\sigma,\lambda} < 8\pi. \tag{5.45}$$

Proof of Lemma 5.3. The computations for Φ_L can be found in Lemma A.6, so that we only have to translate them to $\phi_{\rho,\theta,L}$ here. By assumption, we have $L \geq \frac{1}{2\sigma}$, so that we can indeed apply Lemma A.6 for $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough.

Scale invariance of the Dirichlet energy allows to translate the bounds (A.51) and (A.56) into the bounds (5.28) and (5.29), as well as to obtain the bound

$$\int_{\mathbb{R}^2} |\nabla \phi_{\rho,\sigma,L}|^2 dx < 16\pi. \tag{5.46}$$

The fact that $\mathcal{N}(\phi_{\rho,\sigma,L}) = 1$ follows from $\mathcal{N}(\Phi_L) = 1$ and scale and rotation invariance of \mathcal{N} .

The bound (5.30) follows directly from the estimate (A.52) via rescaling. Similarly, we get the bound (5.31) from the bound (A.57), as well as $\phi_{\rho,\theta,L} + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ from $\Phi_L + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ on account of $\Phi_L \in \mathcal{A}$. Together with $\mathcal{N}(\phi_{\rho,\theta,L}) = 1$ and the estimate (5.46) we therefore obtain $\phi_{\rho,\theta,L} \in \mathcal{A}$ for all $\sigma \in (0, \sigma_0)$ with σ_0 small enough universal.

For $x = (x_1, x_2) \in \mathbb{R}^2$ we define $x^\perp := (-x_2, x_1)$. The fact that $(\Phi'_L)^\perp \cdot \nabla \Phi_{L,3} = 0$ everywhere and the identity (A.53) allow us to calculate the DMI term to be

$$\begin{aligned} \int_{\mathbb{R}^2} 2\phi'_{\rho,\theta,L} \cdot \nabla \phi_{\rho,\theta,L,3} dx &= 2\rho \cos \theta \int_{\mathbb{R}^2} \Phi'_L \cdot \nabla \Phi_{L,3} dx \\ &= 8\pi\rho \cos \theta + O\left(\rho L^{-\frac{1}{2}}\right), \end{aligned} \tag{5.47}$$

taking care of estimate (5.33). The same argument using the identity (A.50) instead of the identity (A.53) gives the identity (5.32).

Due to $\nabla \cdot \phi'_{\rho,\theta,L} = \cos\theta \nabla \cdot \phi'_{\rho,0,L}$ and equation (A.54), the contribution of the volume charges is

$$F_{\text{vol}}(\phi'_{\rho,\theta,L}) = \rho F_{\text{vol}}(\Phi'_L) \cos^2\theta = \frac{3\pi^3}{8} \rho \cos^2\theta + O\left(\rho L^{-\frac{1}{4}}\right). \quad (5.48)$$

As a result of the identity (A.55), the surface charges are simply given by

$$F_{\text{surf}}(\phi_{\rho,\theta,L,3}) = \rho F_{\text{surf}}(\Phi_{L,3}) = \frac{\pi^3}{8} \rho + O\left(\rho L^{-\frac{1}{2}}\right). \quad (5.49)$$

Combined, these two estimates give the identity (5.34).

Taking everything together and recalling (5.23), we obtain

$$\begin{aligned} E_{\sigma,\lambda}(\phi_{\rho,\theta,L}) &= 8\pi + \frac{4\pi}{L^2} + 4\pi\sigma^2\rho^2 \log\left(\frac{4L^2}{e^{2(1+\gamma)}}\right) - \sigma^2\rho g(\lambda, \theta) \\ &+ O\left(\frac{\log^2 L}{L^3}\right) + O\left(\frac{\sigma^2\rho^2 \log^2 L}{L}\right) + O\left(\lambda\sigma^2\rho L^{-\frac{1}{2}}\right) \\ &+ O\left((1-\lambda)\sigma^2\rho L^{-\frac{1}{4}}\right), \end{aligned} \quad (5.50)$$

which for a given $(\rho, \theta, L) \in V_\sigma$ and $\rho \leq 1$ translates into the estimate (5.27). \square

Proof of Proposition 5.4. *Step 1: Minimization in θ .*

We define $\Delta(\lambda, \theta) := \bar{g}(\lambda) - g(\lambda, \theta)$ and for $\lambda < \lambda_c = \frac{3\pi^2}{32+3\pi^2}$ calculate

$$\Delta(\lambda, \theta) = \frac{3\pi^3}{8}(1-\lambda) \left(\cos\theta - \frac{32\lambda}{3\pi^2(1-\lambda)} \right)^2. \quad (5.51)$$

For $\lambda \geq \lambda_c$ we instead have

$$\Delta(\lambda, \theta) = \frac{3\pi^3}{4} \left(\frac{\lambda}{\lambda_c} - 1 \right) (1 - \cos\theta) + \frac{3\pi^3}{8}(1-\lambda)(1 - \cos\theta)^2. \quad (5.52)$$

By inspection we have $\Delta(\lambda, \theta) \geq 0$ for all $\lambda \in [0, 1]$ and $\theta \in [-\pi, \pi)$, and $\Delta(\lambda, \theta) = 0$ if and only if $\theta = \theta_0^\pm$, where θ_0^\pm is given by (2.29). In particular, we get the identity (5.39) and the fact that $-\bar{g}(\lambda)$ is the minimal value of $-g(\lambda, \theta)$ which is achieved at the two minima $\theta = \theta_0^\pm$. Therefore, we have for all $\sigma \in (0, \sigma_0)$, $\lambda \in [0, 1]$, $\rho > 0$, and $L > \frac{1}{4\sigma\sqrt{\pi}}$ that

$$\mathcal{E}_{\sigma,\lambda;K}(\rho, \theta, L) = |\log\sigma|(\sigma L)^{-2} + \frac{4\pi \log(KL^2)}{|\log\sigma|} \rho^2 - \bar{g}(\lambda)\rho + \Delta(\lambda, \theta)\rho. \quad (5.53)$$

Step 2: Minimization in ρ .

We observe that

$$\frac{4\pi \log(KL^2)}{|\log \sigma|} \left(\rho - \frac{\bar{g}(\lambda)|\log \sigma|}{8\pi \log(KL^2)} \right)^2 = \frac{4\pi \log(KL^2)}{|\log \sigma|} \rho^2 - \bar{g}(\lambda)\rho + \frac{\bar{g}^2(\lambda)|\log \sigma|}{16\pi \log(KL^2)}. \tag{5.54}$$

Consequently, the quantity

$$\rho_0(L) := \frac{\bar{g}(\lambda)|\log \sigma|}{8\pi \log(KL^2)} \tag{5.55}$$

minimizes the map $\rho \mapsto \mathcal{E}_{\sigma,\lambda;K}(\rho, \theta_0, L)$, and we have the identity

$$\begin{aligned} \mathcal{E}_{\sigma,\lambda;K}(\rho, \theta, L) &= |\log \sigma| (\sigma L)^{-2} - \frac{\bar{g}^2(\lambda)|\log \sigma|}{16\pi \log(KL^2)} \\ &\quad + \Delta(\lambda, \theta)\rho + 4\pi \frac{\log(KL^2)}{|\log \sigma|} \left(\rho - \frac{\bar{g}(\lambda)|\log \sigma|}{8\pi \log(KL^2)} \right)^2. \end{aligned} \tag{5.56}$$

Step 3: Minimization in L.

We make the substitution $t = K^{-1}L^{-2}$ and minimize

$$f(t) := K \frac{|\log \sigma|}{\sigma^2} t + \frac{\bar{g}^2(\lambda)|\log \sigma|}{16\pi \log t} \tag{5.57}$$

in $0 < t \leq \frac{16\pi}{K}\sigma^2 < 1$ for $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough universal, in view of the assumption on K . Since $f(t) \nearrow 0$ as $t \searrow 0$, the function attains its minimum over this interval. We also observe that $\lim_{\sigma \rightarrow 0} f\left(\frac{16\pi}{K}\sigma^2\right) = \infty$, so that the minimum is achieved for $0 < t < \frac{16\pi}{K}\sigma^2 < 1$ for $\sigma_0 > 0$ small enough.

We calculate

$$f'(t) = K \frac{|\log \sigma|}{\sigma^2} - \frac{\bar{g}^2(\lambda)|\log \sigma|}{16\pi t \log^2 t} \tag{5.58}$$

and note that $0 < t_0 < \frac{16\pi}{K}\sigma^2 < 1$ solves $f'(t_0) = 0$ if and only if

$$-t_0^{\frac{1}{2}} \log \left(t_0^{\frac{1}{2}} \right) = \frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \tag{5.59}$$

for σ_0 small enough. In turn, for $s_0 := \log \left(t_0^{\frac{1}{2}} \right)$ this equation is equivalent to

$$s_0 e^{s_0} = -\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}}. \tag{5.60}$$

Solutions to this equation only exist provided $\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \leq e^{-1}$, which is the case for $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough. Under this condition there are precisely two solutions given by

$$s_{0,i} = W_i \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right) \tag{5.61}$$

for $i = -1$ and $i = 0$, where W_i are the two real-valued branches of the Lambert W -function, see CORLESS et al. [22]. In terms of t_0 , these are

$$t_{0,i} := \exp\left(2W_i\left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right)\right). \tag{5.62}$$

As W_0 is smooth at 0, with $W_0(0) = 0$, we have

$$t_{0,0} = 1 + O(\sigma) > \frac{16\pi}{K}\sigma^2 \tag{5.63}$$

for σ_0 sufficiently small, so that this solution is irrelevant to us. The point $t_{0,-1}$ being the only other critical point of f , we get that it indeed is the minimizer over $0 < t < \frac{16\pi}{K}\sigma^2$ for $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough. Consequently, the minimum is taken at $t_0 := t_{0,-1}$, and exploiting the identity $f'(t_0)t_0 = 0$ the minimal value can be seen to be

$$\begin{aligned} \min_{V_\sigma} \mathcal{E}_{\sigma,\theta;K} &= f(t_0) \\ &= \frac{\bar{g}^2(\lambda)|\log \sigma|}{64\pi} \left(W_{-1}^{-2}\left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right) + 2W_{-1}^{-1}\left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right) \right). \end{aligned} \tag{5.64}$$

In order to determine the behavior of the minimal energy as $\sigma \rightarrow 0$, for $-e^{-1} < s < 0$ with $|s| \ll 1$ we refer to the expansion

$$W_{-1}(s) = \log(-s) - \log|\log(-s)| + O\left(\frac{\log|\log(-s)|}{|\log(-s)|}\right), \tag{5.65}$$

see CORLESS et al. [22, equation (4.19), as well as the discussion following equation (4.20)]. Combined with $\log\left(\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right) = \log \sigma + \log\left(\frac{\bar{g}(\lambda)}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right)$ and (2.28), this gives

$$\begin{aligned} W_{-1}\left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right) &= \log \sigma + \log\left(\frac{\bar{g}(\lambda)}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right) - \log|\log \sigma| \\ &\quad - \log\left(1 + \frac{\log\left(\frac{\bar{g}(\lambda)}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right)}{\log \sigma}\right) \\ &\quad + O\left(\frac{\log|\log \sigma|}{|\log \sigma|}\right) \\ &= -|\log \sigma| - \log|\log \sigma| + \log\left(\frac{\bar{g}(\lambda)}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}}\right) \\ &\quad + O\left(\frac{\log|\log \sigma|}{|\log \sigma|}\right) \end{aligned} \tag{5.66}$$

for σ_0 sufficiently small. With $\frac{1}{1+r} = 1 - r + O(r^2)$ for $r \ll 1$ we obtain

$$W_{-1}^{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}} \right) = -\frac{1}{|\log \sigma|} + \frac{\log |\log \sigma|}{|\log \sigma|^2} - \frac{\log \left(\frac{\bar{g}(\lambda)}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}} \right)}{|\log \sigma|^2} + O \left(\frac{\log^2 |\log \sigma|}{|\log \sigma|^3} \right), \tag{5.67}$$

$$W_{-1}^{-2} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}} \right) = \frac{1}{|\log \sigma|^2} + O \left(\frac{\log |\log \sigma|}{|\log \sigma|^3} \right). \tag{5.68}$$

As a result, we have

$$\min_{V_\sigma} \mathcal{E}_{\sigma,\theta;K} = -\frac{\bar{g}^2(\lambda)}{32\pi} + \frac{\bar{g}^2(\lambda)}{32\pi} \frac{\log |\log \sigma|}{|\log \sigma|} - \frac{\bar{g}^2(\lambda)}{64\pi} \frac{\log \left(\frac{\bar{g}^2(\lambda)}{64\pi e K} \right)}{|\log \sigma|} + O \left(\frac{\log^2 |\log \sigma|}{|\log \sigma|^2} \right). \tag{5.69}$$

Recalling the relation $t = K^{-1}L^{-2}$, the definition (5.62), and using the identity $W_{-1}(s)e^{W_{-1}(s)} = s$ for $s \in (-e^{-1}, 0)$, we also get that the optimal truncation scale is

$$L_0 := K^{-\frac{1}{2}}t_0^{-\frac{1}{2}} = -W_{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}}K^{\frac{1}{2}}} \right) \frac{8\pi^{\frac{1}{2}}}{\bar{g}(\lambda)\sigma} = \frac{8\pi^{\frac{1}{2}}}{\bar{g}(\lambda)} \frac{|\log \sigma|}{\sigma} + O \left(\frac{\log |\log \sigma|}{\sigma} \right). \tag{5.70}$$

Finally, recalling the definition (5.55) the optimal skyrmion radius is given by

$$\rho_0(L_0) = \frac{\bar{g}(\lambda)}{16\pi} + O \left(\frac{\log |\log \sigma|}{|\log \sigma|} \right). \tag{5.71}$$

This concludes the proof of the first part of Proposition 5.4.

Step 4: Proof of stability for L_σ .

Let now $(\rho_\sigma, \theta_\sigma, L_\sigma) \in V_\sigma$ be such that (5.40) holds. Let $t_\sigma := K^{-1}L_\sigma^{-2}$ and note that by (5.24) we have $t_\sigma \in (0, \frac{16\pi}{K}\sigma^2)$, so that $f(t_\sigma)$ is defined. The fact that $\min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K} = f(t_0)$, the representation (5.56), and the assumption (5.40) imply that

$$f(t_\sigma) - f(t_0) \leq \mathcal{E}_{\sigma,\lambda;K}(\rho_\sigma, \theta_\sigma, L_\sigma) - \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K} \leq \frac{\bar{g}^2(\lambda)}{64\pi |\log \sigma|}. \tag{5.72}$$

For $s_\sigma := \frac{t_\sigma}{t_0}$ and $\tilde{f}_\sigma(s) := \frac{64\pi |\log \sigma|}{\bar{g}^2(\lambda)}(f(st_0) - f(t_0))$ with $s \in (0, \frac{16\pi\sigma^2}{Kt_0})$ this translates to

$$\tilde{f}_\sigma(s_\sigma) \leq 1. \tag{5.73}$$

In order to explicitly compute \tilde{f}_σ , we use the definitions (5.57) and (5.62) together with the fact that for all $0 < \tilde{s} < e^{-1}$ we have $W_{-1}(-\tilde{s})e^{W_{-1}(-\tilde{s})} = -\tilde{s}$, obtaining

$$\begin{aligned} \tilde{f}_\sigma(s) &= \frac{64\pi K t_0 |\log \sigma|^2}{\bar{g}^2(\lambda)\sigma^2} (s - 1) + 4|\log \sigma|^2 \left(\frac{1}{\log(st_0)} - \frac{1}{\log t_0} \right) \\ &= \frac{|\log \sigma|^2}{W_{-1}^2 \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right)} (s - 1) \\ &\quad - \frac{2|\log \sigma|^2}{\left(2W_{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right) + \log s \right) W_{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right)} \log s \\ &= \frac{|\log \sigma|^2}{W_{-1}^2 \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right)} \left(s - 1 - \frac{2 \left| W_{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right) \right| \log s}{2 \left| W_{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right) \right| - \log s} \right). \end{aligned} \tag{5.74}$$

Under the assumption $s \in (0, 1)$, this expression can be estimated as

$$\tilde{f}_\sigma(s) \geq \frac{|\log \sigma|^2}{W_{-1}^2 \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right)} \left(-1 + \frac{1}{2} \min \left\{ 2 \left| W_{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right) \right|, |\log(s)| \right\} \right). \tag{5.75}$$

Together with (5.65), (2.28) and $K \in [\frac{K^*}{2}, 2K^*]$ we get for $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough and some universal constant $C > 1$ that for $s \in (0, \frac{1}{C})$ we have $\tilde{f}_\sigma(s) > 1$. As a result of (5.73), we therefore get $s_\sigma \geq \frac{1}{C}$.

