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Hans Knüpfer, Cyrill B. Muratov and Matteo Novaga

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EMERGENCE OF NONTRIVIAL MINIMIZERS FOR THE THREE-DIMENSIONAL OHTA–KAWASAKI ENERGY

HANS KNÜPFER, CYRILL B. MURATOV AND MATTEO NOVAGA

This paper is concerned with the diffuse interface Ohta–Kawasaki energy in three space dimensions, in a periodic setting, in the parameter regime corresponding to the onset of nontrivial minimizers. We identify the scaling in which a sharp transition from asymptotically trivial to nontrivial minimizers takes place as the small parameter characterizing the width of the interfaces between the two phases goes to zero, while the volume fraction of the minority phases vanishes at an appropriate rate. The value of the threshold is shown to be related to the optimal binding energy solution of Gamow's liquid drop model of the atomic nucleus. Beyond the threshold the average volume fraction of the minority phase is demonstrated to grow linearly with the distance to the threshold. In addition to these results, we establish a number of properties of the minimizers of the sharp interface screened Ohta–Kawasaki energy in the considered parameter regime. We also establish rather tight upper and lower bounds on the value of the transition threshold.

1. Introduction and main results

The Ohta–Kawasaki energy is a prototypical energy functional in the studies of spatially modulated phases that appear as a result of the competition of short-range attractive and long-range repulsive forces in physical systems of very different nature. Although it was originally introduced in the context of microphase separation in diblock copolymer melts [Ohta and Kawasaki 1986], Ohta–Kawasaki energy is relevant to a wide range of both soft and hard condensed matter systems (for a discussion of the specific physical systems see, e.g., [Muratov 2002]), as well as to dense nuclear matter at the other extreme of energy and spatial scales [Lattimer et al. 1985; Maruyama et al. 2005]. From the mathematical point of view, the Ohta–Kawasaki energy functional, together with the closely related Thomas–Fermi–Dirac–von Weizsäcker energy [Heisenberg 1934; von Weizsäcker 1935; Lieb 1981; Le Bris and Lions 2005], serves as a paradigm for energy-driven pattern-forming systems with competing interactions [Choksi et al. 2017], which is why the associated variational problem has received an increasing amount of attention in recent years [Choksi 2001; Choksi et al. 2009; Spadaro 2009; Muratov 2010; Choksi and Peletier 2011; Goldman et al. 2013; Goldman et al. 2014; Lu and Otto 2014].

In the macroscopic setting, one considers the Ohta–Kawasaki energy functional for configurations defined on a sufficiently large box with periodic boundary conditions; i.e., for $u \in H^1(\mathbb{T}_\ell)$ one sets

$$\mathcal{E}_\varepsilon(u) := \int_{\mathbb{T}_\ell} \left(\tfrac{1}{2}\varepsilon^2 |\nabla u|^2 + \tfrac{1}{4}(1-u^2)^2 + \tfrac{1}{2}(u - \bar{u}_\varepsilon)(-\Delta)^{-1}(u - \bar{u}_\varepsilon) \right) dx, \tag{1-1}$$

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where $\mathbb{T}_\ell$ is the flat d-dimensional torus with side length $\ell > 0$. Furthermore, $\varepsilon > 0$ is the parameter characterizing interfacial thickness and assumed to be sufficiently small. Also, the parameter $\bar{u}_\varepsilon \in (-1, 1)$ denotes the constant background charge density. In the sequel, we will investigate the limit $\varepsilon \to 0$, assuming that u_ε depends on ε suitably. We note that the physically most relevant dimension is $d = 3$. In order for the last term in (1-1) to be well-defined, the definition in (1-1) needs to be supplemented with the "charge neutrality" constraint:

$$\frac{1}{\ell^d} \int_{\mathbb{T}_\ell} u \, dx = \bar{u}_\varepsilon. \tag{1-2}$$

One then wishes to characterize global energy minimizers of the energy in (2-1) for all ℓ sufficiently large. These global energy minimizers are expected to determine the ground states of the corresponding physical system in a macroscopically large sample.

It is widely believed that as the value of ℓ is increased with all other parameters fixed, the global energy minimizer of $\mathcal{E}_\varepsilon$ should be either constant or spatially periodic, with period approaching a constant independent of ℓ as $\ell \to \infty$. Proving such a *crystallization* result would be one of the main challenges in the theory of energy-driven pattern formation and is currently out of reach (for a recent review, see [Blanc and Lewin 2015]), except for the case $d = 1$, $\bar{u}_\varepsilon \in (-1, 1)$ fixed and $\varepsilon > 0$ sufficiently small [Müller 1993; Ren and Wei 2003; Chen and Oshita 2005] (for some results in that direction in higher dimensions, see [Shirokoff et al. 2015; Morini and Sternberg 2014; Daneri and Runa 2018]). On the other hand, it is known that for $\bar{u}_\varepsilon \in (-1, 1)$ fixed, global energy minimizers are not constant as soon as $\varepsilon \ll 1$ and $\ell \gtrsim 1$ [Choksi 2001; Muratov 2010]. This is in contrast with the case $\bar{u}_\varepsilon \notin (-1, 1)$, for which by direct inspection $u = \bar{u}_\varepsilon$ is the unique global minimizer of the energy. Thus, a transition from the trivial minimizer $u = \bar{u}_\varepsilon$ to a nontrivial, spatially nonuniform minimizer of $\mathcal{E}_\varepsilon$ must occur for $\varepsilon \ll 1$ and $\ell \gtrsim 1$ fixed as the value of $\bar{u}_\varepsilon$ increases from $\bar{u}_\varepsilon = -1$ towards $\bar{u}_\varepsilon = 0$ (in view of the symmetry exhibited by the energy when changing $u \to -u$, it is sufficient to consider only the case $\bar{u}_\varepsilon \le 0$). In fact, for $d \ge 2$ nontrivial minimizers emerge at some $\bar{u}_\varepsilon = -1 + \delta_\varepsilon$ with $\varepsilon \lesssim \delta_\varepsilon \lesssim \varepsilon^{2/3} |\ln \varepsilon|^{1/3}$ [Choksi et al. 2009; Muratov 2010], while for $d = 1$ they emerge for some $\varepsilon \lesssim \delta_\varepsilon \lesssim \varepsilon^{1/2}$ [Choksi et al. 2009; Muratov 2002]. The nature of the transition towards nontrivial minimizers is quite delicate and at present not well understood.

In the absence of general results for nontrivial minimizers of $\mathcal{E}_\varepsilon$ for $\varepsilon \lesssim 1$, $\bar{u}_\varepsilon \in (-1, 1)$ and $\ell \gg 1$ in $d \ge 2$, one can consider different asymptotic regimes that admit further analytical characterization. One such regime was analyzed in [Goldman et al. 2013], where the behavior of the minimizers of $\mathcal{E}_\varepsilon$ was studied in the limit $\varepsilon \to 0$ for $\bar{u}_\varepsilon = -1 + \lambda \varepsilon^{2/3} |\ln \varepsilon|^{1/3}$, with $\lambda > 0$ and $\ell > 0$ fixed, in the case $d = 2$. In this regime, nontrivial minimizers are expected to consist of well-separated "droplets" of the minority phase, i.e., regions where $u \simeq +1$ surrounded by the sea of the majority phase where $u \simeq -1$, separated by narrow domain walls of thickness $\sim \varepsilon$. It was found that there exists an explicit critical value of $\lambda = \lambda_c > 0$ such that the minimizers of $\mathcal{E}_\varepsilon$ are nontrivial for all $\lambda > \lambda_c$, while for $\lambda \le \lambda_c$ the minimizers are "asymptotically trivial", namely, that the energy of the minimizers converges to that of $u = \bar{u}_\varepsilon$, and the minimizer converges to $u = \bar{u}_\varepsilon$ in a certain sense as $\varepsilon \to 0$. Moreover, the threshold value λ_c corresponding to the onset of nontrivial minimizers was found to be independent of ℓ, suggesting that the transition should persist to the macroscopic limit $\ell \to \infty$ with $\varepsilon \ll 1$ and $\bar{u}_\varepsilon$ fixed (i.e., when commuting the order of the $\varepsilon \to 0$ and $\ell \to \infty$ limits). The obtained nontrivial minimizers exhibit a kind of a homogenization limit, with mass distributing uniformly on average throughout the domain. Furthermore, by performing a two-scale expansion of the energy, one can make more precise conclusions about the detailed properties of the minimizers and, in particular, formulate a variational problem in the whole space that determines the placement of the connected components of the minimizers in terms of the so-called renormalized energy, whose minimizers are conjectured to concentrate on the vertices of a hexagonal lattice [Goldman et al. 2014].

Here, we would like to understand how the transition to nontrivial minimizers happens when $\varepsilon \to 0$ and $\ell \gtrsim 1$ in the physical three-dimensional case. Therefore, from now on we fix $d = 3$ throughout the rest of the paper. Once again, in this regime the minimizers are expected to exhibit a two-phase character, with the minority phase occupying a small fraction of space. To this end, we define

$$\bar{u}_\varepsilon := -1 + \lambda \varepsilon^{2/3}, \tag{1-3}$$

where $\lambda > 0$ is fixed. Our main result is the following theorem.

