On an Isoperimetric Problem
with a Competing Nonlocal Term I:
The Planar Case

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Abstract

This paper is concerned with a study of the classical isoperimetric problem modified by an addition of a nonlocal repulsive term. We characterize existence, nonexistence, and radial symmetry of the minimizers as a function of mass in the situation where the nonlocal term is generated by a kernel given by an inverse power of the distance. We prove that minimizers of this problem exist for sufficiently small masses and are given by disks with prescribed mass below a certain threshold when the interfacial term in the energy is dominant. At the same time, we prove that minimizers fail to exist for sufficiently large masses due to the tendency of the low-energy configuration to split into smaller pieces when the nonlocal term in the energy is dominant. In the latter regime, we also establish linear scaling of energy with mass, suggesting that for large masses low-energy configurations consist of many roughly equal-size pieces far apart. In the case of slowly decaying kernels, we give a complete characterization of the minimizers. © 2012 Wiley Periodicals, Inc.

1 Introduction

The isoperimetric problem is a classical problem in the calculus of variations, one formulation of which seeks to find a set of the smallest perimeter enclosing a prescribed volume. By the famous result of De Giorgi, among all sets of finite perimeter in the euclidean space $\mathbb{R}^n$, the solution of this problem is well-known to be a ball [15]. In this paper we are interested in the question how the solution of the isoperimetric problem is affected by an addition of a *repulsive long-range force*. Specifically, for $n \geq 2$ we wish to study the variational problem associated with the energy functional

$$(1.1) \qquad E[u] := \int_{\mathbb{R}^n} |\nabla u| dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x)u(y)}{|x-y|^\alpha} \, dx \, dy,$$

where u is the characteristic function of a subset of $\mathbb{R}^n$ with finite perimeter and mass $m > 0$, and $\alpha \in (0, n)$ is a parameter. More precisely, we look for minimizers of the energy $E[u]$ over all $u \in \mathcal{A}$, where

$$(1.2) \qquad \mathcal{A} := \left\{ u \in BV(\mathbb{R}^n; \{0, 1\}) : \int_{\mathbb{R}^n} u \, dx = m \right\}.$$

The choice of the nonlocal term in (1.1) is motivated by the form appearing in a number of physical problems. In particular, the nonlocal term with $\alpha = 1$ in either two or three space dimensions arises naturally due to Coulombic forces (electrostatic repulsion in the three-dimensional ambient space), in which case the characteristic function u may be associated with the uniform charge density over a subset of either the three-dimensional space or the two-dimensional plane. The underlying physical models can be found, e.g., in [6, 10, 11, 16, 24, 35, 36, 44]. The case $\alpha = 1$ in three dimensions also arises in the studies of models of diblock copolymer melts and related polymer, as well as other systems (see, e.g., [14, 20, 25, 28, 30, 32, 37, 43]).

More generally, the nonlocal term is chosen in this form to have the following four properties [31]:

- the nonlocal term is invariant with respect to translations and rotations,
- the nonlocal term is repulsive,
- the nonlocal term is scale-free, and
- the nonlocal term scales with length faster than volume.

Indeed, the kernel in the nonlocal term depends only on the distance between two points. The repulsive nature of the nonlocal term is due to the fact that for $\alpha > 0$ the kernel is positive and monotonically decreasing with distance. It is also important to note that the quadratic form in $L^2(\mathbb{R}^n)$ generated by the kernel is positive definite. We point out that when the nonlocal term has opposite sign (attractive long-range forces), the minimizers of the considered variational problem are still balls, since the nonlocal term in (2.1) increases with respect to Schwarz symmetrization [29]. Furthermore, the nonlocal term is scale-free in the sense that dilations only result in the appearance of a multiplicative factor in front of the nonlocal term. The scale-free nature of the nonlocal term allows one to reduce the number of free parameters of the problem to a single one, which we choose to be the mass m. Indeed, in this case a coefficient in front of the nonlocal term may be eliminated via a rescaling in space (the coefficient in front of the interfacial term in the energy can be eliminated by rescaling the energy). Finally, the scaling property is ensured by the condition $\alpha < n$. Notice that the nonlocal term is always infinite when $\alpha \geq n$ and if the interior of the support of u is nonempty.

We note that a related energy functional

$$(1.3) \qquad P_\alpha[u] := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^\alpha} \, dx \, dy,$$

where u is the characteristic function of a set $\Omega \subset \mathbb{R}^n$ and the exponent $\alpha \in (n, n + 1)$ leads to a *nonlocal isoperimetric problem* in which P_α gives a generalized notion of the perimeter of Ω [8]. In fact, when $\alpha \to n + 1$ from below, the latter Γ-converges, after a suitable rescaling, to the usual perimeter [3] (see also [9]). In our problem, on the other hand, the nonlocal term has a very different effect. It acts as the square of the negative Sobolev norm of u and therefore favors rapid oscillations of u. This leads to a *competition* between the perimeter and the nonlocal term that can give rise to the appearance of nontrivial energy-minimizing patterns in bounded domains (for overviews, see [27, 31, 32, 41]; for recent studies of multidroplet patterns in systems with Coulombic repulsion, see [12, 13, 33, 39]). Our whole space problem, in turn, appears as a limit problem in the studies of Γ-convergence of the functional in (2.1) in the presence of a small coefficient multiplying the perimeter term [12] (see also [13, 21, 22, 33] for a related problem).

Despite the apparent simplicity of the model, for the problem under consideration even the basic question of existence of minimizers is not completely straightforward. While in the surface energy-dominated regime (small masses, $m \ll 1$) one would naturally expect the minimizers to exist and be in some sense approximations to balls, for the nonlocal energy-dominated regime (large masses, $m \gg 1$) the energy may be lowered by splitting a given configuration into several pieces and moving them far apart. In this situation the minimizers may fail to exist. Our goal is to address these questions analytically.

In this paper, we present a detailed analysis of existence versus nonexistence of minimizers of the considered variational problem in the particular case of two space dimensions. We chose to treat the $n = 2$ case separately, because in two space dimensions many technicalities simplify substantially, allowing one to concentrate on the issues associated with nonlocality and making the analysis more transparent (the general case will be treated elsewhere [26]). Furthermore, in two dimensions the obtained results appear to be optimal in the sense that they cover the whole range of $\alpha \in (0, n)$ and in view of the fact that we are able to solve the variational problem completely for sufficiently small α. In addition, for $n = 2$ the estimates are readily made explicit, and the obtained results are applicable to a number of physical systems, including high-T_c superconductors, magnetic bubble materials, and ferroelectrics [16, 24, 35, 44]. What we prove in the following sections is that for $n = 2$ and $0 < \alpha < 2$ the basic picture presented above is correct: the minimizer of the considered problem exists for small enough m and does not exist for large enough m. Note that the considered problem is different from the one studied in [40], where the nonlocal term has a compactly supported kernel and minimizers exist for all masses. Moreover, we prove that for m sufficiently small, the minimizer is precisely a single disk.

The main ideas of the proofs are as follows. Existence of minimizers for small masses is proved by showing that the members of a minimizing sequence can be chosen to be connected. Nonexistence is proved by showing that for large masses the minimizers must be long and slender, so it is always possible to reduce the

energy by cutting the set in two and moving the pieces far apart. The fact that the minimizer at small masses is a ball is proved by exploiting the good stability properties of the minimizers of the usual isoperimetric problem.

We note that the intricate case of intermediate masses remains largely open. In particular, an interesting open question is whether the minimizer of the considered problem is, in fact, a ball whenever it exists. We prove that this is indeed the case for α sufficiently small. Whether this result also holds for the whole range of $\alpha \in (0, 2)$ does not seem to be clear. Another interesting open question is about the structure of the set of values of m for which the minimizer exists, in particular whether it is an interval. Again, in the case of small α we prove the latter to be the case and compute the precise threshold value of m separating the existence and nonexistence regimes. In the full generality, however, these questions are currently out of reach for the methods of the present paper, since in our analysis we mainly employ the properties of the problem in the regimes dominated by either of the two terms in the energy. New tools that deal with the joint effect of the local and the nonlocal terms need to be developed to further address the finer properties of the minimizers in the considered problem in the regime of intermediate masses. One step in that direction is the precise scaling of minimal energy obtained by us for large masses, using an interpolation inequality relating the interfacial and the nonlocal parts of the energy.

Our paper is organized as follows. In Section 2, we collect all the main results of our paper. In Section 3, we present some background results and the results of explicit computations for several configurations. In Section 4, we prove existence of minimizers for sufficiently small masses. In Section 5, we prove the optimal scaling of the minimal energy for large masses. In Section 6, we prove nonexistence of minimizers for large masses. In Section 7, we prove that minimizers for sufficiently small masses are disks. Finally, in Section 8, we prove that for sufficiently small α, minimizers exist if and only if they are disks and if and only if their mass is less than or equal to an explicit threshold value.

2 Statement of Results

Throughout the rest of this paper we always assume that $n = 2$ and $\alpha \in (0, 2)$. The considered variational problem is then equivalent to minimizing

$$(2.1) \qquad E(\Omega) := |\partial \Omega| + \iint\limits_{\Omega\ \Omega} \frac{1}{|x - y|^\alpha}\, dx\, dy, \quad |\Omega| = m,$$

where Ω is a set of finite perimeter in the plane, $|\Omega|$ denotes the Lebesgue measure of Ω, i.e., $|\Omega| = \mathcal{H}^2(\Omega)$, and $|\partial \Omega|$ denotes the perimeter of Ω, i.e., $|\partial \Omega| = \mathcal{H}^1(\partial \Omega)$ (for definitions, see, e.g., [4]). We note that, when dealing with minimizers of E in (2.1), we can always assume that $\partial \Omega$ is a collection of $C^{2,\beta}$ curves for some $\beta \in (0, 1)$. This is due to the fact that minimizers of E are quasi-minimizers of the perimeter, and therefore the standard regularity theory of minimal surfaces

applies to them (see, e.g., [5, 45]). By a straightforward cutting argument, we also show that minimizers are connected (but not necessarily simply connected). The proposition below collects some basic properties of minimizers:

PROPOSITION 2.1. *Let Ω be a minimizer of E. Then*

(i) *The boundary $\partial\Omega$ of Ω is of class $C^{2,\beta}$ for some $\beta \in (0,1)$, with the regularity constants depending only on m and α.*

(ii) *Ω is bounded and connected. Moreover, Ω contains at most finitely many holes.*

(iii) *The Euler-Lagrange equation for (2.1) is*

$$(2.2) \qquad \kappa(x) + 2v(x) - \mu = 0, \quad v(x) := \int_{\Omega} \frac{1}{|x-y|^{\alpha}} \, dy,$$

where $\kappa(x)$ and $v(x)$ are the curvature and the nonlocal potential at the point $x \in \partial\Omega$, respectively, and $\mu \in \mathbb{R}$ is the Lagrange multiplier due to the mass constraint (the sign of the curvature is chosen to be positive for convex sets).

Note that further $C^{3,\beta}$ regularity can be inferred when $\alpha < 1$, since $v \in C^{1,\beta}(\mathbb{R}^2)$ for some $\beta \in (0,1)$ in that case. The proof of Proposition 2.1 follows from Lemmas 3.1–3.3.

We now state our main results. Concerning the existence of minimizers, as already noted in the introduction, the question is not straightforward, since the minimizing sequences for (2.1) may consist of disconnected pieces moving off to infinity away from each other. What we can establish, however, is that in the regime of small masses, i.e., when the perimeter is the dominant term in the energy, minimizers of E do indeed exist.

THEOREM 2.2 (Existence of Minimizers). *There is $m_1 = m_1(\alpha) > 0$ such that for all $m \leq m_1$ there exists a minimizer of E.*

The proof of this theorem is obtained by using the direct method of calculus of variations by establishing that for small masses minimizing sequences can be localized to a sufficiently large ball. The proof is given in Section 4.

