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Magnetic Domains in Thin Ferromagnetic Films with Strong Perpendicular Anisotropy

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Abstract

We investigate the scaling of the ground state energy and optimal domain patterns in thin ferromagnetic films with strong uniaxial anisotropy and the easy axis perpendicular to the film plane. Starting from the full three-dimensional micromagnetic model, we identify the critical scaling for which the transition from single domain to multidomain ground states such as bubble or maze patterns occurs as the film thickness goes to zero and the lateral extent goes to infinity. Furthermore, we analyze the asymptotic behavior of the energy in these two asymptotic regimes. In the single domain regime, the energy Γ-converges towards a much simpler two-dimensional and local model. In the multidomain regime, we derive the scaling of the minimal energy and deduce a scaling law for the typical domain size.

1. Introduction

Ferromagnetic materials are an important class of solids which have played an indispensable role in data storage technologies of the digital age [22,51,65]. Their utility for technological applications stems from the basic physical property of ferromagnets to exhibit spatially ordered magnetization patterns—magnetic domains—under a variety of conditions [32]. The mechanisms behind the magnetic domain formation can be quite complex, but usually domain patterns may be understood from the energetic considerations based on the micromagnetic modeling framework [9,20,32]. Starting with the early works of Landau and Lifshitz [46] and Kittel [39], ground states of various ferromagnetic systems have been the subject of extensive studies in the physics community (see [32] and references therein), and more recently in the mathematical literature (for a review, see [20]). In particular, within the micromagnetic framework the ground state domain structure of macroscopically thick uniaxial ferromagnetic films is by now fairly well understood mathematically in terms of the energy and length scales, as well as some of the qualitative properties of the domains [14,15,17,40,59]. In contrast, apart from only a handful of studies [16,28,50,54], the vast majority of mathematical treatments of microscopically thin ferromagnetic films deal with the situation in which the magnetization prefers to lie in the film plane (see, for example, [12,13,18,19,26,33–35,41,44,52]; this list is certainly not complete). Thus, one of the fundamental open problems in the theory of uniaxial ferromagnets is to rigorously characterize their ground states in the case of films of vanishing thickness when the magnetization prefers to align normally to the film plane (for various ansatz-based computations in the physics literature, see [21,38,42,57]). This problem is the main subject of the present paper.

Recent advances in nanofabrication allow an unprecedented degree of spatial resolution, with features of only a few atomic layers in thickness and tens of nanometers laterally for planar structures [66], enabling synthesis of ultrathin ferromagnetic films and multilayer structures with novel material properties. Over the last decade, there has been a major focus on films with thickness of only a few atomic layers, primarily due to their promising applications in spintronics [2]. One of the important features of these films is the emergence of perpendicular magnetocrystalline anisotropy due to the increased importance of surface effects [30,36], which favor the magnetization vector to lie along the normal to the film plane. As a result, the magnetization may exhibit either stripe or bubble domain phases depending on the applied external field and other factors [31,56,61,64,68]. We note that studies of magnetic bubble domains in relatively thick films have a long history in the context of magnetic memory devices (see, for example, [42] and the book [48]). However, the occurrence of additional physical effects in ultrathin films, such as spin transfer torque [7,24,37], Dzyaloshinskii-Moriya interaction [5,60] and electric field-controlled perpendicular magnetic anisotropy [23,49] allow for much greater manipulation of the domain patterns, resulting in a renewed attention to bubble domains from experimentalists [37,43,62,63,67]. In particular, the topological characteristics of the bubble domain patterns in these materials are of great current interest [8,24,55]. These considerations further motivate the present study of the basic problem noted at the end of the preceding paragraph.

In this paper, we are interested in deriving a reduced two-dimensional model for ultrathin ferromagnetic films with perpendicular anisotropy and using it to asymptotically characterize the observed ground states and, more generally, all low energy states in films of large spatial extent. Our starting point is the three-dimensional micromagnetic energy functional, coming from the continuum theory of uniaxial bulk ferromagnets [45]. In a periodic setting and after a suitable non-dimensionalization, the micromagnetic energy is

$$\mathscr{E}[m] = \int_{\mathbb{T}_\ell^2 \times (0,t)} \left(|\nabla m|^2 + Q \left(m_1^2 + m_2^2 \right) \right) \mathrm{d}x + \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h|^2 \, \mathrm{d}x. \tag{1.1}$$

Here, $\mathbb{T}_\ell^2 \times (0,t)$ denotes the space occupied by a ferromagnetic sample in the form of a film of thickness t, period ℓ and whose magnetocrystalline easy axis is normal to the film plane, $m \in H^1(\mathbb{T}_\ell^2 \times (0,t); \mathbb{S}^2)$ is the magnetization vector and $h : \mathbb{T}_\ell^2 \times \mathbb{R} \to \mathbb{R}^3$ is the stray field, which is uniquely determined by m via the distributional solution of the static Maxwell's equations

$$\nabla \cdot (h + m) = 0 \quad \text{and} \quad \nabla \times h = 0 \qquad \text{in } \mathbb{T}_\ell^2 \times \mathbb{R}, \tag{1.2}$$

where m has been extended by zero outside $\mathbb{T}_\ell^2 \times (0, t)$. Up to a sign, h equals the Helmholtz projection of m onto the space of gradients. Note that the energy depends on m in a nonlocal way. To emphasize this fact, we sometimes use the notation $\mathscr{H}[m] := h$ to denote the solution of (1.2). Finally, $Q > 0$ is the material quality factor. For an introduction to micromagnetic modeling we refer to, for example, [20,32]. Note that additional physical effects due to the external applied magnetic field and the film surfaces may be easily incorporated and would lead to the same type of a reduced two-dimensional model [54]. Also note that it is well known that the infimum of $\mathscr{E}$ is attained (see, for example, [20]).

Since our focus is on materials with perpendicular magnetic anisotropy, we assume that $Q > 1$ (for a detailed explanation, see the following section). The high magnetocrystalline anisotropy leads to magnetizations that are predominantly perpendicular to the film plane. It is well-known that such materials feature magnetizations that consist of one or many regions of nearly constant magnetization, called *magnetic domains*, separated by interfaces, called *domain walls*. Note that the energy in (1.1) depends on the three dimensionless parameters ℓ, t and Q. To describe such magnetic domains, we investigate the asymptotic behavior of the energy in (1.1) for thin films (that is $t \ll 1$) with large extension in the film plane (that is $\ell \gg 1$). In this work, we identify the critical scaling for the size of the sample where a transition from single domain states to multidomain states occurs. Moreover, we analyze the asymptotic behavior of the energy in the two regimes separated by this transition.

In the subcritical regime, the global minimizers are the single domain states $m = \pm e_3$. We derive the asymptotic behavior of the energy in this regime in the framework of Γ-convergence. The reduced energy turns out to be much simpler than the full energy, in particular, it is two-dimensional and local. In the supercritical regime, which lies beyond the transition towards multidomain configurations, we establish the scaling of the energy (up to a multiplicative constant) and characterize sequences that achieve this scaling. Our analysis shows that the magnetization in this regime consists of several domains and suggests that the typical distance s between domain walls (with all lengths in the units of the so called exchange length, a material parameter [32]) scales as

$$s \sim \frac{e^{2\pi t^{-1}\sqrt{Q-1}}}{\sqrt{Q-1}}.$$

We will show that in the regimes we consider the leading order of the micromagnetic energy, upon rescaling and subtracting a constant, is given by the following two-dimensional functional defined for $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$:

$$F_{\varepsilon,\lambda}[m] = \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon}(1 - m_3^2) \right) \mathrm{d}x - \frac{\lambda}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 \,\mathrm{d}x. \tag{1.3}$$

In (1.3), $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denotes the square flat torus of unit side length representing a rescaling of the two-dimensional footprint of the ferromagnetic film, ε is the renormalized Bloch wall width and λ is the renormalized film thickness (see the following section for the precise definitions). We note that a similar result for a closely related problem of a Ginzburg-Landau energy with dipolar interactions has been obtained in [53], where the meaning of the asymptotic equivalence between the full energy of three-dimensional configurations and the reduced energy of their e_3-averages is discussed in more detail. Also, for ε and λ fixed the infimum of $F_{\varepsilon,\lambda}$ is clearly attained.

The main part of our analysis is concerned with the asymptotic behavior of (1.3) as $\varepsilon \to 0$ for different values of $\lambda > 0$. Note that the last term in (1.3) occurs with a negative sign and hence prefers oscillations of m_3. As it turns out, the value of the parameter λ is crucial—in fact, we will show that the asymptotic behavior changes at $\lambda = \lambda_c$, where

$$\lambda_c = \frac{\pi}{2}, \tag{1.4}$$

which is a singular point in the terminology of [6]. For $\lambda < \lambda_c$ the Γ-limit $F_{*,\lambda} := \Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\lambda}$ measures the length of the interface separating regions with $m = \mathrm{e}_3$ and $m = -\mathrm{e}_3$ (see Theorem 2.5)

$$F_{*,\lambda}[m] = \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_3| \,\mathrm{d}x, & \text{for } m \in L^1(\mathbb{T}^2; \{\pm e_3\}), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.5}$$

Note that the last term in (1.3) leads to a reduction of the interfacial cost by $\frac{\lambda}{\lambda_c}$ compared to the classical result [1] for $\lambda = 0$. On the other hand, for $\lambda > \lambda_c$, the scaling of the minimal energy changes (see Theorem 2.6)

$$\min_{m\in H^1(\mathbb{T}^2;\mathbb{S}^2)} F_{\varepsilon,\lambda}[m] \sim -\frac{\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|} \xrightarrow{\varepsilon\to 0} -\infty,$$

and sequences (m_ε) which achieve the optimal scaling $F_{\varepsilon,\lambda}[m_\varepsilon] \sim \min F_{\varepsilon,\lambda}$ are highly oscillatory in the sense that

$$\int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| \,\mathrm{d}x \sim \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \xrightarrow{\varepsilon\to 0} +\infty.$$

Furthermore, for $\lambda \geqq \lambda_c$, the leading order contributions of all three terms in (1.3) cancel out. The main difficulty in the proof is to find asymptotically optimal estimates for the non-local term.

A reduction of the full three-dimensional micromagnetic energy to a local two-dimensional model in the thin film limit was first established rigorously in [28]. Subsequently, several thin film regimes for magnetically soft materials have been identified and analyzed, see for example [10–12,19,34,41,44,52]. However, since we consider materials with high perpendicular anisotropy, our setting is considerably different.

To conclude our introductory remarks, we note that the behavior of the material changes markedly when the film can no longer be considered to be thin. In [14] the scaling of the ground state energy was identified for the two-dimensional micromagnetic model and in [15] for the three-dimensional model. Magnetizations with optimal energy involve the so-called branching domain patterns which become finer and finer as they approach the boundary of the sample. When the ferromagnetic sample is exposed to a critical external field, a transition between a uniform and a branching domain pattern occurs. The critical field strength and the scaling of the micromagnetic energy for this regime were derived in [40]. In our regime, the thickness of the film is so small that this does not only exclude the branching patterns that occur in bulk samples, but actually forces the magnetization to become constant in the direction normal to the film plane.

Notation: For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we write $x = (x', x_3)$, where $x' = (x_1, x_2)$ is the projection of x onto the film plane. Similarly, in $\mathbb{R}^3$ we write $\nabla = (\nabla', \partial_3)$, where $\nabla' = (\partial_1, \partial_2)$ is the in-plane part of the gradient. The square flat torus with side length $\ell > 0$ is denoted by $\mathbb{T}^2_\ell := (\mathbb{R}^2/\ell\mathbb{Z}^2)$, and we abbreviate $\mathbb{T}^2 := \mathbb{T}^2_1$. We frequently identify functions defined on $\mathbb{T}^2_\ell$ with doubly ℓ-periodic functions defined on $\mathbb{R}^2$. For $u \in L^1(\mathbb{T}^2_\ell \times (0, t))$ we write $\overline{u} \in L^1(\mathbb{T}^2_\ell)$ to denote the e_3-average

$$\overline{u}(x') := \frac{1}{t} \int_0^t u(x', x_3) \mathrm{d}x_3.$$

Moreover, for every $v \in L^1(\mathbb{T}^2_\ell)$ we write $\chi_{(0,t)} v \in L^1(\mathbb{T}^2_\ell \times (0, t))$ to denote the function $(\chi_{(0,t)} v)(x', x_3) = \chi_{(0,t)}(x_3) v(x')$. By $\int_{\mathbb{T}^2_\ell} |\nabla \overline{u}| \, \mathrm{d}x$ we denote the total variation of $\overline{u}$.

The expression $f(x) \lesssim g(x)$ means that there exists a universal constant $C > 0$ such that the inequality $f(x) \leqq C g(x)$ holds for every x. The symbol $\gtrsim$ is defined analogously and we write $\sim$ if both $\lesssim$ and $\gtrsim$ hold.

For future reference, we now fix the constants in the definition of the Fourier series. For $f \in L^2(\mathbb{T}^2_\ell)$, we write

$$\widehat{f}_k := \int_{\mathbb{T}^2_\ell} e^{ik\cdot x} f(x) \, \mathrm{d}x, \qquad \text{where } k \in \frac{2\pi}{\ell} \mathbb{Z}^2.$$

The inverse Fourier transform is then given by

$$f(x) = \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2} e^{-ik\cdot x} \widehat{f}_k \qquad \text{for } x \in \mathbb{T}^2_\ell.$$

Parseval's theorem then states that

$$\int_{\mathbb{T}^2_\ell} f^*(x) g(x) \, \mathrm{d}x = \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2} \widehat{f}^*_k \widehat{g}_k \qquad \text{for } f, g \in L^2(\mathbb{T}^2_\ell),$$

where f^* is the complex conjugate of f. Furthermore, for $s \in (0, 1)$, we write

$$\int_{\mathbb{T}^2_\ell} |\nabla^s u|^2 \, \mathrm{d}x := \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2} |k|^{2s} |\widehat{u}_k|^2.$$

For $s = 1/2$ we will also use the following well-known real space representation of the (square of the) homogeneous $H^{1/2}(\mathbb{T}_\ell^2)$-norm:

$$\int_{\mathbb{T}_\ell^2} |\nabla^{1/2} u|^2 \, \mathrm{d}x = \frac{1}{4\pi} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2} \frac{|u(x+y) - u(x)|^2}{|y|^3} \, \mathrm{d}y \, \mathrm{d}x. \tag{1.6}$$

Lastly, with the usual abuse of notation, for $\varepsilon \to 0$ we will refer to $(m_\varepsilon) \in H^1(\mathbb{T}_\ell^2; \mathbb{S}^2)$ as a sequence, implying the sequence of $m_{\varepsilon_k} \in H^1(\mathbb{T}_\ell^2; \mathbb{S}^2)$ for some sequence of $\varepsilon_k \to 0$ as $k \to \infty$. Similarly, when we refer to the family of functionals $(F_{\varepsilon,\lambda})$ we are always dealing with sequences $F_{\varepsilon_k,\lambda}$.