To handle the denominator in the second term on the right hand side of (5.74) in the case $s \in [1, \frac{16\pi\sigma^2}{Kt_0}]$, note that for all such s we have $s \leq C|\log \sigma|^2$ for $C > 0$ universal by (5.70), (2.28) and $K \in [\frac{K^*}{2}, 2K^*]$. Again using (2.28) and $K \in [\frac{K^*}{2}, 2K^*]$, we therefore get for $\sigma \in (0, \sigma_0)$ with σ_0 sufficiently small that

$$\log s \leq \left| W_{-1} \left(-\frac{\bar{g}(\lambda)\sigma}{8\pi^{\frac{1}{2}} K^{\frac{1}{2}}} \right) \right|. \tag{5.76}$$

Thus, together with (5.65), (2.28) and $K \in [\frac{K^*}{2}, 2K^*]$ we deduce

$$\tilde{f}_\sigma(s) \geq \frac{1}{C} (s - 1 - 2 \log s) \tag{5.77}$$

for $s \in [1, \frac{16\pi\sigma^2}{Kt_0}]$, $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ sufficiently small and $C > 0$ universal. By the assumption (5.73), we thus get $s_\sigma \leq C$, and in total $\frac{1}{C} \leq s_\sigma \leq C$ for some $C > 0$ universal and $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough.

Finally, we can translate the estimate for s_σ back to L_σ using the relations $s_\sigma = \frac{t_\sigma}{t_0}$, $t_\sigma = K^{-1}L_\sigma^{-2}$ and $t_0 = K^{-1}L_0^{-2}$. Therefore, with the help of equation (5.70) and (2.28), we obtain the desired estimate in (5.42).

Step 5: Proof of stability for ρ_σ .

Using $\mathcal{E}_{\sigma,\lambda;K}(\rho_\sigma, \theta_\sigma, L_\sigma) \geq \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K} = \min_{(0, \frac{16\pi}{K}\sigma^2]} f$, the fact that $\Delta(\lambda, \theta_\sigma) \geq 0$, $L_\sigma > \frac{1}{4\sigma\sqrt{\pi}}$ and $K \in [\frac{K^*}{2}, 2K^*]$ in the identity (5.56) we obtain

$$\left(\rho_\sigma - \frac{|\log \sigma|}{\log(K^{\frac{1}{2}}L_\sigma)} \frac{\bar{g}(\lambda)}{16\pi} \right)^2 \leq \frac{C}{|\log \sigma|}. \tag{5.78}$$

Again, using $L_\sigma > \frac{1}{4\sigma\sqrt{\pi}}$ together with $\sigma \in (0, \sigma_0)$ for $\sigma_0 > 0$ sufficiently small and $K \in [\frac{K^*}{2}, 2K^*]$ gains us

$$\left(\frac{|\log \sigma|}{\log(K^{\frac{1}{2}}L_\sigma)} - 1 \right)^2 = \frac{\log^2(K^{\frac{1}{2}}\sigma L_\sigma)}{\log^2(K^{\frac{1}{2}}L_\sigma)} \leq C \frac{\log^2(\sigma L_\sigma) + 1}{\log^2 \sigma}, \tag{5.79}$$

which by the estimate (5.42) can be upgraded to

$$\left| \frac{|\log \sigma|}{\log(K^{\frac{1}{2}}L_\sigma)} - 1 \right| \leq C \frac{\log |\log \sigma|}{\log \sigma} \leq \frac{C}{|\log \sigma|^{1/2}} \tag{5.80}$$

for $\sigma \in (0, \sigma_0)$ and $\sigma_0 > 0$ small enough. In particular, we conclude that

$$\begin{aligned} \left| \rho_\sigma - \frac{\bar{g}(\lambda)}{16\pi} \right| &\leq \left| \rho_\sigma - \frac{|\log \sigma|}{\log(K^{\frac{1}{2}}L_\sigma)} \frac{\bar{g}(\lambda)}{16\pi} \right| + \frac{\bar{g}(\lambda)}{16\pi} \left| \frac{|\log \sigma|}{\log(K^{\frac{1}{2}}L_\sigma)} - 1 \right| \\ &\leq \frac{C}{|\log \sigma|^{1/2}}, \end{aligned} \tag{5.81}$$

which gives the first limit in (5.41).

Step 6: Proof of stability for θ_σ .

Turning towards proving an estimate for θ_σ , as for estimate (5.78) we similarly get from the representation (5.56), the estimate (5.81) and (2.28) that

$$\Delta(\lambda, |\theta_\sigma|) \leq \frac{C}{|\log \sigma|}. \tag{5.82}$$

Therefore, from estimate (5.82) and the form of the function $\Delta(\lambda, \theta)$ computed in equations (5.51) and (5.52) together with the facts that $\cos \theta_0^+ = \frac{32}{3\pi^2} \frac{\lambda}{1-\lambda}$ in the case $\lambda < \lambda_c$ and $\frac{1}{2}(1 - \cos \theta_\sigma)^2 \leq (1 - \cos \theta_\sigma)$, with $\theta_0^+ = 0$, in the case $\lambda \geq \lambda_c$, we obtain

$$\lim_{\sigma \rightarrow 0} \cos |\theta_\sigma| = \cos \theta_0^+. \tag{5.83}$$

As $z \mapsto \arccos z$ is a continuous function from $[-1, 1]$ to $[0, \pi]$, we obtain the second limit in (5.41), which concludes the proof. \square

Proof of Corollary 5.6. For $\sigma \in (0, \sigma_0)$ with $\sigma_0 > 0$ small enough, we use $\rho = \frac{\rho_0}{|\log \sigma|}$, $\theta = \theta_0^+$ and $L = 2\pi^{\frac{1}{2}}L_0$ in Lemma 5.3, where ρ_0, θ_0^+ and L_0 are from the first part of Proposition 5.4. \square

Proof of Lemma 5.7. As the proof of Proposition 5.4 already contains the full details of the minimization, and as this proof closely follows that of [64, Lemma 3.1], we only provide a sketch here. The main step is to find truncations of suitable Belavin–Polyakov profiles such that the sum of the DMI and stray field terms are negative. In the case $\lambda = 1$, we choose $\phi_{\rho,0,L}$, which by equation (5.33) has negative DMI contribution for sufficiently large L depending on ρ . In the case $\lambda < 1$ we choose $\phi_{\rho,\frac{\pi}{2},L}$, as for this function the DMI and volume charge contributions vanish and only the stray field terms contribute a negative term. First minimizing in ρ and then taking L large enough gives the desired statement. \square

5.3. Existence of Minimizers via the Concentration Compactness Principle

We are now in a position to prove existence of minimizers.

Proof of Theorem 2.1. Throughout the proof $C(\sigma, \lambda)$ denotes a generic constant depending on σ and λ that may change from estimate to estimate.

By definition (2.17), there exist $m_n \in \mathcal{D}$ for $n \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} E_{\sigma,\lambda}(m_n) = \inf_{m \in \mathcal{A}} E_{\sigma,\lambda}(m). \tag{5.84}$$

Consider the Borel measures

$$\mu_n(A) := \int_A \left(|\nabla m_n|^2 + |m_n + e_3|^2 \right) dx \tag{5.85}$$

for all Borel sets $A \subset \mathbb{R}^2$. By estimate (5.3), Lemma 5.7 and the assumption $0 < \sigma^2(1 + \lambda)^2 \leq 2$ we have

$$\int_{\mathbb{R}^2} |\nabla m_n|^2 dx < 16\pi \tag{5.86}$$

for n large enough. Hence Lemma A.3, Lemma 5.1 and the estimate (5.4) imply

$$8\pi \leq \mu_n(\mathbb{R}^2) \leq C(\sigma, \lambda). \tag{5.87}$$

Consequently, we may apply the concentration compactness principle [59], see also [78, Section 4.3], to see that the limiting behavior of the sequence μ_n , up to taking subsequences, falls into three alternatives of vanishing, splitting, and compactness.

Case 1: Vanishing.

Here, we have for all $R > 0$ that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^2} \mu_n(B_R(x)) = 0. \tag{5.88}$$

In this setting, the proof of [64, Lemma 4.2] establishes that $m_n + e_3 \rightarrow 0$ in $L^4(\mathbb{R}^2; \mathbb{R}^3)$. The identity $|m_n(x) + e_3|^2 = 2(1 + m_{3,n}(x))$ implies that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} (1 + m_{n,3})^2 \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{1}{4} |m_n + e_3|^4 \, dx = 0. \tag{5.89}$$

Integrating by parts and applying Cauchy–Schwarz inequality, we thus see that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} m'_n \cdot \nabla m_{n,3} \, dx = - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} (m_{n,3} + 1) \nabla \cdot m'_n \, dx = 0. \tag{5.90}$$

Similarly, the interpolation inequality (3.7) implies $F_{\text{surf}}(m_{3,n} + 1) \leq \frac{1}{2} \|m_{3,n} + 1\|_2 \|\nabla m_{3,n}\|_2$, which, combined with the convergence (5.89), yields

$$\lim_{n \rightarrow \infty} F_{\text{surf}}(m_{3,n}) = 0. \tag{5.91}$$

Together with (3.6), the topological bound (A.24) and Lemma 5.7, this then yields a contradiction:

$$\liminf_{n \rightarrow \infty} E_{\sigma,\lambda}(m_n) \geq 8\pi > \inf_{m \in \mathcal{A}} E_{\sigma,\lambda}(m) = \lim_{n \rightarrow \infty} E_{\sigma,\lambda}(m_n). \tag{5.92}$$

This rules out the case of vanishing.

Case 2: Splitting.

In this case, there exists $0 < \eta < 1$ with the following property: For all $\varepsilon > 0$, after a suitable translation depending on ε , there exists $R > 0$ such that for all $\tilde{R} > R$ we have

$$\limsup_{n \rightarrow \infty} \left(\left| \mu_n(B_R(0)) - \eta \mu_n(\mathbb{R}^2) \right| + \left| \mu_n(B_{\tilde{R}}^c(0)) - (1 - \eta) \mu_n(\mathbb{R}^2) \right| \right) \leq \varepsilon. \tag{5.93}$$

Without loss of generality, we may assume $R \geq 1$. Let $\tilde{R} > 32R$. Then the proof of [28, Lemma 8] establishes the existence of $R_n \in (R, 2R)$, $\tilde{R}_n \in \left(\frac{\tilde{R}}{4}, \frac{\tilde{R}}{2}\right)$ and smooth $m_n^{(1)}, m_n^{(2)} : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ such that

$$m_n^{(1)}(x) = m_n(x) \quad \text{for } x \in B_{R_n}(0), \tag{5.94}$$

$$m_n^{(1)}(x) = -e_3 \quad \text{for } x \in B_{2R_n}^c(0), \tag{5.95}$$

$$m_n^{(2)}(x) = m_n(x) \quad \text{for } x \in B_{2\tilde{R}_n}^c(0), \tag{5.96}$$

$$m_n^{(2)}(x) = -e_3 \quad \text{for } x \in B_{\tilde{R}_n}(0) \tag{5.97}$$

and

$$\int_{B_{R_n}^c(0)} \left(\left| \nabla m_n^{(1)} \right|^2 + \left| m_n^{(1)} + e_3 \right|^2 \right) \, dx \leq C(\sigma, \lambda) \varepsilon, \tag{5.98}$$

$$\int_{B_{2\tilde{R}_n}(0)} \left(\left| \nabla m_n^{(2)} \right|^2 + \left| m_n^{(2)} + e_3 \right|^2 \right) \, dx \leq C(\sigma, \lambda) \varepsilon. \tag{5.99}$$

By the pointwise almost everywhere estimate $|m \cdot (\partial_1 m \times \partial_2 m)| \leq C|\nabla m|^2$ and the estimates (5.93), (5.98) and (5.99) we get that $|\mathcal{N}(m_n) - \mathcal{N}(m_n^{(1)}) - \mathcal{N}(m_n^{(2)})| \leq C(\sigma, \lambda)\varepsilon$. Discreteness of the degree then implies for $\varepsilon > 0$ small enough that

$$1 = \mathcal{N}(m_n) = \mathcal{N}(m_n^{(1)}) + \mathcal{N}(m_n^{(2)}). \tag{5.100}$$

Next, combining the estimates (5.98), (5.99) and (5.86), gives

$$\int_{\mathbb{R}^2} \left(|\nabla m_n^{(1)}|^2 + |\nabla m_n^{(2)}|^2 \right) dx \leq \int_{\mathbb{R}^2} |\nabla m_n|^2 dx + C(\sigma, \lambda)\varepsilon < 16\pi + C(\sigma, \lambda)\varepsilon. \tag{5.101}$$

Therefore, for $\varepsilon > 0$ small enough we get, applying the topological bound (A.24) along the way, that

$$8\pi \left(|\mathcal{N}(m_n^{(1)})| + |\mathcal{N}(m_n^{(2)})| \right) \leq \int_{\mathbb{R}^2} \left(|\nabla m_n^{(1)}|^2 + |\nabla m_n^{(2)}|^2 \right) dx < 24\pi. \tag{5.102}$$

Elementary combinatorics using the identity (5.100) consequently give

$$\mathcal{N}(m_n^{(1)}) = 1 \text{ and } \mathcal{N}(m_n^{(2)}) = 0 \tag{5.103}$$

or

$$\mathcal{N}(m_n^{(1)}) = 0 \text{ and } \mathcal{N}(m_n^{(2)}) = 1. \tag{5.104}$$

In the following we will only deal with the first case, as the other one can be handled similarly.

By the estimate (5.87) and the splitting alternative (5.93) for $\varepsilon > 0$ small and n large enough we obtain

$$4\pi(1 - \eta) \leq \frac{1 - \eta}{2} \mu_n(\mathbb{R}^2) \leq \int_{\mathbb{R}^2} \left(|\nabla m_n^{(2)}|^2 + |m_n^{(2)} + e_3|^2 \right) dx. \tag{5.105}$$

Lemma 5.1, along with the bound (5.101), further implies that

$$\begin{aligned} \int_{\mathbb{R}^2} \left(|\nabla m_n^{(2)}|^2 + |m_n^{(2)} + e_3|^2 \right) dx &\leq \int_{\mathbb{R}^2} |\nabla m_n^{(2)}|^2 dx \\ &+ C(\sigma, \lambda) \int_{\mathbb{R}^2} \left| (m_n^{(2)})' \right|^2 dx, \end{aligned} \tag{5.106}$$

as, by the topological lower bound we have $\int_{\mathbb{R}^2} |\nabla m_n^{(1)}|^2 dx \geq 8\pi$, we obtain from estimate (5.101) for $\varepsilon > 0$ small enough that

$$\int_{\mathbb{R}^2} |\nabla m_n^{(2)}|^2 dx < 16\pi. \tag{5.107}$$

Therefore we can apply Lemma 5.2 to get

$$\int_{\mathbb{R}^2} |\nabla m_n^{(2)}|^2 dx + \int_{\mathbb{R}^2} \left| (m_n^{(2)})' \right|^2 dx \leq C(\sigma, \lambda) E_{\sigma, \lambda}(m_n^{(2)}). \tag{5.108}$$

Consequently, concatenating the estimates (5.105), (5.106) and (5.108) we deduce that there exists $\delta > 0$ such that for all n large enough we have

$$E_{\sigma,\lambda} \left(m_n^{(2)} \right) \geq \delta > 0. \tag{5.109}$$

As a result, in order to rule out splitting we only have to prove that

$$\limsup_{n \rightarrow \infty} \left[E_{\sigma,\lambda} \left(m_n^{(1)} \right) + E_{\sigma,\lambda} \left(m_n^{(2)} \right) - E_{\sigma,\lambda} \left(m_n \right) \right] \leq g(\varepsilon, \tilde{R}) \tag{5.110}$$

for some function $g : (0, \infty)^2 \rightarrow (0, \infty)$ with $\lim_{\varepsilon \rightarrow 0} \lim_{\tilde{R} \rightarrow \infty} g(\varepsilon, \tilde{R}) = 0$. Indeed, assuming that the bound (5.110) holds, we can test the infimum $\inf_{\mathcal{A}} E_{\sigma,\lambda}$ with $m_n^{(1)}$ and use the estimates (5.109) and (5.110) to get

$$\begin{aligned} \inf_{m \in \mathcal{A}} E_{\sigma,\lambda}(m) &\leq \liminf_{n \rightarrow \infty} E_{\sigma,\lambda} \left(m_n^{(1)} \right) \\ &\leq \limsup_{n \rightarrow \infty} \left(E_{\sigma,\lambda} \left(m_n^{(1)} \right) + E_{\sigma,\lambda} \left(m_n^{(2)} \right) - \delta \right) \\ &\leq \lim_{n \rightarrow \infty} E_{\sigma,\lambda} \left(m_n \right) + g(\varepsilon, \tilde{R}) - \delta. \end{aligned} \tag{5.111}$$

Then, by first taking \tilde{R} big enough and then $\varepsilon > 0$ small enough we obtain a contradiction.