Theorem 1.1. *Let $\ell > 0$ and $\lambda > 0$, and let $\mathcal{E}_\varepsilon$ be defined in (1-1) with $\bar{u}_\varepsilon$ given by (1-3). Then, there exists a universal constant $\lambda_c > 0$ such that if u_ε is a minimizer of $\mathcal{E}_\varepsilon(u)$ among all $u \in H^1(\mathbb{T}_\ell)$ satisfying (1-2), and $\mu_\varepsilon \in \mathcal{M}^+(\mathbb{T}_\ell)$ is such that $d\mu_\varepsilon(x) = \frac{1}{2}\varepsilon^{-2/3}(1 + \operatorname{sgn} u_\varepsilon(x))\,dx$, we have as $\varepsilon \to 0$:*

(i) *$\mu_\varepsilon \rightharpoonup 0$ in $\mathcal{M}(\mathbb{T}_\ell)$ if $\lambda \le \lambda_c$.*

(ii) *$\mu_\varepsilon \rightharpoonup \bar{\mu}$ in $\mathcal{M}(\mathbb{T}_\ell)$, where $d\bar{\mu} = \frac{1}{2}(\lambda - \lambda_c)\,dx$ if $\lambda > \lambda_c$.*

(iii) *$\varepsilon^{-4/3}\mathcal{E}_\varepsilon(u_\varepsilon) \to \min\{\lambda^2\ell^3, \lambda_c(2\lambda - \lambda_c)\ell^3\}$.*

Thus, the onset of nontrivial minimizers in three space dimensions occurs sooner in terms of $0 < 1 + \bar{u}_\varepsilon \ll 1$ than the corresponding transition in two dimensions. In particular, cylindrical morphologies obtained by trivially extending the two-dimensional minimizers into the third dimension are no longer global energy minimizers. One would, therefore, expect that the emergent nontrivial minimizers consist of a collection of well-separated small droplets of the minority phase surrounded by the sea of the majority phase. Furthermore, the size and the distance between the droplets should scale differently (see [Knüpfer et al. 2016]) from those in two dimensions. Furthermore, in contrast to the latter [Goldman et al. 2013; Goldman et al. 2014] we may no longer conclude that the droplets are nearly spherical.

Concerning the threshold λ_c, its precise value may be expressed in terms of several characteristics of the double-well potential appearing in (1-1) and the solution of the nonlocal isoperimetric problem in the whole space that goes back to [Gamow 1930]; see also [Choksi et al. 2017]. Namely, defining

$$\sigma := \frac{2\sqrt{2}}{3}, \quad \kappa := \frac{1}{\sqrt{2}}, \tag{1-4}$$

which are the *interfacial energy* and the *screening parameter*, respectively, we have (for details, see the following sections)

$$\lambda_c = 2^{-1/3}\sigma^{2/3}\kappa^2 f^*, \tag{1-5}$$

where $f^* > 0$ is the minimum energy per unit volume for Gamow's liquid drop model:

$$f^* := \inf_{u \in \mathrm{BV}(\mathbb{R}^3;\{0,1\})} \frac{\int_{\mathbb{R}^3} |\nabla u|\, dx + \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u(x)u(y)/|x-y|\, dx\, dy}{\int_{\mathbb{R}^3} u\, dx}. \tag{1-6}$$

Note that the minimization in (1-6) is equivalent to minimizing the perimeter plus the Coulombic self-energy over all Borel sets, per unit volume.

Although the precise value of f^* is currently unknown, the characterization in (1-5) allows us to obtain a quantitative estimate for the threshold λ_c in Theorem 1.1, using balls as competitors and a recent quantitative nonexistence result for the Gamow's liquid drop model [Frank et al. 2016].

Theorem 1.2. *With the notation of Theorem 1.1, we have*

$$\frac{3}{4\sqrt[3]{2}} \le \lambda_c \le \frac{3}{2\sqrt[3]{5}}. \tag{1-7}$$

Numerically, the bound in (1-7) appears to be fairly tight: $0.5952 < \lambda_c < 0.8773$, with the lower bound to within 33% of the value of the upper bound. Note that if the conjecture that the minimizers of Gamow's liquid drop model are balls is true, then the upper bound in (1-7) should in fact yield equality.

Our proof relies on our previous results obtained for the three-dimensional sharp interface version of the Ohta–Kawasaki energy [Knüpfer et al. 2016]. Together with the approach from [Muratov 2010, Section 4], the result in Theorem 1.1 is obtained along the lines of the arguments in [Goldman et al. 2013], suitably adapted from the two-dimensional to the three-dimensional case. Note, however, that since the sharp interface energy studied by Knüpfer et al. does not include the effect of *charge screening*, their results cannot be directly combined with those of [Muratov 2010]. In fact, there is no transition from trivial to nontrivial minimizers in the unscreened sharp interface energy. Therefore, as a first step towards the proof one needs to adapt the results of Knüpfer et al. to the case of screened sharp interface energy and obtain an asymptotic characterization of its minimizers as $\varepsilon \to 0$.

As in [Knüpfer et al. 2016], we separate the nonlocal energy into the near-field and far-field contributions, with screening appearing explicitly in the latter. At the same time, the self-interaction energy of the droplets turns out to be still well-approximated by that of Gamow's liquid drop model. Combining the far-field with the near-field contributions to the energy then allows us to establish a Γ-convergence result for the screened sharp interface energy to an energy functional which is quadratic in the limit charge density, with the notion of convergence being the weak* convergence of measures (see, for instance, [Knüpfer et al. 2016, Appendix A]). Along the way, we establish uniform estimates for the connected components of the minimizers similar to those in [Knüpfer et al. 2016], which, in turn, allows us to characterize nonexistence of nontrivial minimizers of the screened sharp interface energy for $\lambda < \lambda_c$ and ε sufficiently small.

Once the Γ-convergence result is established for the screened sharp interface energy, we proceed as in [Goldman et al. 2013] by introducing a piecewise-constant charge density associated with the admissible configurations for the diffuse interface energy that eliminates the small deviations of the charge density from their equilibrium values ± 1 for the double-well potential (for a more detailed explanation of the need of such a step, see the beginning of Section 2.2 in [Goldman et al. 2013]). We then adapt the arguments of [loc. cit., Section 6] to obtain the corresponding Γ-convergence result for the diffuse interface energy to the same quadratic functional in the limit charge density as for the screened sharp interface energy. Finally, explicitly minimizing the limit energy we obtain the main result of our paper contained in Theorem 1.1. Furthermore, we relate the value of the threshold λ_c with the optimal energy per unit mass for Gamow's liquid drop model. In addition, we use recent results in [Frank and Lieb 2015; Frank et al. 2016] characterizing the minimizers of the latter problem to obtain sharp quantitative bounds on the value of the threshold.

To summarize, our paper provides an extension of various recent results for the diffuse interface Ohta–Kawasaki energy to the case of a macroscopic three-dimensional domain, establishing a sharp transition from trivial to nontrivial minimizers in the asymptotic limit of vanishingly thin interfaces. Most of the techniques used in our proofs are adaptations of those that appeared in the earlier studies of this problem in different settings. The main novelty of our results, however, is the way these arguments are combined to yield a nontrivial scaling for the transition to nontrivial minimizers and the limit energy functional for the three-dimensional Ohta–Kawasaki energy. To our knowledge, this is the first sharp asymptotic result for this energy in the regime of strong compositional asymmetry and large number of droplets (for the case of finitely many droplets, see [Choksi and Peletier 2011]). We note that the present lack of knowledge about the minimizers of Gamow's liquid drop model prevents us from going to the next order in a two-scale Γ-expansion to describe local interactions of droplets via a "renormalized energy" [Rougerie and Serfaty 2016]. In particular, it is not known at present whether the minimizer per unit mass exists only for a unique value of the mass (this would be true if minimizers were balls). Thus, further insights into the solution of Gamow's model would be needed to carry out the program realized for the two-dimensional Ohta–Kawasaki energy in [Goldman et al. 2014].

Our paper is organized as follows. In Section 2, we introduce the different energies appearing in our study and state a number of results related to each of the associated variational problems. Also in this section, we prove Theorem 2.2, which gives a quantitative lower bound for the self-interaction energy per unit mass for Gamow's liquid drop model. Then, in Section 3 we state the Γ-convergence result for the sharp interface energy in Theorem 3.2, followed by a proof. Also in Section 3, we provide some further results about the connected components of minimizers of the screened sharp interface energy, see Theorem 3.4 and Corollary 3.5. Finally, in Section 4 we state and prove the corresponding Γ-convergence result for the diffuse interface energy; see Theorem 4.1. The results in Theorems 1.1 and 1.2 are then obtained as simple corollaries of the above theorems.

2. Setting

We now introduce the basic notation used throughout the rest of the paper, together with the assumptions and some technical results.

The diffuse interface energy. We begin by generalizing the diffuse energy functional in (1-1) to one involving an arbitrary symmetric double-well potential $W(u)$:

$$\mathcal{E}_\varepsilon(u) := \int_{\mathbb{T}_\ell} \left(\tfrac{1}{2}\varepsilon^2|\nabla u|^2 + W(u) + \tfrac{1}{2}(u-\bar{u}_\varepsilon)(-\Delta)^{-1}(u-\bar{u}_\varepsilon)\right) dx, \tag{2-1}$$

with $W(u)$ satisfying [Muratov 2010]

(i) $W \in C^2(\mathbb{R})$, $W(u) = W(-u)$, and $W \geq 0$,

(ii) $W(+1) = W(-1) = 0$ and $W''(+1) = W''(-1) > 0$,

(iii) $W''(|u|)$ is monotonically increasing for $|u| \geq 1$, $\lim_{|u|\to\infty} W''(u) = +\infty$, and $|W'(u)| \leq C(1+|u|^q)$ for some $C > 0$ and $1 < q < 5$.

