Skyrmionic bubbles in ultrathin ferromagnetic films

Cyrill B. Muratov

Dipartimento di Matematica
Università di Pisa

in collaboration with M. Piccinini, M. Novaga and V. Slastikov

supported by MUR via PRIN 2022 PNRR project P2022WJW9H

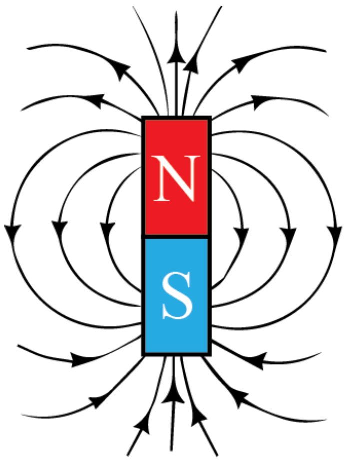
Finanziato dall'Unione europea
NextGenerationEU

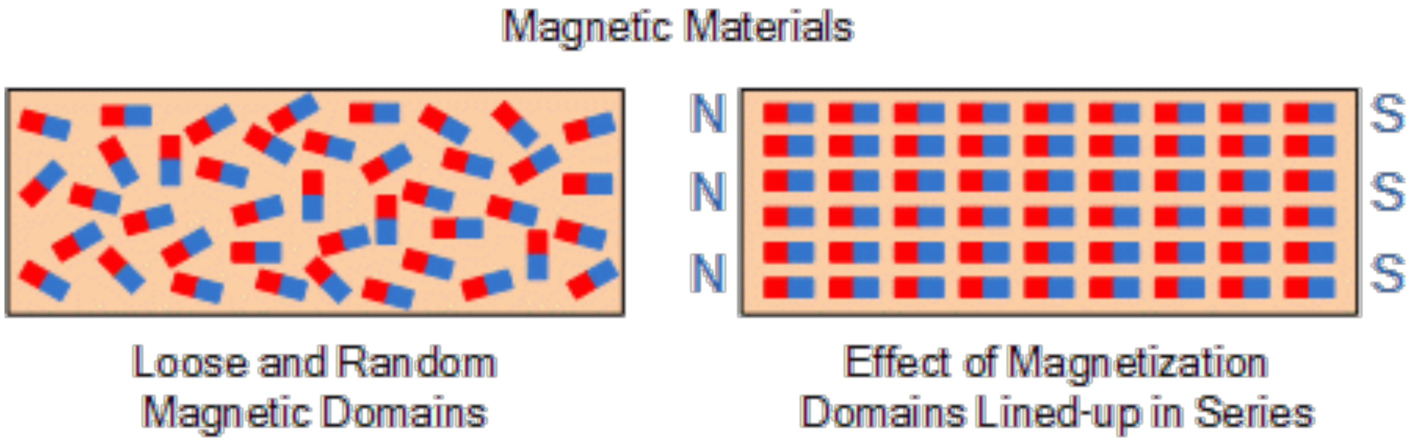
Ministero dell'Università e della Ricerca

Italiadomani
PIANO NAZIONALE DI RIPRESA E RESILIENZA

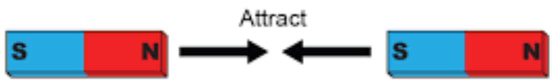
IN SUPREMÆ DIGNITATIS
1343

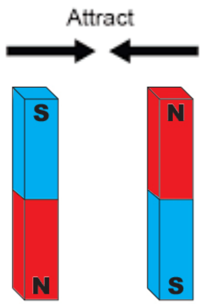
Magnetism and magnets





- spins act as tiny magnetic dipoles
- quantum-mechanical interaction between spins: *exchange*
- in transition metals below the *critical temperature*, exchange results in local spin alignment into the *ferromagnetic state*
- magnetic field mediates long-range attraction/repulsion between magnets