Our next result gives a complete characterization of the minimizers for sufficiently small masses.

THEOREM 2.3 (Disk as the Unique Minimizer). *There is $m_0 = m_0(\alpha) > 0$ such that for all $m \leq m_0$ the unique, up to translations, minimizer of E is $\Omega = B_R(0)$ with $R = (m/\pi)^{1/2}$.*

Since in the case of small masses the energy is dominated by the perimeter, it is expected that the minimizer should be close to a disk. However, our result is stronger, stating that even for small (but positive) masses the minimizer is precisely a disk with mass m. In the proof we proceed by establishing two-sided estimates for the difference of the energy of the minimizer and that of a disk in terms of the

isoperimetric deficit. Note that the proof is particularly intricate in the case of the near-field-dominated regime $\alpha \in [1, 2)$, where the self-interaction of the symmetric difference of the two sets yields a relevant contribution to the energy and has to be estimated carefully. The proof of Theorem 2.3 is given in Section 7.

Our next result shows that in fact the minimizer (global or local) cannot be a disk if m is sufficiently large. Note that these kinds of results have been derived for a number of related problems [13, 23, 24, 32, 34, 41, 46].

PROPOSITION 2.4 (Global and Local Instability of a Disk). *Let $\Omega = B_R(0)$ with $R = (m/\pi)^{1/2}$. There are two constants $m_{c1} < m_{c2}$ given in (3.13) and (3.23) such that the following holds*:

(i) *Ω is not a global minimizer if $m > m_{c1} = m_{c1}(\alpha) > 0$.*
(ii) *Ω is not a local minimizer (with respect to arbitrarily small perturbations of the boundary) if $m > m_{c2} = m_{c2}(\alpha) > 0$.*

Note that by Proposition 2.4 disks cease to be global minimizers before they undergo shape instability. The proof of this proposition is by direct computation and follows from Lemmas 3.6 and 3.7.

While Proposition 2.4 only shows that a minimizer cannot be given by a disk with mass m if m is sufficiently large, the next theorem encompasses a more general result: For sufficiently large masses the energy (2.1) does not have a minimizer. This point was conjectured in [12, remark 4.2] for the case $n = 3$ and $\alpha = 1$. In fact, a more general nonexistence result holds for all $\alpha < 2$ in dimensions $n > 2$ as well [26].

THEOREM 2.5 (Nonexistence). *There exists $m_2 = m_2(\alpha) > 0$ such that there is no minimizer of E for all $m > m_2$.*

The nonattainability of the minimal energy for large masses established by Theorem 2.5 is related to the fact that for $m \gg 1$ it is advantageous for the mass to escape to infinity. The proof of this theorem is given in Section 5 via matching upper and lower bounds for the energy and a cutting argument.

The proof of Theorem 2.5 suggests that for $m \gg 1$ the minimizing sequence may consist asymptotically of disconnected sets of approximately equal masses $m_i \sim 1$ (for a related result in two dimensions, see [12, 21, 33]). In particular, the minimal energy should scale linearly with the mass m for large m. The next theorem supports this picture.

THEOREM 2.6 (Scaling and Equipartition of Energy). *Let $m_0 = m_0(\alpha) > 0$ be as in Theorem 2.3. Then there exist two constants $C, c > 0$ depending only on α such that for all $m > m_0$ we have*

$$(2.3) \qquad\qquad cm \leq \inf E(\Omega) \leq Cm.$$

Furthermore, we have equipartition of energy in the sense that for every $m \geq m_0$ and every configuration Ω with $E(\Omega) \leq Cm$, both terms in the energy obey the same bounds as in (2.3) separately.

Note that Theorem 2.6 shows in particular that for large m the minimal scaling of the energy can only be reached if the interfacial and the nonlocal part of the energy are of the same order. The proof of Theorem 2.6 is given in Section 5. At the core of the proof lies a multiplicative interpolation inequality (5.3), which is derived in the course of the proof of Lemma 5.2.

Theorems 2.2, 2.3, and 2.5 cover the two extremes of the range of values of m but say nothing about what happens at intermediate values of m. Thus, the global structure of minimizers for all masses is currently not available. Nevertheless, when α is sufficiently small, i.e., when the nonlocal interaction is slowly decaying with distance, we have a complete characterization of minimizers:

THEOREM 2.7 (Complete Characterization of Minimizers for Slowly Decaying Kernels). *Let $m_{c1} = m_{c1}(\alpha) > 0$ be given by (3.13). There exists a universal constant $\alpha_0 > 0$ such that for all $\alpha \leq \alpha_0$ we have*:

(i) *For all $m \leq m_{c1}$ there exists a minimizer of E; this minimizer, up to translations, is given by $B_R(0)$ with $R = (m/\pi)^{1/2}$.*
(ii) *For all $m > m_{c1}$ there is no minimizer of E.*

In other words, for sufficiently small values of α the constants in Theorems 2.2, 2.3, and 2.5 obey $m_0 = m_1 = m_2 = m_{c1}$. The proof is presented in Section 8 and relies on the fact that for $\alpha \ll 1$ the nonlocal term in the energy is only weakly sensitive to the shape of the connected components of sets of finite perimeter, allowing us to push the validity of Theorems 2.2 and 2.3 up to $m \gg 1$. At the same time, for $\alpha \ll 1$ the contribution of the nonlocal term to the energy is essentially proportional to m^2 due to nearly equal coupling of different points in connected components of sets of finite perimeter obeying the optimal energy upper bound. Thus Theorem 2.5 remains valid uniformly for all $m \gtrsim 1$. Together, these results imply that the minimizers are disks whenever they exist. To conclude the proof, it then remains to establish that disks are indeed minimizers for all masses up to and including m_{c1} given by Proposition 2.4. The latter relies on the global result of Lemma 3.6(ii) concerning minimizers of E among sets consisting of two arbitrary disks. We note that the arguments in the proof of Theorem 2.7 also imply that as soon as it is known that for a given value of α the minimizers of E can only be disks, we have $m_0 = m_1 = m_2 = m_{c1}$ without the need to assume that α is small. As was already mentioned, however, it is not clear if minimizers are always disks whenever they exist for the whole range of α.

3 Preliminaries

In this section, we present the necessary background for the proofs of the main theorems, as well as the results of some precise ansatz-based calculations. We first outline the proof of the regularity of $\partial\Omega$ for minimizers:

LEMMA 3.1. *Let Ω be a minimizer of E. Then $\partial\Omega$ is of class $C^{2,\beta}$ for some $\beta \in (0, 1)$, with the regularity constants depending only on m and α.*

PROOF. Regularity of $\partial\Omega$ follows from the fact that every minimizer of E is a quasi-minimizer of the interfacial energy. More precisely, we claim that for any set of finite perimeter $\Omega' \subset \mathbb{R}^2$ with $|\Omega'| = m$, we have

(3.1) $$|\partial\Omega| \leq |\partial\Omega'| + c(2-\alpha)^{-\frac{\alpha}{2}} m^{\frac{2-\alpha}{2}} |\Omega \Delta \Omega'|$$

for some universal $c > 0$.[1] Indeed, assuming that (3.1) holds, we can immediately apply [40, theorem 1.4.9] to conclude the uniform $C^{1,\beta}$ regularity of $\partial\Omega$ for any minimizer Ω of E. Furthermore, the boundary satisfies the weak form of the Euler-Lagrange equation (2.2). Noting that clearly $v \in C^{0,\beta}(\mathbb{R}^2)$ for some $\beta \in (0,1)$ and since $\partial\Omega$ is locally a graph of a $C^{1,\beta}$ function, the regularity assertion of the lemma follows by further application of the regularity theory for graphs [19] (for details of the argument, see, e.g., the last paragraph of section 2 in [42], as well as [18]).

It hence remains to show (3.1). By a direct computation, for any $R > 0$ we have

(3.2)
$$\left| \iint_{\Omega'\Omega'} \frac{1}{|x-y|^\alpha} \, dx \, dy - \iint_{\Omega\,\Omega} \frac{1}{|x-y|^\alpha} \, dx \, dy \right|$$
$$\leq 2 \int_{\Omega\Delta\Omega'} \int_{\Omega\cup\Omega'} \frac{1}{|x-y|^\alpha} \, dx \, dy$$
$$\leq 2|\Omega\Delta\Omega'| \sup_{x\in\mathbb{R}^2} \left(\int_{B_R(x)} \frac{1}{|x-y|^\alpha} \, dy + \int_{(\Omega\cup\Omega')\backslash B_R(x)} \frac{1}{|x-y|^\alpha} \, dy \right)$$
$$\leq 2|\Omega\Delta\Omega'| \left(\frac{2\pi R^{2-\alpha}}{2-\alpha} + mR^{-\alpha} \right).$$

The statement then follows from the minimizing property of Ω by choosing $R = (2-\alpha)^{1/2} m^{1/2}$ in (3.2). $\qquad\square$

LEMMA 3.2. *Let Ω be a minimizer of E. Then Ω is connected.*

PROOF. By uniform Hölder estimates of Lemma 3.1, we have $\Omega \subset B_{R_0}(0)$ for some $R_0 > 0$ (where R_0 depends on the configuration). Therefore, if $\Omega^{(1)}$ is a connected component and $\Omega \setminus \Omega^{(1)} \neq \varnothing$, then for any $R > 0$ we can consider a set Ω' obtained by translating $\Omega^{(1)}$ outside $B_{R_0+R}(0)$. The energy of the obtained

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[1] Here and throughout the rest of the paper $A\Delta B := (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of the sets A and B.

set is then

$$E(\Omega') \le E(\Omega) - 2 \int\limits_{\Omega^{(1)}} \int\limits_{\Omega \setminus \Omega^{(1)}} \frac{1}{|x - y|^{\alpha}} \, dx \, dy$$

(3.3)

$$+ 2R^{-\alpha} |\Omega^{(1)}| (m - |\Omega^{(1)}|)$$

$$\le E(\Omega) + 2\left(R^{-\alpha} - (2R_0)^{-\alpha}\right) |\Omega^{(1)}| (m - |\Omega^{(1)}|).$$

But for R sufficiently large, this inequality contradicts the minimizing property of Ω. □

LEMMA 3.3. *Let Ω be a minimizer of E. Then Ω contains at most finitely many holes.*

PROOF. By Lemma 3.1 we can cover $\partial\Omega$ by finitely many balls of equal radius, with the radius depending only on α and m, and such that in each ball $\partial\Omega$ is a graph of a C^1 function. Therefore, since the perimeter of Ω is bounded, $\partial\Omega$ breaks into a finite collection of simple closed curves. □

We now turn to the exact computations related to sets enclosed by ellipses (all algebraic computations below are performed using *Mathematica* 8 software). In the lemma below, we obtain an expression for the energy of such a set.

LEMMA 3.4. *Let Ω_e be a set enclosed by an ellipse of eccentricity e. Then*

$$E(\Omega_e) = \frac{4R}{\sqrt[4]{1 - e^2}} \mathrm{E}(e^2) + \frac{\pi^2 (1 - e^2)^{-\frac{\alpha+2}{4}} \Gamma(2 - \alpha) R^{4-\alpha}}{\Gamma\left(2 - \frac{\alpha}{2}\right) \Gamma\left(3 - \frac{\alpha}{2}\right)}$$

(3.4)

$$\times \left\{ (1 - e^2) {}_2F_1\left(\frac{1}{2}, 1 - \alpha 2; 1; e^2\right) \right.$$

$$\left. + (1 - e^2)^{\alpha/2} {}_2F_1\left(\frac{1}{2}, 1 - \frac{\alpha}{2}; 1; \frac{e^2}{e^2 - 1}\right) \right\},$$

where $\mathrm{E}(x)$ is the complete elliptic integral of the second kind, $_2F_1(a, b; c; z)$ is the hypergeometric function, and $R = (m/\pi)^{1/2}$.