2. Main Results

Our main result is the identification of two thin-film regimes and the derivation of the asymptotic behavior of the energy in these regimes. We will state the results for a suitably rescaled version of $\mathscr{E}(m) - \mathscr{E}(e_3)$, the energy relative to that of the monodomain state.

For $Q > 1$ and $\ell\sqrt{Q-1} > 2$, which is assumed throughout the rest of the paper, it is convenient to introduce the new parameters ε, λ (replacing ℓ, t):

$$\varepsilon := \frac{1}{\ell\sqrt{Q-1}} \qquad \text{and} \qquad \lambda := \frac{t \, \ln\left(\ell\sqrt{Q-1}\right)}{4\sqrt{Q-1}}. \tag{2.1}$$

We rescale the domain of the ferromagnetic film to a fixed domain by means of the anisotropic transformation $\mathbb{T}_\ell^2 \times (0,t) \to \mathbb{T}^2 \times (0,1)$ with $(x_1, x_2, x_3) \mapsto \left(\frac{x_1}{\ell}, \frac{x_2}{\ell}, \frac{x_3}{t}\right)$ and study the energy $E_{\varepsilon,\lambda} : L^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2) \to \mathbb{R} \cup \{+\infty\}$, defined by

$$E_{\varepsilon,\lambda}[m] := \begin{cases} \dfrac{\mathscr{E}[m(\ell\cdot, \ell\cdot, t\cdot)] - \ell^2 t}{2\ell t\sqrt{Q-1}} & \text{for } m \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

noting that $\mathscr{E}(e_3) = \ell^2 t$, as can be verified by an explicit computation. We will show that the energy $E_{\varepsilon,\lambda}$ in (2.2) may be well approximated, in a certain sense, by the reduced two-dimensional energy $F_{\varepsilon,\lambda}$ introduced in (1.3). For both functionals, we study the limit $\varepsilon \to 0$ for fixed $\lambda > 0$, corresponding to the limit of vanishing thickness with suitable rescaling in lateral direction, while keeping Q as a fixed material parameter. For both functionals, we will identify two regimes (recall that λ_c was introduced in (1.4)):

- the subcritical regime: $0 < \lambda < \lambda_c$,
- the supercritical regime: $\lambda_c < \lambda$.

We will show that in the subcritical regime, single domain states appear. Physically this corresponds to relatively small samples where there is not sufficient space to accommodate multidomain states (as imposed by the periodicity). On the other hand, low energy states in the supercritical regime are characterized by the presence of multidomain states. We also give some results about the transition at $\lambda = \lambda_c$.

The asymptotic behavior of $E_{\varepsilon,\lambda}$ in the subcritical regime is characterized in the following theorem:

Theorem 2.1. *(Subcritical regime) Let $Q > 1$ and $\lambda \in (0, \lambda_c)$. Then as $\varepsilon \to 0$ the following holds:*

(i) *Compactness: For every sequence $(m_\varepsilon) \in L^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ with*

$$\limsup_{\varepsilon\to 0} E_{\varepsilon,\lambda}[m_\varepsilon] < +\infty,$$

there exists a sub-sequence and $m \in BV(\mathbb{T}^2; \{\pm e_3\})$ such that

$$m_\varepsilon \to m\chi_{(0,1)} \quad \text{in } L^1\left(\mathbb{T}^2 \times (0,1); \mathbb{R}^3\right);$$

(ii) *Γ-Convergence: The sequence of functionals $(E_{\varepsilon,\lambda})$ Γ-converges towards $F_{*,\lambda}$, in the following sense:*
- *Every sequence $(m_\varepsilon) \in L^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ with $m_\varepsilon \to m\chi_{(0,1)}$ in $L^1(\mathbb{T}^2 \times (0,1); \mathbb{R}^3)$ for some $m \in L^1(\mathbb{T}^2; \{\pm e_3\})$ satisfies*

$$\liminf_{\varepsilon\to 0} E_{\varepsilon,\lambda}[m_\varepsilon] \geqq F_{*,\lambda}[m]. \qquad \textit{(liminf inequality)}$$

- *For every $m \in L^1(\mathbb{T}^2, \{\pm e_3\})$ there exists a sequence $(m_\varepsilon) \in L^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ with $m_\varepsilon \to m\chi_{(0,1)}$ in $L^1(\mathbb{T}^2 \times (0,1); \mathbb{R}^3)$ such that*

$$\limsup_{\varepsilon\to 0} E_{\varepsilon,\lambda}[m_\varepsilon] \leqq F_{*,\lambda}[m]. \qquad \textit{(recovery sequence)}$$

Theorem 2.1 shows in particular that single domain states are preferred asymptotically in the subcritical regime, since the limit energy is local and prefers constant magnetizations. We note that a related model has been studied in a subcritical regime in the context of lipid bilayer membranes, in which the nonlocal energy vanishes in the Γ-limit, but affects the leading order constant of the limit energy [25].

On the other hand, the next theorem shows that the energy leads to pattern formation in the supercritical regime.

Theorem 2.2. *(Supercritical regime) Let $Q > 1$ and $\lambda > \lambda_c$. Then there is $\varepsilon_0 = \varepsilon_0(\lambda, Q) > 0$ such that the minimal energy in (2.2) satisfies the bounds*

$$-\frac{C\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|} \leqq \min E_{\varepsilon,\lambda} \leqq -\frac{c\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|} \tag{2.3}$$

for some universal constants $0 < c < C$ and for any $\varepsilon \in (0, \varepsilon_0)$.

We note that by the direct method of the calculus of variations the minimum energy in the above theorem is indeed attained. Configurations achieving the minimal scaling of energy, including minimizers, can be characterized as follows:

Theorem 2.3. *Let $Q > 1$ and $\lambda > \lambda_c$. Then there is $\varepsilon_0 = \varepsilon_0(\lambda, Q) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, any $\gamma > 0$ and all $m \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ which satisfy*

$$E_{\varepsilon,\lambda}[m] \leqq -\frac{\gamma\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|}, \tag{2.4}$$

we have

$$\text{(i)} \quad \int_{\mathbb{T}^2\times(0,1)} |m - \overline{m}\chi_{(0,1)}|^2 \,\mathrm{d}x \leq C_\gamma \lambda^3 (Q-1)^2 \frac{\varepsilon^{\frac{\lambda_c}{\lambda}}}{|\ln\varepsilon|^3}, \tag{2.5}$$

$$\text{(ii)} \quad \int_{\mathbb{T}^2\times(0,1)} \left(m_1^2 + m_2^2\right) \mathrm{d}x \leqq C_\gamma \varepsilon^{\frac{\lambda_c}{\lambda}}, \tag{2.6}$$

$$\text{(iii)} \quad c_\gamma \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \leqq \int_{\mathbb{T}^2} |\nabla\overline{m}_3| \,\mathrm{d}x \leqq C_\gamma \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}, \tag{2.7}$$

$$\text{(iv)} \quad \int_{\mathbb{T}^2\times(0,1)} \left(\frac{\varepsilon|\nabla m|^2}{2} + \frac{1-m_3^2}{2\varepsilon}\right) \mathrm{d}x \leqq \left(1 + \frac{C_\gamma\lambda}{|\ln\varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla\overline{m}_3| \,\mathrm{d}x \tag{2.8}$$

for some constants $0 < c_\gamma < C_\gamma$ depending only on γ.

We take a moment to interpret the statements (i)–(iv) in Theorem 2.3 above. Item (i) shows that for configurations with the scaling of the minimal energy the magnetization is approximately two-dimensional, that is independent of the thickness variable. Item (ii) implies that the magnetization is mostly perpendicular to the film (that is $m \approx \pm e_3$). Item (iii) is an estimate for the total length of the domain walls which confirms the conjecture from the physics community (see also Remark 2.4). Item (iv) in Theorem 2.3 indicates that domain walls approximate Bloch walls of thickness (in the original, physical variables) proportional to $\varepsilon\ell = 1/\sqrt{Q-1}$ for which the left hand side of estimate (iv) is exactly $\int_{\mathbb{T}^2} |\nabla m_3| \,\mathrm{d}x$. Indeed, (iv) implies that m approximates the optimal Bloch wall profile in an L^2-sense:

$$\int_{\mathbb{T}^2\times(0,1)} \left(\frac{\sqrt{\varepsilon}|\nabla m_3|}{\sqrt{1-m_3^2}} - \frac{\sqrt{1-m_3^2}}{\sqrt{\varepsilon}}\right)^2 \mathrm{d}x \;\leqq\; \frac{2C_\gamma\lambda}{|\ln\varepsilon|} \int_{\mathbb{T}^2} |\nabla\overline{m}_3| \,\mathrm{d}x,$$

with the convention $\frac{|\nabla m_3|}{\sqrt{1-m_3^2}} = 0$ if $|m_3| = 1$.

Remark 2.4. *(Scaling of domain size)* The estimate (2.7), written in terms of the original physical variables, confirms the exponential dependence of the typical distance s between neighboring walls on the inverse thickness t^{-1}, as was already observed in ansatz-based computations in [38] for a two-dimensional sharp interface model. Indeed, in view of Theorem 2.3(iii), the total length of the domain walls w satisfies (for fixed γ)

$$w := \frac{\ell}{2} \int_{\mathbb{T}^2} |\nabla\overline{m}_3| \,\mathrm{d}x \overset{(2.7)}{\sim} \ell\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \overset{(2.1)}{=} \ell^2\sqrt{Q-1}\, e^{-\frac{2\pi\sqrt{Q-1}}{t}}. \tag{2.9}$$

In particular, we have the estimate

$$s := \frac{\ell^2}{w} \overset{(2.9)}{\sim} \frac{e^{2\pi t^{-1}\sqrt{Q-1}}}{\sqrt{Q-1}},$$

where s can be interpreted as the typical domain size. In fact, we expect that the stray field energy induces a repulsive interaction of (nearest) neighboring domain walls and leads to an approximately equidistant spacing of the walls.

We now formulate results analogous to the ones in the previous section, but for the reduced energy $F_{\varepsilon,\lambda}$. The relation between the full energy $E_{\varepsilon,\lambda}$ and the reduced two-dimensional energy $F_{\varepsilon,\lambda}$ will be made rigorous in section 5, and, indeed, most of the results stated above for the energy $E_{\varepsilon,\lambda}$ are corollaries of those for $F_{\varepsilon,\lambda}$ presented below. The reason to formulate our results also in terms of $F_{\varepsilon,\lambda}$ is two-fold: on one hand, the main ideas are easier to understand when they are not obscured by additional difficulties arising from the reduction to a two-dimensional model and the stray-field energy approximation; on the other hand, the energy $F_{\varepsilon,\lambda}$ itself may be considered as a good starting point for modeling ultrathin ferromagnetic layers with perpendicular magnetic anisotropy. The asymptotic behavior of the reduced energy $F_{\varepsilon,\lambda}$ in the subcritical regime is summarized in the following theorem:

Theorem 2.5. *(Subcritical regime) Let $\lambda \in (0, \lambda_c)$ and $F_{\varepsilon,\lambda}$ as defined in (1.3). Then as $\varepsilon \to 0$ the following holds:*

(i) *Compactness: Every sequence (m_ε) in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with $\limsup_{\varepsilon\to 0} F_{\varepsilon,\lambda}[m_\varepsilon] < +\infty$ converges in $L^1(\mathbb{T}^2)$ (up to extracting a subsequence) towards a limit in $BV(\mathbb{T}^2; \{\pm e_3\})$;*
(ii) *Γ-convergence: The family of functionals $(F_{\varepsilon,\lambda})$ Γ-converges with respect to the $L^1(\mathbb{T}^2)$-topology towards $F_{*,\lambda}$, defined in (1.5).*

The next theorem is concerned with the minimal energy and the structure of low energy states in the supercritical regime.

Theorem 2.6. *(Supercritical regime) There are universal constants $0 < \delta < 1 < K$ such that for*

$$0 < \varepsilon < K^{\frac{\lambda}{\lambda_c - \lambda}} \qquad \text{and} \qquad \lambda_c < \lambda < \delta |\ln \varepsilon|,$$

the minimal energy of the family of functionals $(F_{\varepsilon,\lambda})$ satisfies

$$-C\,\frac{\lambda \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln \varepsilon|} \leqq \min F_{\varepsilon,\lambda} \leqq -c\,\frac{\lambda \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln \varepsilon|}$$

for some universal constants $0 < c < C$. Moreover, for all profiles $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ achieving the optimal scaling in the sense that

$$F_{\varepsilon,\lambda}[m] \leqq -\gamma\,\frac{\lambda \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln \varepsilon|}$$

for some $\gamma > 0$, we have

$$\int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x \leqq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_3^2}{2\varepsilon} \right) \mathrm{d}x \leqq \frac{\lambda}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 \, \mathrm{d}x. \tag{2.10}$$

Furthermore, if A and B are any of the three quantities in (2.10), we have

$$c_\gamma \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \leqq A \leq C_\gamma \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \qquad \text{and} \qquad |A - B| \leqq \widetilde{C}_\gamma \frac{\lambda}{|\ln \varepsilon|} A \tag{2.11}$$

for some positive constants c_γ, C_γ and $\widetilde{C}_\gamma$ which depend only on γ.

Under the assumptions of Theorem 2.6, statements analogous to (2.6)–(2.8) in Theorem 2.3 hold as well, they are simple consequences of the stronger statements in (2.10) and (2.11).

The next theorem addresses the structure of minimizers in a neighborhood of the transition.

Theorem 2.7. *(Critical scaling) We have the following:*

(i) *Cross-over of global minimizers: There exist $\varepsilon_0 > 0$ and two constants $0 < \beta_1 < 1 < \beta_2$ such that the minimal energy* $\min F_{\varepsilon,\lambda}$ *is zero and only attained by the constant configurations $m \equiv \pm e_3$ if $\varepsilon \in (0, \varepsilon_0)$ and*

$$\lambda \leqq \lambda_-(\varepsilon) := \lambda_c \left(1 + \frac{\ln \beta_1}{|\ln \varepsilon|} \right). \tag{2.12}$$

On the other hand, the minimal energy is strictly negative and minimizers cannot be constant if $\varepsilon \in (0, \varepsilon_0)$ and

$$\lambda \geqq \lambda_+(\varepsilon) := \lambda_c \left(1 + \frac{\ln \beta_2}{|\ln \varepsilon|} \right); \tag{2.13}$$

(ii) *Γ-convergence: The family of functionals $(F_{\varepsilon,\lambda_c})$ Γ-converges with respect to the $L^1(\mathbb{T}^2)$-topology towards F_{*,λ_c} for $\varepsilon \to 0$;*

(iii) *Lack of compactness: There is a sequence (m_ε) in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ which is not precompact in $L^1(\mathbb{T}^2)$ such that* $\lim_{\varepsilon \to 0} F_{\varepsilon,\lambda_c}[m_\varepsilon] = 0$*;*

(iv) *Compactness upon rescaling: For every $C > 0$, any sequence (m_ε) with*

$$F_{\varepsilon,\lambda_c}[m_\varepsilon] \leqq C |\ln \varepsilon|^{-1}$$

converges in $L^1(\mathbb{T}^2)$ (up to extracting a subsequence) to a limit in $BV(\mathbb{T}^2; \{\pm e_3\})$.