The bounds on the exponent q in condition (iii) are the same as in [Muratov 2010]. In particular, the upper bound $q < 5$ guarantees that minimizers of $\mathcal{E}_\varepsilon$ are bounded and are therefore classical solutions of the Euler–Lagrange equation (2-5). Boundedness of minimizers is also needed to establish the estimate (4-11) in the proof of Theorem 4.1 below.

The energy $\mathcal{E}_\varepsilon$ is well-defined and bounded on the admissible class

$$\mathcal{A}_\varepsilon := \left\{ u \in H^1(\mathbb{T}_\ell) : \frac{1}{\ell^3} \int_{\mathbb{T}_\ell} u \, dx = \bar{u}_\varepsilon \right\}, \tag{2-2}$$

with the nonlocal term interpreted, as usual, with the help of the Green's function $G_0(x)$ solving

$$-\Delta G_0(x) = \delta(x) - \ell^{-3} \quad \text{in } \mathcal{D}'(\mathbb{T}_\ell) \qquad \text{and} \qquad \int_{\mathbb{T}_\ell} G_0(x)\, dx = 0. \tag{2-3}$$

Explicitly, the energy takes the form

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{T}_\ell} \left(\frac{1}{2}\varepsilon^2 |\nabla u|^2 + W(u) \right) dx + \frac{1}{2} \int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} (u(x) - \bar{u}_\varepsilon) G_0(x-y)(u(y) - \bar{u}_\varepsilon)\, dx\, dy, \tag{2-4}$$

noting that the last term in the right-hand side is well-defined by Young's inequality.

Under the above assumptions, every critical point $u \in \mathcal{A}_\varepsilon$ of $\mathcal{E}_\varepsilon$ weakly solves the Euler–Lagrange equation, which can be written as (see [Muratov 2010, Section 4])

$$-\varepsilon^2 \Delta u + W'(u) + v = \Lambda, \qquad -\Delta v = u - \bar{u}_\varepsilon, \tag{2-5}$$

where $v \in H^3(\mathbb{T}_\ell)$ is a zero-average solution of the second equation in (2-5) and $\Lambda \in \mathbb{R}$ is the Lagrange multiplier satisfying

$$\Lambda = \frac{1}{\ell^3} \int_{\mathbb{T}_\ell} W'(u)\, dx, \tag{2-6}$$

as can be seen by integrating the first equation in (2-5) over $\mathbb{T}_\ell$. In particular, we have

$$v(x) = \int_{\mathbb{T}_\ell} G_0(x-y)(u(y) - \bar{u}_\varepsilon)\, dy, \tag{2-7}$$

and $u, v \in C^\infty(\mathbb{T}_\ell)$ are classical solutions of (2-5) [Muratov 2010, Section 4]. We note that, by the direct method of calculus of variations, minimizers of $\mathcal{E}_\varepsilon$ are easily seen to exist for all choices of the parameters.

The sharp interface energy with screening. For $\varepsilon \ll 1$, minimizers of $\mathcal{E}_\varepsilon$ are expected to consist of functions which take values close to ± 1, except for narrow transition regions of width of order ε [Muratov 2010]. As usual, we define the energy of an optimal one-dimensional transition layer connecting $u = \pm 1$

[Modica 1987]:

$$\sigma := \int_{-1}^{1} \sqrt{2W(s)}\, ds > 0. \tag{2-8}$$

We also define the so-called screening parameter

$$\kappa := \frac{1}{\sqrt{W''(1)}} > 0. \tag{2-9}$$

This parameter measures the stiffness of the double-well potential near the wells. Physically, it is related to the effect of charge screening appearing in the sharp interface version of the energy $\mathcal{E}_\varepsilon$, introduced in the sequel (as in the Debye–Hückel theory [Landau and Lifshitz 1980; Muratov 2002; Muratov 2010]). With some obvious modifications, the results of [Muratov 2010, Section 4] apply to $\mathcal{E}_\varepsilon$ defined in (2-1), with the corresponding sharp interface energy E_ε defined as

$$E_\varepsilon(u) := \frac{\varepsilon\sigma}{2} \int_{\mathbb{T}_\ell} |\nabla u|\, dx + \frac{1}{2} \int_{\mathbb{T}_\ell} (u - \bar{u}_\varepsilon)(-\Delta + \kappa^2)^{-1}(u - \bar{u}_\varepsilon)\, dx, \tag{2-10}$$

where u belongs to the admissible class

$$\mathcal{A} := \mathrm{BV}(\mathbb{T}_\ell; \{-1, 1\}). \tag{2-11}$$

Specifically, in the considered scaling regime we have the following relation between the two energies (see the following sections):

$$\frac{\min_{u \in \mathcal{A}_\varepsilon} \mathcal{E}_\varepsilon(u)}{\min_{u \in \mathcal{A}} E_\varepsilon(u)} \to 1 \quad \text{as } \varepsilon \to 0. \tag{2-12}$$

Notice that the neutrality constraint in (1-2) is no longer present in the case of the sharp interface energy.

The energy in (2-10) may be rewritten with the help of the Green's function as

$$E_\varepsilon(u) := \frac{\varepsilon\sigma}{2} \int_{\mathbb{T}_\ell} |\nabla u|\, dx + \frac{1}{2} \int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} (u(x) - \bar{u}_\varepsilon) G(x - y)(u(y) - \bar{u}_\varepsilon)\, dx\, dy, \tag{2-13}$$

where G solves

$$-\Delta G(x) + \kappa^2 G(x) = \delta(x) \quad \text{in } \mathcal{D}'(\mathbb{T}_\ell). \tag{2-14}$$

Notice that G has an explicit representation

$$G(x) = \frac{1}{4\pi} \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \frac{e^{-\kappa|x - \boldsymbol{n}\ell|}}{|x - \boldsymbol{n}\ell|}. \tag{2-15}$$

In particular, we have

$$G(x) \simeq \frac{1}{4\pi|x|} \quad |x| \ll 1, \qquad G(x) \geq c \quad \text{for all } x \in \mathbb{T}_\ell, \tag{2-16}$$

for some $c > 0$ depending on κ and ℓ. Also, integrating (2-14) we get

$$\int_{\mathbb{T}_\ell} G(x)\, dx = \kappa^{-2}. \tag{2-17}$$

The latter allows us to rewrite the energy E_ε in an equivalent form in terms of $\chi \in BV(\mathbb{T}_\ell; \{0, 1\})$, where

$$\chi(x) := \frac{1+u(x)}{2}, \quad x \in \mathbb{T}_\ell, \tag{2-18}$$

as

$$E_\varepsilon(u) = \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2} + \varepsilon\sigma \int_{\mathbb{T}_\ell} |\nabla\chi|\, dx - \frac{2\varepsilon^{2/3}\lambda}{\kappa^2} \int_{\mathbb{T}_\ell} \chi\, dx + 2\int_{\mathbb{T}_\ell}\int_{\mathbb{T}_\ell} G(x-y)\chi(x)\chi(y)\, dx\, dy, \tag{2-19}$$

where we also used (1-2).

We now introduce a version of the energy E_ε written in terms of the rescaling

$$\tilde\chi(x) := \chi\left(\frac{\ell x}{\ell_\varepsilon}\right), \quad x \in \mathbb{T}_{\ell_\varepsilon}, \quad \ell_\varepsilon := \left(\frac{4}{\sigma\varepsilon}\right)^{1/3}\ell. \tag{2-20}$$

With this definition we have $\tilde\chi \in \widetilde{\mathcal{A}}_{\ell_\varepsilon}$, where

$$\widetilde{\mathcal{A}}_{\ell_\varepsilon} := BV(\mathbb{T}_{\ell_\varepsilon}; \{0, 1\}), \tag{2-21}$$

for every $\chi \in \mathcal{A}$, and $E_\varepsilon(\chi) = \widetilde{E}_{\ell_\varepsilon}(\tilde\chi)$, with

$$\widetilde{E}_{\ell_\varepsilon}(\tilde\chi) := \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2} - \frac{\varepsilon^{5/3}\sigma\lambda}{2\kappa^2}\int_{\mathbb{T}_{\ell_\varepsilon}} \tilde\chi\, dx + \left(\frac{\varepsilon^{5/3}\sigma^{5/3}}{4^{2/3}}\right)\left[\int_{\mathbb{T}_{\ell_\varepsilon}} |\nabla\tilde\chi|\, dx + \frac{1}{2}\int_{\mathbb{T}_{\ell_\varepsilon}} \tilde\chi(-\Delta + 4^{-2/3}\varepsilon^{2/3}\sigma^{2/3}\kappa^2)^{-1}\tilde\chi\, dx\right]. \tag{2-22}$$