PROOF. We consider the region Ω_e enclosed by an ellipse whose semi-axes are a and b. Since the area of the ellipse is $\pi ab = \pi R^2$, we have $a = R/\sqrt[4]{1 - e^2}$ and $b = R\sqrt[4]{1 - e^2}$. We also recall that the perimeter of an ellipse is given by the well-known expression

(3.5)
$$|\partial\Omega_e| = \frac{4R}{\sqrt[4]{1 - e^2}} \mathrm{E}(e^2),$$

where $\mathrm{E}(x)$ is the complete elliptic integral of the second kind [1]. To compute the nonlocal part E_{nl} of the energy, we pass to the Fourier space. In terms of the

Fourier transform

(3.6) $$\hat{u}_q = \int_{\mathbb{R}^2} e^{iq\cdot x} u(x)\,dx$$

of $u = \chi_{\Omega_e}$, the characteristic function of Ω_e, the nonlocal part of the energy is given by (see, e.g., [29])

(3.7) $$E_{nl}(\Omega_e) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u(x)u(y)}{|x-y|^\alpha}\,dx\,dy = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |\hat{u}_q|^2 G_q\,dq,$$

where

(3.8) $$G_q = \frac{2^{2-\alpha}\pi\,\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} |q|^{\alpha-2}$$

is the Fourier transform of the kernel in the nonlocal term and $\Gamma(x)$ is the Gamma-function [1]. To proceed, we note that if $x = (x_1, x_2)$, the rescaling $x_1 \to x_1/a$ and $x_2 \to x_2/b$ transforms $u(x)$, after a suitable translation, into the characteristic function of $B_1(0)$. Therefore, with $q = (q_1, q_2)$ one can write explicitly upon integration

(3.9) $$\hat{u}_q = 2\pi R\left(q_1^2\sqrt{1-e^2} + \frac{q_2^2}{\sqrt{1-e^2}}\right)^{-1/2} J_1\left(R\sqrt{q_1^2\sqrt{1-e^2} + \frac{q_2^2}{\sqrt{1-e^2}}}\right),$$

where $J_1(x)$ is the Bessel function of the first kind [1]. Performing another change of variables: $\tilde{q}_1 = q_1\sqrt[4]{1-e^2}$ and $\tilde{q}_2 = q_2/\sqrt[4]{1-e^2}$, and then introducing polar coordinates $\tilde{q}_1 = s\cos t$, $\tilde{q}_2 = s\sin t$, upon integration in s we obtain

(3.10) $$E_{nl}(\Omega_e) = \frac{2^{\frac{2-\alpha}{2}}\pi(1-e^2)^{\frac{2-\alpha}{4}}\Gamma(2-\alpha)R^{4-\alpha}}{\Gamma\left(2-\frac{\alpha}{2}\right)\Gamma\left(3-\frac{\alpha}{2}\right)} \int_0^{2\pi} (2-e^2+e^2\cos 2t)^{\frac{\alpha-2}{2}}\,dt.$$

Finally, performing the integration in t, we obtain

(3.11)
$$E_{nl}(\Omega_e) = \frac{\pi^2(1-e^2)^{-\frac{\alpha+2}{4}}\Gamma(2-\alpha)R^{4-\alpha}}{\Gamma\left(2-\frac{\alpha}{2}\right)\Gamma\left(3-\frac{\alpha}{2}\right)}$$
$$\times \left\{(1-e^2)\,{}_2F_1\left(\frac{1}{2}, 1-\frac{\alpha}{2}; 1; e^2\right)\right.$$
$$\left. + (1-e^2)^{\alpha/2}\,{}_2F_1\left(\frac{1}{2}, 1-\frac{\alpha}{2}; 1; \frac{e^2}{e^2-1}\right)\right\},$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [1]. Combining this formula with (3.5), we obtain the result. $\qquad\square$

Setting $e = 0$, we also obtain the precise formula for the energy of a single ball of mass $m = \pi R^2$.

COROLLARY 3.5. *We have*

$$(3.12) \qquad E(B_R(0)) = 2\pi R + \frac{2\pi^2 \Gamma(2-\alpha)}{\Gamma(2-\frac{\alpha}{2})\Gamma(3-\frac{\alpha}{2})} R^{4-\alpha}.$$

The result in Lemma 3.4 enables us to give the proof of Proposition 2.4 concerning the failure of minimality (either global or local) of $\Omega = B_R(0)$ for $R = (m/\pi)^{1/2}$ large enough. The proof relies on the explicit formula for the energy of elliptical domains (balls, in particular) obtained in Lemma 3.4 and Corollary 3.5. The proof of Proposition 2.4 follows directly from the next two lemmas. Note that it is easy to see that $m_{c1} < m_{c2}$ where m_{c1} and m_{c2} are defined in (3.13) and (3.23).

LEMMA 3.6. *Let*

$$(3.13) \qquad m_{c1}(\alpha) = \pi \left(\frac{(\sqrt{2}-1)\Gamma(2-\frac{\alpha}{2})\Gamma(3-\frac{\alpha}{2})}{\pi(1-2^{\frac{\alpha-2}{2}})\Gamma(2-\alpha)} \right)^{\frac{2}{3-\alpha}}.$$

Then

(i) *the disk of area m is not a global minimizer of E if $m > m_{c1}$;*
(ii) *the disk of area m has lower energy than any two nonoverlapping disks of the same total area if $m \le m_{c1}$.*

PROOF. We first compare the energy of $\Omega = B_R(0)$ with that of

$$\Omega' = B_{R/\sqrt{2}}(re_1) \cup B_{R/\sqrt{2}}(-re_1),$$

where $R = (m/\pi)^{1/2}$, $r > 2R$, and e_1 is the unit vector along the x_1-axis. It is easy to see from (3.12) that

$$E(\Omega') - E(\Omega) \le 2\pi(\sqrt{2}-1)R + \frac{2^{-1-\alpha}m^2}{(r-R)^\alpha}$$

$$(3.14) \qquad\qquad + \frac{\pi^2\Gamma(2-\alpha)}{\Gamma\left(2-\frac{\alpha}{2}\right)\Gamma\left(3-\frac{\alpha}{2}\right)}(2^{\alpha/2}-2)R^{4-\alpha}$$

$$< 0,$$

if $m > m_{c1}$, where m_{c1} is defined in (3.13) and r is sufficiently large, contradicting minimality of Ω.

We now show that it is energetically advantageous to replace a set Ω consisting of two nonoverlapping disks of mass tm and $(1-t)m$, with arbitrary $t \in (0,1)$, by a single disk of mass m whenever $m \le m_{c1}$. Indeed, by positivity of the kernel in the nonlocal part of the energy and (3.12) we have $E(\Omega) > 2\sqrt{\pi}\,m^{1/2}\widetilde{F}(t, m/m_{c1})$, where

$$(3.15) \quad \widetilde{F}(t,\mu) := t^{1/2} + (1-t)^{1/2} + \frac{(\sqrt{2}-1)}{1-2^{\frac{\alpha-2}{2}}}\mu^{\frac{3-\alpha}{2}}\left(t^{\frac{4-\alpha}{2}} + (1-t)^{\frac{4-\alpha}{2}}\right).$$

The statement of the lemma is equivalent to showing that $F(t, \mu) := \widetilde{F}(t, \mu) - \widetilde{F}(0, \mu) \geq 0$ for all $\mu \leq 1$ and, by symmetry, for all $t \in [0, \frac{1}{2}]$. We claim that for fixed t the minimum of F as a function of μ is attained when $\mu = 1$. Indeed, differentiating this expression with respect to μ, we find that

$$(3.16) \quad \frac{\partial F}{\partial \mu} = \frac{\sqrt{2} - 1}{2 - 2^{\alpha/2}}(3 - \alpha)\mu^{\frac{1-\alpha}{2}}F_1(t), \quad F_1(t) := t^{\frac{4-\alpha}{2}} + (1 - t)^{\frac{4-\alpha}{2}} - 1.$$

Clearly, $F_1(t)$ is strictly convex and in view of the fact that $F_1(0) = 0$, we have $F_1(t) < 0$, and hence $\partial F/\partial \mu < 0$ for all $t \in (0, \frac{1}{2}]$. Therefore, it is sufficient to show that $F(t) \geq 0$ for $\mu = 1$.

We now write

$$(3.17) \quad F(t, 1) = \frac{2(\sqrt{2} - 1)}{2 - 2^{\alpha/2}}F_1(t) + F_2(t), \quad F_2(t) := t^{1/2} + (1 - t)^{1/2} - 1.$$

To prove that $F(t, 1)$ is nonnegative for all $\alpha \in (0, 2)$ and $t \in [0, \frac{1}{2}]$ is a tedious exercise in calculus (the reader can readily verify this fact by plotting F as a function of two variables, α and t). Below we sketch the analytical argument. Introducing a cutoff parameter $t_0 = \frac{1}{8}$, it is not difficult to see that for all $t \in [t_0, \frac{1}{2}]$ we have

$$(3.18) \quad F_1(t) \geq 2^{\frac{\alpha-2}{2}} - 1 + 2^{\frac{\alpha-6}{2}}(2^{\alpha}7^{-\alpha/2} + 1)(1 - \frac{\alpha}{2})(2 - \frac{\alpha}{2})(1 - 2t)^2,$$

$$(3.19) \quad F_2(t) \geq \sqrt{2} - 1 - \frac{4\sqrt{2}}{25}(1 - 2t)^2.$$

Then, by inspection we find that $F(t, 1) \geq 0$ for all $t \in [t_0, \frac{1}{2}]$ and all $\alpha \in (0, 2)$.

We now turn to the interval $t \in (0, t_0)$. Here we have the following estimates:

$$(3.20) \quad F_1(t) \geq 2^{\frac{\alpha-2}{2}} - 1 - (2 - \frac{\alpha}{2})t, \quad F_2(t) \geq \sqrt{2} - 1 + \frac{\sqrt{2}(2\sqrt{7} - 1)}{\sqrt{7}}t.$$

Again, by inspection these estimates imply that $F(t, 1) \geq 0$ for all $t \in (0, t_0)$ and all $\alpha \leq 1$. To cover the range $\alpha > 1$, we use a different estimate for F_1:

$$(3.21) \quad F_1(t) \geq \frac{1}{2}(2 - \alpha)(t \ln t + (1 - t)\ln(1 - t)).$$

After some further manipulations, one can obtain the lower bound

$$(3.22) \quad F(t, 1) \geq \left(2\sqrt{2} - \frac{(\sqrt{2} - 1)(2 - \alpha)(1 + \ln 7)}{2 - 2^{\alpha/2}} - \sqrt{\frac{2}{7}}\right)t,$$

which implies that $F(t, 1) \geq 0$ for all $t \in (0, t_0)$, when $\alpha > 1$. $\square$

LEMMA 3.7. *Let*

$$(3.23) \quad m_{c2}(\alpha) = \pi \left(\frac{3\Gamma\left(2 - \frac{\alpha}{2}\right)\Gamma\left(3 - \frac{\alpha}{2}\right)}{\pi\alpha\Gamma(3 - \alpha)}\right)^{\frac{2}{3-\alpha}}.$$

Then the disk of area m is not a local minimizer of E (with respect to arbitrarily small perturbations of the boundary) if $m > m_{c2}$.