We now turn to proving the claim (5.110). The local terms are straightforward to handle using the Cauchy–Schwarz inequality, see for example the proof of Lemma 5.2, and give a contribution of $C(\sigma, \lambda)\varepsilon$ to $g(\varepsilon, \tilde{R})$. Of the nonlocal terms, we first deal with the volume charges by computing

$$\begin{aligned} F_{\text{vol}}(m'_n) &= F_{\text{vol}} \left(\left(m_n^{(1)} \right)' \right) + F_{\text{vol}} \left(\left(m_n^{(2)} \right)' \right) + F_{\text{vol}} \left(m'_n - \left(m_n^{(1)} \right)' - \left(m_n^{(2)} \right)' \right) \\ &\quad + 2F_{\text{vol}} \left(\left(m_n^{(1)} \right)' + \left(m_n^{(2)} \right)', m'_n - \left(m_n^{(1)} \right)' - \left(m_n^{(2)} \right)' \right) \\ &\quad + 2F_{\text{vol}} \left(\left(m_n^{(1)} \right)', \left(m_n^{(2)} \right)' \right). \end{aligned} \tag{5.112}$$

By (3.6), we may discard the term $F_{\text{vol}} \left(m'_n - m_n^{(1)'} - m_n^{(2)'} \right) \geq 0$ right away since we only claim an upper bound in estimate (5.110). The usual interpolation inequality (3.8) for $p = 2$ implies

$$\begin{aligned} &F_{\text{vol}} \left(\left(m_n^{(1)} \right)' + \left(m_n^{(2)} \right)', m'_n - \left(m_n^{(1)} \right)' - \left(m_n^{(2)} \right)' \right) \\ &\leq C \left(\left\| \nabla \left(m_n^{(1)} \right)' \right\|_2 + \left\| \nabla \left(m_n^{(2)} \right)' \right\|_2 \right) \left\| m'_n - \left(m_n^{(1)} \right)' - \left(m_n^{(2)} \right)' \right\|_2. \end{aligned} \tag{5.113}$$

Using (5.102) we obtain that the \dot{H}^1 -norms on the left-hand side are uniformly bounded. The identities (5.94) through (5.97) imply

$$\text{supp} \left(m'_n - \left(m_n^{(1)} \right)' - \left(m_n^{(2)} \right)' \right) \subset \overline{B}_{2\tilde{R}_n}(0) \setminus B_{R_n}(0), \tag{5.114}$$

and on $\overline{B}_{2\tilde{R}_n}(0) \setminus B_{R_n}(0)$ we can estimate each of the terms m'_n , $\left(m_n^{(1)} \right)'$, and $\left(m_n^{(2)} \right)'$ separately by virtue of (5.93), (5.98) and (5.99). Therefore (5.113) gives

$$F_{\text{vol}} \left(\left(m_n^{(1)} \right)' + \left(m_n^{(2)} \right)', m'_n - \left(m_n^{(1)} \right)' - \left(m_n^{(2)} \right)' \right) \leq C(\sigma, \lambda) \varepsilon^{\frac{1}{2}}. \tag{5.115}$$

To estimate the last term in (5.112), we would like to exploit that the sequences $\left(m_n^{(1)} \right)'$ and $\left(m_n^{(2)} \right)'$ have disjoint supports. To this end, we use the real space representation (3.10) and integrate by parts once in each of the two integrals:

$$\begin{aligned} & F_{\text{vol}} \left(\left(m_n^{(1)} \right)', \left(m_n^{(2)} \right)' \right) \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \left(m_n^{(1)}(x) \right)' \nabla \cdot \left(m_n^{(2)}(\tilde{x}) \right)'}{|x - \tilde{x}|} d\tilde{x} dx \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\frac{\left(m_n^{(1)}(x) \right)' \cdot \left(m_n^{(2)}(\tilde{x}) \right)'}{|x - \tilde{x}|^3} \right. \\ &\quad \left. - 3 \frac{\left(m_n^{(1)}(x) \right)' \cdot (\tilde{x} - x) \left(m_n^{(2)}(x) \right)' \cdot (\tilde{x} - x)}{|x - \tilde{x}|^5} \right) d\tilde{x} dx. \end{aligned} \tag{5.116}$$

To extract a quantitative estimate, note that we have

$$\inf \left\{ |x - \tilde{x}| : x \in \text{supp} \left(m_{n,3}^{(1)} + 1 \right), \tilde{x} \in \text{supp} \left(m_{n,3}^{(2)} + 1 \right) \right\} \geq \frac{\tilde{R}}{4} - 4R. \tag{5.117}$$

With $K_{\text{vol}}(z) := \chi \left(|z| \geq \frac{\tilde{R}}{4} - 4R \right) \left(\frac{1}{|z|^3} \text{id} - 3 \frac{z \otimes z}{|z|^5} \right)$ for $z \in \mathbb{R}^2$, Young's inequality for convolutions implies for $\tilde{R} > 32R$ that

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(m_n^{(1)}(x) \right)' \cdot K_{\text{vol}}(\tilde{x} - x) \left(m_n^{(2)}(\tilde{x}) \right)' d\tilde{x} dx \\ & \leq C \left\| \left(m_n^{(1)} \right)' \right\|_2 \left\| \left(m_n^{(2)} \right)' \right\|_2 \|K_{\text{vol}}\|_1 \\ & \leq C(\sigma, \lambda) \tilde{R}^{-1}. \end{aligned} \tag{5.118}$$

For the surface charges we similarly compute

$$\begin{aligned}
 F_{\text{surf}}(m_{n,3}) &= F_{\text{surf}}\left(m_{n,3}^{(1)}\right) + F_{\text{surf}}\left(m_{n,3}^{(2)}\right) \\
 &\quad + F_{\text{surf}}\left(m_{n,3} + m_{n,3}^{(1)} + m_{n,3}^{(2)}, m_{n,3} - m_{n,3}^{(1)} - m_{n,3}^{(2)}\right) \\
 &\quad + 2F_{\text{surf}}\left(m_{n,3}^{(1)}, m_{n,3}^{(2)}\right).
 \end{aligned} \tag{5.119}$$

As in the proof of estimate (5.115), the interpolation inequality (3.7) together with the fact that F_{surf} is invariant under the addition of constant functions gives

$$F_{\text{surf}}\left(m_{n,3} + m_{n,3}^{(1)} + m_{n,3}^{(2)}, m_{n,3} - m_{n,3}^{(1)} - m_{n,3}^{(2)}\right) \leq C(\sigma, \lambda)\varepsilon^{\frac{1}{2}}. \tag{5.120}$$

The real-space representation (3.2) allows us to write the last term in the identity (5.119) as

$$\begin{aligned}
 &F_{\text{surf}}\left(m_{n,3}^{(1)}, m_{n,3}^{(2)}\right) \\
 &= \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left(m_{n,3}^{(1)}(x) - m_{n,3}^{(1)}(\tilde{x})\right)\left(m_{n,3}^{(2)}(x) - m_{n,3}^{(2)}(\tilde{x})\right)}{|x - \tilde{x}|^3} d\tilde{x} dx.
 \end{aligned} \tag{5.121}$$

We may now exploit the fact that $m_{n,3}^{(1)} + 1$ and $m_{n,3}^{(2)} + 1$ have disjoint supports to get

$$F_{\text{surf}}\left(m_{n,3}^{(1)}, m_{n,3}^{(2)}\right) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left(m_{n,3}^{(1)}(x) + 1\right)\left(m_{n,3}^{(2)}(\tilde{x}) + 1\right)}{|x - \tilde{x}|^3} d\tilde{x} dx, \tag{5.122}$$

so that Young’s inequality for convolutions with $K_{\text{surf}}(z) := \frac{\chi\left(|z| \geq \frac{\tilde{R}}{4} - 4R\right)}{|z|^3}$ for $z \in \mathbb{R}^2$ implies

$$\mathcal{F}_{\text{surf}}\left(m_{n,3}^{(1)}, m_{n,3}^{(2)}\right) \leq C \left\|m_{n,3}^{(1)} + 1\right\|_2 \left\|m_{n,3}^{(2)} + 1\right\|_2 \|K_{\text{surf}}\|_1 \leq C(\sigma, \lambda)\tilde{R}^{-1}. \tag{5.123}$$

All together, we see that estimate (5.110) holds, with

$$g(\varepsilon, \tilde{R}) = C(\sigma, \lambda)\left(\varepsilon^{\frac{1}{2}} + \tilde{R}^{-1}\right), \tag{5.124}$$

which rules out splitting.

Case 3: Compactness

As vanishing and splitting have been ruled out, we obtain that after extraction of a subsequence and suitable translations, for every $\varepsilon > 0$ there exists $R > 0$ such that we have

$$\mu_n(B_R^c(0)) \leq \varepsilon \tag{5.125}$$

for all $n \in \mathbb{N}$.

By the Rellich–Kondrachov compactness theorem, [55, Theorem 8.9], there exists $m_\sigma : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ such that $m_n + e_3 \rightharpoonup m_\sigma + e_3$ in $L^2(B_{\tilde{R}}(0); \mathbb{R}^3)$ for all $\tilde{R} > 0$ and $m_n + e_3 \rightharpoonup m_\sigma + e_3$ in $H^1(\mathbb{R}^2; \mathbb{R}^3)$. We first argue that we even have $m_n \rightharpoonup m_\sigma$ in $L^2(\mathbb{R}^2; \mathbb{R}^3)$. Let $\varepsilon > 0$ and let $R > 0$ be such that the tightness estimate (5.125) holds. Then, by lower semi-continuity of the L^2 -norm and the Minkowski inequality, we have

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^2} |m_n - m_\sigma|^2 \, dx \leq 2\varepsilon + \limsup_{n \rightarrow \infty} \int_{B_R(0)} |m_n - m_\sigma|^2 \, dx = 2\varepsilon. \tag{5.126}$$

Therefore, we see $m_n + e_3 \rightharpoonup m_\sigma + e_3$ in $L^2(\mathbb{R}^2; \mathbb{R}^3)$, and in particular we have $m_\sigma + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$.

Next, we argue that

$$\liminf_{n \rightarrow \infty} (E_{\sigma,\lambda}(m_n) - 8\pi\mathcal{N}(m_n)) \geq E_{\sigma,\lambda}(m_\sigma) - 8\pi\mathcal{N}(m_\sigma). \tag{5.127}$$

By the identity (A.23), we obtain for any $n \in \mathbb{N}$ that

$$\begin{aligned} & E_{\sigma,\lambda}(m_n) - 8\pi\mathcal{N}(m_n) \\ &= \int_{\mathbb{R}^2} \left(|\partial_1 m_n + m_n \times \partial_2 m_n|^2 + \sigma^2 |m'_n|^2 - 2\sigma^2 \lambda m'_n \cdot \nabla m_{n,3} \right) \, dx \\ & \quad + \sigma^2 (1 - \lambda) (F_{\text{vol}}(m'_n) - F_{\text{surf}}(m_{n,3})). \end{aligned} \tag{5.128}$$

We have $\partial_1 m_n \rightharpoonup \partial_1 m_\sigma$ and $m_n \times \partial_2 m_n \rightharpoonup m_\sigma \times \partial_2 m_\sigma$ in $L^2(\mathbb{R}^2; \mathbb{R}^3)$, the latter by a weak-times-strong convergence argument. In the first term, we can thus use lower semi-continuity of the L^2 -norm. The anisotropy term converges strongly by our previous argument. By (5.126) and weak convergence of the gradients we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} m'_n \cdot \nabla m_{n,3} \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} m'_\sigma \cdot \nabla m_{n,3} \, dx = \int_{\mathbb{R}^2} m'_\sigma \cdot \nabla m_{\sigma,3} \, dx \tag{5.129}$$

so that also the DMI-term converges. Finally, we see $F_{\text{vol}}(m'_n) \rightarrow F_{\text{vol}}(m'_\sigma)$ and $F_{\text{surf}}(m_{3,n}) \rightarrow F_{\text{surf}}(m_{\sigma,3})$ as $n \rightarrow \infty$ by the interpolation inequalities of Lemma 3.1. Taking all of these things together, we see

$$\begin{aligned} & \liminf_{n \rightarrow \infty} (E_{\sigma,\lambda}(m_n) - 8\pi\mathcal{N}(m_n)) \\ & \geq \int_{\mathbb{R}^2} \left(|\partial_1 m_\sigma + m_\sigma \times \partial_2 m_\sigma|^2 + \sigma^2 |m'_\sigma|^2 - 2\sigma^2 \lambda (m'_\sigma \cdot \nabla) m_{\sigma,3} \right) \, dx \\ & \quad + \sigma^2 (1 - \lambda) (F_{\text{vol}}(m'_\sigma) - F_{\text{surf}}(m_{\sigma,3})) \\ & = E_{\sigma,\lambda}(m_\sigma) - 8\pi\mathcal{N}(m_\sigma), \end{aligned} \tag{5.130}$$

where in the last line we again use the identity (5.128).

Therefore, together with the upper bound of Lemma 5.7 and the observation that $E_{\sigma,\lambda}(m_\sigma) \geq 0$, which follows from the assumption of the theorem and Lemma 5.2, we have

$$0 > \liminf_{n \rightarrow \infty} (E_{\sigma,\lambda}(m_n) - 8\pi) \geq E_{\sigma,\lambda}(m_\sigma) - 8\pi \mathcal{N}(m_\sigma) > -8\pi \mathcal{N}(m_\sigma), \tag{5.131}$$

giving $\mathcal{N}(m_\sigma) > 0$. At the same time, by lower semi-continuity of the Dirichlet energy and the estimate (5.86) we furthermore have

$$\int_{\mathbb{R}^2} |\nabla m_\sigma|^2 dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla m_n|^2 dx < 16\pi. \tag{5.132}$$

Thus the topological bound (A.24) implies $\mathcal{N}(m) = 1$. As we have already shown $m_\sigma + e_3 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ above, we therefore have $m_\sigma \in \mathcal{A}$. Consequently, we have

$$\inf_{\tilde{m} \in \mathcal{A}} E_{\sigma,\lambda}(\tilde{m}) = \liminf_{n \rightarrow \infty} E_{\sigma,\lambda}(m_n) \geq E_{\sigma,\lambda}(m_\sigma) \geq \inf_{\tilde{m} \in \mathcal{A}} E_{\sigma,\lambda}(\tilde{m}), \tag{5.133}$$

which concludes the proof. \square

6. The Conformal Limit

In this section we prove Theorem 2.2. In the spirit of a Γ -convergence argument, we do so by providing ansatz-free lower bounds matching the upper bounds obtained in Corollary 5.6. As the Dirichlet term provides closeness to a Belavin–Polyakov profile $\phi = S\Phi(\rho^{-1}x)$ for $S \in \text{SO}(3)$ and $\rho > 0$ via Theorem 2.4, we have to capture the behavior of the lower order terms as the magnetization approaches ϕ .