Theorem 2.7 suggests that $|\ln \varepsilon| F_{\varepsilon,\lambda_c}$ is the appropriate rescaling for the critical case. Unfortunately, it seems not possible to obtain the Γ-limit of $|\ln \varepsilon| F_{\varepsilon,\lambda_c}$ with our $H^{1/2}$-estimate (3.1) of the following section, because the constant c_* there is not optimal.

We illustrate our results in a phase diagram (Fig. 1). It is not difficult to see that for each $0 < \varepsilon < \frac{1}{2}$ there is a sharp threshold value $\lambda = \lambda_c(\varepsilon) > 0$ at

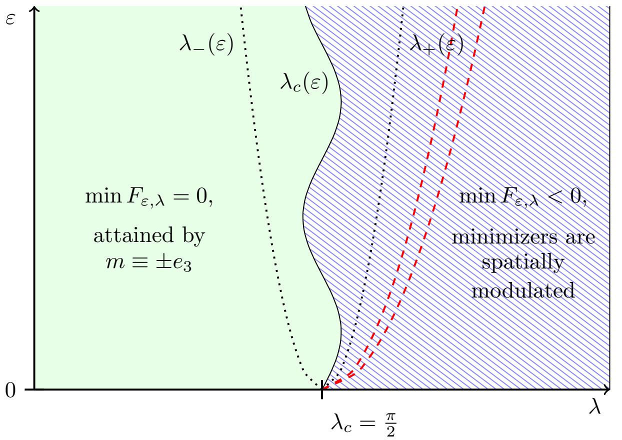

Fig. 1. Sketch of the phase diagram for minimizers of $F_{\varepsilon,\lambda}$ in terms of $\lambda > 0$ and $\varepsilon \ll 1$

which a transition from monodomain states ($m \equiv const$) to multidomain states ($m \not\equiv const$) as global energy minimizers occurs, with $\lambda_c(\varepsilon)$ a Lipschitz-continuous function on $[\delta, 1-\delta]$ for every $0 < \delta < \frac{1}{2}$, see [58]. While we do not know the precise value of $\lambda_c(\varepsilon)$ for $\varepsilon > 0$, we show in Theorem 2.7 that $\lambda_-(\varepsilon) \leqq \lambda_c(\varepsilon) \leqq \lambda_+(\varepsilon)$ and $\lim_{\varepsilon\to 0} \lambda_c(\varepsilon) = \frac{\pi}{2}$, that is the definition above agrees with $\lambda_c := \lambda_c(0) = \frac{\pi}{2}$. Furthermore, global minimizers $m_{\varepsilon,\lambda}$ of $F_{\varepsilon,\lambda}$ with (ε, λ) between any two curves of the form $\lambda(\varepsilon) = \lambda_c + \beta |\ln \varepsilon|^{-1}$ (the dashed curves in the figure) satisfy a uniform bound of the form $c \leqq \int_{\mathbb{T}^2} |\nabla m_{(\varepsilon,\lambda),3}| \,\mathrm{d}x \leqq C$, with constants $C > c > 0$ depending only on the values of $\beta \in \mathbb{R}$ for these curves.

3. A Bound on the Homogeneous $H^{1/2}$-Norm

Since all three terms in $F_{\varepsilon,\lambda}$ contribute in highest order to the limit, it is important to estimate the negative term $\int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 \,\mathrm{d}x$ with precise leading order constant. In this section we will establish an upper bound for the homogeneous $H^{1/2}$-norm which is the key ingredient for the lower bounds (since the nonlocal term in the energy has a negative sign). We will prove the following:

Lemma 3.1. *There is a universal constant $c_* \geqq 1$ such that for every $f \in C^\infty(\mathbb{T}^2)$ and every $0 < \varepsilon < 1$ we have*

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \,\mathrm{d}x \leq \frac{\varepsilon}{2} \int_{\mathbb{T}^2} |\nabla f|^2 \,\mathrm{d}x + \frac{2}{\pi} \ln \left(c_* \max \left\{ 1, \min \left\{ \frac{\|f\|_\infty}{\varepsilon \int_{\mathbb{T}^2} |\nabla f| dx}, \frac{1}{\varepsilon} \right\} \right\} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \,\mathrm{d}x. \quad (3.1)$$

In Lemma 3.1, we improve an inequality established in [18]. Expressed in our setting, the inequality proved in [18, Lemma 1] asserts that for every $\delta > 0$ there exists $M_\delta > 1$ such that for all $\varepsilon \leqq R$ and all $f \in C^\infty(\mathbb{T}^2)$, we have

$$\sum_{k\in 2\pi\mathbb{Z}^2} \min\left\{\frac{1}{\varepsilon}, |k|, R|k|^2\right\} |\widehat{f}_k|^2 \leq (1+\delta)\frac{2}{\pi}\ln\left(\frac{2M_\delta R}{\varepsilon}\right)\|f\|_\infty \int_{\mathbb{T}^2} |\nabla f|\,\mathrm{d}x. \tag{3.2}$$

Note that (3.1) implies for all $\varepsilon \leqq 1$ a similar estimate

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2\,\mathrm{d}x \leqq \frac{\varepsilon}{2}\int_{\mathbb{T}^2} |\nabla f|^2\,\mathrm{d}x + \frac{2}{\pi}\ln\left(\frac{c_*}{\varepsilon}\right)\|f\|_\infty \int_{\mathbb{T}^2} |\nabla f|\,\mathrm{d}x, \tag{3.3}$$

which is weaker than (3.1). Estimate (3.2) is an inequality for a regularized $\mathring{H}^{1/2}$-norm, whereas (3.3) estimates the full $\mathring{H}^{1/2}$-norm, but needs an additional $\mathring{H}^1$-term. It ceases to be optimal for functions which oscillate significantly. Indeed, let $\alpha \in (0, 1)$ and consider functions f_ε with

$$\int_{\mathbb{T}^2} |\nabla f_\varepsilon|\,\mathrm{d}x \gtrsim \varepsilon^{-\alpha}\|f_\varepsilon\|_\infty. \tag{3.4}$$

Then the second term in (3.1) is smaller than the second term in (3.3) by a factor of $(1-\alpha)$ for all f_ε which satisfy (3.4). Asymptotic optimality in the case of strong oscillation is crucial to obtain the results on the supercritical regime. The proof of Lemma 3.1 uses similar ideas as in [18] and is based on a separate treatment of distinct scales. However, our proof does not involve any Fourier Analysis.

Proof. (*Lemma* 3.1) We will show that the following estimates hold for all $f \in C^\infty(\mathbb{T}^2)$ and all $0 < r \leqq R$:

$$\int_{\mathbb{T}^2}\int_{B_r} \frac{|f(x+z)-f(x)|^2}{|z|^3}\,\mathrm{d}z\,\mathrm{d}x \leqq \pi r \int_{\mathbb{T}^2} |\nabla f|^2\,\mathrm{d}x, \tag{3.5}$$

$$\int_{\mathbb{T}^2}\int_{B_R\setminus B_r} \frac{|f(x+z)-f(x)|^2}{|z|^3}\,\mathrm{d}z\,\mathrm{d}x \leqq 8\ln\left(\frac{R}{r}\right)\|f\|_\infty \int_{\mathbb{T}^2} |\nabla f|\,\mathrm{d}x, \tag{3.6}$$

$$\int_{\mathbb{T}^2}\int_{B_R^c} \frac{|f(x+z)-f(x)|^2}{|z|^3}\,\mathrm{d}z\,\mathrm{d}x \leq \frac{2\pi\|f\|_\infty}{R}\min\left\{4\|f\|_\infty, \int_{\mathbb{T}^2} |\nabla f|\,\mathrm{d}x\right\}. \tag{3.7}$$

The claim of the lemma will follow by adding (3.5)–(3.7) and a suitable choice of r and R. Before we start with the proofs of estimates (3.5)–(3.7), we first record an auxiliary inequality for further use. By the Fundamental Theorem of Calculus, Jensen's inequality and Fubini's Theorem we get

$$\int_{\mathbb{T}^2} |f(x+z)-f(x)|^p\,\mathrm{d}x \leqq \int_{\mathbb{T}^2} |\nabla f(x)\cdot z|^p\,\mathrm{d}x \tag{3.8}$$

for all $z \in \mathbb{R}^2$ and all $1 \leqq p < \infty$. In order to prove (3.5), we use Fubini's Theorem and apply (3.8) with $p = 2$ to get

$$\int_{\mathbb{T}^2}\int_{B_r} \frac{|f(x+z)-f(x)|^2}{|z|^3}\,\mathrm{d}z\,\mathrm{d}x \overset{(3.8)}{\leq} \int_{B_r}\int_{\mathbb{T}^2} \frac{|\nabla f(x)\cdot z|^2}{|z|^3}\,\mathrm{d}x\,\mathrm{d}z. \tag{3.9}$$

We apply Fubini's Theorem again and evaluate the integral with respect to z in polar coordinates to get

$$\int_{B_r}\int_{\mathbb{T}^2} \frac{|\nabla f(x)\cdot z|^2}{|z|^3}\,\mathrm{d}x\,\mathrm{d}z = \left(\int_0^r\int_0^{2\pi}\cos^2\phi\,d\phi d\rho\right)\left(\int_{\mathbb{T}^2}|\nabla f(x)|^2\,\mathrm{d}x\right)$$
$$= \pi r\int_{\mathbb{T}^2}|\nabla f|^2\,\mathrm{d}x. \tag{3.10}$$

Together, (3.9) and (3.10) yield the first estimate (3.5).

For the estimate involving intermediate distances (3.6), we use Fubini's Theorem (twice) and (3.8) with $p = 1$ to conclude

$$\int_{\mathbb{T}^2}\int_{B_R\setminus B_r}\frac{|f(x+z)-f(x)|^2}{|z|^3}\,\mathrm{d}z\,\mathrm{d}x \overset{(3.8)}{\leq} 2\|f\|_\infty\int_{\mathbb{T}^2}\int_{B_R\setminus B_r}\frac{|\nabla f(x)\cdot z|}{|z|^3}\,\mathrm{d}z\,\mathrm{d}x. \tag{3.11}$$

Using polar coordinates, we get

$$\int_{B_R\setminus B_r}\frac{|\nabla f(x)\cdot z|}{|z|^3}\,\mathrm{d}z = \int_r^R\int_0^{2\pi}\frac{|\nabla f(x)|\,|\cos\phi|}{\rho}d\phi d\rho = 4\ln\left(\frac{R}{r}\right)|\nabla f(x)|.$$

Inserting this identity into (3.11) yields the claim (3.6).

In order to prove (3.7), we first show

$$\int_{\mathbb{T}^2}|f(x+z)-f(x)|\,\mathrm{d}x \leq \min\left\{2\|f\|_\infty, \frac{1}{2}\int_{\mathbb{T}^2}|\nabla f|\,\mathrm{d}x\right\} \tag{3.12}$$

for all $z\in\mathbb{R}^2$. Indeed, the upper bound of $2\|f\|_\infty$ in (3.12) is trivial. Furthermore, since f is periodic, it is sufficient to show the second upper bound in (3.12) only for $z\in\left(-\frac{1}{2},\frac{1}{2}\right)^2$. Thus the second bound in (3.12) follows from (3.8) with $p=1$

$$\int_{\mathbb{T}^2}|f(x+z)-f(x)|\,\mathrm{d}x \overset{(3.8)}{\leqq}\int_{\mathbb{T}^2}|\nabla f(x)\cdot z|\,\mathrm{d}x \leqq \frac{1}{2}\int_{\mathbb{T}^2}|\nabla f(x)|\,\mathrm{d}x$$

so that the proof of (3.12) is complete. With (3.12) at hand, estimate (3.7) now follows by direct integration.

It remains to prove (3.1), for which we use the real-space representation of the homogeneous $H^{1/2}$-norm

$$\int_{\mathbb{T}^2}|\nabla^{1/2}f|^2\,\mathrm{d}x = \frac{1}{4\pi}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\frac{|f(x+z)-f(x)|^2}{|z|^3}\,\mathrm{d}z\,\mathrm{d}x,$$

see for example [18]. Without loss of generality, we may assume that f is not constant. Adding (3.5)–(3.7) to estimate the right hand side of (3.1), we get

$$\int_{\mathbb{T}^2}|\nabla^{1/2}f|^2\,\mathrm{d}x \leq \frac{r}{4}\int_{\mathbb{T}^2}|\nabla f|^2\,\mathrm{d}x$$
$$+\left(\frac{2}{\pi}\ln\left(\frac{R}{r}\right)+\frac{1}{2R}\min\left\{\frac{4\|f\|_\infty}{\int_{\mathbb{T}^2}|\nabla f|\,\mathrm{d}x},1\right\}\right)\|f\|_\infty\int_{\mathbb{T}^2}|\nabla f|\,\mathrm{d}x. \tag{3.13}$$

For $r := 2\varepsilon$ and $R := \max\left\{2\varepsilon, \min\left\{\frac{4\|f\|_\infty}{\int_{\mathbb{T}^2}|\nabla f|\,\mathrm{d}x},1\right\}\right\}$ the claim (3.1) now follows from (3.13). $\square$

4. Proofs for the Reduced Energy $F_{\varepsilon,\lambda}$

In this section we give the proofs of the theorems involving the reduced energy $F_{\varepsilon,\lambda}$. The proof of Theorem 2.5 is a direct consequence of Lemmas 4.1 and 4.3. Similarly the proof of Theorem 2.6 follows immediately from Lemmas 4.4 and 4.5. Finally, the proof of Theorem 2.7 is presented at the end of this section.

4.1. Proof of Theorem 2.5

In this section, we present the proof of Theorem 2.5. The proof of the lower bound and compactness for the Γ-limit is based on the interpolation result in Lemma 3.1 and is given in Lemma 4.1. The proof of the upper bound follows by explicit construction and is given in Lemma 4.2.