PROOF. We expand the expression in (3.4) for the energy of Ω_e as in Lemma 3.4 in the power series in e at $e = 0$:

$$(3.24) \quad E(\Omega_e) - E(B_R(0)) = \left(\frac{3\pi R}{32} - \frac{\pi^2 \alpha \Gamma(3-\alpha) R^{4-\alpha}}{32\Gamma(2-\frac{\alpha}{2})\Gamma(3-\frac{\alpha}{2})} \right) e^4$$
$$+ O(e^6) < 0,$$

for sufficiently small e if $m > m_{c2}$, where m_{c2} is defined in (3.23). Under this condition the energy decreases upon arbitrarily small distortion of a disk into an ellipse. $\qquad\square$

Lastly, we also need to characterize the nonlocal potential generated by a unit ball and, specifically, its behavior near the boundary, in order to prove Theorem 2.3.

LEMMA 3.8. *Let*

$$(3.25) \quad v^B(x) := \int_{B_1(0)} \frac{1}{|x-y|^\alpha} \, dy.$$

Then

$$(3.26) \quad v^B(x) = \begin{cases} \left(\frac{\pi}{|x|^\alpha} \right) {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha}{2}; 2; \frac{1}{|x|^2} \right), & |x| \geq 1, \\ \left(\frac{2\pi}{2-\alpha} \right) {}_2F_1\left(\frac{\alpha-2}{2}, \frac{\alpha}{2}; 1; |x|^2 \right), & |x| < 1. \end{cases}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function. In particular, if $r = |x| - 1$, we have

$$(3.27) \quad v^B(x) - v_0 = \begin{cases} -\frac{\pi\alpha(2-\alpha)\Gamma(1-\alpha)}{2\Gamma^2(2-\frac{\alpha}{2})} r + O(|r|^{2-\alpha}), & \alpha < 1, \\ -r(2\ln|r|^{-1} - 2 + 3\ln 4) + O(r^2 \ln|r|^{-1}), & \alpha = 1, \\ -\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha-1}{2} \right)}{(2-\alpha)\Gamma\left(\frac{\alpha}{2} \right)} |r|^{1-\alpha} r + O(r), & \alpha > 1, \end{cases}$$

where $\Gamma(x)$ is the Gamma-function and

$$(3.28) \quad v_0 := \frac{\pi\Gamma(2-a)}{\Gamma^2\left(2-\frac{a}{2} \right)}.$$

PROOF. The proof is by an explicit computation. Introducing the Fourier transform $\hat{v}_q^B$ of v^B:

$$(3.29) \quad \hat{v}_q^B = \int_{\mathbb{R}^2} e^{iq \cdot x} v^B(x) dx,$$

and using (3.8) and (3.9) with $e = 0$, we obtain

$$(3.30) \quad \hat{v}_q^B = G_q \hat{u}_q = \frac{2^{3-\alpha} \pi^2 \Gamma\left(1-\frac{\alpha}{2} \right)}{\Gamma\left(\frac{\alpha}{2} \right)} |q|^{\alpha-3} J_1(|q|),$$

where $\widehat{u}_q$ is the Fourier transform of the characteristic function of the unit ball centered at the origin.

Inverting the Fourier transform and integrating over the directions of q with $z = |q|$, we arrive at

$$v^B(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-iq \cdot x} \widehat{v}_q^B \, dq$$

(3.31)

$$= \frac{2^{2-\alpha} \pi \Gamma \left(1 - \frac{\alpha}{2}\right)}{\Gamma \left(\frac{\alpha}{2}\right)} \int_0^\infty z^{\alpha-2} J_1(z) J_0(z|x|) dz,$$

where $J_n(x)$ are the Bessel functions of the first kind. But the right-hand side of (3.31) coincides with the right-hand side of (3.26). Finally, the expansion in (3.27) is an immediate consequence of (3.26). $\qquad\square$

4 Existence of Minimizers for Small Masses

We now prove the existence result in Theorem 2.2. The strategy of the proof is to suitably localize the minimizing sequence for E in (2.1). Existence of minimizers then follows by the usual compactness and lower semicontinuity results for functions of bounded variation [4].

PROOF OF THEOREM 2.2. Let $\{\Omega_k\}_{k=1}^\infty$, with $\Omega_k \subset \mathbb{R}^2$ and $|\Omega_k| = m$, be a minimizing sequence for E. Without loss of generality, we may assume that each Ω_k consists of $N_k < \infty$ disjoint, open, connected components $\Omega_k^{(i)}$ ordered so that $|\Omega_k^{(1)}| \geq \ldots \geq |\Omega_k^{(N_k)}|$ and that the interfaces $\partial \Omega_k$ are smooth. As a first step, we use a ball of radius $R = (m/\pi)^{1/2}$ as a test function to obtain an upper bound for the minimal energy. By comparing the energy of Ω with the energy of the ball $B_R(0)$, we may assume that

(4.1) $$E(\Omega_k) \leq E(B_R(0)) = 2\sqrt{\pi}\, m^{1/2}(1 + Cm^{\frac{3-\alpha}{2}})$$

for some $C > 0$ depending only on α (for the precise constant; see (3.12)).

Suppose now that $N_k > 1$, and hence $|\Omega_k^{(i)}| \leq m/2$ for all $2 \leq i \leq N_k$. By the isoperimetric inequality and by positivity of the nonlocal term in the energy, we have

(4.2) $$2\sqrt{\pi}\left(|\Omega_k^{(i)}|^{1/2} + \left(m - |\Omega_k^{(i)}|\right)^{1/2}\right) \leq E(\Omega_k) \quad \text{for all } 2 \leq i \leq N_k.$$

Squaring both sides of (4.2) and combining it with (4.1), after some algebraic manipulations we obtain

(4.3) $$\left|\Omega_k^{(i)}\right| \leq Cm^{4-\alpha} \quad \text{if } 2 \leq i \leq N_k \text{ and } m \leq 1,$$

for some $C > 0$ depending only on α.

On the other hand, consider a set Ω'_k obtained by erasing $\Omega_k^{(N_k)}$ from Ω_k and then dilating the resulting set by

$$\lambda_k = \sqrt{m \big/ \big(m - |\Omega_k^{(N_k)}|\big)} \in (1, \sqrt{2}],$$

so that $|\Omega'_k| = m$ once again. If $E(\Omega'_k) < E(\Omega_k)$, we replace the set Ω_k by Ω'_k in the minimizing sequence and repeat the above process. Then, after finitely many steps either the set Ω_k is connected or $E(\Omega'_k) \geq E(\Omega_k)$. In the latter case we can write

$$\begin{aligned} E(\Omega'_k) &= \lambda_k \big(|\partial\Omega_k| - |\partial\Omega_k^{(N_k)}|\big) + \lambda_n^{4-\alpha} \int_{\Omega_k \setminus \Omega_k^{((N_k))}} \int_{\Omega_k \setminus \Omega_k^{(N_k)}} \frac{1}{|x-y|^\alpha} \, dx \, dy \\ &\leq \lambda_k^4 E(\Omega_k) - |\partial\Omega_k^{(N_k)}|, \end{aligned}$$
(4.4)

and therefore

$$\big|\partial\Omega_k^{(N_k)}\big| \leq \big(\lambda_k^4 - 1\big) E(\Omega_k) \leq \frac{6\big|\Omega_k^{(N_k)}\big|}{m} E(\Omega_k).$$
(4.5)

Applying again the isoperimetric inequality on the left-hand side of (4.5) and using the fact that by (4.1) we have $E(\Omega_k) \leq C m^{1/2}$ for some $C > 0$ depending only on α and $m \leq 1$, we then conclude that in this case

$$\big|\Omega_k^{(i)}\big| \geq cm \quad \text{if } 1 \leq i \leq N_k \text{ and } m \leq 1$$
(4.6)

for some $c > 0$ depending only on α.

It is easy to see that for sufficiently small m the two inequalities in (4.3) and (4.6) are incompatible. Thus, given a minimizing sequence $\{\Omega_k\}_{k=1}^\infty$, for sufficiently small m it is always possible to construct another minimizing sequence $\{\Omega'_k\}_{k=1}^\infty$ in which each set Ω'_k is connected.

By a suitable translation, one can further assume that the origin belongs to each of Ω'_k. In turn, since the perimeters of Ω'_k are uniformly bounded above, we have $\Omega'_k \Subset B_R(0)$ for some large enough $R > 0$. Therefore, introducing the characteristic functions $u_k \in BV(B_R(0); \{0, 1\})$ of Ω'_k, we get that the functions u_k are equibounded in $BV(B_R(0); \{0, 1\})$. So up to extraction of a subsequence $u_k \to u \in BV(B_R(0); \{0, 1\})$ strongly in $L^1(B_R(0))$ and $u_k \rightharpoonup u$ in $BV(B_R(0); \{0, 1\})$, with the limit independent of R. In particular, $\int_{B_R(0)} u \, dx = m$. Since the perimeter is lower-semicontinuous and the nonlocal term is continuous with respect to the above convergence (the latter follows immediately from (3.2)), we conclude that the set $\Omega = \{u = 1\}$ is a minimizer. $\qquad\square$

5 Scaling of the Minimal Energy

We now consider the opposite extreme, in which the nonlocal term favors splitting the set Ω into smaller disconnected sets. The corresponding scaling of the minimal energy is described by Theorem 2.6, whose proof is an immediate consequence of the following three lemmas. We note that the main point of Theorem

2.6 is the ansatz-free lower bound for large m that matches the upper bound from an ansatz consisting of a collection of equal-size balls far apart. We also note that we only need to prove the bounds in Theorem 2.6 for sufficiently large masses. Indeed, by the isoperimetric inequality and by the positivity of the nonlocal term we have $E(\Omega) \geq |\partial\Omega| \geq 2\sqrt{\pi}\, m^{1/2}$, so $E(\Omega)$ is uniformly bounded away from zero whenever $m \geq c$ for any $c > 0$.

We begin with an ansatz-based upper bound.

LEMMA 5.1. *For every $m \geq 1$ there exists Ω such that $E(\Omega) \leq Cm$ for some $C > 0$ depending only on α.*

PROOF. The proof is by an explicit construction. We take

$$(5.1) \qquad \Omega = \Big(\bigcup_{n=0}^{N-1} B_1(nRe_1) \Big) \cup B_r(NRe_1),$$

where $N = \lfloor (m/\pi) \rfloor$, e_1 is the unit vector along the x_1-direction, $r = \pi^{-1/2}(m - \pi N)^{1/2}$, and $R > 2$; i.e., we take Ω to be a linear chain of nonoverlapping unit balls (except for the last one, whose radius is chosen to accommodate the mass constraint). Then by (3.12) we have

$$(5.2) \qquad E(\Omega) \leq (N+1)E(B_1(0)) + \frac{2\pi^2 N(N+1)}{(R-2)^\alpha} \leq Cm + \frac{4m^2}{(R-2)^\alpha}.$$

So the assertion of the lemma follows by choosing $R = m^{1/\alpha} + 2$. □

We now turn to the ansatz-free lower bound.