Here the main difficulty is the fact that the limiting Belavin–Polyakov profile ϕ from Theorem 2.4 does not necessarily satisfy $\lim_{|x| \rightarrow \infty} \phi(x) = -e_3$, which is a more subtle issue than one might expect. The fundamental problem is that for $r > 0$ the embedding of $H^1(B_r(0))$ to $L^\infty(B_r(0))$ fails logarithmically, and we only have $H^1(B_r(0)) \hookrightarrow \text{BMO}(B_r(0))$ (as a simple result of the Poincaré inequality and the definition [18, (0.5)] of BMO), which in and of itself is not strong enough to control the value at infinity. Indeed, at this stage it is entirely possible that the minimizers exhibit a multi-scale structure: On the scale of the skyrmion radius the profile might approach a tilted Belavin–Polyakov profile, while only on a larger truncation scale decaying to $-e_3$, see for example [28, Step 2b in the proof of Lemma 8] for a construction. Of course such a profile would have a large anisotropy energy, which we exploit in the following Lemma 6.1. The idea is to replace the logarithmic failure of the embedding $H^1 \not\hookrightarrow L^\infty$ with the Moser–Trudinger inequality proved in Lemma 2.5.

Throughout this section we use the abbreviations

$$L := \left(\int_{\mathbb{R}^2} |\nabla m|^2 dx - 8\pi \right)^{-\frac{1}{2}}, \tag{6.1}$$

$$v := \lim_{|x| \rightarrow \infty} \phi(x) = -Se_3, \tag{6.2}$$

provided $\phi = S\Phi(\rho^{-1}(\bullet - x_0))$ for $S \in \text{SO}(3)$, $\rho > 0$ and $x_0 \in \mathbb{R}^2$. Note that this choice of L is consistent with its usage in $\mathcal{E}_{\sigma,\lambda;K^*}$, see definition (5.23), in view of estimate (5.28) and the substitution (5.22). In particular, it can also be thought of as the cut-off length relative to the skyrmion radius.

Lemma 6.1. *There exist universal constants $L_0 > 0$ and $C > 0$ such that for $m \in \mathcal{A}$ with $L \geq L_0$ and the distance-minimizing Belavin–Polyakov profile from Theorem 2.4 given by $\phi(x) = S\Phi(\rho^{-1}(x - x_0))$ with $S \in \text{SO}(3)$, $\rho > 0$ and $x_0 \in \mathbb{R}^2$ we have*

$$\int_{\mathbb{R}^2} |m'|^2 \, dx \geq C|v + e_3|^2 \rho^2 L^2. \tag{6.3}$$

In order to use this bound to rule out v being too far away from $-e_3$ we also need a first lower bound for the DMI and the stray field terms.

Lemma 6.2. *There exist universal constants $L_0 > 0$ and $C > 0$ such that for $m \in \mathcal{A}$ with $L \geq L_0$ and the distance-minimizing Belavin–Polyakov profile from Theorem 2.4 given by $\phi(x) = S\Phi(\rho^{-1}(x - x_0))$ with $S \in \text{SO}(3)$, $\rho > 0$ and $x_0 \in \mathbb{R}^2$ we have*

$$\begin{aligned} & -2\lambda \int_{\mathbb{R}^2} m' \cdot \nabla m_3 \, dx + (1 - \lambda) (F_{\text{vol}}(m') - F_{\text{surf}}(m_3)) \\ & \geq -\frac{1}{2} \int_{\mathbb{R}^2} |m'|^2 \, dx - C\rho (\log L)^{\frac{1}{2}} - CL^{-2}. \end{aligned} \tag{6.4}$$

Armed with these estimates, we obtain that v does indeed converge to e_3 as $\sigma \rightarrow 0$.

Lemma 6.3. *There exist universal constants $C > 0$ and $\sigma_0 > 0$ such that for $\sigma \in (0, \sigma_0)$ the following holds: Let m_σ be a minimizer of $E_{\sigma,\lambda}$ over \mathcal{A} . Let $\rho > 0$, $S \in \text{SO}(3)$ and $x_0 \in \mathbb{R}^2$ be such that $S\Phi(\rho^{-1}(x - x_0))$ is the distance-minimizing Belavin–Polyakov profile from Theorem 2.4 for $m = m_\sigma$. Then we have*

$$L^{-2} \leq 16\pi\sigma^2, \tag{6.5}$$

$$|v + e_3|^2 \leq C \frac{\log^2 L}{L^2}. \tag{6.6}$$

Now that we know that we essentially have pinning of the value at infinity, we turn to proving a more precise lower bound for the anisotropy energy which almost matches the expression (5.30) obtained in Lemma 5.3.

Lemma 6.4. *There exist universal constants $C > 0$ and $L_0 > 0$ such that for $\bar{L} \geq L_0$ the following holds: Let $m \in \mathcal{A}$ be such that there exist $\phi(x) := S\Phi(\rho^{-1}(x - x_0))$ for $x \in \mathbb{R}^2$ with $S \in \text{SO}(3)$, $\rho > 0$ and $x_0 \in \mathbb{R}^2$ satisfying*

$$\int_{\mathbb{R}^2} |\nabla(m - \phi)|^2 \, dx \leq \bar{L}^{-2}, \tag{6.7}$$

and

$$|Se_3 - e_3|^2 < \bar{L}^{-1}. \tag{6.8}$$

Then we have the estimate

$$\int_{\mathbb{R}^2} |m'|^2 dx \geq 4\pi\rho^2 \log(K^*\bar{L}^2) - C\rho^2\bar{L}^{-\frac{2}{3}}, \tag{6.9}$$

where K^* is the constant defined in Eq. (5.26).

We also have another look at the DMI and stray field terms in order to obtain sharper estimates matching those of Lemma 5.3. As therein the expressions depend on the rotation angle θ , we have to replace the profile ϕ obtained in Theorem 2.4 by a rotated one having the correct value at infinity.

Lemma 6.5. *There exist universal constants $C > 0$ and $\sigma_0 > 0$ such that for $\sigma \in (0, \sigma_0)$ and $\lambda \in [0, 1]$ the following holds: Let m_σ be a minimizer of $E_{\sigma,\lambda}$ over \mathcal{A} . Let $S \in \text{SO}(3)$, $\rho > 0$ and $x_0 \in \mathbb{R}^2$ be such that $S\Phi(\rho^{-1}(x - x_0))$ is the distance-minimizing Belavin–Polyakov profile from Theorem 2.4 for $m = m_\sigma$. Then there exists $\theta \in [-\pi, \pi)$ such that with the rotation S_θ of angle θ around the x_3 -axis defined in (2.24) we have*

$$\int_{\mathbb{R}^2} \left| \nabla \left(m_\sigma(x) - S_\theta \Phi(\rho^{-1}(x - x_0)) \right) \right|^2 dx \leq \frac{C \log^2 L}{L^2}, \tag{6.10}$$

$$\int_{\mathbb{R}^2} 2m'_\sigma \cdot \nabla m_{\sigma,3} dx \leq 8\pi\rho \cos \theta + C\sigma |\log \sigma|, \tag{6.11}$$

$$F_{\text{vol}}(m'_\sigma) - F_{\text{surf}}(m_{\sigma,3}) \geq \left(\frac{3\pi^3}{8} \cos^2 \theta - \frac{\pi^3}{8} \right) \rho - C\sigma^{\frac{1}{4}} |\log \sigma|^{-\frac{1}{2}}. \tag{6.12}$$

Furthermore, we have

$$\rho^2 \leq \frac{C}{|\log \sigma|}. \tag{6.13}$$

Proof of Lemma 6.1. The known scaling properties of the L^2 - and \dot{H}^1 -norms allow us, without loss of generality, to set $\rho = 1$. Additionally, we may suppose $x_0 = 0$ by translation invariance. By Lemma 5.1 we notice that

$$\int_{\mathbb{R}^2} |m'|^2 dx \geq \frac{1}{C} \int_{\mathbb{R}^2} |m + e_3|^2 dx \tag{6.14}$$

for some universal constant $C > 0$. It is thus sufficient to estimate the right-hand side from below.

For $R > 0$, by the inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$ for $a, b \in \mathbb{R}^3$ we have

$$\int_{B_R(0)} |m + e_3|^2 dx \geq \int_{B_R(0)} \frac{1}{2} |\phi + e_3|^2 dx - \int_{B_R(0)} |m - \phi|^2 dx. \tag{6.15}$$

As the in-plane components of Φ average to 0 on radially symmetric sets and the out-of-plane component is radial (recall the definition in (2.21)), the first term in the above estimate can be computed explicitly to give

$$\begin{aligned} \int_{B_R(0)} |\phi + e_3|^2 dx &= 2 \int_{B_R(0)} (1 + \phi \cdot e_3) dx \\ &= 2\pi R^2 \left(1 + e_3 \cdot S e_3 \int_{B_R(0)} \frac{1 - |x|^2}{1 + |x|^2} dx \right). \end{aligned} \tag{6.16}$$

In view of $\nu = -S e_3$, see definition (6.2), we have

$$\begin{aligned} \int_{B_R(0)} |\phi + e_3|^2 dx &= \pi R^2 |\nu + e_3|^2 - 2\nu \cdot e_3 \int_{B_R(0)} \frac{2}{|x|^2 + 1} dx \\ &= \pi R^2 |\nu + e_3|^2 - 4\pi \log(1 + R^2) \nu \cdot e_3 \\ &= \left(\pi R^2 - 2\pi \log(1 + R^2) \right) |\nu + e_3|^2 + 4\pi \log(1 + R^2). \end{aligned} \tag{6.17}$$

For $R \geq R_0$ with $R_0 > 0$ big enough we therefore have

$$\int_{B_R(0)} |\phi + e_3|^2 dx \geq \frac{\pi}{2} R^2 |\nu + e_3|^2 + 4\pi \log(R^2). \tag{6.18}$$

In order to control the second term on the right-hand side of estimate (6.15), we make use of the fact that $y(\log(y) - 1)$ for $y > 0$ is the Legendre transformation of the exponential map e^x , i.e., we have the sharp inequality $xy \leq e^x + y(\log(y) - 1)$ for $x \in \mathbb{R}$ and $y > 0$, see for example [12, Chapter 3.3 and Table 3.1]. For $\bar{L} := \|\nabla(m - \phi)\|_2^{-1}$ we consequently get

$$\begin{aligned} \int_{B_R(0)} |m - \phi|^2 dx &= \int_{B_R(0)} |m - \phi|^2 |\nabla\phi|^{-2} |\nabla\phi|^2 dx \\ &\leq \int_{B_R(0)} e^{\frac{2\pi}{3}\bar{L}^2 |m - \phi|^2} |\nabla\phi|^2 dx \\ &\quad + \int_{B_R(0)} \frac{3}{2\pi\bar{L}^2} \left[\log \left(\frac{3|\nabla\phi|^{-2}}{2\pi\bar{L}^2} \right) - 1 \right] dx. \end{aligned} \tag{6.19}$$

Applying Lemma 2.5, specifically our version of the Moser–Trudinger inequality (2.41), where Theorem 2.4 and Lemma A.2 ensure its applicability for $L \geq L_0$ with $L_0 > 0$ sufficiently big, we see that the first term on the right-hand side is universally bounded. For $R \geq R_0$ for $R_0 > 0$ big enough, there furthermore exists a universal constant $C' > 0$ such that $|\nabla\phi(x)| \geq \frac{1}{C'} |R|^{-2}$ for $x \in B_R(0)$. We can therefore also estimate the second term on the right-hand side to see

$$\int_{B_R(0)} |m - \phi|^2 dx \leq C \left[1 + \frac{R^2}{L^2} + \frac{R^2}{L^2} \log \left(\frac{R^4}{L^2} \right) \right]. \tag{6.20}$$

Theorem 2.4 allows us to write this in the form

$$\int_{B_R(0)} |m - \phi|^2 dx \leq C \left[1 + \frac{R^2}{L^2} + \frac{R^2}{L^2} \log \left(\frac{R^4}{L^2} \right) \right]. \tag{6.21}$$

Choosing $R = \eta L$ for a suitable $\eta > 0$ and requiring $L \geq L_0$ for some $L_0 > 0$ sufficiently big we combine the two bounds (6.18) and (6.21) with the one in (6.15) to obtain

$$\int_{B_R(0)} |m + e_3|^2 \geq C |v + e_3|^2 L^2 \tag{6.22}$$

for $C > 0$ universal. \square

Proof of Lemma 6.2. *Step 1: Estimate the DMI term.*

Without loss of generality, we may take $x_0 = 0$. Let $\phi_L(x) := S\Phi_L(\rho^{-1}x)$, where Φ_L was defined in equation (5.19). Estimate (5.29) and the fact that ϕ was chosen according to Theorem 2.4 give

$$\int_{\mathbb{R}^2} |\nabla(\phi_L - m)|^2 dx \leq 2 \int_{\mathbb{R}^2} |\nabla(\phi_L - \phi)|^2 dx + 2 \int_{\mathbb{R}^2} |\nabla(\phi - m)|^2 dx \leq CL^{-2}. \tag{6.23}$$

We calculate

$$\int_{\mathbb{R}^2} m' \cdot \nabla m_3 dx = \int_{\mathbb{R}^2} m' \cdot \nabla(m_3 - \phi_{L,3}) dx + \int_{\mathbb{R}^2} m' \cdot \nabla \phi_{L,3} dx. \tag{6.24}$$

By Young’s inequality and the estimate (6.23), the first term is bounded from above by

$$\int_{\mathbb{R}^2} m' \cdot \nabla(m_3 - \phi_{L,3}) dx \leq \frac{1}{8} \int_{\mathbb{R}^2} |m'|^2 dx + CL^{-2}. \tag{6.25}$$

To estimate the second term, note that $\phi_{L,3}(x) - v_3 = (S(\Phi_L(\rho^{-1}x) + e_3))_3$ for all $x \in \mathbb{R}^2$ by definition (6.2). Using estimate (5.30) to control the in-plane contributions and (5.31) together with $\Phi_3 + 1 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ to control the out-of-plane contributions, we have

$$\int_{\mathbb{R}^2} |\phi_{L,3} - v_3|^2 dx \leq C\rho^2 \log L \tag{6.26}$$

for $L \geq L_0$ with $L_0 > 0$ big enough. Therefore, the function $x \mapsto (\phi_{L,3}(x) - v_3)m'(x)$ is integrable, and we can integrate by parts to obtain

$$\int_{\mathbb{R}^2} m' \cdot \nabla \phi_{L,3} dx = - \int_{\mathbb{R}^2} (\phi_{L,3} - v_3) \nabla \cdot m' dx \leq C\rho(\log L)^{\frac{1}{2}}. \tag{6.27}$$

In total, we obtain

$$2 \int_{\mathbb{R}^2} m' \cdot \nabla m_3 dx \leq \frac{1}{4} \int_{\mathbb{R}^2} |m'|^2 dx + C\rho(\log L)^{\frac{1}{2}} + CL^{-2} \tag{6.28}$$

for all $L \geq L_0$ with $L_0 > 0$ sufficiently big universal.

Step 2: Estimate the nonlocal terms.