Lemma 4.1. *(Lower bound and compactness) Let $\lambda < \lambda_c$ and $F_{\varepsilon,\lambda}$ as defined in* (1.3). *Then any sequence (m_ε) in $H^1(\mathbb{T}^2;\mathbb{S}^2)$ with*

$$\limsup_{\varepsilon\to 0} F_{\varepsilon,\lambda}[m_\varepsilon] < +\infty$$

converges in $L^1(\mathbb{T}^2;\mathbb{R}^3)$ (up to extracting a subsequence) towards a limit in $BV(\mathbb{T}^2;\{\pm e_3\})$. Furthermore, for every sequence (m_ε) in $L^1(\mathbb{T}^2;\mathbb{S}^2)$ with $m_\varepsilon \to m$ for some m in $L^1(\mathbb{T}^2;\mathbb{R}^3)$ we have

$$\liminf_{\varepsilon\to 0} F_{\varepsilon,\lambda}[m_\varepsilon] \geqq \begin{cases} \left(1-\dfrac{\lambda}{\lambda_c}\right)\displaystyle\int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x, & \textit{if } m \in BV(\mathbb{T}^2;\{\pm e_3\}), \\ +\infty & \textit{otherwise.} \end{cases} \tag{4.1}$$

Proof. We first show that for all $\lambda > 0$ and all sufficiently small $\varepsilon > 0$ we have

$$F_{\varepsilon,\lambda}[m] \geqq \left(1-\frac{\lambda|\ln c\varepsilon|}{\lambda_c|\ln\varepsilon|}\right)\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x, \qquad \text{for all } m \in H^1\left(\mathbb{T}^2;\mathbb{S}^2\right), \tag{4.2}$$

for some universal constant $c > 0$. For this, it is sufficient to use Lemma 3.1 for m_3 in the form (3.3). Recalling that $\|m_3\|_\infty \leqq 1$ and $\lambda_c = \frac{\pi}{2}$, we get

$$\frac{\lambda}{|\ln\varepsilon|}\int_{\mathbb{T}^2}|\nabla^{1/2}m_3|^2\,\mathrm{d}x \overset{(3.3)}{\leqq} \frac{\lambda}{|\ln\varepsilon|}\int_{\mathbb{T}^2}\frac{\varepsilon}{2}|\nabla m_3|^2\,\mathrm{d}x + \frac{\lambda}{\lambda_c}\frac{\ln(c_*/\varepsilon)}{|\ln\varepsilon|}\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x. \tag{4.3}$$

We also use the well-known estimate

$$|\nabla m_3| \leqq \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}\left(1-m_3^2\right), \tag{4.4}$$

which is obtained by differentiating $|m|^2 = 1$ and applying Young's inequality. The claimed lower bound in (4.2) then follows from (4.3) and (4.4), since

$$F_{\varepsilon,\lambda}[m] \overset{(4.3)}{\geqq} \left(1-\frac{\lambda}{|\ln\varepsilon|}\right)\int_{\mathbb{T}^2}\left(\frac{\varepsilon}{2}|\nabla m|^2+\frac{1}{2\varepsilon}(1-m_3^2)\right)\mathrm{d}x$$
$$-\frac{\lambda}{\lambda_c}\frac{\ln\frac{c_*}{\varepsilon}}{|\ln\varepsilon|}\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x$$
$$\overset{(4.4)}{\geqq}\left(1-\frac{\lambda}{\lambda_c}\frac{\ln\frac{e^{\lambda_c}c_*}{\varepsilon}}{|\ln\varepsilon|}\right)\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x. \tag{4.5}$$

Now, let (m_ε) be a sequence in $H^1(\mathbb{T}^2;\mathbb{S}^2)$ with $\limsup_{\varepsilon\to0}F_{\varepsilon,\lambda}[m_\varepsilon]<+\infty$. From the argument in (4.5), $|m_\varepsilon|=1$ and $\lambda<\lambda_c$ we obtain

$$0=\limsup_{\varepsilon\to0}\varepsilon F_{\varepsilon,\lambda}[m_\varepsilon]\geqq\frac{1}{2}\left(1-\frac{\lambda'}{\lambda_c}\right)\limsup_{\varepsilon\to0}\int_{\mathbb{T}^2}\left(m_{\varepsilon,1}^2+m_{\varepsilon,2}^2\right)\mathrm{d}x$$

for any $\lambda'\in(\lambda,\lambda_c)$ and $\varepsilon\ll1$, implying that the first two components $m_{\varepsilon,1}$ and $m_{\varepsilon,2}$ of m_ε converge to zero in $L^2(\mathbb{T}^2)$ as $\varepsilon\to0$. Again, as a consequence of $|m_\varepsilon|=1$ this further implies that $m_\varepsilon\to m$ in $L^2(\mathbb{T}^2;\mathbb{R}^3)$ and, upon extraction of a subsequence, we have $m_\varepsilon(x)\to m(x)$ for almost everywhere $x\in\mathbb{T}^2$, with $m(x)=\pm e_3$. Moreover, (4.2) yields a uniform bound for $m_{\varepsilon,3}$ in BV, which implies that $m\in BV(\mathbb{T}^2;\{\pm e_3\})$. Finally, (4.1) follows directly from (4.2), the fact that $\lim_{\varepsilon\to0}\frac{\lambda|\ln c\varepsilon|}{\lambda_c|\ln\varepsilon|}=\frac{\lambda}{\lambda_c}<1$ and lower semi-continuity of the BV-seminorm. □

Before we begin with the construction of the upper bound, we define a family of asymptotically optimal profiles and record some of their properties.

Lemma 4.2. *(Family of asymptotically optimal profiles) For $R\in(0,\infty]$ and $0<\varepsilon<R$, let $\xi_{\varepsilon,R}\in C^1(\mathbb{R})$ be given by $\xi_{\varepsilon,R}(x)=\operatorname{sign}(x)$ for $|x|\geqq R$ and*

$$\xi_{\varepsilon,R}(x):=\sin\left(\frac{\pi}{2}\frac{\arcsin(\tanh(x/\varepsilon))}{\arcsin(\tanh(R/\varepsilon))}\right)\qquad |x|<R. \tag{4.6}$$

Then for some universal $C,c,a>0$ we have

$$\frac{1}{2}\int_{-R}^{R}\left(\frac{\varepsilon|\xi'_{\varepsilon,R}|^2}{1-\xi^2_{\varepsilon,R}}+\frac{1-\xi^2_{\varepsilon,R}}{\varepsilon}\right)\mathrm{d}x\ \leqq\ 2+Ce^{-aR/\varepsilon}, \tag{4.7}$$
$$\int_{-X}^{X}\int_{-X}^{X}\frac{|\xi_{\varepsilon,R}(x)-\xi_{\varepsilon,R}(y)|^2}{|x-y|^2}\mathrm{d}x\,\mathrm{d}y\geqq\ 8\ln\left(\frac{cX}{\varepsilon}\right)\qquad \textit{for } X\geqq2\varepsilon. \tag{4.8}$$

Proof. We begin by observing that $\xi_{\varepsilon,R}(x)$ tends to the optimal one-dimensional Modica-Mortola profile $\xi_{\varepsilon,\infty}(x)=\tanh(x/\varepsilon)$ associated with the left-hand side of (4.7) when $R\to\infty$. Introducing $\theta_{\varepsilon,R}:=\arcsin\xi_{\varepsilon,R}$, we have for all $|x|<R$ that

$$\frac{\varepsilon|\xi'_{\varepsilon,R}(x)|^2}{1-\xi^2_{\varepsilon,R}(x)}+\frac{1-\xi^2_{\varepsilon,R}(x)}{\varepsilon}\overset{(4.6)}{=}\varepsilon|\theta'_{\varepsilon,R}(x)|^2+\frac{\cos^2\theta_{\varepsilon,R}(x)}{\varepsilon}$$
$$\leqq2\xi'_{\varepsilon,\infty}(x)+C\varepsilon^{-1}e^{-aR/\varepsilon} \tag{4.9}$$

for some universal $a>0$ and $C>0$, and thus (4.7) follows from integrating (4.9), possibly modifying the values of a and C.

It remains to prove (4.8). By symmetry of $\xi_{\varepsilon,R}$ we have

$$\int_{-X}^{X}\int_{-X}^{X} \frac{|\xi_{\varepsilon,R}(x)-\xi_{\varepsilon,R}(y)|^2}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x = 2\int_{-X}^{0}\int_{-X}^{X} \frac{|\xi_{\varepsilon,R}(x)-\xi_{\varepsilon,R}(y)|^2}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x$$
$$\geqq 2\int_{-X}^{-\varepsilon}\int_{0}^{X} \frac{|\xi_{\varepsilon,R}(x)-\xi_{\varepsilon,R}(y)|^2}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x$$
$$= 2\int_{-X}^{-\varepsilon}\int_{0}^{X} \frac{4}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x - 2\int_{-X}^{-\varepsilon}\int_{0}^{X} \frac{4-|\xi_{\varepsilon,R}(x)-\xi_{\varepsilon,R}(y)|^2}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x. \tag{4.10}$$

The first term on the right-hand side yields

$$\int_{-X}^{-\varepsilon}\int_{0}^{X} \frac{1}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x = \ln\left(\frac{\varepsilon+X}{2\varepsilon}\right) \geqq \ln\left(\frac{X}{2\varepsilon}\right).$$

Thus, the argument is concluded by showing that the second term on the right hand side of (4.10) is bounded independently of ε, X or R. Indeed, using the fact that $|\mathrm{sign}(x)-\xi_{\varepsilon,R}(x)| \leqq Ce^{-a|x|/\varepsilon}$ for all $x \in \mathbb{R}$, we get

$$\int_{-X}^{-\varepsilon}\int_{0}^{X} \frac{4-|\xi_{\varepsilon,R}(x)-\xi_{\varepsilon,R}(y)|^2}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x \lesssim \int_{1}^{\infty}\int_{0}^{\infty} \frac{e^{-ax}+e^{-ay}}{|x+y|^2}\,\mathrm{d}x\,\mathrm{d}y \lesssim 1,$$

which yields the desired result. □

For the special case $\lambda = 0$, the Γ-convergence and in particular the construction of a recovery sequence is a classical result, relying on the optimal one-dimensional transition profiles to smooth out the jump discontinuity in the limit configuration [1]. As it turns out, this construction also works for $\lambda > 0$, where $F_{\varepsilon,\lambda}$ is nonlocal. We will use a construction based on the nearly optimal profile $\xi_{\varepsilon,R}$ from Lemma 4.2. As the calculations for the local part of the energy are well-known, our focus is on the contribution of the homogeneous $H^{1/2}$-norm. Recall that we need to prove a lower bound for the $H^{1/2}$-norm in order to obtain an upper bound for $F_{\varepsilon,\lambda}$.

Lemma 4.3. *(Recovery sequence) Let $\lambda \leqq \lambda_c$ and $m \in L^1(\mathbb{T}^2;\mathbb{S}^2)$. Then there is a sequence (m_ε) in $H^1(\mathbb{T}^2;\mathbb{S}^2)$ with*

$$\limsup_{\varepsilon\to 0} F_{\varepsilon,\lambda}[m_\varepsilon] \leqq F_{*,\lambda}[m],$$

where $F_{\varepsilon,\lambda}$ is given by (1.3), and $F_{*,\lambda}$ is given by (1.5).

Proof. It is sufficient to prove the limsup inequality under the additional assumption that $m = (\chi_A - \chi_{\mathbb{T}^2\setminus A})e_3$ for a set $A \subset \mathbb{T}^2$ with smooth boundary. By standard density results (see for example [47, Prop. 12.20]) and a diagonal argument, the limsup inequality then extends to arbitrary $A \subset \mathbb{T}^2$ with finite perimeter for $\lambda < \lambda_c$ or to measurable $A \subset \mathbb{T}^2$ for the $\lambda = \lambda_c$ case. Since $F_{*,\lambda}[m] = +\infty$ for $m \notin BV(\mathbb{T}^2,\{\pm e_3\})$ when $\lambda < \lambda_c$ or for $m \notin L^1(\mathbb{T}^2,\{\pm e_3\}))$ when $\lambda = \lambda_c$, this yields the claim.

Our strategy is to adapt the optimal profiles $\xi_{\varepsilon,R}$ from Lemma 4.2 to the two-dimensional setting by means of the signed distance function d, given by $d(x) := \operatorname{dist}(x, A^c) - \operatorname{dist}(x, A)$. Without loss of generality, we may assume $0 < |A| < 1$ (otherwise take $m_\varepsilon \equiv \pm e_3$). To fix the notation, let $\nu : \partial A \to \mathbb{R}^2$ denote the smooth inward normal to A and $\tau : \partial A \to \mathbb{R}^2$, $\tau = \nu^\perp$ denote a smooth tangent vector field to ∂A obtained by a counter-clockwise $90°$ rotation of ν. As ∂A is assumed to be smooth, there exists a tubular neighborhood $(\partial A)_R = \bigcup_{x\in\partial A} B_R(x) \subset \mathbb{T}^2$ for some $R > 0$ such that the projection $p : (\partial A)_R \to \partial A$, $p(x) := \operatorname{argmin}_{y\in\partial A} |x - y|$ is single-valued and hence well-defined. Furthermore, the projection p and the signed distance function d are smooth on $(\partial A)_R$ and the identity $x = p(x) + d(x)\nu(p(x))$ holds for all $x \in (\partial A)_R$, see for example [27, Lemma 14.16].