LEMMA 5.2. *For every admissible Ω we have $E(\Omega) \geq cm$ for some universal $c > 0$.*

PROOF. The proof of the lower bound can be obtained by retracing the steps in the proof of [33, lemma B.1]. Here we present a simpler proof, which does not rely on Fourier techniques and the properties of special functions. The result is obtained from the following interpolation inequality:

$$(5.3) \qquad \int_{\mathbb{R}^2} u^2 \, dx \leq C \Big(\|u\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\nabla u| dx \Big)^{\frac{2-\alpha}{3-\alpha}} \Big(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{u(x)u(y)}{|x-y|^\alpha} \, dx \, dy \Big)^{\frac{1}{3-\alpha}}$$

for some universal $C > 0$, which is valid for any $u \in BV(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Indeed, for any admissible set Ω, let u be the characteristic function of Ω. Applying (5.3), we then have

$$(5.4) \qquad \begin{aligned} m &\leq C \Big(\int_{\mathbb{R}^2} |\nabla u| dx \Big)^{\frac{2-\alpha}{3-\alpha}} \Big(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{u(x)u(y)}{|x-y|^\alpha} \, dx \, dy \Big)^{\frac{1}{3-\alpha}} \\ &\leq C E^{\frac{2-\alpha}{3-\alpha}}(\Omega) E^{\frac{1}{3-\alpha}}(\Omega) = CE(\Omega). \end{aligned}$$

For the proof of (5.3), we only need to take into account the nonlocal interaction on intermediate-length scales of order $R > 0$, which will be determined later. With the change of variables $z = y - x$, we have

$$\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{u(x)u(y)}{|x-y|^\alpha}\,dx\,dy$$

(5.5)

$$\geq \int\limits_{\mathbb{R}^2}\int\limits_{B_{2R}(0)\setminus B_R(0)} \frac{u(x)u(x+z)}{|z|^\alpha}\,dz\,dx$$

$$= \int\limits_{\mathbb{R}^2}\int\limits_{B_{2R}(0)\setminus B_R(0)} \frac{|u(x)|^2}{|z|^\alpha}\,dz\,dx$$

$$+ \int\limits_{\mathbb{R}^2}\int\limits_{B_{2R}(0)\setminus B_R(0)} \frac{u(x)(u(x+z)-u(x))}{|z|^\alpha}\,dz\,dx.$$

Using the fact that $R \leq |z| \leq 2R$ and that $|B_{2R}(0)\setminus B_R(0)| = 3\pi R^2$, we hence get

$$\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{u(x)u(y)}{|x-y|^\alpha}\,dx\,dy$$

(5.6)

$$\geq CR^{2-\alpha}\int\limits_{\mathbb{R}^2} u^2\,dx$$

$$- R^{-\alpha}\int\limits_{\mathbb{R}^2}\int\limits_{B_{2R}(0)\setminus B_R(0)}\int_0^1 |z||u(x)||\nabla u(x+tz)|dt\,dz\,dx$$

$$\geq CR^{2-\alpha}\int\limits_{\mathbb{R}^2} u^2\,dx - C'R^{3-\alpha}\|u\|_{L^\infty(\mathbb{R}^2)}\int\limits_{\mathbb{R}^2} |\nabla u(x)|dx$$

for some universal $C, C' > 0$ (recall that $\alpha \in (0, 2)$), where we argued by approximating u with smooth functions, noting that by an argument similar to the one used in (3.2) the nonlocal term is continuous in the L^1-topology. Therefore

$$(5.7) \quad \int\limits_{\mathbb{R}^2} u^2\,dx \leq CR\|u\|_{L^\infty}\int\limits_{\mathbb{R}^2} |\nabla u|dx + CR^{\alpha-2}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{u(x)u(y)}{|x-y|^\alpha}\,dx\,dy,$$

for some universal $C > 0$. Estimate (5.3) then follows by minimizing the right-hand side of (5.7) in R; i.e., we choose

$$(5.8) \quad R = \left(\|u\|_{L^\infty(\mathbb{R}^2)}\int\limits_{\mathbb{R}^2} |\nabla u|dx\right)^{-\frac{1}{3-\alpha}}\left(\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{u(x)u(y)}{|x-y|^\alpha}\,dx\,dy\right)^{\frac{1}{3-\alpha}}.$$

This concludes the proof of (5.3). □

The following lemma strengthens the lower bound for configurations that satisfy the linear scaling of the energy:

LEMMA 5.3. *Let* $m \geq 1$ *and suppose that* Ω *satisfies* $E(\Omega) \leq Cm$ *for some* $C > 0$. *Then there is another constant* $c > 0$ *depending only on* C *such that*

$$(5.9) \qquad |\partial\Omega| \geq cm \quad \text{and} \quad \iint_{\Omega\,\Omega} \frac{1}{|x-y|^\alpha}\,dx\,dy \geq cm.$$

PROOF. The proof follows directly from $E \leq Cm$ and (5.3). $\square$

6 Nonexistence of Minimizers for Large Masses

We now present the proof of Theorem 2.5. We begin with a basic estimate (from above and below) of the diameter of a minimizer Ω of E.

LEMMA 6.1. *Let* $m \geq 1$, *let* Ω *be a minimizer of* E, *and let* $d := \mathrm{diam}(\Omega)$. *Then*

$$(6.1) \qquad cm^{1/\alpha} \leq d \leq Cm$$

for some $c, C > 0$ *depending only on* α.

PROOF. We first recall that by Proposition 2.1 the set Ω is regular and connected. Therefore, we have $2d \leq |\partial\Omega| \leq E(\Omega) \leq Cm$ for some $C > 0$ depending only on α, in view of Lemma 5.1.

On the other hand,

$$(6.2) \qquad \frac{m^2}{d^\alpha} \leq \iint_{\Omega\,\Omega} \frac{1}{|x-y|^\alpha}\,dx\,dy \leq E(\Omega) \leq Cm,$$

which yields the second inequality. $\square$

Let us note that as an immediate consequence of Lemma 6.1 we get nonexistence of minimizers for large masses when $\alpha < 1$. We call this regime *far-field dominated*, as opposed to the opposite regime ($\alpha \geq 1$), which we call *near-field dominated*. We will see again in Section 7 that this distinction also plays a role for minimizers at small masses.

COROLLARY 6.2 (Nonexistence in the Far-Field-Dominated Case). *Let* $\alpha < 1$. *Then there exists* $m_2 = m_2(\alpha) > 0$ *such that there are no minimizers of* E *for all* $m > m_2$.

PROOF. By Lemma 5.1 and 6.1 every minimizer Ω has to satisfy $E \leq Cm$ and $E \geq \mathrm{diam}(\Omega) \geq cm^{1/\alpha}$. For $\alpha < 1$ and sufficiently large m, both inequalities cannot be satisfied at the same time. $\square$

We now turn to completing the proof of Theorem 2.5, which in view of Corollary 6.2 amounts to the proof of the following proposition.

PROPOSITION 6.3 (Nonexistence in the Near-Field-Dominated Case). *Let $\alpha \geq 1$. Then there exists $m_2 = m_2(\alpha) > 0$ such that there are no minimizers of E for all $m > m_2$.*

PROOF. We argue by contradiction. Let Ω be a minimizer of E for some $m \geq 1$. Introducing $d := \mathrm{diam}(\Omega)$, let $x_1, x_2 \in \partial\Omega$ be such that $|x_1 - x_2| = d$. For every $s \in (0, d)$, define $T(s)$ to be the line perpendicular to $x_1 - x_2$ and located at distance s from x_1. The line $T(s)$ cuts the set Ω into two nonempty parts. We define the part of Ω that is closer to x_1 as Ω_s. We also define $V(s) := |\Omega_s|$ and $A(s) := |\Omega \cap T(s)|$. Note that $A \in L^\infty(0, d)$ (since the diameter of Ω is bounded), and by Cavalieri's principle we have

$$(6.3) \qquad V(s) = \int_0^s A(s')ds' \quad \forall s \in (0, d).$$

Also, without loss of generality we may assume that $V(d/2) \leq m/2$. In particular, this implies that for all $s \in (0, d/2)$,

$$(6.4) \qquad E(\Omega) \geq |\partial\Omega| + \iint_{\Omega_s \, \Omega_s} \frac{1}{|x - y|^\alpha} \, dx \, dy$$
$$+ \int_{\Omega\setminus\Omega_s} \int_{\Omega\setminus\Omega_s} \frac{1}{|x - y|^\alpha} \, dx \, dy + \frac{mV(s)}{d^\alpha}.$$

Now, consider a new set $\Omega' = (T_R\Omega_s) \cup (\Omega \setminus \Omega_s)$, where T_R denotes a translation by distance $R > 0$ along the vector $\overrightarrow{x_2x_1}$; i.e., the set Ω' is obtained by cutting Ω with $T(s)$ and moving the resulting pieces distance R apart. We have

$$E(\Omega') \leq |\partial\Omega| + 2A(s) + \iint_{\Omega_s \, \Omega_s} \frac{1}{|x - y|^\alpha} \, dx \, dy$$
$$+ \int_{\Omega\setminus\Omega_s} \int_{\Omega\setminus\Omega_s} \frac{1}{|x - y|^\alpha} \, dx \, dy + \frac{2mV(s)}{R^\alpha}.$$

Therefore, from the minimizing property of Ω we obtain

$$(6.5) \qquad 2A(s) \geq \frac{mV(s)}{2d^\alpha} \quad \forall s \in (0, d/2)$$

for large enough R, or, equivalently,

$$(6.6) \qquad \frac{dV}{ds} \geq \frac{mV}{4d^\alpha} \quad \text{for a.e. } s \in (0, d/2).$$

Integrating this expression from $s \in (0, d/2)$ to $d/2$, we then conclude that

$$(6.7) \qquad V(s) \leq \frac{m}{2} \exp\left(-\frac{m(d - 2s)}{8d^\alpha}\right) \quad \forall s \in (0, d/2).$$

In particular, by Lemma 6.1

$$(6.8) \qquad V(s) \leq \frac{m}{2} \exp\left(-\frac{1}{16} m d^{1-\alpha}\right) \leq m \exp\left(-c m^{2-\alpha}\right) \quad \forall s \in (0, d/4]$$

for some $c > 0$ depending only on α; i.e., $V(s)$ becomes uniformly small for $s \in (0, d/4]$ and $m \gg 1$.

Let us now show that the latter is impossible. We consider a different set Ω'' obtained by erasing Ω_s from Ω and dilating the resulting set $\Omega \setminus \Omega_s$ by a factor $\lambda_s = \sqrt{m/(m - V(s))} > 1$ to make Ω'' admissible. By the minimizing property of Ω and positivity of the kernel in the nonlocal term, we have

$$(6.9) \qquad E(\Omega) \leq E(\Omega'') \leq \lambda_s^{4-\alpha} E(\Omega \setminus \Omega_s) \leq \lambda_s^4 (E(\Omega) - |\partial \Omega_s| + 2A(s)),$$

where we argued as in (4.4). Therefore, by the isoperimetric inequality and Lemma 5.1 we have

$$(6.10) \qquad 2\sqrt{\pi} \, V^{1/2}(s) \leq |\partial \Omega_s| \leq C(V(s) + A(s))$$

for some $C > 0$ depending only on α. In view of (6.8), there exists $m_2 \geq 1$ such that $CV \leq \sqrt{\pi} \, V^{1/2}$ for all $s \in (0, d/4]$ and all $m > m_2$. Therefore, for these values of m (6.10) implies

$$(6.11) \qquad \frac{dV}{ds} \geq c V^{1/2} \quad \text{for a.e. } s \in (0, d/4),$$

with some $c > 0$ depending only on α. Integrating this inequality from 0 to $s \in (0, d/4]$, we then find that

$$(6.12) \qquad V(s) \geq c s^2 \quad \forall s \in (0, d/4]$$

for some $c > 0$ depending only on α. But by Lemma 6.1 this contradicts (6.8) at $s = d/4$. $\qquad \square$

7   Shape of Minimizers for Small Masses

We now turn to the proof of Theorem 2.3. Here it is convenient first to rescale length in such a way that the rescaled set Ω has a fixed mass. Let us define a positive parameter

$$(7.1) \qquad \varepsilon := \left(\frac{m}{\pi}\right)^{\frac{3-\alpha}{2}}.$$

Then the renormalized energy

$$(7.2) \qquad E_\varepsilon(\Omega) := |\partial \Omega| + \varepsilon \iint_{\Omega\,\Omega} \frac{1}{|x - y|^\alpha} \, dx \, dy, \quad |\Omega| = \pi,$$

is related to the original energy as

$$(7.3) \qquad E(\Omega) = (m/\pi)^{1/2} E_\varepsilon(\Omega_\varepsilon),$$

where Ω_ε is obtained by dilating the set Ω by a factor of $(m/\pi)^{-1/2}$. We note that by virtue of Theorem 2.2, the minimizers of E_ε exist for all $\varepsilon \leq \varepsilon_1(\alpha)$, where ε_1 is related to m_1 via (7.1). Furthermore, the regularity result in Proposition 2.1, with constants depending on ε and α, holds for the minimizers of E_ε.