The volume charges are simply estimated by $F_{\text{vol}}(m') \geq 0$, see Lemma 3.1. For the surface charges, we exploit bilinearity of F_{surf} and the fact that F_{surf} is invariant under addition of constants, see (3.2), to get

$$F_{\text{surf}}(m_3) - F_{\text{surf}}(\phi_{L,3}) = F_{\text{surf}}(m_3 + \phi_{L,3} + 1 - \nu_3, m_3 - \phi_{L,3}). \quad (6.29)$$

The inequality (3.7) together with (6.23) then implies

$$|F_{\text{surf}}(m_3) - F_{\text{surf}}(\phi_{L,3})| \leq C (\|\phi_{L,3} - \nu_3\|_2 + \|m_3 + 1\|_2) L^{-1}. \quad (6.30)$$

By (3.7) and (5.28) we furthermore get

$$F_{\text{surf}}(\phi_{L,3}) = F_{\text{surf}}(\phi_{L,3} - \nu_3) \leq C \|\phi_{L,3} - \nu_3\|_2. \quad (6.31)$$

Thus we can combine (6.30) and (6.31), estimating the $\|\phi_{L,3} - \nu_3\|_2$ and $\|m_3 + 1\|_2$ contributions by (6.26) and (5.2), respectively, and applying Young's inequality, to get

$$F_{\text{surf}}(m_3) \leq \frac{1}{4} \int_{\mathbb{R}^2} |m'|^2 dx + C\rho(\log L)^{\frac{1}{2}} + CL^{-2}, \quad (6.32)$$

provided $L \geq L_0$ for $L_0 > 0$ sufficiently big. \square

Proof of Lemma 6.3. Combining the a priori bound (5.3) with the upper bound of Lemma 5.7 we see that

$$L^{-2} = \int_{\mathbb{R}^2} |\nabla m_\sigma|^2 dx - 8\pi \leq 16\pi\sigma^2 \quad (6.33)$$

for all $\sigma > 0$, which is the desired estimate (6.5). Therefore, for all $\sigma < \sigma_0$ small enough universal we may apply Lemmas 6.1 and 6.2, the latter together with the bound (6.33) to control the CL^{-2} -term, to get

$$\begin{aligned} & \frac{|\log \sigma|}{\sigma^2} (E_{\sigma,\lambda}(m_\sigma) - 8\pi) \\ & \geq |\log \sigma| (\sigma L)^{-2} + |\log \sigma| \left(C_1 |v + e_3|^2 L^2 \rho^2 - C_2 \left(\rho(\log L)^{\frac{1}{2}} + \sigma^2 \right) \right) \end{aligned} \quad (6.34)$$

for two universal constants $C_1, C_2 > 0$. By (2.28), we may define $\tilde{\rho} := \frac{C_2 |\log \sigma| (\log L)^{\frac{1}{2}}}{g(\lambda)} \rho$. Recalling the definition (5.24), observe that for θ_0^+ as in the first part of Proposition 5.4 we have $(\tilde{\rho}, \theta_0^+, L) \in V_\sigma$ by estimate (6.33). We thus get from Corollary 5.6, (2.28) and (5.39) that

$$|\log \sigma| (\sigma L)^{-2} + \tilde{C}_1 \frac{|v + e_3|^2 L^2}{|\log \sigma| \log L} \tilde{\rho}^2 - g(\lambda, \theta_0^+) \tilde{\rho} \leq \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K^*} + \tilde{C}_2 \sigma^{\frac{1}{4}} |\log \sigma|, \quad (6.35)$$

for some $\tilde{C}_1, \tilde{C}_2 > 0$ universal.

Towards a contradiction, assume that

$$\tilde{C}_1 |\nu + e_3|^2 L^2 \geq 16\pi \log^2 L. \tag{6.36}$$

Then we have, for all $\sigma < \sigma_0$ small enough, the universal

$$\tilde{C}_1 \frac{|\nu + e_3|^2 L^2}{|\log \sigma| \log L} \tilde{\rho}^2 \geq \frac{4\pi \log(2K^* L^2)}{|\log \sigma|} \tilde{\rho}^2, \tag{6.37}$$

by estimate (6.33). Recalling the definition (5.23) and that $(\tilde{\rho}, \theta_0^+, L) \in V_\sigma$, we therefore obtain from (5.39) and the bound (6.35) that

$$\min_{V_\sigma} \mathcal{E}_{\sigma, \lambda; 2K^*} \leq \mathcal{E}_{\sigma, \lambda; 2K^*}(\tilde{\rho}, \theta_0^+, L) \leq \min_{V_\sigma} \mathcal{E}_{\sigma, \lambda; K^*} + \tilde{C}_2 \sigma^{\frac{1}{4}} |\log \sigma|. \tag{6.38}$$

This evidently contradicts the identity

$$\min_{V_\sigma} \mathcal{E}_{\sigma, \lambda; 2K^*} - \min_{V_\sigma} \mathcal{E}_{\sigma, \lambda; K^*} = \frac{\tilde{g}^2(\lambda) \log 2}{64\pi |\log \sigma|} + O\left(\frac{\log^2 |\log \sigma|}{|\log \sigma|^2}\right), \tag{6.39}$$

resulting from the expansion (5.38). \square

Proof of Lemma 6.4. *Step 1: Write the problem in Fourier space.*

The strategy is, essentially, to relax the unit length constraint on m and carry out the resulting quadratic minimization in Fourier space. Without loss of generality, we may assume $\rho = 1$ and $x_0 = 0$.

By assumption, we have $m' \in L^2(\mathbb{R}^2; \mathbb{R}^2)$, together with

$$\int_{\mathbb{R}^2} |\nabla(m' - (S\Phi)')|^2 dx \leq \bar{L}^{-2} \tag{6.40}$$

with $|Se_3 - e_3|^2 \leq \bar{L}^{-1}$ for $S \in \text{SO}(3)$. Letting $h := \text{Im}(\mathcal{F}m') \in L^2(\mathbb{R}^2; \mathbb{R}^2)$, where \mathcal{F} denotes the Fourier transform defined via (3.1), Plancherel’s identity implies

$$\int_{\mathbb{R}^2} |m'|^2 dx \geq \int_{\mathbb{R}^2} |h|^2 \frac{dk}{(2\pi)^2}. \tag{6.41}$$

In order to express the constraint (6.40) in Fourier space, let $S' \in \mathbb{R}^{2 \times 2}$ be defined by $S'_{ij} := S_{ij}$ for $i, j = 1, 2$. Notice that by Lemma A.5 we have that $\mathcal{F}(\nabla\Phi_3)$ is purely imaginary, while $\mathcal{F}(\nabla\Phi')$ is purely real. Therefore, in view of (A.46) we have $\text{Re}(\mathcal{F}(\partial_i(S\Phi)_j(k))) = \mathcal{F}(\partial_i(S'\Phi')_j(k)) = k_i(S'g)_j$ for $i, j = 1, 2$ and almost all $k \in \mathbb{R}^2$, where $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as

$$g(k) := -4\pi K_1(|k|) \frac{k}{|k|}. \tag{6.42}$$

Furthermore, we have $\mathcal{F}(\partial_i m'_j)(k) = ik_i \mathcal{F}(m'_j)$ for all $i, j = 1, 2$ and almost all $k \in \mathbb{R}^2$, so that we obtain $\operatorname{Re} \left(\mathcal{F}(\partial_i m'_j)(k) \right) = -k_i h_j$. Only keeping the real parts, Plancherel's identity and the assumption (6.40) then give

$$\int_{\mathbb{R}^2} |k|^2 |h + S'g|^2 \frac{dk}{(2\pi)^2} \leq \int_{\mathbb{R}^2} |\mathcal{F}(\nabla(m' - (S\Phi)'))|^2 \frac{dk}{(2\pi)^2} \leq \bar{L}^{-2}. \quad (6.43)$$

Step 2: Introduce the expected minimizer into the quadratic expressions.

Let $\mu > 0$ be a proxy for the Lagrange multiplier associated to the minimization of the right-hand side of (6.41) under the constraint (6.43). By (A.44) and (A.43) we have $\frac{\mu|k|^2}{1+\mu|k|^2} S'g \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ and $\frac{|k|}{1+\mu|k|^2} S'g \in L^2(\mathbb{R}^2; \mathbb{R}^2)$. Therefore, we may calculate

$$\begin{aligned} \int_{\mathbb{R}^2} |h|^2 \frac{dk}{(2\pi)^2} &= \int_{\mathbb{R}^2} \left| h + \frac{\mu|k|^2}{1+\mu|k|^2} S'g \right|^2 \frac{dk}{(2\pi)^2} \\ &\quad - 2 \int_{\mathbb{R}^2} \frac{\mu|k|^2}{1+\mu|k|^2} S'g \cdot \left(h + \frac{\mu|k|^2}{1+\mu|k|^2} S'g \right) \frac{dk}{(2\pi)^2} \\ &\quad + \int_{\mathbb{R}^2} \frac{\mu^2|k|^4}{(1+\mu|k|^2)^2} |S'g|^2 \frac{dk}{(2\pi)^2}, \end{aligned} \quad (6.44)$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} |k|^2 |h + S'g|^2 \frac{dk}{(2\pi)^2} &= \int_{\mathbb{R}^2} |k|^2 \left| h + \frac{\mu|k|^2}{1+\mu|k|^2} S'g \right|^2 \frac{dk}{(2\pi)^2} \\ &\quad + 2 \int_{\mathbb{R}^2} \frac{|k|^2}{1+\mu|k|^2} S'g \cdot \left(h + \frac{\mu|k|^2}{1+\mu|k|^2} S'g \right) \frac{dk}{(2\pi)^2} \\ &\quad + \int_{\mathbb{R}^2} \frac{|k|^2}{(1+\mu|k|^2)^2} |S'g|^2 \frac{dk}{(2\pi)^2}. \end{aligned} \quad (6.45)$$

Multiplying (6.45) by $\mu > 0$ and rearranging the terms we get by estimate (6.43) that

$$\begin{aligned} &- 2 \int_{\mathbb{R}^2} \frac{\mu|k|^2}{1+\mu|k|^2} S'g \cdot \left(h + \frac{\mu|k|^2}{1+\mu|k|^2} S'g \right) \frac{dk}{(2\pi)^2} \\ &\geq \int_{\mathbb{R}^2} \frac{\mu|k|^2}{(1+\mu|k|^2)^2} |S'g|^2 \frac{dk}{(2\pi)^2} - \mu \bar{L}^{-2}. \end{aligned} \quad (6.46)$$

Plugging this into the first identity (6.44), we obtain

$$\int_{\mathbb{R}^2} |h|^2 \frac{dk}{(2\pi)^2} \geq \int_{\mathbb{R}^2} \frac{\mu|k|^2}{1+\mu|k|^2} |S'g|^2 \frac{dk}{(2\pi)^2} - \mu \bar{L}^{-2}. \quad (6.47)$$

Step 3: Conclusion.

Since $S \in \operatorname{SO}(3)$, we have $(S'^T S' - \operatorname{id})_{ij} = -S_{3,i} S_{3,j}$ for $i, j = 1, 2$. Therefore, with $v_i := S_{3,i}$ for $i = 1, 2$ we obtain that the symmetric 2×2 matrix $S'^T S' - \operatorname{id}$ has

the eigenvalues $\lambda_1 := -|v|^2$ and $\lambda_2 := 0$ with eigenvectors v and v^\perp , respectively. By (6.8) we have $|v|^2 \leq |Se_3 - e_3|^2 < \bar{L}^{-1}$, so that we can calculate

$$|S'g'|^2 = |g'|^2 + g' \cdot (S'^T S' - \text{id})g' \geq |g'|^2 (1 - |\lambda_1|) \geq |g'|^2 (1 - \bar{L}^{-1}). \tag{6.48}$$

Recalling the definition (6.42), we can thus upgrade the estimate (6.47) to

$$\int_{\mathbb{R}^2} |h|^2 \frac{dk}{(2\pi)^2} \geq 16\pi^2 (1 - \bar{L}^{-1}) \int_{\mathbb{R}^2} \frac{\mu|k|^2}{1 + \mu|k|^2} K_1^2(|k|) \frac{dk}{(2\pi)^2} - \mu\bar{L}^{-2}. \tag{6.49}$$

Then by Lemma A.7 we can rewrite the above inequality as

$$\int_{\mathbb{R}^2} |h|^2 \frac{dk}{(2\pi)^2} \geq 4\pi (1 - \bar{L}^{-1}) \log \left(\frac{4\mu}{e^{2\gamma} + 1} \right) - \mu\bar{L}^{-2} - C(1 - \bar{L}^{-1})\mu^{-\frac{1}{3}} \tag{6.50}$$

for some $C > 0$ and all μ sufficiently large universal. For $\bar{L} \geq \bar{L}_0$ with $\bar{L}_0 > 0$ big enough, the right-hand side of (6.50) is maximized by $\mu = 4\pi\bar{L}^2$ to the leading order in $\bar{L}^{-1} \ll 1$. Plugging in this value of μ into (6.50) then yields

$$\int_{\mathbb{R}^2} |h|^2 \frac{dk}{(2\pi)^2} \geq 4\pi \log \left(\frac{16\pi}{e^{2\gamma} + 1} \bar{L}^2 \right) - 4\pi - C\bar{L}^{-\frac{2}{3}} \tag{6.51}$$

for some $C > 0$ universal, which is the desired estimate. \square

Proof of Lemma 6.5. *Step 1: Preliminary bounds on the radius.*

For $\sigma \in (0, \sigma_0)$ and $\sigma_0 > 0$ small enough, the lower bound (6.9) of Lemma 6.4, which is applicable due to Lemma 6.3 and Theorem 2.4, gives

$$2\pi\rho^2 \log(L^2) \leq \int_{\mathbb{R}^2} |m'_\sigma|^2 dx. \tag{6.52}$$

From the topological bound (A.24) and the estimate (5.4) we get

$$\begin{aligned} \frac{\sigma^2}{2} \int_{\mathbb{R}^2} |m'_\sigma|^2 dx &\leq \int_{\mathbb{R}^2} |\nabla m_\sigma|^2 dx - 8\pi + \frac{\sigma^2}{2} \int_{\mathbb{R}^2} |m'_\sigma|^2 dx \\ &\leq E_{\sigma,\lambda}(m_\sigma) - 8\pi + \sigma^2 \frac{(1 + \lambda)^2}{2} \int_{\mathbb{R}^2} |\nabla m_\sigma|^2 dx. \end{aligned} \tag{6.53}$$

Therefore, Lemma 5.7 and the bound $\int_{\mathbb{R}^2} |\nabla m_\sigma|^2 dx < 16\pi$ resulting from $m_\sigma \in \mathcal{A}$, see definition (2.15), imply

$$\int_{\mathbb{R}^2} |m'_\sigma|^2 dx \leq C \tag{6.54}$$

for some $C > 0$ universal. In particular, from the estimates (6.52) and (6.5) we get the bound (6.13).

Step 2: Estimate the DMI term.