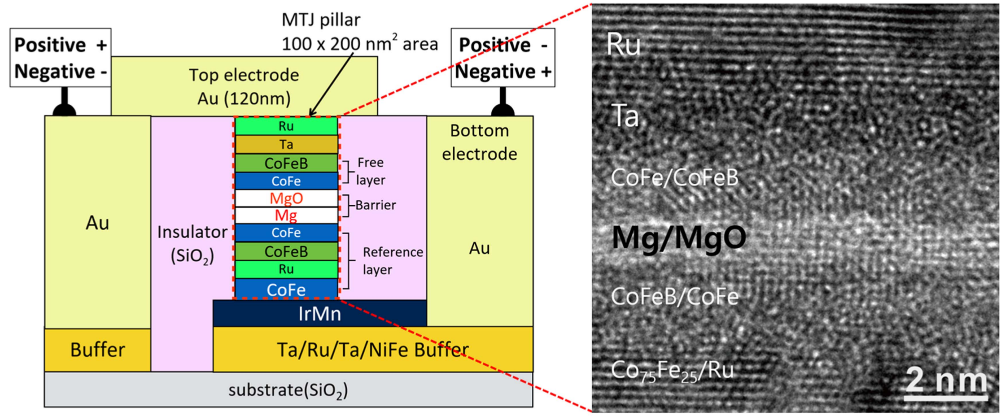




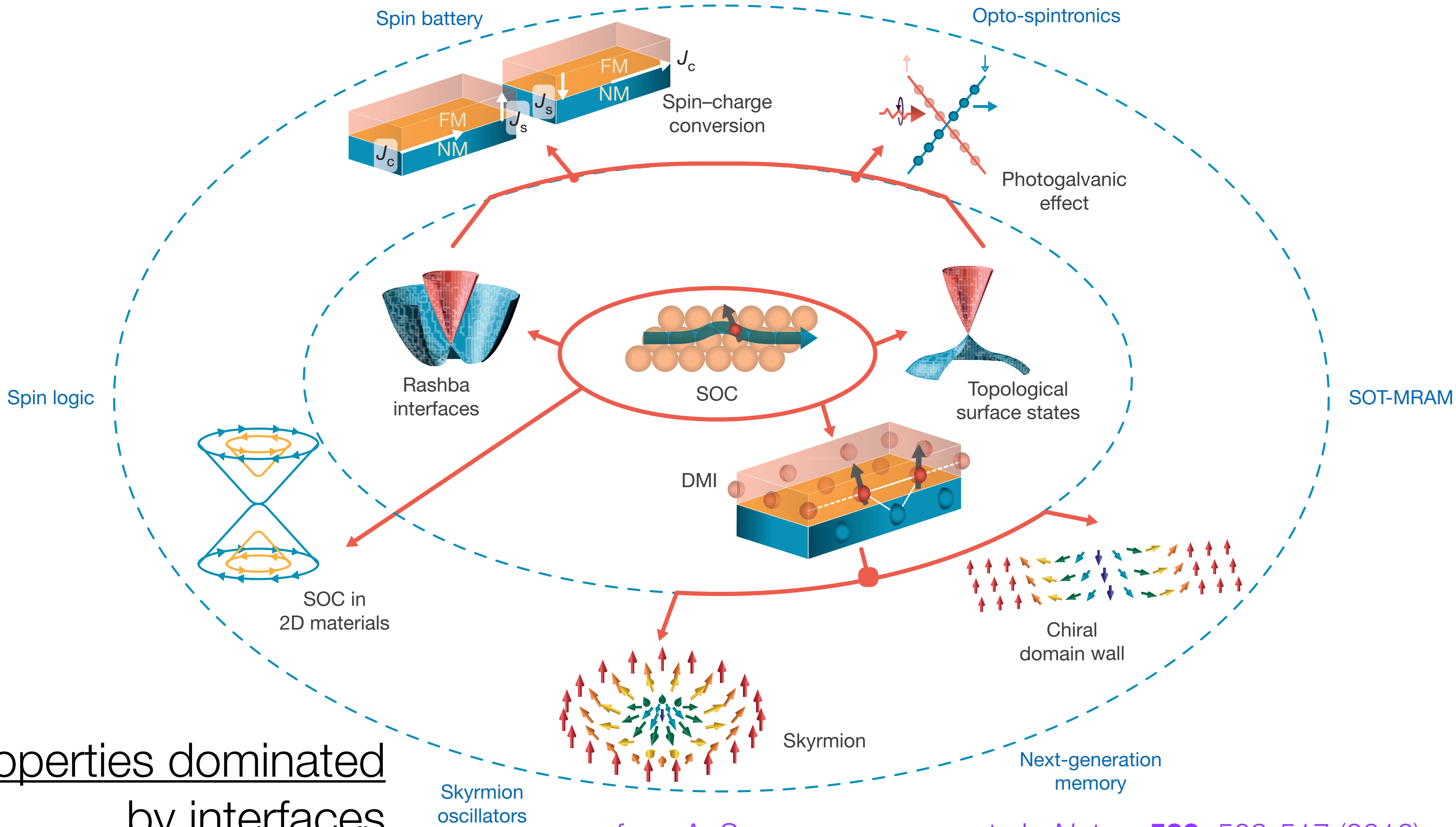
images borrowed from:
Tom Whyntie. (2016), zenodo.com
mammothmemory.net