With the necessary notation at hand, we define the recovery sequence by

$$m_\varepsilon(x) \;:=\; \xi_{\varepsilon,R}(d(x))e_3 + \sqrt{1 - \xi^2_{\varepsilon,R}(d(x))}\,\tau(p(x)), \tag{4.11}$$

which is Lipschitz continuous and piecewise smooth. It is easy to see that $m_\varepsilon \to m$ in $L^1(\mathbb{T}^2)$,

and by the co-area formula we have for all $\varepsilon \ll 1$,

$$\begin{aligned}
&\int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon}(1 - m^2_{\varepsilon,3}) \right) \mathrm{d}x \\
&\quad = \int_{(\partial A)_R} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(d)|^2}{2(1 - \xi^2_{\varepsilon,R}(d))} + \frac{1}{2\varepsilon}(1 - \xi^2_{\varepsilon,R}(d)) \right) \mathrm{d}x \\
&\quad = \int_{-R}^{R} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(s)|^2}{2(1 - \xi^2_{\varepsilon,R}(s))} + \frac{1}{2\varepsilon}(1 - \xi^2_{\varepsilon,R}(s)) \right) \mathscr{H}^1(\{d(x) = s\})\, \mathrm{d}s.
\end{aligned} \tag{4.12}$$

Inserting the estimate for the 1-d profile from Lemma 4.2, we obtain

$$\begin{aligned}
&\int_{-R}^{R} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(s)|^2}{2(1 - \xi^2_{\varepsilon,R}(s))} + \frac{1}{2\varepsilon}(1 - \xi^2_{\varepsilon,R}(s)) \right) \mathscr{H}^1(\{d(x) = s\})\, \mathrm{d}s \\
&\overset{(4.7)}{\leqq} 2\mathscr{H}^1(\{d(x) = 0\}) + CR\left(1 + e^{-aR/\varepsilon}\right),
\end{aligned} \tag{4.13}$$

where $C > 0$ is a constant depending only on A, and $a > 0$ is universal. Thus

$$\limsup_{R\to 0}\, \limsup_{\varepsilon\to 0} \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon}\left(1 - m^2_{\varepsilon,3}\right) \right) \mathrm{d}x \leqq 2\mathscr{H}^1(\partial A). \tag{4.14}$$

We now turn to the estimate of the nonlocal term in the energy F. As for the local terms, our strategy is to use the one-dimensional estimates from Lemma 4.2. Invoking the co-area formula twice and inserting (4.11), we get

$$\begin{aligned}
&\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|m_{\varepsilon,3}(x) - m_{\varepsilon,3}(y)|^2}{|x - y|^3}\, \mathrm{d}^2x\, \mathrm{d}^2y \\
&\geqq \int_{-R}^{R} \int_{\{x:\, d(x)=\rho'\}} \int_{-R}^{R} \int_{\{y:\, d(y)=\rho\}} \frac{|\xi_{\varepsilon,R}(\rho') - \xi_{\varepsilon,R}(\rho)|^2}{|x - y|^3} d\mathscr{H}^1(y) d\rho d\mathscr{H}^1(x)\, \mathrm{d}\rho'.
\end{aligned} \tag{4.15}$$

We claim that the integrals over curves tangential to the boundary may be estimated as follows: for every $\delta > 0$, there is an $R_{\delta,A}$ such that

$$\int_{\{x:\, d(x)=\rho'\}} \int_{\{y:\, d(y)=\rho\}} \frac{1}{|x-y|^3} d\mathscr{H}^1(y) d\mathscr{H}^1(x) \geqq (1-\delta) \frac{2\mathscr{H}^1(\partial A)}{(\rho-\rho')^2} \quad (4.16)$$

for all $R \leqq R_{\delta,A}$ and all $\rho \neq \rho' \in (-R, R)$. Assuming for a moment that (4.16) holds, we conclude by inserting (4.16) into (4.15) and applying the one-dimensional estimate (4.8)

$$\begin{aligned} &\frac{\lambda}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_{\varepsilon,3}|^2 \, \mathrm{d}^2 x \\ &\overset{(4.15),(4.16)}{\geqq} (1-\delta) \frac{\lambda \mathscr{H}^1(\partial A)}{2\pi |\ln \varepsilon|} \int_{-R}^{R} \int_{-R}^{R} \frac{|\xi_{\varepsilon,R}(\rho) - \xi_{\varepsilon,R}(\rho')|^2}{|\rho-\rho'|^2} \mathrm{d}\rho' \mathrm{d}\rho \\ &\overset{(4.8)}{\geqq} (1-\delta) 2\mathscr{H}^1(\partial A) \frac{\lambda}{\lambda_c} \frac{\ln(cR/\varepsilon)}{|\ln \varepsilon|}. \end{aligned}$$

Since δ was arbitrary, we obtain

$$\liminf_{R\to 0} \liminf_{\varepsilon \to 0} \frac{\lambda}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_{\varepsilon,3}|^2 \, \mathrm{d}^2 x \geqq \frac{2\lambda}{\lambda_c} \mathscr{H}^1(\partial A). \quad (4.17)$$

Together, (4.14) and (4.17) imply the limsup inequality by a standard diagonal argument.

It remains to prove (4.16), for which we fix $x \in (\partial A)_R$ with $d(x) = \rho'$ and pass to curvilinear coordinates in a neighborhood of $\tilde{x} := p(x) \in \partial A$. More precisely, let the curve $\gamma : (-R^{1/2}, R^{1/2}) \to \partial A$ be a parametrization by arclength of a neighborhood of $\tilde{x}$ in ∂A with $\gamma(0) = \tilde{x}$. Then, for all $R \leqq R_A$ with some $R_A > 0$ the function $\Psi(\sigma, \rho) := \gamma(\sigma) + \nu(\gamma(\sigma))\rho$ is a diffeomorphism from $(-R^{1/2}, R^{1/2}) \times (-R, R)$ onto its image, which we denote by $\Gamma_{\tilde{x}}$. The choice of $R^{1/2}$ will become clear later. Note that due to compactness of ∂A, we may choose R_A independent of $\tilde{x}$. A transformation of variables then yields

$$\int_{\{y:\, d(y)=\rho\} \cap \Gamma_{p(x)}} \frac{1}{|x-y|^3} d\mathscr{H}^1(y) = \int_{-R^{1/2}}^{R^{1/2}} \frac{(1+\kappa(\gamma(\sigma))\rho)}{|\Psi(0,\rho') - \Psi(\sigma,\rho)|^3} d\sigma, \quad (4.18)$$

where $\kappa(\tilde{y})$ denotes the signed curvature of ∂A at $\tilde{y}$ (negative if A is convex). Since the curvature of ∂A is bounded, there is, for any $\delta > 0$, an $R_{\delta,A} > 0$ such that for all $R \leqq R_{\delta,A}$ we have

$$|\kappa| R \leqq \delta \quad \text{and} \quad |\Psi(0,\rho') - \Psi(\sigma,\rho)| \leq (1+\delta)\sqrt{\sigma^2 + (\rho-\rho')^2}. \quad (4.19)$$

We conclude that, for any $\tilde{\delta} > 0$, there is an $\tilde{R}_{\tilde{\delta},A} > 0$ such that for all $R \leqq \tilde{R}_{\tilde{\delta},A}$ and all $\rho, \rho' \in (-R, R)$ we have

$$\begin{aligned}
&\int_{\{y:\,d(y)=\rho\}\cap\Gamma_{p(x)}} \frac{1}{|x-y|^3} d\mathscr{H}^1(y) \\
&\overset{(4.18),(4.19)}{\geqq} (1-\tilde{\delta}) \int_{-R^{1/2}}^{R^{1/2}} \frac{1}{\left(\sigma^2+(\rho-\rho')^2\right)^{3/2}} \mathrm{d}\sigma \\
&= \frac{2R^{1/2}(1-\tilde{\delta})}{(\rho-\rho')^2\sqrt{R+(\rho-\rho')^2}} \geqq \frac{2(1-2\tilde{\delta})}{(\rho-\rho')^2}.
\end{aligned}$$

Integrating this estimate over x and again invoking smoothness of ∂A, we obtain (4.16). $\square$

4.2. *Proof of Theorem 2.6*

We begin with the proof of the lower bound in Theorem 2.6, which is the subject of Lemma 4.4. The proof of Theorem 2.6 is completed with the construction of the upper bound, carried out in Lemma 4.5.

Lemma 4.4. *(Lower bound) Let $F_{\varepsilon,\lambda}$ be defined in* (1.3)*. Then there is a universal constant $\delta > 0$ such that for all $\varepsilon < \frac{1}{2}$ and all*

$$\lambda_c \leqq \lambda < \delta |\ln\varepsilon|, \tag{4.20}$$

the family of functionals $(F_{\varepsilon,\lambda})$ is bounded below by

$$\min F_{\varepsilon,\lambda} \gtrsim -\frac{\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|}. \tag{4.21}$$

Moreover, the profiles achieving the optimal scaling can be characterized as follows: For any $\gamma > 0$ and all $m \in H^1(\mathbb{T}^2;\mathbb{S}^2)$ which satisfy

$$F_{\varepsilon,\lambda}[m] \leqq -\frac{\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|}\gamma, \tag{4.22}$$

there holds

$$\int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x \leqq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-m_3^2)\right) \mathrm{d}x \leqq \frac{\lambda}{|\ln\varepsilon|}\int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2\,\mathrm{d}x. \tag{4.23}$$

Furthermore, if there is $c_\gamma > 0$, depending only on γ, such that for any

$$A, B \in \left\{ \int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x,\ \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1-m_{\varepsilon,3}^2}{2\varepsilon}\right) \mathrm{d}x,\ \frac{\lambda}{|\ln\varepsilon|}\int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2\,\mathrm{d}x \right\},$$

we have

$$A \sim c_\gamma \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \quad \text{and} \quad |A-B| \lesssim \frac{c_\gamma\lambda}{|\ln\varepsilon|}A. \tag{4.24}$$

Proof. By (3.1), we may bound the energy from below by

$$F_{\varepsilon,\lambda}[m] \overset{(3.1)}{\geq} \left(1 - \frac{\lambda}{|\ln\varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-m_3^2)\right) \mathrm{d}x \\ - \frac{\lambda}{\lambda_c|\ln\varepsilon|} \ln\left(c_* \max\left\{1, \min\left\{\frac{1}{\varepsilon\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x}, \frac{1}{\varepsilon}\right\}\right\}\right) \int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x. \tag{4.25}$$

Without loss of generality, we may assume that $\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x > 0$. We first consider the case $\min\left\{\frac{1}{\varepsilon\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x}, \frac{1}{\varepsilon}\right\} \leqq 1$, for which, with the help of (4.4), the estimate in (4.25) turns into

$$F_{\varepsilon,\lambda}[m] \geqq \left(1 - \frac{C\lambda}{|\ln\varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x \overset{(4.20)}{\geqq} (1 - C\delta) \int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x \tag{4.26}$$

for some universal $C > 0$. For $\delta < 1/C$, the right hand side of (4.26) is positive and the lower bound follows. Hence, we may assume $\min\left\{\frac{1}{\varepsilon\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x}, \frac{1}{\varepsilon}\right\} > 1$, so that (4.25) implies

$$F_{\varepsilon,\lambda}[m] \geqq \left(1 - \frac{\lambda}{|\ln\varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-m_3^2)\right) \mathrm{d}x \\ - \frac{\lambda}{\lambda_c|\ln\varepsilon|} \ln\left(\frac{c_*}{\varepsilon\int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x}\right) \int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x. \tag{4.27}$$

With the abbreviation

$$D_\varepsilon[m] := \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-m_3^2)\right) \mathrm{d}x - \int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x,$$

and inserting $\mu := \varepsilon^{\frac{\lambda-\lambda_c}{\lambda}} \int_{\mathbb{T}^2}|\nabla m_3|\,\mathrm{d}x$ and $c_{**} := c_* e^{\lambda_c}$ into the lower bound in (4.27), we get

$$F_{\varepsilon,\lambda}[m] \geqq \left(1 - \frac{\lambda}{|\ln\varepsilon|}\right) D_\varepsilon[m] - \frac{\lambda}{\lambda_c} \frac{\ln\left(\frac{c_{**}}{\mu}\right)}{|\ln\varepsilon|} \mu\, \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}. \tag{4.28}$$

Since $\sup_{\mu>0} \mu\ln(c_{**}/\mu) = c_{**}/e$, and since $D_\varepsilon[m] \geqq 0$ by (4.4), the lower bound in (4.21) follows.

In order to prove (4.24), we first note that (4.4) and $F_{\varepsilon,\lambda}[m] \leqq 0$ yield (4.23). Furthermore, combining the lower bound (4.28) with the upper bound (4.22) yields $\mu\ln(c_{**}/\mu) \gtrsim 1$, which in turn implies $\mu \sim 1$. Hence,

$$\int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x = \mu\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \sim \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}.$$

The proof of (4.24) is then completed by noting that for $\delta > 0$ sufficiently small universal, $\mu \sim 1$ and in view of (4.28), we have

$$\frac{\lambda}{|\ln\varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|\,\mathrm{d}x - \int_{\mathbb{T}^2} |\nabla m_3|\,\mathrm{d}x = -F_{\varepsilon,\lambda}[m] + D_\varepsilon[m] \overset{(4.28)}{\lesssim} \frac{\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|}.$$

□

Lemma 4.5. *(Upper bound) There is a universal constant $K > 1$ such that for any ε, λ with*

$$\lambda_c < \lambda \quad \text{and} \quad 0 < \varepsilon < K^{\frac{\lambda}{\lambda_c - \lambda}}, \tag{4.29}$$

there is $m_\varepsilon \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ which satisfies

$$F_{\varepsilon,\lambda}[m_\varepsilon] \lesssim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\ln \varepsilon|}.$$

Proof. We make an ansatz with N_ε transitions equally separated by $1/N_\varepsilon$-sized regions of approximately constant magnetization. More precisely, we take the transitions as solutions of the optimal Modica-Mortola profile $\xi_{\varepsilon,\infty}$, given in Lemma 4.2, and define

$$m_\varepsilon(x_1, x_2) = \begin{cases} \xi_{\varepsilon,\infty}\Big(\frac{x_1 - \frac{1}{2N_\varepsilon}}{\varepsilon}\Big) e_3 + \sqrt{1 - \xi_{\varepsilon,\infty}^2\Big(\frac{x_1 - \frac{1}{2N_\varepsilon}}{\varepsilon}\Big)}\, e_2, & x_1 \in \big[0, \frac{1}{N_\varepsilon}\big], \\ \xi_{\varepsilon,\infty}\Big(\frac{\frac{3}{2N_\varepsilon} - x_1}{\varepsilon}\Big) e_3 + \sqrt{1 - \xi_{\varepsilon,\infty}^2\Big(\frac{\frac{3}{2N_\varepsilon} - x_1}{\varepsilon}\Big)}\, e_2, & x_1 \in \big[\frac{1}{N_\varepsilon}, \frac{2}{N_\varepsilon}\big], \end{cases}$$

extended periodically to $\mathbb{T}^2$ for N_ε to be fixed later. Applying Lemma 4.2 with $X = \frac{1}{2N_\varepsilon}$ and using symmetries of m_ε, we get