Without loss of generality, we may assume $x_0 = 0$. By Lemma 6.3 and the fact that $v = -Se_3$, see definition (6.2), we obtain $|Se_3 - e_3|^2 \leq CL^{-2} \log^2 L$ for σ_0 small enough. On the other hand, by the Euler rotation theorem the matrix S admits a representation $S = RS_\theta$, where S_θ is defined by (2.24) for some $\theta \in [-\pi, \pi)$ and $R \in \text{SO}(3)$ is a composition of rotations around the x_1 - and x_2 -axes. It is not difficult to see that

$$|R - \text{id}| \leq C|Re_3 - e_3| = C|Se_3 - e_3| \leq C'L^{-1} \log L \tag{6.55}$$

for some universal $C, C' > 0$. Therefore, by the properties of the Frobenius norm we get

$$|S - S_\theta|^2 = |R - \text{id}|^2 \leq CL^{-2} \log^2 L \tag{6.56}$$

for some universal $C > 0$. Together with Theorem 2.4, Lemma 6.3 and definition (6.1), we deduce that the function $\phi(x) := S_\theta \Phi(\rho^{-1}x)$ for $x \in \mathbb{R}^2$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla(m_\sigma - \phi)|^2 dx &\leq C \int_{\mathbb{R}^2} |\nabla(m_\sigma - S\Phi(\rho^{-1}x))|^2 dx + C|S - S_\theta|^2 \\ &\leq C \frac{\log^2 L}{L^2} \end{aligned} \tag{6.57}$$

for some $C > 0$ and $\sigma_0 > 0$ small enough, both universal, which is estimate (6.10). By the identity (5.32) we have

$$\int_{\mathbb{R}^2} 2\phi' \cdot \nabla\phi_3 dx = 8\pi\rho \cos\theta. \tag{6.58}$$

Therefore, the bound (6.11) for the DMI term follows once we control

$$\begin{aligned} &\int_{\mathbb{R}^2} (m'_\sigma \cdot \nabla m_{\sigma,3} - \phi' \cdot \nabla\phi_3) dx \\ &= \int_{\mathbb{R}^2} m'_\sigma \cdot \nabla(m_{\sigma,3} - \phi_3) dx + \int_{\mathbb{R}^2} (m'_\sigma - \phi') \cdot \nabla(\phi_3 + 1) dx \\ &= \int_{\mathbb{R}^2} m'_\sigma \cdot \nabla(m_{\sigma,3} - \phi_3) dx - \int_{\mathbb{R}^2} (\phi_3 + 1)\nabla \cdot (m'_\sigma - \phi'), \end{aligned} \tag{6.59}$$

where the decay of $\phi_3 + 1$ at infinity is sufficiently strong to erase the boundary term in the integration by parts. By explicit calculation and (6.13), we have

$$\int_{\mathbb{R}^2} |\phi_3 + 1|^2 dx \leq C\rho^2 \leq \frac{C}{|\log\sigma|}. \tag{6.60}$$

Consequently, the Cauchy–Schwarz inequality applied to the right-hand side of (6.59) and the estimates (6.54), (6.60), and (6.57) imply

$$\left| \int_{\mathbb{R}^2} (\phi' \cdot \nabla\phi_3 - m'_\sigma \cdot \nabla m_{\sigma,3}) dx \right| \leq C \frac{\log L}{L}. \tag{6.61}$$

The bound (6.5) then gives the desired estimate (6.11) for the DMI term.

Step 3: Estimate the stray field terms.

We consider $\phi_L := S_\theta \Phi_L(\rho^{-1}x)$, with Φ_L as defined in equation (5.19), and note that we still have

$$\int_{\mathbb{R}^2} |\nabla(m_\sigma - \phi_L)|^2 dx \leq C \frac{\log^2 L}{L^2} \tag{6.62}$$

by the bounds (6.57) and (5.29). In Lemma 5.3 we also already computed

$$\begin{aligned} F_{\text{vol}}(\phi'_L) - F_{\text{surf}}(\phi_{L,3}) &\geq \left(\frac{3\pi^3}{8} \cos^2 \theta - \frac{\pi^3}{8} \right) \rho - C\rho L^{-\frac{1}{4}} \\ &\geq \left(\frac{3\pi^3}{8} \cos^2 \theta - \frac{\pi^3}{8} \right) \rho - C\sigma^{\frac{1}{4}} |\log \sigma|^{-\frac{1}{2}}, \end{aligned} \tag{6.63}$$

where in the last step we used estimates (6.5) and (6.13).

As the nonlocal terms are bilinear, we have

$$F_{\text{vol}}(m'_\sigma) - F_{\text{vol}}(\phi'_L) = F_{\text{vol}}(m'_\sigma + \phi'_L, m'_\sigma - \phi'_L) \tag{6.64}$$

and the interpolation inequality (3.8) for $p = 2$ together with estimate (6.62) imply

$$|F_{\text{vol}}(m'_\sigma) - F_{\text{vol}}(\phi'_L)| \leq C (\|\phi'_L\|_2 + \|m'_\sigma\|_2) \frac{\log L}{L}. \tag{6.65}$$

The fact that $\|\phi'_L\|_2 \leq C\rho(\log L)^{\frac{1}{2}}$, see (5.30), together with the estimates (6.54), (6.13) and (6.5) thus gives

$$|F_{\text{vol}}(m'_\sigma) - F_{\text{vol}}(\phi'_L)| \leq C\sigma |\log \sigma|. \tag{6.66}$$

A similar argument exploiting the estimates (3.7), and (6.60), as well as Lemma 5.1 gives

$$|F_{\text{surf}}(m_{\sigma,3}) - F_{\text{surf}}(\phi_{L,3})| \leq C\sigma |\log \sigma|. \tag{6.67}$$

Combining the last two estimates with (6.63) yields (6.12). \square

6.1. Convergence to Shrinking Belavin–Polyakov Profiles via Stability of the Reduced Energy $\mathcal{E}_{\sigma,\lambda;K}$

Having completed the preparatory work in the form of the previously presented statements, we now proceed to prove Theorem 2.2.

Proof of Theorem 2.2. By Theorem 2.4 and definition (6.1), there exist $S \in \text{SO}(3)$, $\rho_\sigma > 0$ and $x_\sigma \in \mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} \left| \nabla \left(m_\sigma(x) - S\Phi(\rho_\sigma^{-1}(x - x_\sigma)) \right) \right|^2 dx \leq CL^{-2}. \tag{6.68}$$

Without loss of generality, we choose $x_\sigma = 0$. For $\varepsilon > 0$ to be chosen sufficiently small later and for $\sigma \in (0, \sigma_0)$ for $\sigma_0 > 0$ small enough depending only on ε , we

combine the bound (6.5) and the local version of the stability result in Lemma 4.4 to improve the above estimate to

$$\int_{\mathbb{R}^2} \left| \nabla \left(m_\sigma(x) - S\Phi(\rho_\sigma^{-1}x) \right) \right|^2 dx \leq \left(\frac{3}{2} + \varepsilon \right) L^{-2}. \tag{6.69}$$

Existence of $\theta_\sigma \in [-\pi, \pi)$ with

$$\int_{\mathbb{R}^2} |\nabla(m_\sigma(x) - S_{\theta_\sigma}\Phi(\rho_\sigma^{-1}x))|^2 dx \leq C \frac{\log^2 L}{L^2} \tag{6.70}$$

follows from Lemma 6.5, and $(|\log \sigma| \rho_\sigma, \theta_\sigma, L) \in V_\sigma$ is a result of estimate (6.5).

Recalling the definition (5.23) of $\mathcal{E}_{\sigma,\lambda;K}$, for $\bar{L} := \left(\frac{3}{2} + \varepsilon\right)^{-\frac{1}{2}} L$ and $K := \left(\frac{3}{2} + \varepsilon\right)^{-1} K^*$ we have by Lemmas 6.3, 6.4 and 6.5:

$$\mathcal{E}_{\sigma,\lambda;K}(|\log \sigma| \rho_\sigma, \theta_\sigma, L) \leq \frac{|\log \sigma|}{\sigma^2} (E_{\sigma,\lambda}(m_\sigma) - 8\pi) + C\sigma^{\frac{1}{4}} |\log \sigma|^{\frac{1}{2}}. \tag{6.71}$$

Corollary 5.6 gives

$$\frac{|\log \sigma|}{\sigma^2} (E_{\sigma,\lambda}(m_\sigma) - 8\pi) \leq \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K^*} + C\sigma^{\frac{1}{4}} |\log \sigma|. \tag{6.72}$$

For $\varepsilon \leq \frac{1}{2}$ we have $K \geq \frac{1}{2} K^*$, so that the expansion (5.38) implies

$$\min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K^*} \leq \min_{V_\sigma} \mathcal{E}_{\sigma,\lambda;K} + \frac{\bar{g}^2(\lambda) \log\left(\frac{K^*}{K}\right)}{64\pi |\log \sigma|} + C \frac{\log^2 |\log \sigma|}{|\log \sigma|^2}. \tag{6.73}$$

Concatenating the estimates (6.71), (6.72) and (6.73), we get for $\sigma \in (0, \sigma_0)$ for $\sigma_0 > 0$ sufficiently small that

$$\begin{aligned} \mathcal{E}_{\sigma,\lambda;K}(|\log \sigma| \rho_\sigma, \theta_\sigma, L) &\leq \min_{V_\sigma} (\mathcal{E}_{\sigma,\lambda;K}) + \frac{\bar{g}^2(\lambda) \log\left(\frac{3}{2} + \varepsilon\right)}{64\pi |\log \sigma|} \\ &\quad + C \frac{\log^2 |\log \sigma|}{|\log \sigma|^2}, \end{aligned} \tag{6.74}$$

where we also used the definition of K .

Since $\log \frac{3}{2} < 1$, for $\varepsilon > 0$ and $\sigma_0 > 0$ small enough universal we deduce

$$\mathcal{E}_{\sigma,\lambda;K}(|\log \sigma| \rho_\sigma, \theta_\sigma, L) \leq \min_{V_\sigma} (\mathcal{E}_{\sigma,\lambda;K}) + \frac{\bar{g}^2(\lambda)}{64\pi |\log \sigma|}. \tag{6.75}$$

Consequently, part 2 of Proposition 5.4 then implies the desired convergences for ρ_σ and θ_σ . Furthermore, the bounds (6.70) and (5.42) give

$$\int_{\mathbb{R}^2} |\nabla(m_\sigma(x) - S_{\theta_\sigma}\Phi(\rho_\sigma^{-1}x))|^2 dx \leq C\sigma^2.$$

Finally, the estimate

$$\left| \frac{|\log \sigma|}{\sigma^2} (E_{\sigma,\lambda}(m_\sigma) - 8\pi) - \left(-\frac{\bar{g}^2(\lambda)}{32\pi} + \frac{\bar{g}^2(\lambda) \log |\log \sigma|}{32\pi |\log \sigma|} \right) \right| \leq \frac{C}{|\log \sigma|} \tag{6.76}$$

follows from estimates (6.71), (6.72) and the expansion (5.38) of Proposition 5.4. \square

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A. Appendix

Here, we first provide a concise set-up of the differential geometry necessary for our argument. In a second section, we prove the topological bound (1.4) and prove that all extremizers are in fact Belavin–Polyakov profiles. Both sections are included for the convenience of readers who may be unfamiliar with the presented material, and we do not claim originality of the definitions and results. Finally, we will present a number of calculations involving Bessel functions.

A.1. Sobolev Spaces on the Sphere

Let $u : \mathbb{S}^2 \rightarrow \mathbb{R}^n$ for $n \geq 1$ be a smooth map, which we may extend to a smooth map on $\mathbb{R}^3 \setminus \{0\}$ by setting $U(x) := u\left(\frac{x}{|x|}\right)$. Following [3, Definition 7.25], we consider its gradient

$$\nabla u(y) := (\partial_{\tau_1} U)(y)\tau_1(y) + (\partial_{\tau_2} U)(y)\tau_2(y) \tag{A.1}$$

for $y \in \mathbb{S}^2$ and $\{\tau_1(y), \tau_2(y)\}$ an orthonormal basis of the tangent space $T_y\mathbb{S}^2 := \{v \in \mathbb{R}^3 : v \cdot y = 0\}$. Due to the chain rule in \mathbb{R}^n , we recover the standard notion of gradient in Riemannian geometry. We will also need the tangential divergence for smooth functions $\xi : \mathbb{S}^2 \rightarrow \mathbb{R}^3$, for $y \in \mathbb{S}^2$ defined as

$$\nabla \cdot \xi(y) := (\partial_{\tau_1} \xi)(y) \cdot \tau_1(y) + (\partial_{\tau_2} \xi)(y) \cdot \tau_2(y), \tag{A.2}$$

where again $\Xi(x) := \xi\left(\frac{x}{|x|}\right)$ for $x \in \mathbb{R}^3 \setminus \{0\}$ and $\{\tau_1(y), \tau_2(y)\}$ is an orthonormal basis of $T_y\mathbb{S}^2$, see [3, Definition 7.27 and Remark 7.28]. The Laplace–Beltrami operator for a smooth map $u : \mathbb{S}^2 \rightarrow \mathbb{R}$ then is $\Delta u := \nabla \cdot \nabla u$, see [46, (2.1.16)].

We define the space $H^1(\mathbb{S}^2)$ as the completion of $C^\infty(\mathbb{S}^2)$ with respect to the norm

$$\|u\|_{H^1(\mathbb{S}^2)} := \left(\int_{\mathbb{S}^2} (|\nabla u|^2 + |u|^2) \, d\mathcal{H}^2 \right)^{\frac{1}{2}}. \tag{A.3}$$

Let $H^1(\mathbb{S}^2; \mathbb{R}^3)$ be defined analogously for \mathbb{R}^3 -valued maps and set

$$H^1(\mathbb{S}^2; \mathbb{S}^2) := \left\{ \tilde{m} \in H^1(\mathbb{S}^2; \mathbb{R}^3) : \tilde{m}(y) \in \mathbb{S}^2 \text{ for } \mathcal{H}^2\text{-a.e. } y \in \mathbb{S}^2 \right\}, \tag{A.4}$$

$$H^1(\mathbb{S}^2; T\mathbb{S}^2) := \left\{ \xi \in H^1(\mathbb{S}^2; \mathbb{R}^3) : \xi(y) \in T_y\mathbb{S}^2 \text{ for } \mathcal{H}^2\text{-a.e. } y \in \mathbb{S}^2 \right\}, \tag{A.5}$$

where $T\mathbb{S}^2 := \bigcup_{y \in \mathbb{S}^2} \{y\} \times T_y\mathbb{S}^2$ is the tangent bundle of \mathbb{S}^2 . The weak gradient ∇u for $u \in H^1(\mathbb{S}^2)$ and weak divergence $\nabla \cdot \xi$ for $\xi \in H^1(\mathbb{S}^2; \mathbb{R}^3)$ exist as measurable maps characterized by the following integration-by-parts formula:

Lemma A.1. *Let $u \in H^1(\mathbb{S}^2)$ and $\xi \in H^1(\mathbb{S}^2; \mathbb{R}^3)$. Then we have*

$$\int_{\mathbb{S}^2} \xi \cdot \nabla u \, d\mathcal{H}^2(y) = \int_{\mathbb{S}^2} (2u \xi \cdot y - u \nabla \cdot \xi) \, d\mathcal{H}^2(y), \tag{A.6}$$

and this identity determines ∇u and $\nabla \cdot \xi$ up to sets of \mathcal{H}^2 -measure zero. Furthermore, for smooth maps $\zeta, \xi : \mathbb{S}^2 \rightarrow \mathbb{R}$ we have

$$\int_{\mathbb{S}^2} \nabla \xi \cdot \nabla \zeta \, d\mathcal{H}^2 = - \int_{\mathbb{S}^2} \zeta \Delta \xi \, d\mathcal{H}^2. \tag{A.7}$$

We furthermore remark that, following BREZIS and NIRENBERG [18], we can define the Brouwer degree for maps $\tilde{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$, and even for functions of vanishing oscillation, in the following way: For $y \in \mathbb{S}^2$, let $y \mapsto (\tau_1(y), \tau_2(y))$ be an orthonormal frame of $T_y\mathbb{S}^2$ which is smooth except in a single point. For maps $\tilde{m} \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ we use the integral formula (see also definition (2.44))

$$\mathcal{N}_{\mathbb{S}^2}(\tilde{m}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \det(\nabla \tilde{m}) \, d\mathcal{H}^2, \tag{A.8}$$

where for $y \in \mathbb{S}^2$ we define $\det(\nabla \tilde{m}(y)) := \det M(y)$ with $M_{ij}(y) := \tau_j(\tilde{m}(y)) \cdot [(\tau_i(y) \cdot \nabla) \tilde{m}(y)]$ for $i, j = 1, 2$ to be the determinant of the linear map $v \mapsto (v \cdot \nabla) \tilde{m}$ from $T_y\mathbb{S}^2$ to $T_{\tilde{m}(y)}\mathbb{S}^2$, expressed in the ordered bases $(\tau_1(y), \tau_2(y))$ and $(\tau_1(\tilde{m}(y)), \tau_2(\tilde{m}(y)))$. It can be seen that this definition is independent of the frame (τ_1, τ_2) ; in fact, for certain choices of τ_1 and τ_2 this is part of the proof of the representation

$$\mathcal{N}(m) = \mathcal{N}_{\mathbb{S}^2} \left(m \circ \phi^{-1} \right), \tag{A.9}$$

for any $\phi \in \mathcal{B}$, found in Lemma A.2. The degree can then be extended as a continuous map to $H^1(\mathbb{S}^2; \mathbb{S}^2)$ by approximation with smooth maps provided by a result of SCHOEN and UHLENBECK [75]. In particular, the above representation (A.8)

holds true, as the 2×2 determinant is a quadratic function, and thus the integral is continuous in the strong H^1 -topology, see also [18, Property 4].