Magnetic materials for spintronics

atomically thin multilayers with strong spin-orbit coupling (SOC):



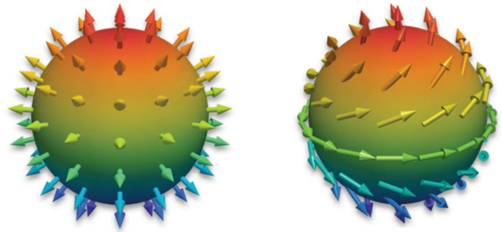
C.-M. Choi et al., *Semicond. Sci. Technol.* 32, 105007 (2017)



properties dominated by interfaces

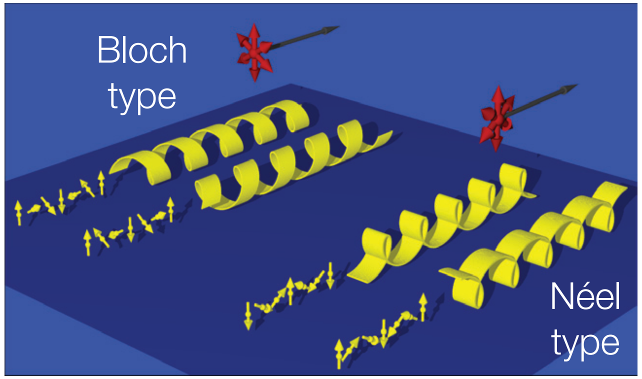
from A. Soumyanarayanan et al., *Nature* **539**, 509-517 (2016)

Topological spin textures

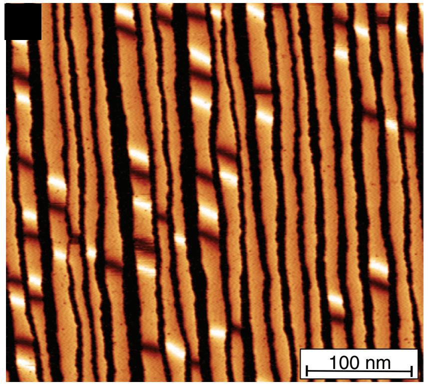


T. Lancaster, *Contemp. Phys.* **60**, 246-261 (2019)

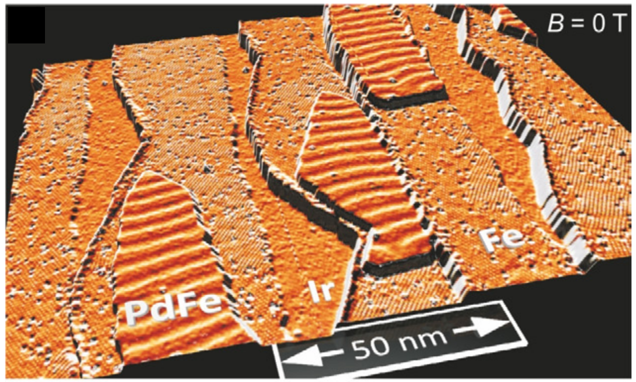
spin spirals and chiral domain walls from **Dzyaloshinskii-Moriya interaction** (DMI):



2ML Fe on W(110)

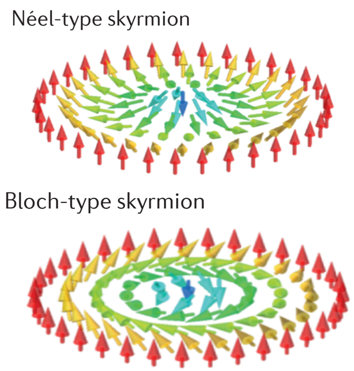


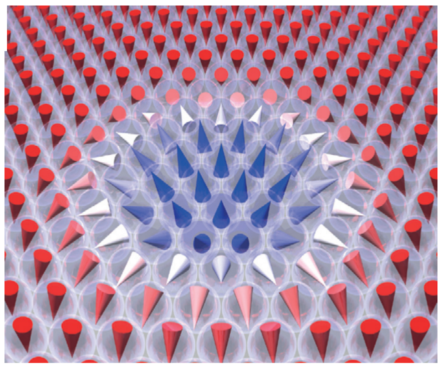
Pd/Fe bilayer on Ir(111)