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon}\left(1 - m_{\varepsilon,3}^2\right)\right) \mathrm{d}x \leqq 2N_\varepsilon, \tag{4.30}$$

and, for all $\varepsilon < \frac{1}{4N_\varepsilon}$, we have

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla^{1/2} m_{\varepsilon,3}|^2 \,\mathrm{d}x &\overset{(1.6)}{=} \frac{1}{4\pi} \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|m_{\varepsilon,3}(x_1, 0) - m_{\varepsilon,3}(y_1, 0)|^2}{(|x_1 - y_1|^2 + s^2)^{3/2}} \,\mathrm{d}s \,\mathrm{d}x_1 \,\mathrm{d}y_1 \\ &\geqq \frac{1}{2\pi} \sum_{k=1}^{N_\varepsilon} \int_{\frac{k-1}{N_\varepsilon}}^{\frac{k}{N_\varepsilon}} \int_{\frac{k-1}{N_\varepsilon}}^{\frac{k}{N_\varepsilon}} \frac{|m_{\varepsilon,3}(x_1, 0) - m_{\varepsilon,3}(y_1, 0)|^2}{|x_1 - y_1|^2} dx_1 \,\mathrm{d}y_1 \\ &= \frac{N_\varepsilon}{4\lambda_c} \int_{-\frac{1}{2N_\varepsilon}}^{\frac{1}{2N_\varepsilon}} \int_{-\frac{1}{2N_\varepsilon}}^{\frac{1}{2N_\varepsilon}} \frac{|\xi_{\varepsilon,\infty}(x) - \xi_{\varepsilon,\infty}(y)|^2}{|x - y|^2} \,\mathrm{d}x \,\mathrm{d}y \overset{(4.8)}{\geq} 2N_\varepsilon \frac{\ln(\frac{c}{2\varepsilon N_\varepsilon})}{\lambda_c} \end{aligned} \tag{4.31}$$

for some universal $c > 0$. To obtain the upper bound, we combine estimates (4.30) and (4.31) and optimize in $N_\varepsilon \in \mathbb{N}$. With $K^{-1} := \frac{1}{8}\min\{1, c\}$, the choice $N_\varepsilon := 2\big\lfloor K^{-1} \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \big\rfloor$ is admissible because $N_\varepsilon \geqq 2$ by (4.29) and $\varepsilon N_\varepsilon \leqq 2K^{-1} \leqq \frac{1}{4}$. Since $0 < \varepsilon < 1$, we get

$$F_{\varepsilon,\lambda}[m_\varepsilon] \leqq 2N_\varepsilon \left(1 - \frac{\lambda \ln\left(\frac{c}{2\varepsilon N_\varepsilon}\right)}{\lambda_c |\ln \varepsilon|}\right) \leqq -\frac{C\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\ln \varepsilon|} \tag{4.32}$$

for some universal $C > 0$, which is the desired estimate. □

4.3. Proof of Theorem 2.7

We begin with the proof of (i). Inserting (2.12) into the lower bound (4.2), we get, for sufficiently small $\varepsilon > 0$,

$$F_{\varepsilon,\lambda}[m] \geqq \left(\frac{|\ln \varepsilon| \ln(c/\beta_1) + \ln(c)\ln(\beta_1)}{|\ln \varepsilon|^2} \right) \int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x.$$

For $\beta_1 < c$ and $\varepsilon \ll 1$, the bracket above is positive, so that the minimal value $F_{\varepsilon,\lambda} = 0$ is only attained for $m \equiv \pm e_3$. On the other hand, since $\varepsilon^{\frac{\lambda_+(\varepsilon)-\lambda_c}{\lambda_+(\varepsilon)}} \le \frac{2}{\beta_2}$ for sufficiently small $\varepsilon > 0$, $m = \pm e_3$ is not a minimizer by Lemma 4.5.

We turn to the proof of (ii): We first note that the construction of the recovery sequence was already carried out in Lemma 4.3. For the proof of the lower bound, we claim that

$$\int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x \lesssim \max\left\{1, |\ln \varepsilon| F_{\varepsilon,\lambda_c}[m]\right\}. \tag{4.33}$$

For the proof of (4.33), it is enough to show that there are constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x \geqq C \quad \Longrightarrow \quad F_{\varepsilon,\lambda_c}[m] \gtrsim \frac{1}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x. \tag{4.34}$$

Indeed, by (3.1), we may bound the energy from below by

$$\begin{aligned} F_{\varepsilon,\lambda_c}[m_\varepsilon] &\overset{(3.1)}{\geq} \left(1 - \frac{\lambda_c}{|\ln \varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon}(1 - m_{\varepsilon,3}^2) \right) \mathrm{d}x \\ &\quad - \frac{\ln\left(c_* \max\left\{1, \min\left\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| \, \mathrm{d}x}, \frac{1}{\varepsilon}\right\}\right\}\right)}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| \, \mathrm{d}x. \end{aligned} \tag{4.35}$$

We first consider the case $\min\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| \, \mathrm{d}x}, \frac{1}{\varepsilon}\} \leqq 1$, when (4.35) turns into

$$F_{\varepsilon,\lambda_c}[m_\varepsilon] \geqq \left(1 - \frac{\lambda_c + \ln(c_*)}{|\ln \varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| \, \mathrm{d}x \gtrsim \int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x.$$

For the remaining case, we have $\min\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x}, \frac{1}{\varepsilon}\} \geqq 1$, and (4.35) implies

$$\begin{aligned} F_{\varepsilon,\lambda_c}[m_\varepsilon] &\geqq \left(1 - \frac{\lambda_c}{|\ln \varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon}(1 - m_{\varepsilon,3}^2) \right) \mathrm{d}x \\ &\quad - \frac{1}{|\ln \varepsilon|} \ln\left(\frac{c_*}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| \, \mathrm{d}x} \right) \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| \, \mathrm{d}x \\ &\overset{(4.4)}{\geqq} - \frac{1}{|\ln \varepsilon|} \ln\left(\frac{c_{**}}{\int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x} \right) \int_{\mathbb{T}^2} |\nabla m_3| \, \mathrm{d}x, \end{aligned}$$

inserting $c_{**} := c_* e^{\lambda_c}$. The estimate (4.34) follows with the choice $C = 2c_{**}$.

Let now $m_\varepsilon \to m$ in $L^1(\mathbb{T}^2)$ for some $m \in L^1(\mathbb{T}^2; \mathbb{R}^3)$. Lemma 4.4 yields

$$\liminf_{\varepsilon \to 0} F_{\varepsilon,\lambda_c}[m_\varepsilon] \geqq 0,$$

which proves the lower bound if $m \in L^1(\mathbb{T}^2; \{\pm e_3\})$. Otherwise, we may assume $\int_{\mathbb{T}^2}(1-m_{\varepsilon,3}^2)\,\mathrm{d}x \gtrsim 1$. For sufficiently small ε, estimates (3.1) and (4.33) then yield

$$\begin{aligned} 1 &\lesssim \int_{\mathbb{T}^2}(1-m_{\varepsilon,3}^2)\,\mathrm{d}x \lesssim \varepsilon\left(F_{\varepsilon,\lambda}[m_\varepsilon] + \frac{\lambda_c}{|\ln\varepsilon|}\int_{\mathbb{T}^2}|\nabla^{1/2}m_{\varepsilon,3}|^2\,\mathrm{d}x\right) \\ &\overset{(3.1)}{\lesssim} \varepsilon\Big(F_{\varepsilon,\lambda}[m_\varepsilon] + \int_{\mathbb{T}^2}|\nabla m_{\varepsilon,3}|\,\mathrm{d}x\Big) \overset{(4.33)}{\lesssim} \varepsilon\left(1+|\ln\varepsilon|F_{\varepsilon,\lambda_c}[m_\varepsilon]\right), \end{aligned} \tag{4.36}$$

which implies $\liminf_{\varepsilon\to 0} F_{\varepsilon,\lambda_c}[m_\varepsilon] = +\infty$, concluding the proof of Γ-convergence.

To prove item (iii), we use the construction in Lemma 4.5; however choosing $N_\varepsilon := \lfloor \ln(|\ln\varepsilon|)\rfloor$ this time. Analogous to (4.32), we get for $\varepsilon \ll 1$ that

$$F_{\varepsilon,\lambda}[m_\varepsilon] \leqq 2N_\varepsilon\left(1 - \frac{\ln(\frac{c}{2\varepsilon N_\varepsilon})}{|\ln\varepsilon|}\right) \lesssim \frac{N_\varepsilon \ln N_\varepsilon}{|\ln\varepsilon|} \longrightarrow 0 \quad \text{for } \varepsilon \to 0.$$

It hence remains to show that m_ε is not compact in the strong L^1-topology. Since $\int_{\mathbb{T}^2}|m_\varepsilon|^2\,\mathrm{d}x = 1$, any possible limit $\tilde{m}$ of (a subsequence of) m_ε then would need to satisfy $\int_{\mathbb{T}^2}|\tilde{m}|^2\,\mathrm{d}x = 1$. However, since $\varepsilon N_\varepsilon \to 0$ as $\varepsilon \to 0$, clearly $m_\varepsilon \rightharpoonup 0$ in $L^2(\mathbb{T}^2)$, leading to a contradiction.

Finally, item (iv) follows directly from (4.36), (4.33) and the compact embedding $BV(\mathbb{T}^2) \hookrightarrow L^1(\mathbb{T}^2)$.

5. Stray Field Estimates and Reduction of the Full Energy

In this section, we establish a basic lower bound for the full micromagnetic energy $\mathscr{E}$ by an expression in which the stray field energy is represented, up to an additive constant, by an effective anisotropy term minus a multiple of the square of the $\mathring{H}^{1/2}(\mathbb{T}_\ell^2)$-norm of the average of the out-of-plane component of the magnetization, provided the exchange stiffness is slightly reduced. Importantly, this lower bound becomes asymptotically sharp in the limit of vanishing film thickness. We note that in the context of ferromagnetic films in which the magnetization lies mostly in the film plane, related results have been obtained in [10–12,41].

Proposition 5.1. *(Reduction of the energy) Let $t > 0$, $\ell > 0$ and $Q > 1$, and let $m \in H^1(\mathbb{T}_\ell^2 \times (0,t); \mathbb{R}^3)$. Then there is a universal constant $C > 0$ such that $\mathscr{E}$ may be bounded below as follows:*

$$\begin{aligned} \mathscr{E}[m] \geqq \ell^2 t + \left(1 - Ct^2\right)\int_{\mathbb{T}_\ell^2\times(0,t)}|\nabla m|^2\,\mathrm{d}x + (Q-1)\int_{\mathbb{T}_\ell^2\times(0,t)}(1-m_3^2)\,\mathrm{d}x \\ - \frac{t^2}{2}\int_{\mathbb{T}_\ell^2}|\nabla^{1/2}\overline{m}_3|^2\,\mathrm{d}x, \end{aligned} \tag{5.1}$$

where $\overline{m}(x') = \frac{1}{t}\int_0^t m(x',x_3)\,\mathrm{d}x_3$.

Also note that for two-dimensional magnetizations, that is, for configurations of the form $m = \overline{m}\chi_{(0,t)}$, the estimate in (5.1) also holds in the reversed direction if $-C$ is replaced by C. We remark that a similar sharp estimate for the three-dimensional dipolar energy holds for thin three-dimensional domains in the whole space [53]. For the proof of Proposition 5.1, which is deferred until the end of this section, we need several estimates presented in the sequel.

We begin with an observation that since the thickness t of the film is small, the exchange energy strongly penalizes oscillations of the magnetization in the normal direction of the film. Hence the averaged magnetization $\overline{m}$ is a good approximation of m, which can be made rigorous by the following Poincaré-type inequality which holds for all $m \in H^1(\mathbb{T}_\ell^2 \times (0,t); \mathbb{R}^3)$:

$$\int_{\mathbb{T}_\ell^2\times(0,t)} |m - \chi_{(0,t)}\overline{m}|^2 \,\mathrm{d}x \lesssim t^2 \int_{\mathbb{T}_\ell^2\times(0,t)} |\partial_3 m|^2 \,\mathrm{d}x. \tag{5.2}$$

We next show that for thin films the difference between the stray field energy $\mathscr{E}_s[m] := \int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m]|^2 \,\mathrm{d}x$ and

$$\mathscr{E}_s^0[m] := \int_{\mathbb{T}_\ell^2\times(0,t)} m_3^2 \,\mathrm{d}x - \frac{t^2}{2}\int_{\mathbb{T}_\ell^2} |\nabla^{1/2}\overline{m}_3|^2 \,\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{T}_\ell^2} |\nabla^{-1/2}\nabla\cdot\overline{m}'|^2 \,\mathrm{d}x \tag{5.3}$$

may be estimated by the exchange energy at lower order. We state the result in the form of a theorem, as it is of independent interest. In fact, our result provides a universal stray field energy expansion for thin films in a periodic setting and thus contains all previously obtained asymptotic estimates for specific thin film regimes [10–12,19,41]. Note that our result is slightly stronger than what is necessary to prove Proposition 5.1.

Theorem 5.2. *Let $t > 0$, $\ell > 0$ and let $m \in H^1(\mathbb{T}_\ell^2 \times (0,t); \mathbb{R}^3)$. With the notation*

$$D := \int_{\mathbb{T}_\ell^2\times(0,t)} |\nabla m|^2 \,\mathrm{d}x,$$

the stray field energy then satisfies

(i) $\displaystyle \left|\int_{\mathbb{T}_\ell^2\times\mathbb{R}} \left(|\mathscr{H}[m]|^2 - |\mathscr{H}[m_3e_3]|^2 - |\mathscr{H}[m - m_3e_3]|^2\right)\mathrm{d}x\right| \lesssim t^2 D,$

(ii) $\displaystyle \left|\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m]|^2 - |\mathscr{H}[\overline{m}\chi_{(0,t)}]|^2 \,\mathrm{d}x\right| \lesssim t^2 D.$

Furthermore, the contributions due to m_3 and $m - m_3e_3$ may be approximated by

(iii) $\displaystyle \left|\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m_3e_3]|^2 \,\mathrm{d}x - \int_{\mathbb{T}_\ell^2\times(0,t)} m_3^2 \,\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{T}_\ell^2} |\nabla^{1/2}\overline{m}_3|^2 \,\mathrm{d}x\right| \lesssim t^2 D,$

(iv) $\displaystyle \left|\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m - m_3e_3]|^2 \,\mathrm{d}x - \frac{t^2}{2}\int_{\mathbb{T}_\ell^2} |\nabla^{-1/2}\nabla\cdot(\overline{m} - \overline{m}_3e_3)|^2 \,\mathrm{d}x\right| \lesssim t^2 D.$

In particular, if $\mathscr{E}_s^0$ is defined in (5.3), then $|\mathscr{E}_s[m] - \mathscr{E}_s^0[m]| \lesssim t^2 D$.