Using the above definitions, we describe how the various quantities behave under reparametrization by ϕ^{-1} with $\phi \in \mathcal{B}$. In particular, we prove that the harmonic map problem is invariant under this operation.

Lemma A.2. *Let $\phi \in \mathcal{B}$ and let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be measurable. Then the map $x \mapsto u(x)|\nabla\phi(x)|^2$ is integrable on \mathbb{R}^2 if and only if $u \circ \phi^{-1}$ is integrable on \mathbb{S}^2 , and we have*

$$\int_{\mathbb{R}^2} u|\nabla\phi|^2 dx = 2 \int_{\mathbb{S}^2} u \circ \phi^{-1} d\mathcal{H}^2. \tag{A.10}$$

Furthermore, we have $u \in H^1_{\mathbb{W}}(\mathbb{R}^2)$, where the space $H^1_{\mathbb{W}}(\mathbb{R}^2)$ is defined in (4.4), if and only if $u \circ \phi^{-1} \in H^1(\mathbb{S}^2)$, and for every $u, v \in H^1_{\mathbb{W}}(\mathbb{R}^2)$ there holds

$$\int_{\mathbb{R}^2} \nabla u \cdot \nabla v dx = \int_{\mathbb{S}^2} \nabla(u \circ \phi^{-1}) \cdot \nabla(v \circ \phi^{-1}) d\mathcal{H}^2. \tag{A.11}$$

We also have $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ if and only if $\tilde{m} := m \circ \phi^{-1} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$, in which case we additionally have $\mathcal{N}(m) = \mathcal{N}_{\mathbb{S}^2}(\tilde{m})$. In particular, we have that $m \in \mathcal{C}$ is a minimizer of F if and only if $\tilde{m} \in \mathcal{C}_{\mathbb{S}^2}$ is a minimizer of $F_{\mathbb{S}^2}$.

Proof of Lemma A.1. The fact that ∇u and $\nabla \cdot \xi$ are determined up to sets of \mathcal{H}^2 -measure zero is a standard fact in analogy to uniqueness of weak derivatives of functions defined in the Euclidean space. By approximation, it is sufficient to prove the formula for smooth functions u and ξ . Using the definitions (A.1) and (A.2) it is straightforward to check the identity

$$\nabla \cdot (u\xi) = \nabla u \cdot \xi + u \nabla \cdot \xi. \tag{A.12}$$

We therefore have

$$\int_{\mathbb{S}^2} \xi \cdot \nabla u d\mathcal{H}^2 = \int_{\mathbb{S}^2} (\nabla \cdot (u\xi) - u \nabla \cdot \xi) d\mathcal{H}^2. \tag{A.13}$$

By the divergence theorem on manifolds [3, Theorem 7.34] we have

$$\int_{\mathbb{S}^2} \nabla \cdot (u\xi) d\mathcal{H}^2 = \int_{\mathbb{S}^2} u \xi \cdot ((\nabla \cdot y)y) d\mathcal{H}^2(y) = \int_{\mathbb{S}^2} 2u \xi \cdot y d\mathcal{H}^2(y), \tag{A.14}$$

where $-(\nabla \cdot y)y = -2y$ has the significance of being the mean curvature vector at $y \in \mathbb{S}^2$, see [3, Definition 7.32]. This proves the identity (A.6), from which the formula (A.7) easily follows. \square

Proof of Lemma A.2. For all $x \in \mathbb{R}^2$ we have the identities

$$\partial_i \phi(x) \cdot \partial_i \phi(x) = \frac{1}{2} |\nabla \phi(x)|^2 > 0, \tag{A.15}$$

$$\partial_1 \phi(x) \cdot \partial_2 \phi(x) = 0 \tag{A.16}$$

for $i = 1, 2$, and thus the map $x \mapsto \left(\frac{\sqrt{2}}{|\nabla\phi(x)|} \partial_1\phi(x), \frac{\sqrt{2}}{|\nabla\phi(x)|} \partial_2\phi(x) \right)$ provides a smooth orthonormal frame for $T_{\phi(x)}\mathbb{S}^2$ for $x \in \mathbb{R}^2$. Equation (A.10) and the corresponding equivalence are straightforward results of the area formula [3, Theorem 2.71] and the fact that $\frac{1}{2}|\nabla\phi|^2 = (\det \nabla\phi^T \nabla\phi)^{\frac{1}{2}}$ is the Jacobian of ϕ .

Let $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ be smooth functions and let $x \in \mathbb{R}^2$. The chain rule implies

$$\partial_i u(x) = \partial_i \phi(x) \cdot \left(\nabla \left(u \circ \phi^{-1} \right) (\phi(x)) \right) \tag{A.17}$$

for $i = 1, 2$. As a result, we obtain

$$\nabla u(x) \cdot \nabla v(x) = \frac{1}{2} |\nabla\phi(x)|^2 \nabla \left(u \circ \phi^{-1} \right) (\phi(x)) \cdot \nabla \left(v \circ \phi^{-1} \right) (\phi(x)). \tag{A.18}$$

Since smooth functions are dense with respect to the \dot{H}^1 -topology in both spaces $H^1_w(\mathbb{R}^2)$ and $H^1(\mathbb{S}^2)$, as can be easily seen via convolutions, we obtain equation (A.11). For $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ we thus have $\tilde{m} := m \circ \phi^{-1} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$.

In order to show $\mathcal{N}(m) = \mathcal{N}_{\mathbb{S}^2}(\tilde{m})$, we first define the orthogonal frame

$$(\tau_1(y), \tau_2(y)) := \left[\left(\frac{\sqrt{2}}{|\nabla\phi|} \partial_1\phi, \frac{\sqrt{2}}{|\nabla\phi|} \partial_2\phi \right) \circ \phi^{-1} \right] (y) \tag{A.19}$$

for $y \in \mathbb{S}^2$, which is smooth except in the single point $v := \lim_{|x| \rightarrow \infty} \phi(x)$. We may, therefore, calculate

$$\mathcal{N}(m) = \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot (\partial_1 m \times \partial_2 m) \, dx \tag{A.20}$$

$$= \frac{1}{4\pi} \int_{\mathbb{S}^2} \tilde{m} \cdot [(\tau_1(y) \cdot \nabla)\tilde{m} \times (\tau_2(y) \cdot \nabla)\tilde{m}] \, d\mathcal{H}^2(y), \tag{A.21}$$

due to the chain rule (A.17), the area formula and the fact that $\frac{1}{2}|\nabla\phi|^2$ is the Jacobian of ϕ . For almost all $y \in \tilde{m}^{-1}(v)$ we have $\nabla m(y) = 0$ by standard statements about weak derivatives. Therefore, for $i = 1, 2$ we can almost everywhere express $(\tau_i \cdot \nabla)\tilde{m}(y)$ in the basis $\{\tau_1(\tilde{m}(y)), \tau_2(\tilde{m}(y))\}$ to get

$$\mathcal{N}(m) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \tilde{m} \cdot (\tau_1(\tilde{m}) \times \tau_2(\tilde{m})) \det(\nabla\tilde{m}) \, d\mathcal{H}^2 = \mathcal{N}_{\mathbb{S}^2}(\tilde{m}), \tag{A.22}$$

by virtue of $z \cdot (\tau_1(z) \times \tau_2(z)) = 1$ for all $z \in \mathbb{S}^2 \setminus \{v\}$ according to (A.19). \square

A.2. The Topological Bound and Energy Minimizing Harmonic Maps of Degree 1

In this section, we prove the topological bound (1.4) and characterize the corresponding minimizers for the convenience of the reader. The following statement is an amalgam of results due to BELAVIN and POLYAKOV [6], LEMAIRE [52] and WOOD [80], see the discussion in Sect. 1. Our approach below is to reduce the problem to that of the solutions of an H-system treated by BREZIS and CORON [17].

Lemma A.3. *For all $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ we have*

$$|\nabla m|^2 \pm 2m \cdot (\partial_1 m \times \partial_2 m) = |\partial_1 m \mp m \times \partial_2 m|^2 \geq 0, \tag{A.23}$$

almost everywhere, as well as

$$\int_{\mathbb{R}^2} |\nabla m|^2 \, dx \geq 8\pi |\mathcal{N}(m)|. \tag{A.24}$$

The functions with $\mathcal{N} = 1$ achieving equality, i.e., energy minimizing harmonic maps of degree 1, are given by the set of Belavin–Polyakov profiles \mathcal{B} , see definition (2.20). We furthermore have the representation

$$\mathcal{B}_{\mathbb{S}^2} = \left\{ \Phi \circ f \circ \Phi^{-1} : f(x) := \frac{ax + b}{cx + d} \text{ for } a, b, c, d \in \mathbb{C} \text{ with } ad - bc \neq 0 \right\} \tag{A.25}$$

for the set $\mathcal{B}_{\mathbb{S}^2}$ of minimizing harmonic maps of degree 1 from \mathbb{S}^2 to itself, see definition (2.45).

We also briefly state a version of [54, Lemma 9] in our setting relating the energy excess to the Hamiltonian, see Sect. 4.1, which will come in handy a number of times.

Lemma A.4. ([54, Lemma 9]). *For $m \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ and $\phi \in \mathcal{B}$ we have the identity*

$$F(m) - 8\pi = \int_{\mathbb{R}^2} \left(|\nabla(m - \phi)|^2 - (m - \phi)^2 |\nabla\phi|^2 \right) \, dx. \tag{A.26}$$

Proof of Lemma A.3. The inequality (A.23) is a result of completing the square, and the topological bound (A.24) then follows by integration.

Let $\phi \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ be such that $\mathcal{N}(\phi) = 1$ and

$$\int_{\mathbb{R}^2} |\nabla\phi|^2 \, dx = 8\pi. \tag{A.27}$$

Then equation (A.23) implies $\partial_1\phi = -\phi \times \partial_2\phi$ almost everywhere. Together with the fact that $\phi \cdot \partial_i\phi = 0$ for $i = 1, 2$ almost everywhere, we also have $\phi \times \partial_1\phi = -\phi \times (\phi \times \partial_2\phi) = \partial_2\phi$ and

$$2\partial_1\phi \times \partial_2\phi = \partial_1\phi \times (\phi \times \partial_1\phi) - (\phi \times \partial_2\phi) \times \partial_2\phi = |\nabla\phi|^2\phi. \tag{A.28}$$

Because ϕ is evidently an energy minimizing harmonic map, it satisfies (2.23) distributionally, and thus the map $\tilde{\phi} := -\phi$ satisfies

$$\Delta\tilde{\phi} = 2\partial_1\phi \times \partial_2\phi, \quad \int_{\mathbb{R}^2} |\nabla\tilde{\phi}|^2 \, dx = 8\pi. \tag{A.29}$$

Thus [17, Lemma A.1] implies for almost all $x \in \mathbb{C}$ that

$$\tilde{\phi}(x) = \Phi_n \left(\frac{P(x)}{Q(x)} \right) \tag{A.30}$$

for complex polynomials P and Q of degree 1 such that $\frac{P}{Q}$ is irreducible, where $\Phi_n(x) := -\Phi(x)$ for $x \in \mathbb{C}$ and $\Phi_n(\infty) := e_3$ is the stereographic projection with respect to the north pole and division by zero is taken to evaluate to infinity. Therefore, we get the representation

$$\phi(x) = \Phi\left(\frac{P(x)}{Q(x)}\right). \tag{A.31}$$

for the smooth representative of ϕ . Let $a, b, c, d \in \mathbb{C}$ such that $P(x) = ax + b$ and $Q(x) = cx + d$ for $x \in \mathbb{C}$. As P and Q are irreducible, they must be linearly independent polynomials. Consequently, we have

$$ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0, \tag{A.32}$$

and the representation (A.25) follows from Lemma A.2.

Let $S \in \text{SO}(3)$ be such that $\lim_{|x| \rightarrow \infty} S\phi(x) = -e_3$. Because $S\phi$ also satisfies $\mathcal{N}(S\phi) = 1$ and $\int_{\mathbb{R}^2} |\nabla S\phi|^2 dx = 8\pi$, there exist $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{C}$ with $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0$ and

$$S\phi(x) = \Phi\left(\frac{\tilde{a}x + \tilde{b}}{\tilde{c}x + \tilde{d}}\right) \tag{A.33}$$

for all $x \in \mathbb{C}$. From $\lim_{|x| \rightarrow \infty} S\phi(x) = -e_3$ it follows that $\lim_{|x| \rightarrow \infty} \left|\frac{\tilde{a}x + \tilde{b}}{\tilde{c}x + \tilde{d}}\right| = \infty$, and thus $\tilde{c} = 0$. Therefore, $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0$ implies $\tilde{a} \neq 0$ and $\tilde{d} \neq 0$. Without loss of generality we may assume $\tilde{d} = 1$, so that for all $x \in \mathbb{C}$ we have

$$S\phi(x) = \Phi(\tilde{a}x + \tilde{b}). \tag{A.34}$$

With $\rho := |\tilde{a}|^{-1}$ and $x_0 := -\frac{\tilde{b}}{\tilde{a}}$ we get for all $x \in \mathbb{C}$ that

$$S\phi(\rho x + x_0) = \Phi\left(\frac{\tilde{a}}{|\tilde{a}|}x\right). \tag{A.35}$$

Since we evidently have $\left|\frac{\tilde{a}}{|\tilde{a}|}\right| = 1$, there exists $\theta \in [-\pi, \pi)$ such that

$$\frac{\tilde{a}}{|\tilde{a}|} = (\cos \theta, \sin \theta). \tag{A.36}$$

The symmetry properties of Φ , see definition (2.21), immediately imply

$$S\phi(\rho x + x_0) = \Phi\left(\frac{\tilde{a}}{|\tilde{a}|}x\right) = S_\theta \Phi(x), \tag{A.37}$$

where S_θ was defined in equation (2.24). Consequently, for all $x \in \mathbb{C}$ we obtain

$$\phi(x) = S^{-1} S_\theta \Phi\left(\rho^{-1}(x - x_0)\right), \tag{A.38}$$

concluding the proof. \square

Proof of Lemma A.4. We follow the arguments in the proof of [54, Lemma 9]. A straightforward algebraic computation gives

$$\int_{\mathbb{R}^2} (|\nabla m|^2 - |\nabla \phi|^2) \, dx = \int_{\mathbb{R}^2} (|\nabla(m - \phi)|^2 + 2\nabla\phi : \nabla(m - \phi)) \, dx. \tag{A.39}$$

By inspecting the definition (2.21) of Φ we see that $|\nabla\Phi(x)| = O(|x|^{-2})$ as $x \rightarrow \infty$. Consequently, we may integrate by parts in the second term on the right-hand side and use the fact that ϕ solves the harmonic map equation $\Delta\phi + |\nabla\phi|^2\phi = 0$, see equation (2.23), to obtain

$$\int_{\mathbb{R}^2} (|\nabla m|^2 - |\nabla\phi|^2) \, dx = \int_{\mathbb{R}^2} (|\nabla(m - \phi)|^2 + 2\phi \cdot (m - \phi)|\nabla\phi|^2) \, dx. \tag{A.40}$$

The fact $|m - \phi|^2 = -2\phi \cdot (m - \phi)$ gives the claim. \square

A.3. Integrals Involving Belavin–Polyakov Profiles

Here we collect the results of a number of computations involving the original and truncated Belavin–Polyakov profiles Φ and Φ_L , respectively. As they involve dealing with modified Bessel functions of the second kind, specifically K_0 and K_1 , we begin by collecting some of the well-known properties of these functions (for definitions, etc., see [2, Section 9.6]).