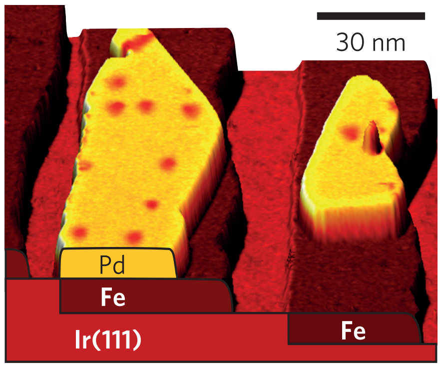


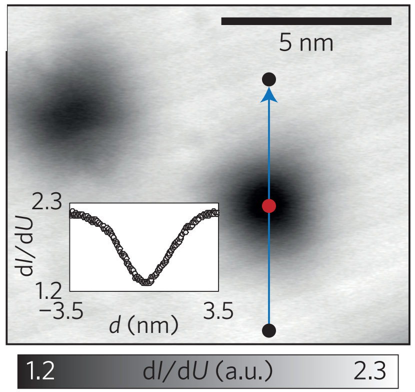
K. von Bergmann et al., J. Phys.: Condens. Matter **26**, 394002 (2014)

magnetic skyrmions:



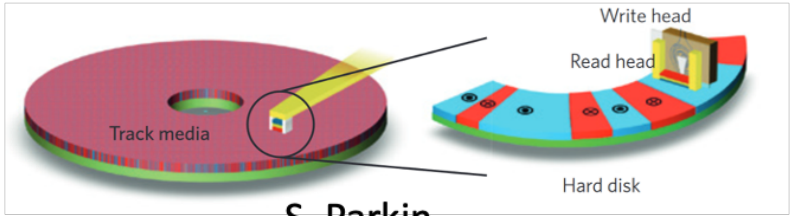




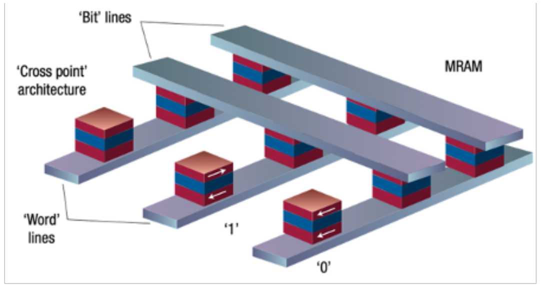


C. Hanneken et al., *Nature Nanotechnol.* **10**, 1039–1042 (2015)

Magnetic memory applications

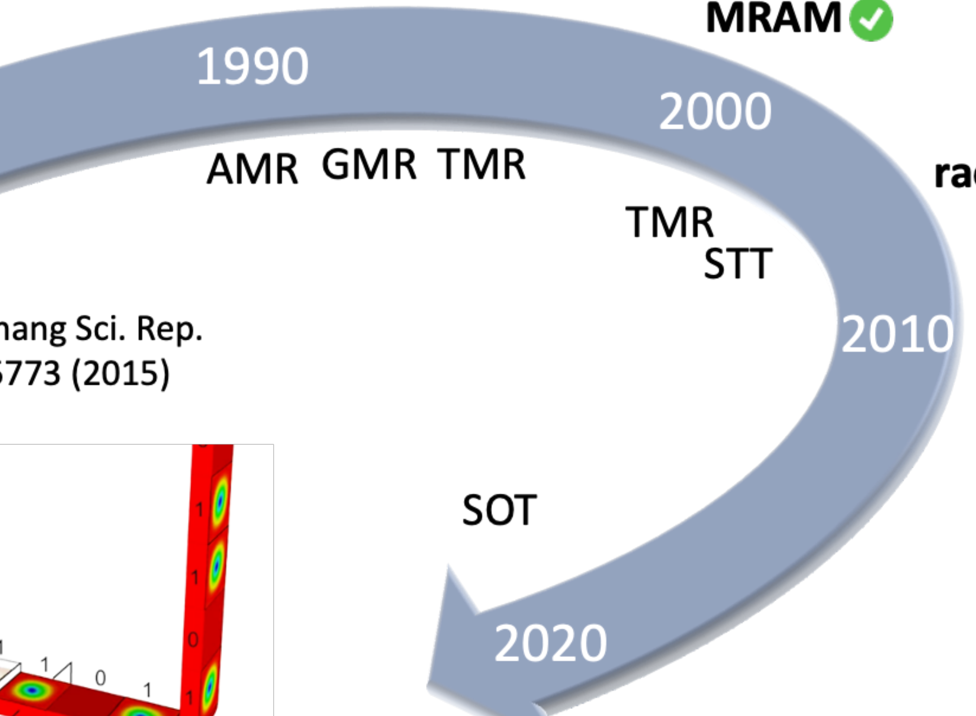


S. Parkin



HDD ✓

MRAM ✓



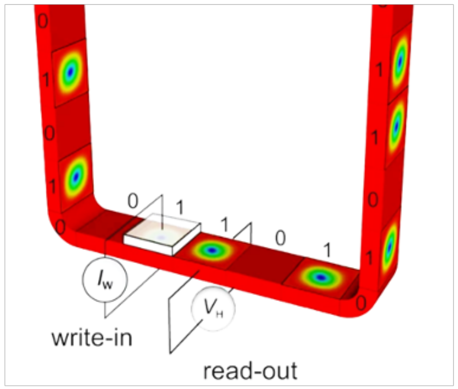
AMR GMR TMR

TMR
STT

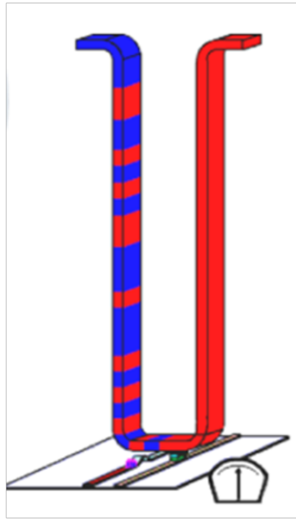
racetrack memory ?

S. Zhang Sci. Rep.
5 15773 (2015)

SOT

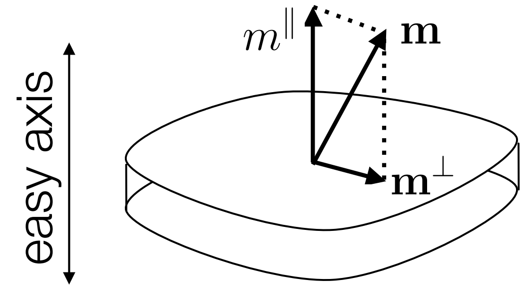


Skyrmion racetrack ?



H. Fangohr

Micromagnetics of ultrathin films



atomically thin extended ferromagnetic films with PMA and DMI

state variable $\mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^2$ — *normalized magnetization per unit area*

micromagnetic energy: $\mathbf{m} = (\mathbf{m}^{\perp}, m^{\|})$ $\quad m^{\|}(\mathbf{r}) \to -1 \text{ as } |\mathbf{r}| \to \infty$