Introducing G_ε, which solves

$$-\Delta G_\varepsilon(x) + 4^{-2/3}\kappa^2\varepsilon^{2/3}\sigma^{2/3} G_\varepsilon(x) = \delta(x) \quad \text{in } \mathbb{T}_{\ell_\varepsilon}, \tag{2-23}$$

we can then express the energy $\widetilde{E}_{\ell_\varepsilon}$ as

$$\widetilde{E}_{\ell_\varepsilon}(\tilde\chi) := \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2} - \frac{\varepsilon^{5/3}\sigma\lambda}{2\kappa^2}\int_{\mathbb{T}_{\ell_\varepsilon}} \tilde\chi\, dx + \left(\frac{\varepsilon^{5/3}\sigma^{5/3}}{4^{2/3}}\right)\left[\int_{\mathbb{T}_{\ell_\varepsilon}} |\nabla\tilde\chi|\, dx + \frac{1}{2}\int_{\mathbb{T}_{\ell_\varepsilon}} G_\varepsilon(x-y)\tilde\chi(x)\tilde\chi(y)\, dx\, dy\right]. \tag{2-24}$$

Note that, as in (2-15), we have the following representation for G_ε:

$$G_\varepsilon(x) = \frac{1}{4\pi}\sum_{\boldsymbol{n}\in\mathbb{Z}^3} \frac{e^{-4^{-1/3}\varepsilon^{1/3}\sigma^{1/3}\kappa|x-\boldsymbol{n}\ell_\varepsilon|}}{|x-\boldsymbol{n}\ell_\varepsilon|}. \tag{2-25}$$

The whole space energy. As was shown by us in [Knüpfer et al. 2016], in the absence of screening, i.e., with $\kappa = 0$ and $u \in \mathcal{A}$ also satisfying (1-2), the asymptotic behavior of the minimizers of E_ε in (2-10) with $\bar u_\varepsilon$ satisfying (1-3) can be expressed in terms of those for the energy defined on the whole of $\mathbb{R}^3$:

$$\widetilde{E}_\infty(\tilde\chi) := \int_{\mathbb{R}^3} |\nabla\tilde\chi|\, dx + \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\tilde\chi(x)\tilde\chi(y)}{|x-y|}\, dx\, dy, \tag{2-26}$$

which is well-defined in the admissible class

$$\widetilde{\mathcal{A}}_\infty := \mathrm{BV}(\mathbb{R}^3; \{0, 1\}). \tag{2-27}$$

In particular, the optimal self-energy per unit volume of the minority phase is

$$f^* := \inf_{\tilde{\chi} \in \widetilde{\mathcal{A}}_\infty} \frac{\widetilde{E}_\infty(\tilde{\chi})}{\int_{\mathbb{R}^3} \tilde{\chi}\, dx}. \tag{2-28}$$

Note that within the nuclear physics context, this is precisely the dimensionless form of the celebrated Gamow's liquid drop model of the atomic nucleus [Gamow 1930] (for a recent mathematical overview, see [Choksi et al. 2017]). In particular, the value of f^* corresponds to the energy per nucleon in the tightest bound nucleus.

The relationship between E_ε and $\widetilde{E}_\infty$ can be seen formally by passing to the limit $\varepsilon \to 0$ in (2-24) with $\tilde{\chi}$ taken to be the characteristic function of a fixed bounded set restricted to $\mathbb{T}_{\ell_\varepsilon}$. Then we have

$$\left(\frac{4^{2/3}}{\varepsilon^{5/3}\sigma^{5/3}}\right)\left(\widetilde{E}_{\ell_\varepsilon}(\tilde{\chi}) - \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2}\right) \to -\frac{\lambda f^*}{\lambda_c}\int_{\mathbb{R}^3} \tilde{\chi}\, dx + \widetilde{E}_\infty(\tilde{\chi}), \tag{2-29}$$

where $\tilde{\chi}$ was extended by zero to the whole of $\mathbb{R}^3$ and λ_c is defined in (1-5).

The following result was recently established about minimizers of the problem in the whole space [Knüpfer et al. 2016; Frank et al. 2016].

Theorem 2.1. *There exists a bounded, connected open set $F^* \subset \mathbb{R}^3$ with smooth boundary such that*

$$f^* = \frac{\widetilde{E}_\infty(\tilde{\chi}_{F^*})}{|F^*|}, \tag{2-30}$$

where χ_{F^*} is the characteristic function of the set F^*.

It has been conjectured that the minimizer of $\widetilde{E}_\infty$ with fixed mass is given by a ball whenever such a minimizer exists [Choksi and Peletier 2011]. Therefore, taking a ball of radius R as a test function in (2-28) and optimizing in R, one obtains an estimate

$$f^* \le 3^{5/3} \cdot 2^{-2/3} \cdot 5^{-1/3}. \tag{2-31}$$

The conjecture above would imply that the inequality in (2-31) is in fact an equality. Proving such a result is a difficult hard analysis problem that currently appears to be out of reach. Nevertheless, we can establish a first quantitative lower bound for the value of f^*, using equipartition of energy of F^* established in [Frank and Lieb 2015] and a quantitative upper bound on $|F^*|$ obtained in [Frank et al. 2016]. Note that the resulting lower bound equals about 67% of the upper bound in (2-31). This is one of the main results of the present paper.

Theorem 2.2. *We have*

$$f^* \ge \frac{3^{5/3}}{4}. \tag{2-32}$$

Proof. Let F^* be a minimizer from Theorem 2.1, and write

$$f^* = \frac{P(F^*) + V(F^*)}{|F^*|}, \tag{2-33}$$

where $P(F^*)$ is the perimeter of F^* and $V(F^*)$ is the Coulombic self-energy of F^*. By the result from [Frank and Lieb 2015], the energy exhibits equipartition of energy in the sense that

$$V(F^*) = \tfrac{1}{2} P(F^*), \tag{2-34}$$

which can be easily seen by considering the sets λF^* as competitors for f^* and taking advantage of the homogeneity of P and V with respect to dilations. Thus, we have

$$f^* = \frac{3P(F^*)}{2|F^*|}. \tag{2-35}$$

Therefore, applying the isoperimetric inequality yields

$$f^* \geq \left(\frac{243\pi}{2|F^*|} \right)^{1/3}. \tag{2-36}$$

The proof is then concluded by recalling the quantitative upper bound $|F^*| \leq 32\pi$ from [Frank et al. 2016]. □

The limit energy. For $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$, define

$$E_0(\mu) := \frac{\lambda^2 \ell^3}{2\kappa^2} - \frac{2}{\kappa^2}(\lambda - \lambda_c) \int_{\mathbb{T}_\ell} d\mu + 2 \int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} G(x-y)\, d\mu(x)\, d\mu(y). \tag{2-37}$$

Note that $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$ implies that μ is a nonnegative Radon measure with bounded Coulombic energy:

$$\int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} G(x-y)\, d\mu(x)\, d\mu(y) < \infty, \tag{2-38}$$

where G is the screened Coulombic kernel from (2-15). The converse is also true; i.e., a positive Radon measure with bounded Coulombic energy defines a bounded linear functional on $H^1(\mathbb{T}_\ell)$. This fact is a consequence of the following lemma, whose proof is a straightforward adaptation of the proof of [Goldman et al. 2013, Lemma 3.2] in two dimensions (see [Knüpfer et al. 2016, Appendix A]). In particular, it allows us to extend the definition of E_0 to arbitrary positive Radon measures on $\mathbb{T}_\ell$, with $E_0(\mu) < +\infty$ if and only if $\mu \in H^{-1}(\mathbb{T}_\ell)$.

Lemma 2.3. *Let $\mu \in \mathcal{M}^+(\mathbb{T}_\ell)$ and let (2-38) hold. Then*:

(i) $\mu \in H^{-1}(\mathbb{T}_\ell)$, *in the sense that it can be extended to a bounded linear functional over $H^1(\mathbb{T}_\ell)$.*

(ii) *If*

$$v(x) := \int_{\mathbb{T}_\ell} G(x-y)\, d\mu(y), \tag{2-39}$$

then $v \in H^1(\mathbb{T}_\ell)$*. Furthermore,* v *solves*

$$-\Delta v + \kappa^2 v = \mu \tag{2-40}$$

weakly in $H^1(\mathbb{T}_\ell)$*, and*

$$\nabla v(x) = \int_{\mathbb{T}_\ell} \nabla G(x-y)\, d\mu(y), \tag{2-41}$$

in the sense of distributions.