Expressed in terms of the rescaled problem, Theorem 2.3 takes the following form:

PROPOSITION 7.1. *There exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the minimizer of E_ε is a unit disk.*

The proof proceeds differently for the far-field-dominated ($\alpha < 1$) and near-field-dominated ($\alpha \geq 1$) regimes. But before we turn to the proof, let us establish a number of basic properties of the minimizers of E_ε that we will need in our analysis. Recall that for any set of finite perimeter in $\mathbb{R}^2$, the isoperimetric deficit is given by

$$(7.4) \qquad D(\Omega) := \frac{|\partial\Omega|}{2\pi} - 1.$$

We begin with a basic estimate of the isoperimetric deficit of minimizers.

LEMMA 7.2. *Let Ω be a minimizer of E_ε, and let $D(\Omega)$ be the isoperimetric deficit of Ω. Then for some $C > 0$ depending only on α we have*

$$(7.5) \qquad D(\Omega) \leq C\varepsilon.$$

PROOF. The proof is obtained by testing E_ε with a unit disk. The assertion follows immediately from the minimizing property of Ω, positivity of the nonlocal term, and the fact that by (3.12),

$$(7.6) \qquad |\partial\Omega| \leq E_\varepsilon(\Omega) \leq E_\varepsilon(B_1(0)) = 2\pi + C\varepsilon. \qquad \square$$

We next establish that for small values of ε the minimizers are necessarily convex and hence simply connected.

LEMMA 7.3. *Let Ω be a minimizer of E_ε. Then there exists $\varepsilon_2 = \varepsilon_2(\alpha) > 0$ such that Ω is convex for all $\varepsilon \leq \varepsilon_2$.*

PROOF. The Euler-Lagrange equation for the minimizers of E_ε is given by

$$(7.7) \qquad \kappa(x) + 2\varepsilon v(x) - \mu = 0, \quad v(x) := \int_\Omega \frac{1}{|x-y|^\alpha}\, dy,$$

where, as in (2.2), $\kappa(x)$ and $v(x)$ denote the curvature and potential at $x \in \partial\Omega$, respectively, and $\mu \in \mathbb{R}$ is the Lagrange multiplier. To estimate μ, let us integrate (7.7) over the outer boundary $\partial\Omega_o$ of Ω, which is justified by Proposition 2.1. After dividing by $|\partial\Omega_o| > 0$, we obtain

$$(7.8) \qquad \mu = \frac{2\pi}{|\partial\Omega_o|} + 2\varepsilon\bar{v}, \quad \bar{v} := \frac{1}{|\partial\Omega_o|}\int_{\partial\Omega_o} v(x)\,d\mathcal{H}^1(x).$$

Now, let Ω_o be the set enclosed by $\partial\Omega_o$, so that $\Omega \subseteq \Omega_o$ and $\partial\Omega_o \subseteq \partial\Omega$. In particular, we have $\pi = |\Omega| \leq |\Omega_o|$, and by the isoperimetric inequality $2\sqrt{\pi}\,|\Omega_o|^{1/2} \leq |\partial\Omega_o| \leq |\partial\Omega|$. Lemma 7.2 thus implies

$$(7.9) \qquad\qquad 2\pi \leq |\partial\Omega_o| \leq 2\pi + C\varepsilon$$

for some $C > 0$ depending only on α. Similarly, we have for every $x \in \Omega$

$$(7.10) \qquad 0 \leq v(x) \leq \int_{B_1(x)} \frac{1}{|x-y|^\alpha}\,dy + \int_{\Omega\backslash B_1(x)} \frac{1}{|x-y|^\alpha}\,dy \leq C$$

$$\text{and} \quad 0 \leq \overline{v} \leq C$$

for some $C > 0$ depending only on α. Inserting (7.9) and (7.10) into (7.8), we obtain that $|\mu - 1| \leq C\varepsilon$ for some $C > 0$. Substituting this estimate, together with (7.10), into (7.7), we then conclude that $|\kappa(x) - 1| \leq C\varepsilon$. Thus, for all small enough ε we have $\kappa(x) \geq 0$ for all $x \in \partial\Omega$, which proves the statement. $\qquad\square$

The next lemma is key to the analysis of the small-ε regime and is based on a Bonnesen-type inequality for convex sets with small isoperimetric deficit [7] (for a review, see [38]). In view of Lemma 7.2, the latter is the case for the minimizers of E_ε when ε is sufficiently small. We will use a version of the result that was proved by Fuglede [17], which connects the isoperimetric deficit to the spherical deviation of the set Ω from a unit ball centered at the barycenter of Ω to prove this lemma.

LEMMA 7.4. *Let Ω be a minimizer of E_ε, and let $x_0 \in \mathbb{R}^2$ be the barycenter of Ω. Then there exists $\varepsilon_3 = \varepsilon_3(\alpha) > 0$ such that for all $\varepsilon \leq \varepsilon_3$*

    (i) *There exists $\delta > 0$ satisfying*

$$(7.11) \qquad\qquad \delta \leq C\sqrt{D(\Omega)},$$

        with some universal $C > 0$ such that $B_{1-\delta}(x_0) \subset \Omega \subset B_{1+\delta}(x_0)$.
    (ii) *Let $\rho : \mathbb{R} \to (-\delta, \delta)$ be such that $r = 1 + \rho(\theta)$ defines the graph of $\partial\Omega$ in polar coordinates (r, θ) centered at x_0. Then*

$$(7.12) \qquad\qquad D(\Omega) \leq C\|\rho\|^2_{H^1(0,2\pi)}$$

    for some universal $C > 0$.

PROOF. When ε is sufficiently small, the minimizer Ω of E exists, has small isoperimetric deficit by Lemma 7.2, and is convex by Lemma 7.3. The result then follows from [17, theorem 1.3 and footnote 4]. $\qquad\square$

We can now proceed to the conclusion of the proof of Theorem 2.3. We start with the far-field-dominated case.

PROPOSITION 7.5 (Minimizer Is a Disk, Far-Field-Dominated Regime). *Let $\alpha < 1$. Then there exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that for all $\varepsilon \leq \varepsilon_0$, the unique, up to translations, minimizer of E_ε is $\Omega = B_1(0)$.*

PROOF. If ε is sufficiently small, there exists a minimizer Ω of E. Furthermore, the set Ω satisfies the conclusions of Lemma 7.4. Since Ω is a minimizer, we have $E(\Omega) \leq E(B_1(x_0))$, where x_0 is the barycenter of Ω, which is equivalent to

$$(7.13) \quad D(\Omega) \leq \frac{\varepsilon}{2\pi}\left(\int_{B_1(x_0)}\int_{B_1(x_0)} \frac{1}{|x-y|^\alpha}\,dx\,dy - \iint_{\Omega\,\Omega} \frac{1}{|x-y|^\alpha}\,dx\,dy\right).$$

Let u and u^B be the characteristic functions of Ω and $B_1(x_0)$, respectively, and let v^B be as in (3.25). Then, since the nonlocal kernel is positive definite (as can be seen from (3.7) and (3.8)), and since $\int_{\mathbb{R}^2}(u^B - u)dx = 0$, we have

$$
\begin{aligned}
(7.14) \quad & \int_{B_1(x_0)}\int_{B_1(x_0)} \frac{1}{|x-y|^\alpha}\,dx\,dy - \int_\Omega\int_\Omega \frac{1}{|x-y|^\alpha}\,dx\,dy \\
&= 2\int_{\mathbb{R}^2} v^B(x-x_0)(u^B(x)-u(x))dx \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(u^B(x)-u(x))(u^B(y)-u(y))}{|x-y|^\alpha}\,dx\,dy \\
&\leq 2\int_{\mathbb{R}^2} (v^B(x-x_0)-v_0)(u^B(x)-u(x))dx,
\end{aligned}
$$

where v_0 is given by (3.28). Thus

$$
\begin{aligned}
(7.15) \quad & \int_{B_1(x_0)}\int_{B_1(x_0)} \frac{1}{|x-y|^\alpha}\,dx\,dy - \iint_{\Omega\,\Omega} \frac{1}{|x-y|^\alpha}\,dx\,dy \\
&\leq 2\int_{\Omega\triangle B_1(x_0)} |v^B(x-x_0)-v_0|dx \\
&\leq 2\|v^B(\cdot-x_0)-v_0\|_{L^\infty(\Omega\triangle B_1(x_0))}|\Omega\triangle B_1(x_0)| \\
&\leq C\delta\|v^B-v_0\|_{L^\infty(B_{1+\delta}(0)\setminus B_{1-\delta}(0))}
\end{aligned}
$$

for some universal $C > 0$. On the other hand, by Lemma 3.8, we have $|v^B - v_0| \leq C\delta$ in $B_{1+\delta}(0) \setminus B_{1-\delta}(0)$ for some $C > 0$ depending only on α. Combining this inequality with the estimates (7.11), (7.13), and (7.15), we get

$$(7.16) \qquad\qquad c\delta^2 \leq D(\Omega) \leq C\varepsilon\delta^2$$

for some universal $c > 0$ and some $C > 0$ depending only α. Therefore, as long as ε is small enough, we have $D(\Omega) = 0$, which implies that $\Omega = B_1(x_0)$. $\square$

We note that the above proof fails in the near-field-dominated regime, $\alpha \geq 1$, since in this case v^B fails to be in $C^1(\mathbb{R}^2)$, as can be seen from (3.27) (in fact, the

radial derivative of v^B gets singular at $\partial B_1(0)$). Therefore, a more delicate analysis of the contribution of the deviation of Ω from a ball to the nonlocal part of the energy is necessary. In fact, we need to prove some cancellations in the difference of the two nonlocal energies (related to the minimizer and the corresponding ball of the same area) to obtain an analogue of (7.16). For this, we will make a more detailed use of the Euler-Lagrange equation.

It remains to prove the following proposition:

PROPOSITION 7.6 (Minimizer Is a Disk, Near-Field-Dominated Regime). *Let α ≥ 1. Then there exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that for all $\varepsilon \leq \varepsilon_0$, the unique, up to translations, minimizer of E_ε is $\Omega = B_1(0)$.*

PROOF. The main point here is to obtain the inequality in the right-hand side of (7.16) from (7.13). The conclusion then follows as in the proof of Proposition 7.5. We begin by writing

$$\int\limits_{B_1(x_0)} \int\limits_{B_1(x_0)} \frac{1}{|x-y|^\alpha} \, dx \, dy - \iint\limits_{\Omega\,\Omega} \frac{1}{|x-y|^\alpha} \, dx \, dy$$

$$= \int\limits_{B_1(x_0)\backslash\Omega} (v^B(x-x_0) + v(x) - 2v_0) dx$$

(7.17)

$$- \int\limits_{\Omega\backslash B_1(x_0)} (v^B(x-x_0) + v(x) - 2v_0) dx$$

$$= \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \int\limits_{B_1(x_0)\backslash\Omega} \left(v(x) - v^B(x-x_1(x))\right) dx$$

$$- \int\limits_{\Omega\backslash B_1(x_0)} \left(v(x) - v^B(x-x_1(x))\right) dx,$$

(7.18)

$$\mathrm{II} = \int\limits_{B_1(x_0)\backslash\Omega} \left(v^B(x-x_1(x)) + v^B(x-x_0) - 2v_0\right) dx$$

$$- \int\limits_{\Omega\backslash B_1(x_0)} \left(v^B(x-x_1(x)) + v^B(x-x_0) - 2v_0\right) dx,$$

and

(7.19) $x_1(x) := x_0 + (|x| - 1) \dfrac{x - x_0}{|x - x_0|};$

i.e., $x_1(x)$ is the center of a ball whose center is shifted from x_0 in the direction of x in such a way that $x \in \partial B_1(x_1(x))$. Introducing polar coordinates as in Lemma

7.4(ii), we have (with a slight abuse of notation)

$$(7.20) \qquad \mathrm{I} = \int_0^{2\pi} \int_1^{1+\rho(\theta)} \left(v^B(r - \rho(\theta)) - v(r, \theta)\right) r \, dr \, d\theta,$$

$$(7.21) \qquad \mathrm{II} = \int_0^{2\pi} \int_1^{1+\rho(\theta)} \left(2v_0 - v^B(r) - v^B(r - \rho(\theta))\right) r \, dr \, d\theta.$$

Let us estimate the term in (7.21) first. In the following we will only explicitly consider the case $\alpha > 1$; the case $\alpha = 1$ is treated analogously. In view of (3.27), with $s = r - 1$ we have

$$
\begin{aligned}
\mathrm{II} &= \int_0^{2\pi} \int_0^{\rho(\theta)} \left(2v_0 - v^B(1 + s) - v^B(1 + s - \rho(\theta))\right)(1 + s) ds \, d\theta \\
(7.22) \\
&= C \int_0^{2\pi} \int_0^{\rho(\theta)} (|s|^{1-\alpha} s - |\rho(\theta) - s|^{1-\alpha}(\rho(\theta) - s)) ds \, d\theta + O(\delta^2),
\end{aligned}
$$

for some $C > 0$ depending only on α, where

$$(7.23) \qquad \delta := \|\rho\|_{L^\infty(\mathbb{R})}.$$

However, the integral in the second line of (7.22) is identically 0, so we have $\mathrm{II} = O(\delta^2)$.