$$\boxed{E(\mathbf{m}) = E_{\mathrm{ex}}(\mathbf{m}) + E_{\mathrm{a}}(\mathbf{m}) + E_{\mathrm{Z}}(\mathbf{m}) + E_{\mathrm{DMI}}(\mathbf{m}) + E_{\mathrm{s}}(\mathbf{m})}$$

where:

$$E_{\mathrm{ex}}(\mathbf{m}) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r, \quad E_{\mathrm{a}}(\mathbf{m}) = Q \int_{\mathbb{R}^2} |\mathbf{m}^{\perp}|^2 d^2 r, \quad E_{\mathrm{Z}}(\mathbf{m}) = -2 \int_{\mathbb{R}^2} h(1 + m^{\|}) d^2 r,$$

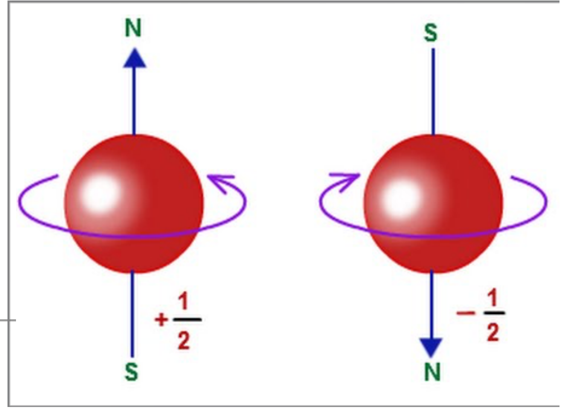
$$E_{\mathrm{DMI}}(\mathbf{m}) = \kappa \int_{\mathbb{R}^2} \left(m^{\|} \nabla \cdot \mathbf{m}^{\perp} - \mathbf{m}^{\perp} \cdot \nabla m^{\|} \right) d^2 r$$

dimensionless parameters:

$$\boxed{Q = \frac{K_{\mathrm{u}}}{K_{\mathrm{d}}}, \qquad \kappa = \frac{D}{\sqrt{A K_{\mathrm{d}}}}, \qquad h = \frac{H}{M_{\mathrm{s}}},}$$

unit of length:

$$\ell = \sqrt{A / K_{\mathrm{d}}}, \qquad K_{\mathrm{d}} = \tfrac{1}{2} \mu_0 M_{\mathrm{s}}^2$$