(ii) *If* v *is as in* (ii)*, we have* $\kappa^2 \int_{\mathbb{T}_\ell} v\, dx = \int_{\mathbb{T}_\ell} d\mu$ *and*

$$\int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} G(x-y)\, d\mu(x)\, d\mu(y) = \int_{\mathbb{T}_\ell} (|\nabla v|^2 + \kappa^2 v^2)\, dx. \tag{2-42}$$

According to Lemma 2.3, the energy E_0 may be equivalently rewritten in terms of the associated potential v in (2-39) as

$$E_0(\mu) = \frac{\lambda^2 \ell^3}{2\kappa^2} - 2(\lambda - \lambda_c) \int_{\mathbb{T}_\ell} v\, dx + 2 \int_{\mathbb{T}_\ell} (|\nabla v|^2 + \kappa^2 v^2)\, dx, \tag{2-43}$$

and minimizing $E_0(\mu)$ over $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$ is the same as minimizing the right-hand side of (2-43) with respect to all $v \in H^1(\mathbb{T}_\ell)$ such that $v \geq 0$ in $\mathbb{T}_\ell$ and $-\Delta v + \kappa^2 v \in \mathcal{M}^+(\mathbb{T}_\ell)$. By inspection, the latter is minimized by $v = \bar{v}$, where

$$\bar{v} = \begin{cases} 0, & \lambda \leq \lambda_c, \\ (1/(2\kappa^2))(\lambda - \lambda_c), & \lambda > \lambda_c. \end{cases} \tag{2-44}$$

In terms of the measures, we can state this result as follows:

Proposition 2.4. *The energy* $E_0(\mu)$ *is minimized by a unique measure* $\bar{\mu}$ *among all* $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$*, with* $\bar{\mu} = 0$ *for all* $\lambda \leq \lambda_c$*, and* $d\bar{\mu} = \frac{1}{2}(\lambda - \lambda_c)dx$ *for* $\lambda > \lambda_c$*, respectively. Moreover, we have*

$$E_0(\bar{\mu}) = \begin{cases} \lambda^2 \ell^3/(2\kappa^2), & \lambda \leq \lambda_c, \\ \lambda_c(2\lambda - \lambda_c)\ell^3/(2\kappa^2), & \lambda > \lambda_c, \end{cases} \tag{2-45}$$

and (2-40) *is solved by* $v(x) = \bar{v}$.

3. Sharp interface energy E_ε

We now consider the sharp-interface functional E_ε defined in (2-10) in the limit $\varepsilon \to 0$ with $\bar{u}_\varepsilon$ given by (1-3) and positive σ, λ, κ, ℓ fixed. For a given sequence $(u_\varepsilon) \in \mathcal{A}$, we introduce a measure μ_ε that is continuous with respect to the Lebesgue measure on $\mathbb{T}_\ell$ and whose density is an appropriately rescaled characteristic function of the minority phase:

$$d\mu_\varepsilon(x) := \tfrac{1}{2}\varepsilon^{-2/3}(1 + u_\varepsilon(x))dx. \tag{3-1}$$

Note that by definition the measure μ_ε is nonnegative. We also introduce the potential v_ε via

$$-\Delta v_\varepsilon + \kappa^2 v_\varepsilon = \mu_\varepsilon \quad \text{in } \mathbb{T}_\ell. \tag{3-2}$$

Our first result establishes compactness of sequences with bounded energy after a suitable rescaling.

Theorem 3.1 (equicoercivity). *Let $(u_\varepsilon) \in \mathcal{A}$ be such that*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-4/3} E_\varepsilon(u_\varepsilon) < +\infty, \tag{3-3}$$

and let μ_ε and v_ε be defined in (3-1) and (3-2), respectively. Then, up to extraction of a subsequence, we have

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{T}_\ell), \qquad v_\varepsilon \rightharpoonup v \quad \text{in } H^1(\mathbb{T}_\ell), \tag{3-4}$$

as $\varepsilon \to 0$, for some $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$ and $v \in H^1(\mathbb{T}_\ell)$ satisfying

$$-\Delta v + \kappa^2 v = \mu \quad \text{in } \mathbb{T}_\ell. \tag{3-5}$$

Proof. Inserting (3-1) into (2-10) and dropping the perimeter term, following the argument of [Muratov 2010] we arrive at (see also (2-19))

$$E_\varepsilon(u_\varepsilon) \geq \frac{\lambda^2\ell^3}{2\kappa^2} - \frac{2\lambda}{\kappa^2}\int_{\mathbb{T}_\ell} d\mu_\varepsilon(x) + 2\int_{\mathbb{T}_\ell}\int_{\mathbb{T}_\ell} G(x-y)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y), \tag{3-6}$$

where we used (2-17) and (1-3) and took into account the translational invariance of the problem in $\mathbb{T}_\ell$. By (2-16) we get

$$E_\varepsilon(u_\varepsilon) \geq -\frac{2\lambda}{\kappa^2}\mu_\varepsilon(\mathbb{T}_\ell) + 2c\mu_\varepsilon^2(\mathbb{T}_\ell), \tag{3-7}$$

where we again recall that μ_ε is nonnegative by definition. It then follows that

$$\mu_\varepsilon(\mathbb{T}_\ell) < C \tag{3-8}$$

for some constant $C > 0$ independent of ε, which implies that $\mu_\varepsilon \rightharpoonup \mu$ up to a subsequence by the Banach–Alaoglu theorem (see [Brezis 2011, Theorem 3.16]). The considerations above together with Lemma 2.3(iii) and (3-3) show that

$$\int_{\mathbb{T}_\ell} (|\nabla v_\varepsilon|^2 + \kappa^2 v_\varepsilon^2)\, dx = \int_{\mathbb{T}_\ell}\int_{\mathbb{T}_\ell} G(x-y)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y) < C, \tag{3-9}$$

and upon extraction of a further subsequence by the Banach–Alaoglu theorem we get $v_\varepsilon \rightharpoonup v$ in $H^1(\mathbb{T}_\ell)$. Finally, (3-5) follows by passing to the limit in (3-2). □

We now proceed to the main result of this section which establishes the Γ-limit of the screened sharp interface energy, similar to its two-dimensional analog in [Goldman et al. 2013, Theorem 1] (see also [Knüpfer et al. 2016, Theorem 3.3] for the unscreened case).

Theorem 3.2 (Γ-convergence of E_ε). *As $\varepsilon \to 0$ we have*

$$\varepsilon^{-4/3} E_\varepsilon \xrightarrow{\Gamma} E_0, \tag{3-10}$$

with respect to the weak convergence of measures. More precisely, we have:

(i) *Lower bound*: *Suppose that* $(u_\varepsilon) \in \mathcal{A}$ *and let* μ_ε *be defined as in* (3-1), *and suppose that*

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{T}_\ell), \tag{3-11}$$

as $\varepsilon \to 0$, *for some* $\mu \in \mathcal{M}_\ell{}^+(\mathbb{T}_\ell)$. *Then*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-4/3} E_\varepsilon(u_\varepsilon) \geq E_0(\mu). \tag{3-12}$$

(ii) *Upper bound*: *Given* $\mu \in \mathcal{M}^+(\mathbb{T}_\ell)$, *there exists* $(u_\varepsilon) \in \mathcal{A}$ *such that for the corresponding* μ_ε *as in* (3-1) *we have*

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{T}_\ell), \tag{3-13}$$

as $\varepsilon \to 0$, *and*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-4/3} E_\varepsilon(u_\varepsilon) \leq E_0(\mu). \tag{3-14}$$

Proof. Assume first that $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$, so that $E_0(\mu) < +\infty$. As in the proof of Propositions 5.1 and 5.2 in [Knüpfer et al. 2016], we separate the contributions of the near-field and far-field interaction; i.e., for $0 < \rho \leq \frac{1}{4}$ we write

$$G_\rho(x) = \eta_\rho(x) G(x), \qquad H_\rho(x) := G(x) - G_\rho(x), \tag{3-15}$$

where $\eta_\rho(x)$ is a smooth cutoff function depending on $|x|$ which is monotonically increasing from 0 to 1 as $|x|$ goes from 0 to ρ, with $\eta_\rho(x) = 0$ for all $|x| < \frac{1}{2}\rho$ and $\eta_\rho(x) = 1$ for all $|x| > \rho$. With the help of (2-19), for any $u_\varepsilon \in \mathcal{A}$ we decompose the energy as $E_\varepsilon = E_\varepsilon^{(1)} + E_\varepsilon^{(2)}$, where

$$\begin{aligned} \varepsilon^{-4/3} E_\varepsilon^{(1)}(u_\varepsilon) &= \frac{\lambda^2 \ell^3}{2\kappa^2} - \frac{2\lambda}{\kappa^2} \int_{\mathbb{T}_\ell} d\mu_\varepsilon(x) + 2 \int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} G_\rho(x-y)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y), \\ \varepsilon^{-4/3} E_\varepsilon^{(2)}(u_\varepsilon) &= \varepsilon^{-1/3} \sigma \int_{\mathbb{T}_\ell} |\nabla \chi_\varepsilon|\, dx + 2\varepsilon^{-4/3} \int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} H_\rho(x-y) \chi_\varepsilon(x) \chi_\varepsilon(y)\, dx\, dy, \end{aligned} \tag{3-16}$$

where χ_ε is as in (2-18) with u replaced with u_ε. The term $E_\varepsilon^{(1)}$ is continuous with respect to the weak convergence of measures; hence

$$\int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} G_\rho(x-y)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y) \to \int_{\mathbb{T}_\ell} \int_{\mathbb{T}_\ell} G_\rho(x-y)\, d\mu(x)\, d\mu(y) \quad \text{as } \varepsilon \to 0. \tag{3-17}$$