We now turn to estimating (7.20), which can be written as

$$(7.24) \quad \mathrm{I} = \int_0^{2\pi} \int_0^{\rho(\theta)} \int_{\theta-\pi}^{\theta+\pi} \int_{\rho(\theta')}^{\rho_B(\theta,\theta')} d^{-\alpha}(s, s', \theta, \theta')(1 + s)(1 + s') ds' \, d\theta' \, ds \, d\theta,$$

where $d(s, s', \theta, \theta')$ is the distance between the points with polar coordinates $(1 + s, \theta)$ and $(1 + s', \theta')$, and $\rho_B(\theta, \theta')$ solves

$$(7.25) \qquad 1 = (1 + \rho_B)^2 + \rho^2 - 2\rho(1 + \rho_B) \cos(\theta - \theta'),$$

and for each θ simply describes the polar graph $r(\theta') = 1 + \rho_B(\cdot, \theta')$ of a circle shifted by $\rho(\theta)$ in the direction of θ from the origin. Clearly, $\rho_B(\theta, \cdot) \in C^\infty(\mathbb{R})$ for sufficiently small δ, and furthermore for all $\theta, \theta' \in \mathbb{R}$ we have

$$(7.26) \qquad |\rho_B(\theta, \theta')| \le \delta, \quad |\rho_B(\theta, \theta') - \rho(\theta)| \le C\delta|\theta - \theta'|^2,$$

for some universal $C > 0$. In addition, for small enough δ we have

$$(7.27) \qquad d(s, s', \theta, \theta') \ge c|\theta - \theta'|$$

for some universal $c > 0$. Combining estimates (7.26) and (7.27), we obtain

$$
\begin{aligned}
|\mathrm{I}| &\leq C \left| \int_0^{2\pi} \int_0^{\rho(\theta)} \int_{\theta-\pi}^{\theta+\pi} \int_{\rho(\theta')}^{\rho_B(\theta,\theta')} |\theta - \theta'|^{-\alpha} \, ds' \, d\theta' \, ds \, d\theta \right| \\
&\leq C \Bigg(\left| \int_0^{2\pi} \int_0^{\rho(\theta)} \int_{\theta-\pi}^{\theta+\pi} \int_{\rho(\theta)}^{\rho_B(\theta,\theta')} |\theta - \theta'|^{-\alpha} \, ds' \, d\theta' \, ds \, d\theta \right| \\
&\qquad + \left| \int_0^{2\pi} \int_0^{\rho(\theta)} \int_{\theta-\pi}^{\theta+\pi} \int_{\rho(\theta')}^{\rho(\theta)} |\theta - \theta'|^{-\alpha} \, ds' \, d\theta' \, ds \, d\theta \right| \Bigg) \\
&\leq C'\delta \left(\delta + \|\rho_\theta\|_{L^\infty(\mathbb{R})} \int_0^{2\pi} \int_{\theta-\pi}^{\theta+\pi} |\theta - \theta'|^{1-\alpha} \, d\theta' \, d\theta \right) \\
&\leq C''\delta(\delta + \|\rho_\theta\|_{L^\infty(\mathbb{R})}),
\end{aligned}
\tag{7.28}
$$

for some $C, C', C'' > 0$ depending only on α, where here and below the subscript θ denotes a derivative with respect to θ. So, in order to conclude that $I = O(\delta^2)$ as well, it remains to show that

$$
\|\rho_\theta\|_{L^\infty(\mathbb{R})} \leq C\delta \tag{7.29}
$$

for some $C > 0$ depending only on α.

To obtain (7.29), we write the Euler-Lagrange equation for $\rho(\theta)$ in polar coordinates. Using the well-known formula for the curvature in polar coordinates, we can write (7.7) in the form

$$
\frac{(1 + \rho)^2 + 2\rho_\theta^2 - (1 + \rho)\rho_{\theta\theta}}{\{(1 + \rho)^2 + \rho_\theta^2\}^{3/2}} = \frac{2\pi}{|\partial\Omega|} - 2\varepsilon(v - \bar{v}), \tag{7.30}
$$

where $v = v(1 + \rho(\theta), \theta)$. In fact, by continuity of $\rho(\theta)$ there exists θ^* such that $\bar{v} = v(1 + \rho(\theta^*), \theta^*)$. Now, by [17, lemma 2.2], for sufficiently small δ we also have

$$
\|\rho_\theta\|_{L^\infty(\mathbb{R})} \leq C\delta^{1/2} \tag{7.31}
$$

with some universal $C > 0$. Therefore, subtracting 1 from both sides of (7.30), after a straightforward calculation we obtain

$$
\|\rho_{\theta\theta}\|_{L^\infty(\mathbb{R})} \leq C(\delta + D(\Omega) + \varepsilon\|v - \bar{v}\|_{L^\infty(\mathbb{R})}). \tag{7.32}
$$

On the other hand, arguing as in (7.28), we have

$$
\begin{aligned}
|v(\rho(\theta), \theta) - v_0| &= \left| \int_{\theta-\pi}^{\theta+\pi} \int_{\rho(\theta')}^{\rho_B(\theta,\theta')} d^{-\alpha}(\rho(\theta), s', \theta, \theta')(1 + s') ds' \, d\theta' \right| \\
&\leq C \left(\delta + \|\rho_\theta\|_{L^\infty(\mathbb{R})} \int_{\theta-\pi}^{\theta+\pi} |\theta - \theta'|^{1-\alpha} \, d\theta' \right) \\
&\leq C'(\delta + \|\rho_\theta\|_{L^\infty(\mathbb{R})}),
\end{aligned}
\tag{7.33}
$$

for some $C, C' > 0$ depending only on α. In particular, since the same estimate holds for $\theta = \theta^*$, we have

$$(7.34) \qquad \|v - \bar{v}\|_{L^\infty(\mathbb{R})} \leq C(\delta + \|\rho_\theta\|_{L^\infty(\mathbb{R})})$$

for some $C > 0$ depending only on α.

Finally, using Lemma 7.4(ii), (7.32), and (7.34), we conclude that

$$(7.35) \qquad \|\rho_{\theta\theta}\|_{L^\infty(\mathbb{R})} \leq C(\delta + \varepsilon\|\rho_\theta\|_{L^\infty(\mathbb{R})})$$

for some $C > 0$ depending only on α. Observe that $\rho_\theta(\theta) = \int_{\theta_0}^\theta \rho_{\theta\theta}(\theta')d\theta'$ for some $\theta_0 \in \mathbb{R}$. Therefore, using the smallness of ε, from (7.35) we immediately obtain (7.29). $\qquad \square$

8 Complete Characterization in the Case of Small α

In this section, we present the proof of Theorem 2.7. The proof is a slight modification of the proof of Proposition 7.5, and so we find it more convenient to work with the energy in (7.2) (but now without the smallness assumption on ε). The proof also requires a refinement of the nonexistence result from Section 6.

In terms of E_ε, the result we wish to obtain is a consequence of the following proposition.

PROPOSITION 8.1. *There exists $\alpha_0 > 0$ such that for all $\alpha \leq \alpha_0$ the minimizer of E_ε, if it exists, is given by $\Omega = B_1(x_0)$ for some $x_0 \in \mathbb{R}^2$.*

The proof follows from a sequence of lemmas.

LEMMA 8.2. *Let $\Omega \subset \mathbb{R}^2$ be a set of finite perimeter, and let $|\Omega| = \pi$. Then*

$$(8.1) \qquad E_\varepsilon(\Omega) = |\partial\Omega| + \varepsilon\pi^2 + \alpha \iint\limits_{\Omega\,\Omega} g(x - y)dx\, dy,$$

where

$$(8.2) \qquad |g(x - y)| \leq C\varepsilon\frac{|\ln|x - y||}{|x - y|^\alpha}$$

and where the constant $C > 0$ depends only on $d := \operatorname{diam}(\Omega)$.

PROOF. Applying the Taylor formula to the exponential function, we get

$$(8.3) \qquad |x - y|^{-\alpha} - 1 = e^{-\alpha \ln|x-y|} - 1 = -\alpha|x - y|^{-\alpha\theta} \ln|x - y|$$

for some $\theta = \theta(x-y) \in (0, 1)$. The statement then follows with $C = \max\{1, d^2\}$. $\qquad \square$

Our next lemma establishes nonexistence of minimizers of E_ε for sufficiently large ε *uniformly* in α (as long as $\alpha \leq \alpha_0 < 1$ for some fixed α_0).

LEMMA 8.3. *For every $\alpha_0 \in (0, 1)$ there exists $\varepsilon_2 > 0$ (depending only on α_0) such that for every $\alpha \in (0, \alpha_0]$ there is no minimizer of E_ε for any $\varepsilon > \varepsilon_2$.*

PROOF. We prove the statement for the original energy E, which amounts to existence of $m_2 = m_2(\alpha_0) > 0$ such that there is no minimizer of E for all $m > m_2$ and $\alpha \in (0, \alpha_0]$. By Lemma 5.1, for a minimizer Ω of E we have $E(\Omega) \leq Cm$ for all $m \geq 1$, where the dependence of the constant $C > 0$ on α is via $E(B_1(0))$. By continuous dependence of $E(B_1(0))$ on $\alpha \in [0, \alpha_0]$ (see (3.12)), we can, in fact, choose $C \geq 1$ to depend only on α_0. Therefore, arguing as in the proof of Lemma 6.1, we have $m^{2-\alpha}/C^\alpha \leq m^2/d^\alpha \leq Cm$ or, equivalently, $m \leq C^{(1+\alpha)/(1-\alpha)} \leq C^{2/(1-\alpha_0)}$. $\square$

We next prove that minimizers of E_ε must have small isoperimetric deficit for sufficiently small α.