Stray field energy

electron spins are magnetic dipoles

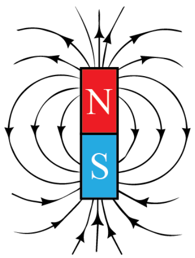
in a thin film the stray field is due to the *bulk* and *surface* magnetic charges:

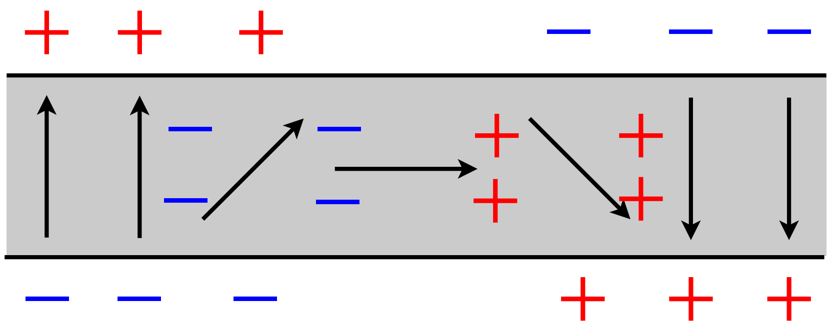
$$E_{\mathrm{s}}(\mathbf{m}) \simeq -\int_{\mathbb{R}^2} |\mathbf{m}^\perp|^2 d^2r + \frac{\delta}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\cdot\mathbf{m}^\perp(\mathbf{r})\,\nabla\cdot\mathbf{m}^\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\,d^2r' - \frac{\delta}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(m^\parallel(\mathbf{r}) - m^\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\,d^2r'$$

Dietze and Thomas, 1961; Garcia-Cervera, 1999; De Simone et al., 2000; M, 2019; Knüpfer, M and Nolte, 2019

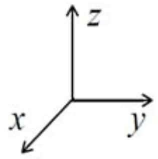
here $\delta \ll 1$ is the *effective* film thickness

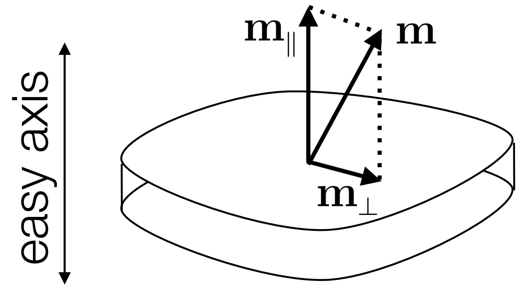
$$|\mathbf{m}^\perp|^2 = 1 - |m^\parallel|^2$$





$$\rho = -\nabla\cdot\mathbf{m}$$





The minimal model

use the local approximation for the stray field

Winter, 1961; Gioia and James, 1997

two-dimensional micromagnetic energy:

Bogdanov and Yablonskii, 1989
Rohart and Thiaville, 2013
Bernand-Mantel, M and Simon, 2020

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left(|\nabla \mathbf{m}|^2 - 2\kappa \mathbf{m}^{\perp} \cdot \nabla m^{\parallel} + (Q-1)|\mathbf{m}^{\perp}|^2 \right) d^2 r \qquad Q > 1$$

for $|\kappa| < \sqrt{Q-1}$ the ground state is $\mathbf{m} = \pm \hat{\mathbf{z}}$. Indeed, for $\mathbf{m} \neq \pm \hat{\mathbf{z}}$

$$E(\mathbf{m}) \geq \|\nabla \mathbf{m}\|_2^2 - 2|\kappa| \cdot \|\mathbf{m}^{\perp}\|_2 \|\nabla m^{\parallel}\|_2 + (Q-1)\|\mathbf{m}^{\perp}\|_2^2 > E(\pm \hat{\mathbf{z}})$$

specify a non-trivial *topological degree*: $m^{\parallel}(\mathbf{r}) \to -1 \text{ as } |\mathbf{r}| \to \infty$

$$\mathcal{N}(\mathbf{m}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathbf{m} \cdot (\partial_1 \mathbf{m} \times \partial_2 \mathbf{m}) \, d^2 r \in \mathbb{Z}$$