The proof of the lower bound for $E_\varepsilon^{(2)}$ follows with similar arguments as in the proof of [Knüpfer et al. 2016, Theorem 3.3]. After the rescaling in (2-20), one can write

$$\varepsilon^{-4/3} E_\varepsilon^{(2)}(u_\varepsilon) = \left(\frac{\varepsilon^{1/3} \sigma^{5/3}}{4^{2/3}} \right) \left[\int_{\mathbb{T}_{\ell_\varepsilon}} |\nabla \tilde{\chi}_\varepsilon|\, dx + \frac{1}{2} \int_{\mathbb{T}_{\ell_\varepsilon}} \int_{\mathbb{T}_{\ell_\varepsilon}} \widetilde{H}_\rho^\varepsilon(x-y) \tilde{\chi}_\varepsilon(x) \tilde{\chi}_\varepsilon(y)\, dx\, dy \right], \tag{3-18}$$

where $\tilde{\chi}_\varepsilon(x) := \chi_\varepsilon(x\ell/\ell_\varepsilon)$ and

$$\widetilde{H}_\rho^\varepsilon(x) := (1 - \eta_\rho(x\ell/\ell_\varepsilon)) G_\varepsilon(x). \tag{3-19}$$

Observe that by (2-25) and monotonicity of $\eta_\rho(x)$ in $|x|$ we have

$$\widetilde{H}_\rho^\varepsilon(x) \geq (1-\rho) \Gamma_{\rho_0}^{\#}(x),$$

where $\Gamma^{\#}_{\rho_0}(x) := (1 - \eta_{\rho_0}(x))\Gamma^{\#}(x)$ and $\Gamma^{\#}(x) := 1/(4\pi|x|)$ is the restriction of the Newton potential on the torus, for any $\rho_0 > 0$ and all ε small enough depending only on κ, σ and ρ_0. The rest of the proof of the lower bound, as well as the proof of the upper bound, follows exactly as in [Knüpfer et al. 2016].

Finally, if $\mu \notin H^{-1}(\mathbb{T}_\ell)$, then $E_0(\mu) = +\infty$ and the upper bound is trivial, while the lower bound follows via a contradiction argument from the compactness result established in Theorem 3.1. □

As a direct consequence of Theorems 3.1 and 3.2, we have the following characterization of the minimizers of the sharp interface energy in the limit $\varepsilon \to 0$.

Corollary 3.3. *Let $(u_\varepsilon) \in \mathcal{A}$ be minimizers of E_ε. Let μ_ε be defined in (3-1) and let v_ε be the solution of (3-2). Then as $\varepsilon \to 0$, we have*

$$\mu_\varepsilon \rightharpoonup \bar{\mu} \quad \text{in } \mathcal{M}(\mathbb{T}_\ell), \qquad v_\varepsilon \rightharpoonup \bar{v} \quad \text{in } H^1(\mathbb{T}_\ell), \tag{3-20}$$

where $\bar{\mu}$ and $\bar{v}$ are as in Proposition 2.4.

We note that for $\lambda \gg \lambda_c$ the minimum energy per unit volume for minimizers in Corollary 3.3 approaches asymptotically that of the unscreened sharp interface energy studied in [Knüpfer et al. 2016], indicating that the presence of an additional screening does not affect the limit behavior of the energy at higher densities than those appearing in (1-3). We would thus expect that the same result would still hold for the sharp interface energy even for $1 + \bar{u}_\varepsilon = o(1)$ as $\varepsilon \to 0$, consistently with a recent result for the sharp interface energy without screening [Emmert et al. 2018].

We conclude by proving an analog of [Knüpfer et al. 2016, Theorem 3.6] that provides uniform bounds on the diameter of the connected components of minimizers of E_ε as $\varepsilon \to 0$, and convergence of most of the connected components to minimizers of Gamow's model per unit mass.

Theorem 3.4 (minimizers: droplet structure). *For $\lambda > 0$, let $(u_\varepsilon) \in \mathcal{A}$ be regular representatives of minimizers of E_ε, and assume that the sets $\{u_\varepsilon = +1\}$ are nonempty for ε sufficiently small. Let N_ε be the number of connected components of the set $\{u_\varepsilon = +1\}$, let $\chi_{\varepsilon,k} \in \mathrm{BV}(\mathbb{R}^3; \{0, 1\})$ be the characteristic function of the k-th connected component of the support of the periodic extension of $\{u_\varepsilon = +1\}$ to the whole of $\mathbb{R}^3$ modulo translations in $\mathbb{Z}^3$, and let $x_{\varepsilon,k} \in \mathrm{supp}(\chi_{\varepsilon,k})$. Then there exists $\varepsilon_0 > 0$ such that the following properties hold*:

(i) *There exist constants $C, c > 0$ depending only on σ, κ, λ and ℓ such that for all $\varepsilon \le \varepsilon_0$ we have*

$$0 < v_\varepsilon \le C \quad \text{and} \quad \int_{\mathbb{R}^3} \chi_{\varepsilon,k}\, dx \ge c\varepsilon, \tag{3-21}$$

where v_ε solves (3-2). Moreover we have

$$\mathrm{supp}(\chi_{\varepsilon,k}) \subseteq B_{C\varepsilon^{1/3}}(x_{\varepsilon,k}). \tag{3-22}$$

(ii) *If $\lambda > \lambda_c$, where λ_c is given by (1-5), there exist constants $C, c > 0$ as above such that for all $\varepsilon \le \varepsilon_0$ we have*

$$c(\lambda - \lambda_c)\varepsilon^{-1/3} \le N_\varepsilon \le C(\lambda - \lambda_c)\varepsilon^{-1/3}. \tag{3-23}$$

Moreover, there exists $\widetilde{N}_\varepsilon \le N_\varepsilon$ *with* $\widetilde{N}_\varepsilon / N_\varepsilon \to 1$ *as* $\varepsilon \to 0$ *and a subsequence* $\varepsilon_n \to 0$ *such that for every* $k_n \le \widetilde{N}_{\varepsilon_n}$ *the following holds: after possibly relabeling the connected components, we have*

$$\tilde{\chi}_n \to \tilde{\chi} \quad \text{in } L^1(\mathbb{R}^3), \tag{3-24}$$

where $\tilde{\chi}_n(x) := \chi_{\varepsilon_n, k_n}(\varepsilon_n^{1/3}(x + x_{\varepsilon_n, k_n}))$, *and* $\tilde{\chi} \in \widetilde{\mathcal{A}}_\infty$ *is a minimizer of the right-hand side of (2-28).*

Proof. The proof can be obtained as in [Knüpfer et al. 2016, Theorem 3.6], with some simplifications due to the absence of a volume constraint. We outline the necessary modifications below. As stated above, the constants in the estimates below depend on σ, κ, λ and ℓ, and may change from line to line.

For $\tilde{u}_\varepsilon(x) := u_\varepsilon(\ell x / \ell_\varepsilon)$, we define $F \subset \mathbb{T}_{\ell_\varepsilon}$ to be the set $\{\tilde{u}_\varepsilon = +1\}$, which by our assumption is nonempty for ε sufficiently small. Then we can write $v_\varepsilon(x) = (\sigma/4)^{2/3} v_F(\ell x / \ell_\varepsilon)$, where $v_F(x) := \int_{\mathbb{T}_{\ell_\varepsilon}} G_\varepsilon(x - y) \chi_F(y)\, dy$, and G_ε is defined in (2-25). The first step in the proof is to obtain an L^∞-bound on the potential v_F analogous to the one in [Knüpfer et al. 2016, Lemma 6.3]:

$$0 < v_F \le C \varepsilon^{-2/9}. \tag{3-25}$$

Observe that by strict positivity of G_ε we clearly have $v_F > 0$. On the other hand, the upper bound follows exactly as in [Knüpfer et al. 2016, Lemma 6.3], due to the fact that $G_\varepsilon(x) \le C/|x|$ for some $C > 0$, since in view of (2-15) we have

$$G_\varepsilon(x) = \frac{\ell}{\ell_\varepsilon} G\left(\frac{\ell x}{\ell_\varepsilon}\right) = \frac{1}{4\pi} \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \frac{e^{-\kappa \ell |(x/\ell_\varepsilon) - \boldsymbol{n}\ell|}}{|x - \boldsymbol{n}\ell|}. \tag{3-26}$$

Next, we need to estimate the gradient of v_F pointwise in terms v_F itself, as in [Knüpfer et al. 2016, Lemma 6.5], which relies on [Knüpfer et al. 2016, equation (6.15)]. It is easy to see that the latter estimate still holds in the present setting, with the constants depending on κ and ℓ. The proof then follows as in [Knüpfer et al. 2016], with a few simplifications due to positivity of G. Also, since v_F satisfies

$$-\Delta v_F + \left(\tfrac{1}{4}\varepsilon\sigma\right)^{2/3} \kappa^2 v_F = \left(\tfrac{1}{4}\sigma\right)^{2/3} \chi_F \quad \text{in } \mathbb{T}_{\ell_\varepsilon}, \tag{3-27}$$

by the positivity of v_F we have that v_F is subharmonic outside $\bar{F}$. Thus, v_F attains its global maximum in $\mathbb{T}_{\ell_\varepsilon}$ for some $\bar{x} \in \bar{F}$, and the analog of [Knüpfer et al. 2016, equation (6.19)] holds true:

$$v_F(x) \ge \tfrac{3}{4} v_F(\bar{x}) - C \quad \text{for all } x \in B_r(\bar{x}), \tag{3-28}$$

for some $C > 0$ and $r > 0$.