LEMMA 8.4. *Let $\alpha_0 \in (0, 1)$, let $\alpha \in (0, \alpha_0]$, and let Ω be a minimizer of E_ε. Then $D(\Omega) \leq C\alpha$ for some $C > 0$ depending only on α_0.*

PROOF. By Lemma 8.3, we have $\varepsilon \leq \varepsilon_2(\alpha_0)$, which implies, in particular, that $\mathrm{diam}(\Omega) \leq \frac{1}{2}|\partial\Omega| \leq \frac{1}{2}E(B_1(0)) \leq C$ for some universal $C > 0$. The result then follows immediately from (8.1) and (8.2) by an estimate analogous to the one in (3.2). $\square$

The result in Lemma 8.4 implies that we can use the same ideas as in Section 7 (in the far-field-dominated case), replacing ε with α and taking advantage of the smallness of α to prove radial symmetry of minimizers. In particular, we have the analogue of Lemma 7.3:

LEMMA 8.5. *There exists $\alpha_0 \in (0, 1)$ such that for every $\alpha \in (0, \alpha_0]$ any minimizer Ω of E_ε is convex.*

Similarly, the analogue of Lemma 7.4 is the following:

LEMMA 8.6. *There exists $\alpha_0 \in (0, 1)$ such that for every $\alpha \in (0, \alpha_0]$ any minimizer Ω of E_ε satisfies $B_{1-\delta}(x_0) \subset \Omega \subset B_{1+\delta}(x_0)$, where x_0 is the barycenter of Ω for some $\delta \leq C\sqrt{D(\Omega)}$ with some universal $C > 0$.*

PROOF OF PROPOSITION 8.1. We argue as in the proof of Proposition 7.5. Repeating the steps of that proof with the help of Lemmas 8.4, 8.5, and 8.6, we obtain

$$(8.4) \qquad\qquad c\delta^2 \leq D(\Omega) \leq C\alpha\delta^2$$

for some universal $c, C > 0$, where the second inequality in (8.4) follows from the fact that the potential v^B given by (3.25) obeys

$$(8.5) \qquad |\nabla v^B(x)| \leq \alpha \int_{B_1(0)} \frac{1}{|x-y|^{1+\alpha}}\, dy \leq C\alpha \quad \forall x \in \mathbb{R}^2$$

for some universal $C > 0$, provided that α_0 is sufficiently small. The proof is then completed by observing that (8.4) implies $D(\Omega) = 0$ for α_0 sufficiently small. $\square$

PROOF OF THEOREM 2.7. Clearly, in view of Proposition 8.1 and Lemma 3.6(i) there are no minimizers for all $m > m_{c1}$ and $\alpha \leq \alpha_0$. It hence remains to show that there exists a minimizer for every $m \leq m_{c1}$. The assertion of Theorem 2.7 then follows by Proposition 8.1.

Suppose that $m \leq m_{c1}$ and consider a minimizing sequence $\{\Omega_k\}_{k=1}^{\infty}$. Then $E(\Omega_k) \to e(m) := \inf_{|\Omega|=m} E(\Omega)$ as $k \to \infty$, and by an approximation argument we may assume that all sets Ω_k consist of $N_k < \infty$ disjoint, open, connected components. In fact, Ω_k can be chosen so that N_k is independent of k. Indeed, by Theorem 2.3 we can lower the energy by replacing all the connected components whose mass is less than m_0 with balls of the same mass translated sufficiently far apart (as in the proof of Lemma 3.2). Then, if more than one ball is present in the resulting set, by Lemma 3.6(ii) we can further lower the energy by merging these balls, two at a time, and translating the resulting balls further apart.

In view of the above argument we may assume that $\Omega_k = \bigcup_{j=1}^{N} \Omega_k^{(j)}$ with $N_k \leq N$ for some $N \leq 1 + (m/m_0)$, and the sets $\Omega_k^{(j)}$ are connected (some of $\Omega_k^{(j)}$ are empty if there are less than N connected components). After taking a subsequence, we may assume that for each $1 \leq j \leq N$ we have $E(\Omega_k^{(j)}) \to e_j$, $|\Omega_k^{(j)}| \to \mu_j$ for some constants $e_j \geq 0$, and $\mu_j \geq 0$ as $k \to \infty$, and, furthermore, by compactness each set $\Omega_k^{(j)}$ "converges" to a set $\Omega^{(j)}$ as $k \to \infty$ after a suitable translation. More precisely, if $u_k^{(j)}$ are the characteristic functions of $\Omega_k^{(j)}$ translated to contain the origin, then $u_k^{(j)} \to u^{(j)}$ in $BV(\mathbb{R}^2)$ as $k \to \infty$, where $u^{(j)}$ is the characteristic function of $\Omega^{(j)}$. Furthermore, since the sets $\Omega_k^{(j)}$ are either connected and uniformly bounded or empty, we have $|\Omega^{(j)}| = \mu_j$.

Observe that since $\sum_{j=1}^{N} \mu_j = m$, we have (see also [12, remark 4.1])

$$(8.6) \qquad e(m) \leq \sum_{j=1}^{N} e(\mu_j).$$

Indeed, if $\Omega_j \subset \mathbb{R}^2$ are such that $|\Omega_j| = \mu_j$ and $E(\Omega_j) < e(\mu_j) + \delta$ for some $\delta > 0$, we can construct a set Ω' with $|\Omega'| = m$ and $E(\Omega') < \sum_{j=1}^{N} e(\mu_j) + 2\delta$ by taking Ω' to be a union of Ω_j translated sufficiently far apart. The result then follows by arbitrariness of δ. At the same time, we have

$$(8.7) \qquad \sum_{j=1}^{N} E(\Omega^{(j)}) \leq e(m).$$

Indeed, by lower semicontinuity of E with respect to the weak BV-convergence and by positivity of the kernel in the nonlocal term in the energy we have

$$(8.8) \quad \sum_{j=1}^{N} E(\Omega^{(j)}) \leq \sum_{j=1}^{N} e_j = \lim_{k \to \infty} \sum_{j=1}^{N} E(\Omega_k^{(j)}) \leq \lim_{k \to \infty} E(\Omega_k) = e(m).$$

We now claim that $E(\Omega^{(j)}) = e(\mu_j)$. Indeed, clearly $E(\Omega^{(j)}) \geq e(\mu_j)$ for all $1 \leq j \leq N$. On the other hand, by (8.6) and (8.7) we get

$$(8.9) \quad e(m) \leq \sum_{j=1}^{N} e(\mu_j) \leq \sum_{j=1}^{N} E(\Omega^{(j)}) \leq e(m),$$

so that all inequalities in (8.9) turn into equalities (compare also with [12, lemma 4.4(3)]). Again, since $e(\mu_j) \leq E(\Omega^{(j)})$ for each $1 \leq j \leq N$, we get $e(\mu_j) = E(\Omega^{(j)})$ as well.

Thus, each set $\Omega^{(j)}$ is a minimizer of E with prescribed mass μ_j. Therefore, by Proposition 8.1 for each $1 \leq j \leq N$, the set $\Omega^{(j)}$ is either a ball or is empty. Then, repeating the argument at the beginning of the proof, with the help of Lemma 3.6(ii) we conclude that $E(B_R(0)) \leq \sum_{j=1}^{N} E(\Omega^{(j)})$, where $R = (m/\pi)^{1/2}$, and hence $B_R(0)$ is a minimizer by (8.7). □

Remark 8.7. It is easy to see that to the leading order in α the nonlocal part of the energy in (8.1) is generated by the kernel $g(x - y) \simeq \varepsilon \ln |x - y|^{-1}$, which appears in the studies of the sharp interface version of the Ohta-Kawasaki energy in two dimensions [12, 21, 33, 47]. In this respect the result of Proposition 8.1 is closely related to the rigidity result obtained in [33, prop. 3.5].

Finally, let us point out that the fact that the minimizers in Theorem 2.7 exist if and only if $m \leq m_{c1}$, where m_{c1} is given by (3.13), does not rely on smallness of α and would remain valid as long as minimizers of E are disks whenever they exist. This can also be seen from the following general result, which says, essentially, that any set of finite perimeter can be replaced by a set with lower energy consisting of a union of finitely many disjoint sets, each of which is a minimizer of E.

PROPOSITION 8.8. *Let $\Omega \subset \mathbb{R}^2$ be a set of finite perimeter. Then there exists a set*

$$(8.10) \quad \Omega' = \bigcup_{i=1}^{N} \overline{\Omega}_i, \quad \overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing \; \forall i \neq j,$$

with $N < \infty$ and $|\Omega'| = |\Omega|$ such that $E(\Omega') \leq E(\Omega)$ and $E(\overline{\Omega}_i) = \inf_{m=|\Omega_i|} E$.

PROOF. If Ω is a minimizer of E, there is nothing to prove. So assume that it is not. Then Ω may be uniquely decomposed into $N \leq \infty$ connected components denoted by Ω_i whose boundaries $\partial \Omega_i$ are Jordan curves that are essentially disjoint (up to negligible sets; see, e.g., [2, cor. 1 and theorem 8]). By positivity and decay of the kernel in the nonlocal part of the energy, we have

$$(8.11) \quad E(\Omega) > \sum_{\substack{\Omega_i = \overline{\Omega}_i}} \inf_{m=m_i} E + \sum_{\substack{\Omega_i \neq \overline{\Omega}_i}} \inf_{m=m_i} E, \quad m_i := |\Omega_i|,$$

where $\overline{\Omega}_i$ are minimizers of E with mass m_i whenever such a minimizer exists. The strict inequality in (8.11) follows from the fact that for $N = 1$ the set Ω is not a minimizer, while for $N > 1$ the energy can be reduced by spreading different connected components sufficiently far apart (as, e.g., in the proof of Lemma 5.1).

Suppose that $\Omega_i \neq \overline{\Omega}_i$. If the minimizer of E exists for mass $m = m_i$, we replace Ω_i with $\Omega'_i = \overline{\Omega}_i$. By the minimizing property of $\overline{\Omega}_i$, we then have $E(\overline{\Omega}_i) = \inf_{m=m_i} E$. Alternatively, if the minimum of E is not attained at $m = m_i$, there exists $\delta_0 > 0$ such that if Ω'_i has mass m_i and $E(\Omega'_i) < \inf_{m=m_i} E + \delta_0$, then Ω'_i is *disconnected*. Indeed, if not, there exists a minimizing sequence consisting of $\Omega_k \subset \mathbb{R}^2$ with $|\Omega_k| = m_i$ and each Ω_k connected. Then by the argument in the proof of Theorem 2.2, the minimum of E is attained, contradicting our assumption.

We can therefore replace all sets $\Omega_i \neq \overline{\Omega}_i$ for those values of m_i at which the minimum of E with $m = m_i$ is not attained with disconnected sets Ω'_i such that

$$(8.12) \qquad E(\Omega'_i) < \inf_{m=m_i} E + \delta$$

for some $\delta \in (0, \delta_0)$ to be specified later. Observe that by Theorem 2.2 for all those we have $m_i > m_1 = m_1(\alpha) > 0$. Therefore, the number of the components $\Omega_i \neq \overline{\Omega}_i$ corresponding to m_i at which the minimum of E at $m = m_i$ is not attained is bounded above in terms of $|\Omega|$. We now apply to each such component the algorithm in the proof of Theorem 2.2 to lower energy by erasing the smallest connected component of Ω'_i and rescaling the resulting set back to mass m_i. In view of the fact that $\delta < \delta_0$, this process must terminate before only one connected component remains. Then, arguing as in the proof of Theorem 2.2, we conclude that the mass of each remaining connected component of Ω'_i is bounded below by some $c > 0$ depending only on $E(\Omega)$ and α.

We are now able to choose $\delta > 0$ sufficiently small and construct a new set Ω' with $|\Omega'| = |\Omega|$ and $E(\Omega') < E(\Omega)$ by taking the union of all connected components of the sets Ω'_i constructed above, suitably translated to be sufficiently far apart. In this process the mass of each connected component of Ω' that is distinct from a minimizer is bounded above by $\max_{\Omega_i \neq \overline{\Omega}_i} |\Omega_i| - c$. Repeatedly applying this process, we then find that after finitely many iterations all connected components are minimizers. $\qquad\square$

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