Recall that $K_0(r)$ and $K_1(r)$ are positive, monotonically decreasing functions of $r > 0$. They have the following asymptotic expansions as $r \rightarrow 0$:

$$K_0(r) = |\log r| + \log 2 - \gamma + O(r^2|\log r|), \tag{A.41}$$

$$K_1(r) = \frac{1}{r} + O(r|\log r|), \tag{A.42}$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant, while as $r \rightarrow \infty$ we have

$$K_{0,1}(r) = \sqrt{\frac{\pi}{2r}} e^{-r} (1 + O(r^{-1})). \tag{A.43}$$

Finally, we will need the following basic upper bound:

$$K_1(r) < \frac{r_0 K_1(r_0)}{r} \quad \forall r > r_0 > 0, \tag{A.44}$$

which easily follows from the strong maximum principle for the differential equation

$$r^2 K_1'' + r K_1' - (r^2 + 1) K_1 = 0 \tag{A.45}$$

satisfied by K_1 , the asymptotics in (A.43), and the facts that $g(r) := \frac{r_0 K_1(r_0)}{r}$ is a strict supersolution for the above equation.

We next express the Fourier transforms of several quantities involving the Belavin–Polyakov profile Φ and use them to compute its nonlocal energies. We also compute the contribution of Φ to the DMI energy.

Lemma A.5. *We have*

$$\mathcal{F}(\nabla\Phi')(k) = -4\pi K_1(|k|) \frac{k}{|k|} \otimes k, \tag{A.46}$$

$$\mathcal{F}(\Phi_3 + 1)(k) = 4\pi K_0(|k|). \tag{A.47}$$

Furthermore, it holds that

$$F_{\text{vol}}(\Phi') = \frac{3}{8}\pi^3, \tag{A.48}$$

$$F_{\text{surf}}(\Phi_3 + 1) = \frac{1}{8}\pi^3, \tag{A.49}$$

$$\int_{\mathbb{R}^2} 2\Phi' \cdot \nabla\Phi_3 \, dx = 8\pi. \tag{A.50}$$

Having obtained the above formulas, we are in a position to derive the formulas that are useful in obtaining an upper bound for the energy of a truncated Belavin–Polyakov profile.

Lemma A.6. *There exist universal constants $C > 0$ and $L_0 > 0$ such that for all $L \geq L_0$ the truncation Φ_L defined in (5.19) satisfies $\Phi_L \in \mathcal{A}$ and the estimates*

$$\int_{\mathbb{R}^2} |\nabla\Phi_L|^2 \, dx - 8\pi \leq \frac{4\pi}{L^2} + \frac{C \log^2 L}{L^3}, \tag{A.51}$$

$$\int_{\mathbb{R}^2} |\Phi'_L|^2 \, dx \leq 4\pi \log\left(\frac{4L^2}{e^{2(1+\gamma)}}\right) + \frac{C \log^2 L}{L}, \tag{A.52}$$

$$\int_{\mathbb{R}^2} 2\Phi'_L \cdot \nabla\Phi_{L,3} \, dx = 8\pi + O\left(L^{-\frac{1}{2}}\right), \tag{A.53}$$

$$F_{\text{vol}}(\Phi'_L) = \frac{3}{8}\pi^3 + O\left(L^{-\frac{1}{4}}\right), \tag{A.54}$$

$$F_{\text{surf}}(\Phi_{3,L}) = \frac{1}{8}\pi^3 + O\left(L^{-\frac{1}{2}}\right), \tag{A.55}$$

$$\int_{\mathbb{R}^2} |\nabla(\Phi_L - \Phi)|^2 \, dx \leq CL^{-2}, \tag{A.56}$$

$$\int_{\mathbb{R}^2} |\Phi_{3,L} - \Phi_3|^2 \, dx \leq CL^{-1}. \tag{A.57}$$

Lastly, a direct computation allows to establish an estimate for an integral appearing in the lower bound of the anisotropy energy in Sect. 6. Here and everywhere below the integrals and the series expansions have been carried out using MATHEMATICA 11.2.0.0 software. We have also verified these computations explicitly by hand, but the details are too tedious to be presented here.

Lemma A.7. *As $\mu \rightarrow \infty$, we have*

$$\int_0^\infty \frac{\mu r^3}{1 + \mu r^2} K_1^2(r) \, dr \geq \frac{1}{2} \log\left(\frac{4\mu}{e^{2\gamma+1}}\right) + O\left(\frac{\log^2 \mu}{\mu}\right). \tag{A.58}$$

Proof of Lemma A.5. For a radial function $U(x) := u(|x|)$ with $u \in L^1(\mathbb{R}^+, r dr)$ it is well known that the Fourier transform of U reduces to the Hankel transform

$$\mathcal{F}(U)(k) = 2\pi \int_0^\infty u(r) J_0(|k|r) r dr, \tag{A.59}$$

see for example [14, p. 336], where J_0 is the zeroth order Bessel function of the first kind. Due to $\Phi_3(x) + 1 = \frac{2}{|x|^2+1}$, $\nabla \cdot \Phi'(x) = -\frac{4}{(1+|x|^2)^2}$ and [14, Table 13.2], this allows us to compute

$$\mathcal{F}(\Phi_3 + 1)(k) = 4\pi K_0(|k|), \tag{A.60}$$

$$\mathcal{F}(\nabla \cdot \Phi')(k) = -4\pi |k| K_1(|k|). \tag{A.61}$$

As $\Phi'(x) = -\nabla \log(1 + |x|^2)$, there exists a tempered distribution H such that $\mathcal{F}(\Phi') = ikH$, and from equation (A.61) we get

$$|k|^2 H = 4\pi |k| K_1(|k|). \tag{A.62}$$

Therefore, we have

$$\mathcal{F}(\nabla \Phi')(k) = -k \otimes k H = -4\pi K_1(|k|) \frac{k}{|k|} \otimes k. \tag{A.63}$$

Inserting the expressions (A.61) and (A.60) into the representations (3.3) and (3.9), respectively, we obtain

$$F_{\text{surf}}(\Phi_3 + 1) = 4\pi \int_0^\infty s^2 K_0^2(s) ds = \frac{1}{8}\pi^3, \tag{A.64}$$

$$F_{\text{vol}}(\Phi') = 4\pi \int_0^\infty s^2 K_1^2(s) ds = \frac{3}{8}\pi^3. \tag{A.65}$$

Lastly, (A.50) follows by direct computation. \square

Proof of Lemma A.6. *Step 1: Proof of estimate (A.51).*

For $L > 1$, we first note that f_L is piecewise smooth, so in view of (A.43) we have $\Phi_L + e_3 \in H^1(\mathbb{R}^2; \mathbb{S}^2)$. A direct computation also shows that $\mathcal{N}(\Phi_L) = 1$, as it should. Therefore, admissibility of Φ_L for large enough L would follow, as soon as we establish (A.51).

In the following, all estimates and expansions are valid for $L \geq L_0$ with $L_0 > 0$ sufficiently large. We begin by observing that

$$|\nabla \Phi_L(x)|^2 = \frac{(f'_L)^2(|x|)}{1 - f_L^2(|x|)} + \frac{f_L^2(|x|)}{|x|^2} \tag{A.66}$$

and thus an explicit calculation gives

$$\int_{B_{\sqrt{L}}(0)} |\nabla \Phi_L|^2 dx = \frac{8\pi L}{1 + L}. \tag{A.67}$$

With the help of (A.44) we then obtain for all $r > L^{\frac{1}{2}}$:

$$f_L(r) < \frac{L^{\frac{1}{2}}}{r} f\left(L^{\frac{1}{2}}\right) \leq \frac{2}{r}. \tag{A.68}$$

Consequently, we have for $r > L^{\frac{1}{2}}$ that

$$\frac{1}{1 - f_L^2(r)} = 1 + f_L^2(r) + \frac{f_L^4(r)}{1 - f_L^2(r)} \leq 1 + \frac{4}{r^2} + \frac{C}{r^4} \leq 1 + \left(1 + \frac{C}{L}\right) \frac{4}{r^2}. \tag{A.69}$$

We can insert this estimate into the identity (A.66) and compute for all $x \in B_{\sqrt{L}}^c(0)$:

$$|\nabla\Phi_L(x)|^2 \leq \frac{4L^2K_1^2\left(\frac{|x|}{L}\right) + (|x|^2 + 4(1 + CL^{-1}))\left(K_0\left(\frac{|x|}{L}\right) + K_2\left(\frac{|x|}{L}\right)\right)^2}{L(L + 1)^2K_1^2\left(L^{-\frac{1}{2}}\right)|x|^2}, \tag{A.70}$$

where K_2 is the modified Bessel function of the second kind. Integrating in radial coordinates and then expanding in the powers of L^{-1} yields

$$\int_{B_{\sqrt{L}}^c(0)} |\nabla\Phi_L|^2 dx \leq \frac{8\pi}{L} - \frac{4\pi}{L^2} + O\left(\frac{\log^2 L}{L^3}\right), \tag{A.71}$$

which together with equation (A.67) finally implies (A.51). In particular, $\Phi_L \in \mathcal{A}$ for all $L \geq L_0$ with some $L_0 > 0$ sufficiently large.

Step 2: Estimate the rates of convergence of Φ_L to Φ in several norms.

We start with an L^2 -estimate for the out-of-plane components. For $r \geq L^{\frac{1}{2}}$, by the estimate (A.68) we have

$$\begin{aligned} \left(\sqrt{1 - f^2(r)} - \sqrt{1 - f_L^2(r)}\right)^2 &= \frac{(f_L^2(r) - f^2(r))^2}{\left(\sqrt{1 - f^2(r)} + \sqrt{1 - f_L^2(r)}\right)^2} \\ &\leq \frac{C}{r^2} (f_L(r) - f(r))^2. \end{aligned} \tag{A.72}$$

Thus the right-hand side decays as r^{-4} for $r \rightarrow \infty$, and we have

$$\int_{\mathbb{R}^2} |\Phi_3 - \Phi_{L,3}|^2 dx \leq CL^{-1}. \tag{A.73}$$

Similarly, together with (A.68), $f(r) \leq \frac{2}{r}$ for $r > 0$ and the fact that $|\nabla\Phi(x)|^2 \leq C|x|^{-4}$ for $x \in \mathbb{R}^2$ we obtain

$$\int_{\mathbb{R}^2} |\Phi_L - \Phi|^2 |\nabla\Phi|^2 dx \leq CL^{-2}. \tag{A.74}$$

Combining this with Lemma A.4 and the estimate (A.51) we get the bound (A.56), meaning

$$\int_{\mathbb{R}^2} |\nabla(\Phi_L - \Phi)|^2 dx \leq CL^{-2}. \tag{A.75}$$

To handle the volume charges, we need L^p -estimates for $p \neq 2$ in view of the fact that $\Phi' \notin L^2(\mathbb{R}^2; \mathbb{R}^2)$. To this end, we can use estimates (A.68) and $f(r) \leq \frac{2}{r}$ for $r > 0$ to obtain

$$\int_{\mathbb{R}^2} |\Phi'_L - \Phi'|^4 dx \leq CL^{-1}. \tag{A.76}$$

Additionally, we will need a matching $L^{\frac{4}{3}}$ -estimate for $\nabla\Phi'_L$ in order to apply Lemma 3.1 later. For $x \in B^c_{\sqrt{L}}(0)$ we use (A.66) and (A.68) to calculate

$$|\nabla\Phi'_L(x)| \leq C \left(|f'_L(|x|)| + \frac{f_L(|x|)}{|x|} \right). \tag{A.77}$$

By the identity $K'_1(r) = -K_0(r) - \frac{K_1(r)}{r}$ for all $r > 0$, the estimate (A.68) and the expansions (A.41) and (A.42) we have for all $x \in B^c_{\sqrt{L}}(0)$:

$$|\nabla\Phi'_L(x)| \leq C \left(\frac{1}{L^2} + \frac{1}{|x|^2} \right) e^{-\frac{|x|}{L}}. \tag{A.78}$$

Integrating this bound we arrive at

$$\int_{B^c_{\sqrt{L}}(0)} |\nabla\Phi'_L(x)|^{\frac{4}{3}} dx \leq CL^{-\frac{2}{3}}. \tag{A.79}$$

The remaining integral over $B_{\sqrt{L}}(0)$ is bounded since $|\nabla\Phi'| \in L^{\frac{4}{3}}(\mathbb{R}^2)$ and we get

$$\int_{\mathbb{R}^2} |\nabla\Phi'_L|^{\frac{4}{3}} dx \leq C. \tag{A.80}$$

Step 3: Estimate the anisotropy, DMI and stray field contributions.

First compute the contribution to the anisotropy energy from the core region:

$$\int_{B_{\sqrt{L}}(0)} |\Phi'_L|^2 dx = 4\pi \left(\frac{1}{L+1} + \log(L+1) - 1 \right). \tag{A.81}$$

Next, evaluate the contribution of the tail region:

$$\int_{B^c_{\sqrt{L}}(0)} |\Phi'_L|^2 dx = \frac{4\pi L^2 \left(K_0^2 \left(L^{-\frac{1}{2}} \right) + 2L^{\frac{1}{2}} K_1 \left(L^{-\frac{1}{2}} \right) K_0 \left(L^{-\frac{1}{2}} \right) - K_1^2 \left(L^{-\frac{1}{2}} \right) \right)}{(L+1)^2 K_1^2 \left(L^{-\frac{1}{2}} \right)}. \tag{A.82}$$

Combining these two expressions and expanding in L^{-1} yields (A.52).

As Φ'_L decays exponentially at infinity by virtue of estimate (A.43) and $\Phi_3 + 1$ decays as r^{-2} , we can integrate by parts in the difference of the DMI terms

$$\begin{aligned} & \int_{\mathbb{R}^2} (\Phi'_L \cdot \nabla \Phi_{L,3} - \Phi' \cdot \nabla \Phi_3) \, dx \\ &= \int_{\mathbb{R}^2} ((\Phi_3 + 1) \nabla \cdot \Phi' - (\Phi_{L,3} + 1) \nabla \cdot \Phi'_L) \, dx \\ &= \int_{\mathbb{R}^2} ((\Phi_3 + 1) \nabla \cdot (\Phi' - \Phi'_L) - (\Phi_{L,3} - \Phi_3) \nabla \cdot \Phi'_L) \, dx. \end{aligned} \tag{A.83}$$

By the facts that $\Phi_3 + 1 \in L^2(\mathbb{R}^2)$, the estimates (A.75) and (A.73), and the Cauchy–Schwarz inequality we deduce

$$\left| \int_{\mathbb{R}^2} (\Phi'_L \cdot \nabla \Phi_{L,3} - \Phi' \cdot \nabla \Phi_3) \, dx \right| \leq CL^{-\frac{1}{2}}. \tag{A.84}$$

Together with (A.50), this then yields (A.53).

Similarly, by Lemma A.5 we only need to estimate the error terms in the stray field contributions to prove estimates (A.54) and (A.55). By bilinearity and the estimates, (3.8) with $p = 4$, (A.76), (A.80) and $|\nabla \Phi'| \in L^{\frac{4}{3}}(\mathbb{R}^2; \mathbb{R}^2)$ we get

$$\begin{aligned} |F_{\text{vol}}(\Phi'_L) - F_{\text{vol}}(\Phi')| &\leq |F_{\text{vol}}(\Phi'_L + \Phi', \Phi'_L - \Phi')| \\ &\leq C \|\Phi'_L - \Phi'\|_4 \|\nabla(\Phi'_L + \Phi')\|_{\frac{4}{3}} \\ &\leq CL^{-\frac{1}{4}}. \end{aligned} \tag{A.85}$$

A similar calculation for the surface term gives

$$|F_{\text{surf}}(\Phi_{L,3}) - F_{\text{surf}}(\Phi_3)| \leq C \|\Phi_{L,3} - \Phi_3\|_2 \|\nabla(\Phi_{L,3} + \Phi_3)\|_2 \tag{A.86}$$

We can now apply the interpolation inequality (3.7) together with the estimates (A.51) and (A.73) in order to obtain

$$|F_{\text{surf}}(\Phi_{L,3}) - F_{\text{surf}}(\Phi_3)| \leq CL^{-\frac{1}{2}}, \tag{A.87}$$

concluding the proof. \square

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