Proceeding as in [Knüpfer et al. 2016, Lemma 6.7 and Proposition 6.2], we establish a lower density estimate for F: given $x_0 \in \bar{F}$ and letting F_0 be the connected component of F containing x_0, we have

$$|F_0 \cap B_r(x_0)| \ge c r^3 \quad \text{for all } r \le C \min(1, \|v_F\|_\infty^{-1}) \le C \varepsilon^{2/9}, \tag{3-29}$$

for some $c, C > 0$, where the last inequality follows from (3-25). The assertion in (3-21) then follows as in [Knüpfer et al. 2016, Theorem 6.9] from (3-25), (3-28) and (3-29). The idea of the proof in [Knüpfer et al. 2016] is to find a suitable competitor F' which is obtained by cutting from F a ball of radius independent of ε, centered at the point where the potential v_F attains its maximum. Compared to [Knüpfer et al. 2016], the proof here is simpler since we don't have a volume constraint, so that we can allow competitors with smaller volume than F. Arguing by contradiction, if the maximum of v_F is large, then necessarily the density of F in the ball has to be small, otherwise the energy of F' would be less than the energy of F. However, this contradicts the density estimate in (3-29). Finally, exactly as in [Knüpfer et al. 2016, Lemma 6.11], the bound on the potential and the density estimate (3-29) also imply the diameter bound

$$\operatorname{diam}(F_0) \le C \tag{3-30}$$

for some constant $C > 0$, which gives (3-22). This concludes the proof of part (i).

The proof of part (ii) follows as in the proof of [Knüpfer et al. 2016, Theorem 3.6], with the exception that the estimate on N_ε in (3-23) now follows from (3-21) and the fact that, recalling Corollary 3.3,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{T}_\ell} d\mu_\varepsilon = \bar{\mu}(\mathbb{T}_\ell) = \tfrac{1}{2}(\lambda - \lambda_c), \tag{3-31}$$

where $\bar{\mu}$ is as in Proposition 2.4. □

The results obtained in Theorem 3.4 allow us to establish a sharp transition from trivial to nontrivial minimizers at the level of the sharp interface energy near $\lambda = \lambda_c$ for all $\varepsilon \ll 1$.

Corollary 3.5. *There exists $\varepsilon_0 = \varepsilon_0(\sigma, \kappa, \lambda, \ell) > 0$ such that if λ_c is given by (1-5), then*:

(i) *For any $\lambda < \lambda_c$ and $\varepsilon < \varepsilon_0$ we have that $u = -1$ is the unique minimizer of E_ε in $\mathcal{A}$.*

(ii) *For $\lambda > \lambda_c$ and $\varepsilon < \varepsilon_0$ we have that $u = -1$ is not a minimizer of E_ε in $\mathcal{A}$.*

Proof. Since the statement in (ii) follows immediately from Corollary 3.3, we only need to demonstrate (i). The strategy is analogous to the one used in the proof of [Muratov 2010, Proposition 3.2]. For $\lambda < \lambda_c$, let u_ε be a minimizer of E_ε over $\mathcal{A}$, and assume, by contradiction, that $u_\varepsilon \neq -1$ for a sequence of $\varepsilon \to 0$. Let $\chi_{\varepsilon,k}$ be as in Theorem 3.4. By (2-15) we have

$$E_\varepsilon(u_\varepsilon) \ge \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2} + \sum_{k=1}^{N_\varepsilon}\biggl(\varepsilon\sigma \int_{\mathbb{R}^3} |\nabla \chi_{\varepsilon,k}|\,dx - \frac{2\varepsilon^{2/3}\lambda}{\kappa^2}\int_{\mathbb{R}^3} \chi_{\varepsilon,k}\,dx \\ + \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|}\chi_{\varepsilon,k}(x)\chi_{\varepsilon,k}(y)\,dx\,dy\biggr). \tag{3-32}$$

At the same time, since by Theorem 3.4 the diameter of the support of $\chi_{\varepsilon,k}$ is bounded above by $C\varepsilon^{1/3}$, for every $\delta > 0$ we have $e^{-\kappa|x-y|} \ge 1-\delta$ for all ε sufficiently small and all $x, y \in \operatorname{supp}(\chi_{\varepsilon,k})$. Introducing $\tilde{\chi}_{\varepsilon,k}(x) := \chi_{\varepsilon,k}(\ell_\varepsilon x/\ell)$ as in (2-20), we can then write

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &\ge \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2} - \frac{\varepsilon^{5/3}\sigma\lambda}{2\kappa^2}\sum_{k=1}^{N_\varepsilon}\int_{\mathbb{R}^3}\tilde{\chi}_{\varepsilon,k}\,dx \\ &\quad + \frac{\varepsilon^{5/3}\sigma^{5/3}}{4^{2/3}}\sum_{k=1}^{N_\varepsilon}\biggl(\int_{\mathbb{R}^3}|\nabla\tilde{\chi}_{\varepsilon,k}|\,dx + \frac{1-\delta}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\tilde{\chi}_{\varepsilon,k}(x)\tilde{\chi}_{\varepsilon,k}(y)}{|x-y|}\,dx\,dy\biggr) \qquad (3\text{-}33) \\ &\ge \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2} - \frac{\varepsilon^{5/3}\sigma\lambda}{2\kappa^2}\sum_{k=1}^{N_\varepsilon}\int_{\mathbb{R}^3}\tilde{\chi}_{\varepsilon,k}\,dx + \frac{\varepsilon^{5/3}\sigma^{5/3}(1-\delta)}{4^{2/3}}\sum_{k=1}^{N_\varepsilon}\widetilde{E}_\infty(\tilde{\chi}_{\varepsilon,k}). \end{aligned}$$

Now we substitute the definitions of f^* and λ_c in (2-28) and (1-5), respectively, into (3-33). This yields

$$E_\varepsilon(u_\varepsilon) \geq \frac{\varepsilon^{4/3}\lambda^2\ell^3}{2\kappa^2} + \frac{\varepsilon^{5/3}\sigma((1-\delta)\lambda_c - \lambda)}{2\kappa^2} \sum_{k=1}^{N_\varepsilon} \int_{\mathbb{R}^3} \tilde{\chi}_{\varepsilon,k}\, dx. \tag{3-34}$$

In particular, for $\lambda < \lambda_c$ one can choose δ small enough so that $E_\varepsilon(u_\varepsilon) > \varepsilon^{4/3}\lambda^2\ell^3/(2\kappa^2) = E_\varepsilon(-1)$ for all ε sufficiently small, contradicting minimality of u_ε. □

4. Diffuse interface energy $\mathcal{E}_\varepsilon$

We now consider the diffuse-interface functional $\mathcal{E}_\varepsilon$ defined in (1-1) in the limit $\varepsilon \to 0$ with, as before, $\bar{u}_\varepsilon$ given by (1-3) and positive σ, λ, κ, ℓ fixed.

Let

$$d\mu_\varepsilon^0(x) := \tfrac{1}{2}\varepsilon^{-2/3}(1 + u_\varepsilon^0(x))\, dx, \tag{4-1}$$

where

$$u_\varepsilon^0(x) := \begin{cases} +1 & \text{if } u_\varepsilon(x) > 0, \\ -1 & \text{if } u_\varepsilon(x) \leq 0, \end{cases} \tag{4-2}$$

and let v_ε^0 satisfy

$$-\Delta v_\varepsilon^0 + \kappa^2 v_\varepsilon^0 = \mu_\varepsilon^0 \quad \text{in } \mathbb{T}_\ell. \tag{4-3}$$

With this notation, we are now in the position to state the main technical result of this paper.

Theorem 4.1 (equicoercivity and Γ-convergence of $\mathcal{E}_\varepsilon$). *For $\lambda > 0$ and $\ell > 0$, let $\mathcal{E}_\varepsilon$ be defined by (2-1) with W satisfying the assumptions of Section 2, let $\bar{u}_\varepsilon$ given by (1-3), and let σ and κ be given by (2-8) and (2-9), respectively. Then, as $\varepsilon \to 0$ we have*

$$\varepsilon^{-4/3}\mathcal{E}_\varepsilon \xrightarrow{\Gamma} E_0(\mu), \tag{4-4}$$

where $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$. More precisely, we have:

(i) *Compactness and lower bound: Let $(u_\varepsilon) \in \mathcal{A}_\varepsilon$ be such that $\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{T}_\ell)} \leq 1$ and*

$$\limsup_{\varepsilon \to 0} \varepsilon^{-4/3}\mathcal{E}_\varepsilon(u_\varepsilon) < +\infty. \tag{4-5}$$

Then, up to extraction of a subsequence, we have

$$\mu_\varepsilon^0 \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{T}_\ell), \qquad v_\varepsilon^0 \rightharpoonup v \quad \text{in } H^1(\mathbb{T}_\ell), \tag{4-6}$$

as $\varepsilon \to 0$, where $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$ and $v \in H^1(\mathbb{T}_\ell)$ satisfy

$$-\Delta v + \kappa^2 v = \mu \quad \text{in } \mathbb{T}_\ell. \tag{4-7}$$

Moreover, we have

$$\liminf_{\varepsilon \to 0} \varepsilon^{-4/3}\mathcal{E}_\varepsilon(u_\varepsilon) \geq E_0(\mu). \tag{4-8}$$