Proof. It is sufficient to argue for $m \in C_c^\infty(\mathbb{T}_\ell^2 \times \mathbb{R}; \mathbb{R}^3)$, since the general case follows by an approximation argument, as we now explain. For every $m \in H^1(\mathbb{T}_\ell^2 \times (0,t); \mathbb{R}^3)$, extended by zero to the rest of $\mathbb{T}_\ell^2 \times \mathbb{R}$, there is a sequence $(m_n)_{n\in\mathbb{N}}$ with $m_n \in C_c^\infty(\mathbb{T}_\ell^2 \times \mathbb{R}; \mathbb{R}^3)$ such that $\|m - m_n\|_{L^2(\mathbb{T}_\ell^2\times\mathbb{R})} \to 0$ and $\|\nabla m - \nabla m_n\|_{L^2(\mathbb{T}_\ell^2\times(0,t))} \to 0$. It remains to check that all terms in (i)–(iv) are continuous. By Jensen's inequality we have $\|\overline{m}_n - \overline{m}\|_{L^2(\mathbb{T}_\ell^2)} \to 0$. Moreover, clearly $t\|\nabla\overline{m}_n\|^2_{L^2(\mathbb{T}_\ell^2)} \lesssim \|\nabla m_n\|^2_{L^2(\mathbb{T}_\ell^2\times(0,t))}$. Hence the convergence follows from the elliptic estimate $\|\mathscr{H}[m_n - m]\|_{L^2(\mathbb{T}_\ell^2\times\mathbb{R})} \leqq \|m_n - m\|_{L^2(\mathbb{T}_\ell^2\times\mathbb{R})}$ and by interpolation for the terms involving fractional derivatives.

We write the stray field energy in terms of the magnetostatic potential ϕ:

$$\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m]|^2 \,\mathrm{d}x = -\int_{\mathbb{T}_\ell^2\times\mathbb{R}} \phi\, \nabla\cdot m \,\mathrm{d}x, \qquad \Delta\phi = \nabla\cdot m \text{ in } \mathscr{D}'(\mathbb{T}_\ell^2 \times \mathbb{R}).$$

Upon passing to Fourier series with respect to the in-plane variables, we get

$$\int_{\mathbb{T}_\ell^2\times\mathbb{R}} \phi\, \nabla\cdot m \,\mathrm{d}x = \frac{1}{\ell^2}\int_{\mathbb{R}} \sum_k \widehat{\phi}_k^*(z) \left(\partial_z \widehat{m}_{3,k}(z) - ik\cdot\widehat{m}_k'(z)\right) \,\mathrm{d}z, \tag{5.4}$$

where the Fourier coefficients $\widehat{\phi}_k : \mathbb{R} \to \mathbb{C}$, $k \in \frac{2\pi}{\ell}\mathbb{Z}^2$, of ϕ solve

$$\partial_z^2\widehat{\phi}_k - |k|^2\widehat{\phi}_k = \partial_z\widehat{m}_{3,k} - ik\cdot\widehat{m}_k'.$$

We introduce the fundamental solution (for a closely related approach, see [10,11])

$$H_k(z) = \begin{cases} \frac{1}{|k|}e^{-|k||z|} & \text{for } k \neq 0, \\ -|z| & \text{for } k = 0, \end{cases}$$

which, using the notation $\delta(z)$ for the Dirac measure at $z = 0$, satisfies

$$-\partial_z^2 H_k + |k|^2 H_k = 2\delta(z) \qquad \text{in } \mathscr{D}'(\mathbb{R}). \tag{5.5}$$

The fundamental solution allows us to rewrite $\widehat{\phi}_k(z)$ as

$$\widehat{\phi}_k(z) = -\frac{1}{2}\int_{\mathbb{R}} H_k(z-z')\left(\partial_z\widehat{m}_{3,k}(z') - ik\cdot\widehat{m}_k'(z')\right)dz',$$

which, by (5.4), leads to the following expression for the stray field energy:

$$\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m]|^2 \,\mathrm{d}x = \frac{1}{2\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \sum_k (\partial_z\widehat{m}_{3,k}(z) - ik\cdot\widehat{m}_k'(z))^* \times H_k(z-z')(\partial_z\widehat{m}_{3,k}(z') - ik\cdot\widehat{m}_k'(z'))\,\mathrm{d}z\,\mathrm{d}z'. \tag{5.6}$$

To prove (i), we need to show that the mixed terms in (5.6) of the form

$$I := \frac{1}{\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \sum_k \partial_z\widehat{m}_{3,k}^*(z) H_k(z-z')(ik\cdot\widehat{m}_k'(z'))\,\mathrm{d}z\,\mathrm{d}z' \tag{5.7}$$

satisfy $|I| \lesssim t^2 D$. Integrating by parts in (5.7) and writing $m = \chi_{(0,t)}\overline{m} + u$ with the usual notation $\overline{m}(x') = \frac{1}{t}\int_0^t m(x', x_3)\,\mathrm{d}x_3$, we get

$$\begin{aligned} I &= -\frac{1}{\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_k \widehat{m}^*_{3,k}(z)\partial_z H_k(z-z')(ik\cdot\widehat{m}'_k(z'))\,\mathrm{d}z\,\mathrm{d}z' \\ &= -\frac{1}{\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_k (\chi_{(0,t)}(z)\widehat{\overline{m}}^*_{3,k} + \widehat{u}^*_{3,k}(z))\partial_z H_k(z-z') \\ &\qquad \times \left(ik\cdot\chi_{(0,t)}(z')\widehat{\overline{m}}'_k + ik\cdot\widehat{u}'_k(z')\right)\,\mathrm{d}z\,\mathrm{d}z'. \end{aligned} \tag{5.8}$$

Since $\partial_z H_k(z) = -\frac{z}{|z|}e^{-|k||z|}$ is anti-symmetric in z, we have $\int_0^t\int_0^t \partial_z H_k(z-z')\,\mathrm{d}z\,\mathrm{d}z' = 0$ so that upon expanding (5.8), the term involving $\widehat{\overline{m}}_{3,k}$ and $\widehat{\overline{m}}'_k$ vanishes. Using $|\partial_z H_k| \leqq 1$, (5.8) then can be estimated by

$$|I| \leqq \frac{1}{\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_k \left(|\widehat{u}_{3,k}(z)|\,|k\cdot\widehat{m}'_k(z')| + |\chi_{(0,t)}(z)\widehat{\overline{m}}_{3,k}|\,|k\cdot\widehat{u}'_k(z')|\right)\,\mathrm{d}z\,\mathrm{d}z'. \tag{5.9}$$

We note that passing to Fourier series in the in-plane variables commutes with taking e_3-averages. Thus $\widehat{u}_{j,k}$ has e_3-average zero for all $j = 1, 2, 3$. By the fundamental theorem of calculus, we thus get

$$|\widehat{u}_{j,k}(z)| \leqq \int_0^t |\partial_z\widehat{m}_{j,k}(\tau)|d\tau \qquad \text{for all } z\in(0,t) \text{ and } j = 1,2,3. \tag{5.10}$$

Inserting (5.10) into (5.9), we get

$$|I| \lesssim \sum_{n,j=1}^3 \frac{t}{\ell^2}\int_0^t\int_0^t \sum_k |\partial_z\widehat{m}_{j,k}(z)|\,|k|\,|\widehat{m}_{n,k}(z')|\,\mathrm{d}z\,\mathrm{d}z'.$$

By Young's inequality and Parseval's identity, we conclude

$$|I| \lesssim \sum_{n,j=1}^3 \frac{t}{\ell^2}\int_0^t\int_0^t \sum_k \left(|\partial_z\widehat{m}_{j,k}(z)|^2 + |k|^2|\widehat{m}_{n,k}(z')|^2\right)\,\mathrm{d}z\,\mathrm{d}z' \lesssim t^2 D,$$

completing the proof of (i).

Assuming that (iii) and (iv) hold, estimate (ii) is obtained as follows. Applying (i) to m and $\chi_{(0,t)}\overline{m}$, we get

$$\begin{aligned} \Bigg| &\int_{\mathbb{T}^2_\ell\times\mathbb{R}} |\mathscr{H}[m]|^2\,\mathrm{d}x - \int_{\mathbb{T}^2_\ell\times\mathbb{R}} |\mathscr{H}[\chi_{(0,t)}\overline{m}]|^2\,\mathrm{d}x \\ &- \int_{\mathbb{T}^2_\ell\times\mathbb{R}} |\mathscr{H}[m_3e_3]|^2\,\mathrm{d}x + \int_{\mathbb{T}^2_\ell\times\mathbb{R}} |\mathscr{H}[\chi_{(0,t)}\overline{m}_3e_3]|^2\,\mathrm{d}x \\ &- \int_{\mathbb{T}^2_\ell\times\mathbb{R}} |\mathscr{H}[m - m_3e_3]|^2\,\mathrm{d}x + \int_{\mathbb{T}^2_\ell\times\mathbb{R}} |\mathscr{H}[\chi_{(0,t)}(\overline{m} - \overline{m}_3e_3)]|^2\,\mathrm{d}x\Bigg| \lesssim t^2 D, \end{aligned} \tag{5.11}$$

where we also have used that

$$\int_{\mathbb{T}_\ell^2\times(0,t)} |\nabla(\overline{m}\chi_{(0,t)})|^2\,\mathrm{d}x = t\int_{\mathbb{T}_\ell^2} |\nabla\overline{m}|^2\,\mathrm{d}x \leqq D.$$

Applying (iii) and (iv) to (5.11), we get

$$\Big|\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m]|^2\,\mathrm{d}x - \int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[\chi_{(0,t)}\overline{m}]|^2\,\mathrm{d}x \\ - \int_{\mathbb{T}_\ell^2\times(0,t)} m_3^2\,\mathrm{d}x + \int_{\mathbb{T}_\ell^2\times(0,t)} \chi_{(0,t)}\overline{m}_3^2\,\mathrm{d}x\Big| \lesssim t^2 D,$$

which yields the claim with the help of (5.2) and in view of the fact that $\int_{\mathbb{T}_\ell^2\times(0,t)}(m_3^2-\chi_{(0,t)}\overline{m}_3^2)\,\mathrm{d}x = \int_{\mathbb{T}_\ell^2\times(0,t)} |m_3-\chi_{(0,t)}\overline{m}_3|^2\,\mathrm{d}x$.

We turn to the proof of (iii). Integrating by parts twice and by (5.5), with the help of Parseval's identity we get

$$\begin{aligned}\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m_3e_3]|^2\,\mathrm{d}x &\overset{(5.6)}{=} -\frac{1}{2\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_k \widehat{m}_{3,k}^*(z)\partial_z^2 H_k(z-z')\widehat{m}_{3,k}(z')\,\mathrm{d}z\,\mathrm{d}z' \\ &\overset{(5.5)}{=} \int_{\mathbb{R}\times(0,t)} m_3^2\,\mathrm{d}x - \frac{1}{2\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_k \widehat{m}_{3,k}^*(z)|k|e^{-|k||z-z'|}\widehat{m}_{3,k}(z')\,\mathrm{d}z\,\mathrm{d}z'.\end{aligned} \tag{5.12}$$

Since $|1-e^{-|k||z|}| \leqq |k|t$ for $z\in(-t,t)$, the last integral above

$$J := \frac{1}{2\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_k \widehat{m}_{3,k}^*(z)|k|e^{-|k||z-z'|}\widehat{m}_{3,k}(z')\,\mathrm{d}z\,\mathrm{d}z'$$

may be estimated, with the help of Cauchy-Schwarz inequality, as follows:

$$\begin{aligned}\Big|J - \frac{t^2}{2\ell^2}\sum_k |k||\widehat{\overline{m}}_{3,k}(z)|^2\Big| &\lesssim \frac{t}{\ell^2}\int_0^t\int_0^t\sum_k |\widehat{m}_{3,k}(z)||k|^2|\widehat{m}_{3,k}(z')|\,\mathrm{d}z\,\mathrm{d}z' \\ &\lesssim \frac{t^2}{\ell^2}\int_0^t\sum_k |k|^2|\widehat{m}_{3,k}(z)|^2\,\mathrm{d}z,\end{aligned}$$

which by Parseval's identity is equivalent to

$$\Big|J - \frac{t^2}{2}\int_{\mathbb{T}_\ell^2}|\nabla^{1/2}\overline{m}_3|^2\,\mathrm{d}x\Big| \lesssim t^2\int_{\mathbb{T}_\ell^2\times(0,t)}|\nabla' m_3|^2\,\mathrm{d}x \;\leqq\; t^2 D. \tag{5.13}$$

Assertion (iii) then follows from (5.12) together with (5.13).

We continue with the proof of (iv). By (5.6) we have

$$\begin{aligned}&\int_{\mathbb{T}_\ell^2\times\mathbb{R}} |\mathscr{H}[m-m_3e_3]|^2\,\mathrm{d}x \\ &\overset{(5.6)}{=} \frac{1}{2\ell^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_k (k\cdot\widehat{m}_k'(z))^* H_k(z-z')k\cdot\widehat{m}_k'(z')\,\mathrm{d}z\,\mathrm{d}z'.\end{aligned}$$

Since $|1-e^{-|k||z|}| \leqq |k|t$ for $z \in (0,t)$, we may insert $|H_k(z-z') - \frac{1}{|k|}| \leqq t$ for $k \neq 0$ above. Using Cauchy-Schwarz inequality, this yields

$$\left| \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |\mathscr{H}[m - m_3 e_3]|^2 \, \mathrm{d}x - \frac{t^2}{2\ell^2} \sum_{k \neq 0} \frac{|k \cdot \widehat{\overline{m}}_k'|^2}{|k|} \right| \lesssim \frac{t^2}{2\ell^2} \int_{\mathbb{R}} \sum_k |k \cdot \widehat{m}_k'(z)|^2 \, \mathrm{d}z$$
$$\lesssim t^2 D,$$

which completes the proof of (iv) and of the theorem. □

We are now ready to give the proof of Proposition 5.1.