Brezis and Coron, 1983

sharp topological lower bound:

$$\int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 d^2 r \geq 8\pi \, |\mathcal{N}(\mathbf{m})|$$

$$|\nabla m|^2 \pm 2m \cdot (\partial_1 m \times \partial_2 m) = |\partial_1 m \mp m \times \partial_2 m|^2$$

Compact magnetic skyrmions

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left(|\nabla \mathbf{m}|^2 - 2\kappa \mathbf{m}^\perp \cdot \nabla m^\parallel + (Q-1)|\mathbf{m}^\perp|^2 \right) d^2 r$$

consider the topologically non-trivial admissible class

$$\mathcal{A} := \left\{ \mathbf{m} \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2) : \mathcal{N}(\mathbf{m}) = 1, \ \mathbf{m} + \hat{\mathbf{z}} \in L^2(\mathbb{R}^2; \mathbb{R}^3) \right\}$$

note that the last condition simply selects the limit at infinity, since

$$\min \left\{ \int_{\mathbb{R}^2} |m^\parallel + 1|^2 \, d^2 r, \int_{\mathbb{R}^2} |m^\parallel - 1|^2 \, d^2 r \right\} \leq \frac{1}{4\pi} \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 \, d^2 r \int_{\mathbb{R}^2} |\mathbf{m}^\perp|^2 \, d^2 r$$

we have the following *nearly optimal* existence result: Bernand-Mantel, M and Simon, 2020

Theorem 1. *Let $Q > 1$ and let $\kappa \in \mathbb{R}$ be such that $0 < |\kappa| < \frac{1}{\sqrt{2}}\sqrt{Q-1}$. Then there exists $\mathbf{m} \in \mathcal{A}$ such that*

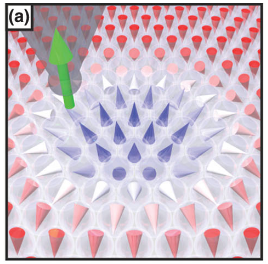
$$E(\mathbf{m}) = \inf_{\widetilde{\mathbf{m}} \in \mathcal{A}} E(\widetilde{\mathbf{m}}).$$

adapting arguments of
Melcher, 2014
Döring and Melcher, 2017
see also Greco, 2019

Note: no minimizers if $\kappa = 0$ or $|\kappa| > \frac{4}{\pi}\sqrt{Q-1}$

(Derrick-Pohozaev) (*E* unbounded below: stripes)

Skyrmions as degree 1 energy minimizers

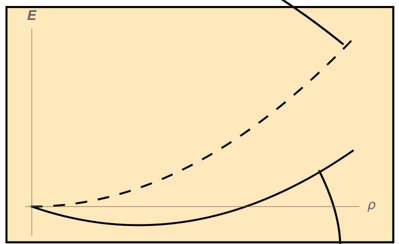


exchange + anisotropy: let $\mathbf{m}_\rho(\mathbf{r}) = \mathbf{m}(\rho^{-1}\mathbf{r})$

$$E(\mathbf{m}_\rho) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 \, d^2r + \rho^2(Q-1)\int_{\mathbb{R}^2} |\mathbf{m}^\perp|^2 \, d^2r$$

achieves minimum when $\rho \to 0$ => no minimizer

goes back to Derrick, 1964; Pokhozhaev, 1965; Berestycki and Lions, 1983



exchange + anisotropy + DMI:

$$E(\mathbf{m}_\rho) = \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 \, d^2r - 2\rho\kappa \int_{\mathbb{R}^2} \mathbf{m}^\perp \cdot \nabla m^\parallel \, d^2r + \rho^2(Q-1)\int_{\mathbb{R}^2} |\mathbf{m}^\perp|^2 \, d^2r$$

choosing $\mathbf{m}$ as the *truncated* Belavin-Polyakov profile: $\min E(\mathbf{m}) < 8\pi$

- implies existence of global energy minimizers with non-trivial degree

Bogdanov and Yablonskii, 1989; Bogdanov, Kudinov and Yablonskii, 1989; Ivanov et al., 1990