(ii) *Upper bound*: *Given $\mu \in \mathcal{M}^+(\mathbb{T}_\ell) \cap H^{-1}(\mathbb{T}_\ell)$ and $v \in H^1(\mathbb{T}_\ell)$ solving (4-7), there exist $(u_\varepsilon) \in \mathcal{A}_\varepsilon$ such that for the corresponding μ_ε^0, v_ε^0 as in (4-1) and (4-3) we have*

$$\mu_\varepsilon^0 \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{T}_\ell), \qquad v_\varepsilon^0 \rightharpoonup v \quad \text{in } H^1(\mathbb{T}_\ell), \tag{4-9}$$

as $\varepsilon \to 0$, and

$$\limsup_{\varepsilon \to 0} \varepsilon^{-4/3} \mathcal{E}_\varepsilon(u_\varepsilon) \le E_0(\mu). \tag{4-10}$$

Proof. As in [Muratov 2010], the basic strategy is to relate the minimization problem for $\mathcal{E}_\varepsilon$ to that for E_ε and apply the results in Theorems 3.1 and 3.2. The proof relies on the fact, first observed in [Muratov 2010], that the energy $\mathcal{E}_\varepsilon$ is asymptotically equivalent to E_ε in the following sense: for any $\delta > 0$ and $u_\varepsilon \in \mathcal{A}_\varepsilon$ satisfying some mild technical conditions (see below) there is $\tilde{u}_\varepsilon \in \mathcal{A}$ such that

$$E_\varepsilon[\tilde{u}_\varepsilon] \le (1+\delta)\mathcal{E}_\varepsilon(u_\varepsilon) \tag{4-11}$$

for all $\varepsilon \ll 1$, and, conversely, for any $\tilde{u}_\varepsilon \in \mathcal{A}$, again, satisfying some mild technical conditions, there is $u_\varepsilon \in \mathcal{A}_\varepsilon$ such that

$$\mathcal{E}_\varepsilon[u_\varepsilon] \le (1+\delta) E_\varepsilon(\tilde{u}_\varepsilon) \tag{4-12}$$

for all $\varepsilon \ll 1$. The proof proceeds as in the two-dimensional case [Goldman et al. 2013, Theorem 1], with modifications appropriate to three space dimensions. We outline the key differences below.

For (4-11) to hold, we need to verify the assumptions of [Muratov 2010, Proposition 4.2], which are equivalent to checking that $\|u_\varepsilon\|_\infty \to 1$, $\mathcal{E}_\varepsilon(u_\varepsilon) \to 0$ and $\|v_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$, where $v_\varepsilon(x) := \int_{\mathbb{T}_\ell} G_0(x-y)(u_\varepsilon(y) - \bar{u}_\varepsilon)\, dy$. The first and second conditions are clearly satisfied by the assumptions of the theorem. To check the third condition, we note that the nonlocal part of the energy may be written in terms of v_ε as

$$\frac{1}{2}\int_{\mathbb{T}_\ell}\int_{\mathbb{T}_\ell} (u_\varepsilon(x) - \bar{u}_\varepsilon) G_0(x-y)(u_\varepsilon(y) - \bar{u}_\varepsilon)\, dx\, dy = \frac{1}{2}\int_{\mathbb{T}_\ell} |\nabla v_\varepsilon|^2\, dx. \tag{4-13}$$

Since $\int_{\mathbb{T}_\ell} v_\varepsilon(x)\, dx = 0$ we have by Poincaré's inequality that the right-hand side of (4-13) is bounded below by a multiple of $\|v_\varepsilon\|_2^2$. In turn, the latter is bounded below by a multiple of $\|v_\varepsilon\|_\infty^5$, in view of the fact that, by elliptic regularity [Gilbarg and Trudinger 1983], we have $\|\nabla v_\varepsilon\|_\infty \le C$ for some $C > 0$ depending only on ℓ, for all $\varepsilon \ll 1$. Therefore, from $\mathcal{E}_\varepsilon(u_\varepsilon) \to 0$ we also obtain that $\|v_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$. Thus, by (4-11) $\tilde{u}_\varepsilon$ satisfies the assumptions of Theorem 3.1, and so there exists $\mu \in \mathcal{M}(\mathbb{T}_\ell)$ such that, upon extraction of subsequences, $\mu_\varepsilon \rightharpoonup \mu$ in $\mathcal{M}(\mathbb{T}_\ell)$, where the measure μ_ε is defined by (3-1) with u_ε replaced by $\tilde{u}_\varepsilon$. For those subsequences, Theorem 3.2 holds true for μ_ε as well.

Now, from the construction of $\tilde{u}_\varepsilon$ in the proof of [Muratov 2010, Lemma 4.1] we know that $\tilde{u}_\varepsilon(x) = u_\varepsilon^0(x)$ for all $x \in \mathbb{T}_\ell$ such that $|u_\varepsilon(x)| > 1 - \delta^2$. Hence from the bound on $\mathcal{E}_\varepsilon(u_\varepsilon)$ and the assumptions on W we get that $\|\tilde{u}_\varepsilon - u_\varepsilon^0\|_1 \le C\varepsilon^{4/3}\delta^{-4}$ for some $C > 0$ and all $\varepsilon \ll 1$. This implies that $\mu_\varepsilon^0 \rightharpoonup \mu$ in $\mathcal{M}(\mathbb{T}_\ell)$ as well $\varepsilon \to 0$. Together with the conclusions of Theorems 3.1 and 3.2, this gives the compactness and the lower bound statement of Theorem 4.1, in view of the arbitrariness of δ.

For (4-12) to hold, we need to verify the assumptions of [Muratov 2010, Proposition 4.3] on $\tilde{u}_\varepsilon \in \mathcal{A}$, namely, that the connected components of the support of $\{\tilde{u}_\varepsilon = +1\}$ are smooth and at least ε^α apart for some $\alpha \in [0, 1)$ and have boundaries whose curvature is bounded by $\varepsilon^{-\alpha}$, and that $\|\tilde{v}_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$, where $\tilde{v}_\varepsilon(x) := \int_{\mathbb{T}_\ell} G(x-y)(\tilde{u}_\varepsilon(y) - \bar{u}_\varepsilon)\,dy$. Clearly the first two assumptions hold true for the recovery sequence in the proof of Theorem 3.2 with any $\alpha \in \left(\frac{1}{3}, 1\right)$, provided that $\varepsilon \ll 1$. The third assumption is satisfied for all $\varepsilon \ll 1$, in view of the fact that the nonlocal part of the sharp interface energy can be written as

$$\frac{1}{2}\int_{\mathbb{T}_\ell}\int_{\mathbb{T}_\ell} (\tilde{u}_\varepsilon(x) - \bar{u}_\varepsilon) G_0(x-y)(\tilde{u}_\varepsilon(y) - \bar{u}_\varepsilon)\,dx\,dy = \frac{1}{2}\int_{\mathbb{T}_\ell} (|\nabla \tilde{v}_\varepsilon|^2 + \kappa^2 \tilde{v}_\varepsilon^2)\,dx, \tag{4-14}$$

and the desired estimate follows from $E_\varepsilon(\tilde{u}_\varepsilon) \to 0$ just like in the case of the diffuse interface energy. Thus, the proof of the upper bound is concluded by taking the functions u_ε appearing in (4-12), associated with the recovery sequence $(\tilde{u}_\varepsilon)$ from Theorem 3.2, once again, in view of arbitrariness of δ. □

Similarly to the sharp interface energy, as a direct consequence of Theorem 4.1 we have the following asymptotic characterization of the minimizers of the diffuse interface energy.

Corollary 4.2. *Under the assumptions of Theorem 4.1, let $(u_\varepsilon) \in \mathcal{A}_\varepsilon$ be minimizers of $\mathcal{E}_\varepsilon$. Let μ_ε^0 be defined in (4-1) and v_ε^0 be the solution of (4-3). Then as $\varepsilon \to 0$, we have*

$$\mu_\varepsilon^0 \rightharpoonup \bar{\mu} \quad \text{in } \mathcal{M}(\mathbb{T}_\ell), \qquad v_\varepsilon^0 \rightharpoonup \bar{v} \quad \text{in } H^1(\mathbb{T}_\ell), \tag{4-15}$$

where $\bar{\mu}$ and $\bar{v}$ are as in Proposition 2.4.

We emphasize that the limit behavior of the minimal energy obtained in Corollary 4.2 *differs* from that of the unscreened sharp interface energy one would naively associate with $\mathcal{E}_\varepsilon$. In particular, the minimal energy exhibits a threshold behavior, contrary to that of the minimizers of the unscreened sharp interface energy studied in [Knüpfer et al. 2016; Emmert et al. 2018].

Proof of Theorems 1.1 and 1.2. The statement of Theorem 1.1 is simply the restatement of Theorem 4.1 that does not specify the precise values of the constants appearing there. In turn, the statement of Theorem 1.2 uses the explicit values of σ and κ given by (1-4) for (1-1), together with the bounds on f^* obtained in (2-31) and (2-32). □

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HANS KNÜPFER: hans.knuepfer@math.uni-heidelberg.de
Institut für Angewandte Mathematik, Universität Heidelberg, Heidelberg, Germany

CYRILL B. MURATOV: muratov@njit.edu
Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ, United States

MATTEO NOVAGA: matteo.novaga@unipi.it
Dipartimento di Matematica, Università di Pisa, Pisa, Italy