Proof. (*Proposition* 5.1) We invoke Theorem 5.2 to obtain a lower bound for the stray field energy. Combining Theorem 5.2(i) with (iii) and neglecting the non-negative term $\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |\mathscr{H}[m - m_3 e_3]|^2 \, \mathrm{d}x$, we get

$$\begin{aligned} \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |\mathscr{H}[m]|^2 \, \mathrm{d}x &\geq \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |\mathscr{H}[m_3 e_3]|^2 \, \mathrm{d}x - Ct^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 \, \mathrm{d}x \\ &\geqq \int_{\mathbb{T}_\ell^2 \times (0,t)} m_3^2 \, \mathrm{d}x - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \overline{m}_3|^2 \, \mathrm{d}x - Ct^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 \, \mathrm{d}x \end{aligned} \tag{5.14}$$

for some universal $C > 0$. Inserting (5.14) into the energy E yields (5.1). □

6. Proofs for the Full Energy $E_{\varepsilon,\lambda}$

The proofs for the full energy $E_{\varepsilon,\lambda}$ are based on the corresponding arguments for the reduced energy $F_{\varepsilon,\lambda}$. Under mild assumptions on ℓ, t, Q, weaker than those of Theorems 2.1 and 2.2, Lemma 3.1, and Theorem 5.2 yield the following estimates for the full energy $E_{\varepsilon,\lambda}$ (given in (2.2)):

Lemma 6.1. *Let $Q > 1$ and $\lambda > 0$. There is $\varepsilon_0 = \varepsilon_0(\lambda, Q) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $m \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ we have*

$$\begin{aligned} E_{\varepsilon,\lambda}[m] \geq & \left(1 - \frac{C\lambda^2(Q-1)}{|\ln \varepsilon|^2} - \frac{\lambda}{|\ln \varepsilon|}\right) \int_{\mathbb{T}^2 \times (0,1)} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon}(1 - m_3^2)\right) \mathrm{d}x \\ & - \frac{\lambda}{\lambda_c |\ln \varepsilon|} \ln \left(c_* \max \left\{1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla \overline{m}_3| \, \mathrm{d}x}, \frac{1}{\varepsilon} \right\} \right\} \right) \int_{\mathbb{T}^2} |\nabla \overline{m}_3| \, \mathrm{d}x, \\ & + \frac{C |\ln \varepsilon|^2}{\varepsilon \lambda^2 (Q-1)^2} \int_{\mathbb{T}^2 \times (0,1)} |\partial_3 m|^2 \, \mathrm{d}x. \end{aligned} \tag{6.1}$$

Furthermore, for any $\overline{m} \in H^1(\mathbb{T}^2; \mathbb{S}^2)$, we have

$$\begin{aligned} E_{\varepsilon,\lambda}[\overline{m}\chi_{(0,1)}] \leqq & \left(1 + \frac{C\lambda^2(Q-1)}{|\ln \varepsilon|^2}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla \overline{m}|^2 + \frac{1}{2\varepsilon}(1 - \overline{m}_3^2)\right) \mathrm{d}x \\ & - \frac{\lambda}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} \overline{m}_3|^2 \, \mathrm{d}x. \end{aligned} \tag{6.2}$$

Proof. The lower bound for $\mathscr{E}$ in Proposition 5.1 implies

$$E_{\varepsilon,\lambda}[m] = \frac{\mathscr{E}[m(\ell\cdot,\ell\cdot,t\cdot)] - \ell^2 t}{2\ell t\sqrt{Q-1}}$$
$$\geqq (1-Ct^2)\int_{\mathbb{T}^2\times(0,1)}\left(\frac{|\nabla' m|^2}{2\ell\sqrt{Q-1}} + \frac{\ell}{2t^2\sqrt{Q-1}}|\partial_3 m|^2\right)\mathrm{d}x$$
$$+ \frac{\ell\sqrt{Q-1}}{2}\int_{\mathbb{T}^2\times(0,1)}\left(1-m_3^2\right)\mathrm{d}x - \frac{t}{4\sqrt{Q-1}}\int_{\mathbb{T}^2}|\nabla^{1/2}\overline{m}_3|^2\,\mathrm{d}x.$$

In terms of ε, λ (defined in (2.1)), this turns into

$$E_{\varepsilon,\lambda}[m]$$
$$\geqq \left(1 - \frac{C_1\lambda^2(Q-1)}{|\ln\varepsilon|^2}\right)\int_{\mathbb{T}^2\times(0,1)}\left(\frac{\varepsilon|\nabla' m|^2}{2} + \frac{1-m_3^2}{2\varepsilon} + \frac{C_2|\ln\varepsilon|^2|\partial_3 m|^2}{\varepsilon\lambda^2(Q-1)^2}\right)\mathrm{d}x$$
$$- \frac{\lambda}{|\ln\varepsilon|}\int_{\mathbb{T}^2}|\nabla^{1/2}\overline{m}_3|^2\,\mathrm{d}x \tag{6.3}$$

for some universal $C_1, C_2 > 0$. Therefore, with the help of Jensen's inequality and Lemma 3.1, we obtain

$$E_{\varepsilon,\lambda}[m] \geqq X + \frac{C_2|\ln\varepsilon|^2}{\varepsilon\lambda^2(Q-1)^2}\left(1 - \frac{C_1\lambda^2(Q-1)}{|\ln\varepsilon|^2}\right)\int_{\mathbb{T}^2\times(0,1)}|\partial_3 m_3|^2\,\mathrm{d}x,$$

where X denotes the first two lines on the right hand side of (6.1). By choosing $\varepsilon_0(\lambda, Q) > 0$ sufficiently small, (6.1) follows. The proof for the upper bound (6.2) is simpler and analogous to the arguments that led to (6.3). $\square$

6.1. Proof of Theorem 2.1

We apply the lower bound of Lemma 6.1 directly and extend the corresponding arguments to $F_{\varepsilon,\lambda}$ in the proof of Theorem 2.5. Note that it would also be possible to invoke the lower bound for $F_{\varepsilon,\lambda}$ on slices $\{x_3 = \text{const}\}$ to obtain the lower bound for the full energy $E_{\varepsilon,\lambda}$. We will not pursue this option, since this approach would get rather technical in view of the fact that $C^\infty(\mathbb{T}^2\times(0,1);\mathbb{S}^2)$ is not dense in $H^1(\mathbb{T}^2\times(0,1);\mathbb{S}^2)$, see for example [3,4,29].

Consequences of the lower bound in (6.1): By assumption we have $Q > 1$ and $0 < \lambda < \lambda_c$. We also note that for $\varepsilon < 1$ we have

$$\ln\left(c_*\max\left\{1, \min\left\{\frac{1}{\varepsilon\int_{\mathbb{T}^2}|\nabla\overline{m}_3|\,\mathrm{d}x}, \frac{1}{\varepsilon}\right\}\right\}\right) \leqq \ln\left(\frac{c_*}{\varepsilon}\right), \tag{6.4}$$

where c_* is as in Lemma 3.1. Inserting (6.4) into the lower bound in (6.1) and using (4.4) and Jensen's inequality, we deduce that for any $\gamma > 0$ and sufficiently small $\varepsilon \leqq \varepsilon_0(\gamma, Q)$, we have

$$E_{\varepsilon,\lambda}[m] \geqq \left(1 - \frac{\lambda}{\lambda_c} - \gamma\right)\int_{\mathbb{T}^2\times(0,1)}\left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-m_3^2)\right)\mathrm{d}x$$
$$+ \frac{C|\ln\varepsilon|^2}{\varepsilon\lambda^2(Q-1)^2}\int_{\mathbb{T}^2\times(0,1)}|\partial_3 m|^2\,\mathrm{d}x. \tag{6.5}$$

Applying (4.4) to (6.5) and again using Jensen's inequality yields

$$E_{\varepsilon,\lambda}[m] \geqq \left(1 - \frac{\lambda}{\lambda_c} - \gamma\right) \int_{\mathbb{T}^2} |\nabla' \overline{m}_3| \, \mathrm{d}x \tag{6.6}$$

for sufficiently small ε. With the choice $\gamma := \frac{1}{2}(1 - \frac{\lambda}{\lambda_c})$, (6.5) and (5.2) imply

$$\begin{aligned} &\int_{\mathbb{T}^2 \times (0,1)} (1 - m_3^2) \ \mathrm{d}x \lesssim \frac{\varepsilon}{\lambda_c - \lambda} E_{\varepsilon,\lambda}[m], \\ &\int_{\mathbb{T}^2 \times (0,1)} |m - \overline{m} \chi_{(0,1)}|^2 \, \mathrm{d}x \overset{(5.2)}{\lesssim} \int_{\mathbb{T}^2 \times (0,1)} |\partial_3 m|^2 \, \mathrm{d}x \overset{(6.5)}{\lesssim} \frac{\varepsilon \lambda^2 (Q-1)^2}{|\ln \varepsilon|^2} E_{\varepsilon,\lambda}[m]. \end{aligned} \tag{6.7}$$

Compactness: In order to prove compactness, we consider a sequence $m_\varepsilon \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ with $E_{\varepsilon,\lambda}[m_\varepsilon] \leqq C$. Since $\lambda < \lambda_c$, (6.6) yields a uniform bound on $\overline{m}_{\varepsilon,3}$ in $BV(\mathbb{T}_2)$, which implies that $\overline{m}_{\varepsilon,3} \to \overline{m}_3$ in $L^1(\mathbb{T}^2)$ as $\varepsilon \to 0$ for a subsequence and some $\overline{m}_3 \in BV(\mathbb{T}^2)$. In view of (6.7) and by application of the triangle inequality, we also get $m_\varepsilon \to \chi_{(0,1)} \overline{m}_3 e_3$ in $L^1(\mathbb{T}^2 \times (0,1); \mathbb{R}^3)$, with $\overline{m}_3(x)$ taking values ± 1 for almost everywhere $x \in \mathbb{T}^2$.
Liminf inequality: Let $m_\varepsilon \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ with $m_\varepsilon \to m$ in $L^1(\mathbb{T}^2 \times (0,1))$. By Jensen's inequality, we also have $\overline{m}_\varepsilon \to \overline{m}$ in $L^1(\mathbb{T}^2; \mathbb{R}^3)$. By lower semicontinuity of the BV seminorm, the liminf inequality follows from (6.6) by arbitrariness of γ.
Recovery sequence: It remains to prove the upper bound for the Γ-convergence. As it turns out, we may use the recovery sequence for the reduced energy $F_{\varepsilon,\lambda}$ also for the full energy $E_{\varepsilon,\lambda}$ (up to thickening). Let $\lambda \leqq \lambda_c$, $\overline{m} \in BV(\mathbb{T}^2; \{\pm e_3\})$ and let $\overline{m}_\varepsilon \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ denote the recovery sequence for $F_{\varepsilon,\lambda}$ from Lemma 4.3. We set

$$m_\varepsilon(x', x_3) := \chi_{(0,1)}(x_3) \overline{m}_\varepsilon(x') \qquad \text{for } (x', x_3) \in \mathbb{T}^2 \times (0,1).$$

From the upper bound in (6.2), we obtain

$$\begin{aligned} E_{\varepsilon,\lambda}[m_\varepsilon] &\leqq \left(1 + \frac{C\lambda^2(Q-1)}{|\ln \varepsilon|^2}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla \overline{m}_\varepsilon|^2 + \frac{1}{2\varepsilon}(1 - \overline{m}_{\varepsilon,3}^2)\right) \mathrm{d}x \\ &\qquad - \frac{\lambda}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} \overline{m}_{\varepsilon,3}|^2 \, \mathrm{d}x \\ &= F_{\varepsilon,\lambda}[\overline{m}_\varepsilon] + \frac{C\lambda^2(Q-1)}{|\ln \varepsilon|^2} \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla \overline{m}_\varepsilon|^2 + \frac{1}{2\varepsilon}\left(1 - \overline{m}_{\varepsilon,3}^2\right)\right) \mathrm{d}x. \end{aligned} \tag{6.8}$$

From (4.12) and (4.13) in the proof of Lemma 4.3, we note that the second term on the right hand side of (6.8) vanishes in the limit $\varepsilon \to 0$. The estimate then follows upon applying Lemma 4.3 to (6.8).

6.2. *Proof of Theorem 2.2*

Lower bound in (2.3)*:* As in the argument that lead from (4.25) to (4.27), we reduce (6.1) to the estimate

$$
\begin{aligned}
E_{\varepsilon,\lambda}[m] \geqq & \left(1 - \frac{\lambda}{|\ln\varepsilon|} - \frac{C\lambda^2(Q-1)}{|\ln\varepsilon|^2}\right) \int_{\mathbb{T}^2\times(0,1)} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-m_3^2)\right) \mathrm{d}x \\
& - \frac{\lambda}{\lambda_c|\ln\varepsilon|} \ln\left(\frac{c_*}{\varepsilon\int_{\mathbb{T}^2}|\nabla\overline{m}_3|\,\mathrm{d}x}\right) \int_{\mathbb{T}^2} |\nabla\overline{m}_3|\,\mathrm{d}x \\
& + \frac{C|\ln\varepsilon|^2}{\varepsilon\lambda^2(Q-1)^2} \int_{\mathbb{T}^2\times(0,1)} |\partial_3 m|^2\,\mathrm{d}x. \qquad (6.9)
\end{aligned}
$$

We introduce the notation

$$
D_\varepsilon[m] := \int_{\mathbb{T}^2\times(0,1)} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-m_3^2)\right)\mathrm{d}x - \int_{\mathbb{T}^2} |\nabla\overline{m}_3|\,\mathrm{d}x,
$$

and set

$$
\mu := \varepsilon^{\frac{\lambda-\lambda_c}{\lambda}} \int_{\mathbb{T}^2} |\nabla\overline{m}_3|\,\mathrm{d}x, \qquad (6.10)
$$

and $c_{**} := c_* e^{2\lambda_c} > 1$ (a universal constant, as in the proof of Theorem 2.6). For ε sufficiently small, as in the argument leading to (4.28), (6.9) then can be written as

$$
\begin{aligned}
E_{\varepsilon,\lambda}[m] \geqq & \left(1 - \frac{2\lambda}{|\ln\varepsilon|}\right) D_\varepsilon[m] - \ln\left(\frac{c_{**}}{\mu}\right) \frac{\mu\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{\lambda_c|\ln\varepsilon|} \\
& + \frac{C|\ln\varepsilon|^2}{\varepsilon\lambda^2(Q-1)^2} \int_{\mathbb{T}^2\times(0,1)} |\partial_3 m|^2\,\mathrm{d}x. \qquad (6.11)
\end{aligned}
$$

Minimizing in $\mu > 0$, (6.11) yields $\mu \sim 1$, and the lower bound in (2.3) follows by positivity of $D_\varepsilon[m]$ in view of (4.4).
Upper bound in (2.3)*:* Let $\overline{m}_\varepsilon$ denote the function constructed in Lemma 4.5 and let $m_\varepsilon(x', x_3) := \chi_{(0,1)}(x_3)\overline{m}_\varepsilon(x')$. We choose $N_\varepsilon = 2\lfloor \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \rfloor$, noting that $N_\varepsilon > 0$ for ε sufficiently small. We insert (4.30) and (4.31) into (6.2) to deduce that for ε sufficiently small we have

$$
E_{\varepsilon,\lambda}(m_\varepsilon) \leqq 2N_\varepsilon \left(1 - \frac{\lambda\ln(\frac{c}{2\varepsilon N_\varepsilon})}{\lambda_c|\ln\varepsilon|}\right) \leqq -\frac{C\lambda\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\ln\varepsilon|}.
$$

6.3. *Proof of Theorem 2.3*

Let μ be defined by (6.10). Then (2.4) and (6.11) imply $\mu \sim 1$ and hence (2.7) in (iii). In turn, inserting (2.4) and (iii) into (6.11) implies (2.8) in (iv). Poincaré's inequality together with (6.11) and (2.4) yield (2.5) in (i). Finally, we deduce (2.6) in (ii) from (iii) and (iv).

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