Melcher, 2014; Li and Melcher, 2018; Bernand-Mantel, M, Simon, 2020

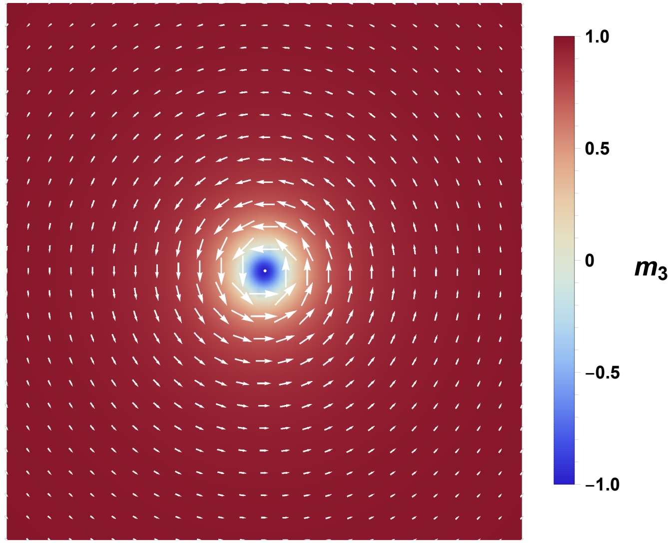
- quantitative rigidity of degree 1 harmonic maps allows to quantify closeness to BP profiles for small $|\kappa|$

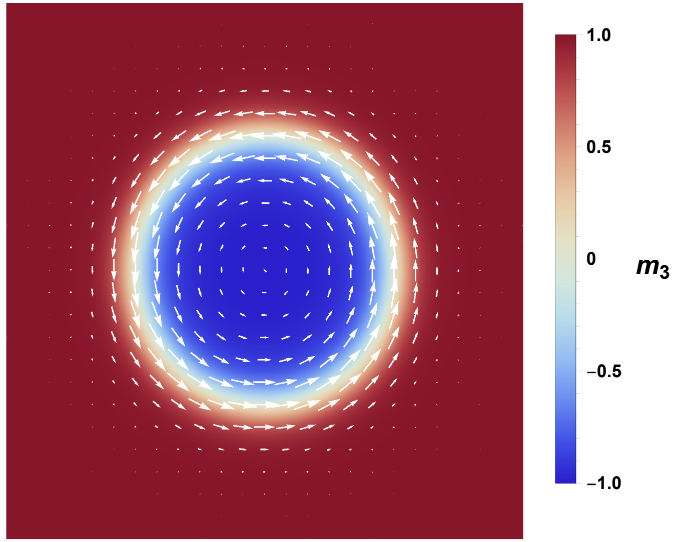
Bernand-Mantel, M, Slastikov, 2022

Magnetic skyrmions: full model

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}^\perp|^2 - 2\kappa \mathbf{m}^\perp \cdot \nabla m^\parallel \right\} d^2 r + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}^\perp(\mathbf{r}) \nabla \cdot \mathbf{m}^\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2 r \, d^2 r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m^\parallel(\mathbf{r}) - m^\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2 r \, d^2 r'$$

compact skyrmion vs. skyrmionic bubble





for skyrmionic bubble, the stray field energy *diverges* with radius:

$$E_{\text{surf}}(\mathbf{m}_R) \sim -\delta R \ln R$$

M and Simon, 2019

hence $\mathbf{m} \in \mathcal{A} \quad \not\Rightarrow \quad E(\mathbf{m})$ bounded below

no hope to construct solutions as absolute minimizers with prescribed degree

Clamped case

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}^\perp|^2 - 2\kappa \mathbf{m}^\perp \cdot \nabla m^\parallel - 2h(m^\parallel + 1) \right\} d^2r$$
$$+ \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}^\perp(\mathbf{r}) \nabla \cdot \mathbf{m}^\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\, d^2r' - \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m^\parallel(\mathbf{r}) - m^\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r'$$

fix $\mathbf{m} = -\hat{\mathbf{z}}$ outside $\Omega_\varepsilon = \varepsilon^{-1}\Omega$ for fixed bdd domain Ω with smooth bdry

perform the Modica-Mortola rescaling $Q > 1$

$$E_\varepsilon(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ \varepsilon |\nabla \mathbf{m}|^2 + \frac{Q-1}{\varepsilon}|\mathbf{m}^\perp|^2 - 2\kappa \mathbf{m}^\perp \cdot \nabla m^\parallel - 2h(m^\parallel + 1) \right\} d^2r$$
$$+ \frac{\delta_\varepsilon}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}^\perp(\mathbf{r}) \nabla \cdot \mathbf{m}^\perp(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2r\, d^2r' - \frac{\delta_\varepsilon}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m^\parallel(\mathbf{r}) - m^\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r'$$

remark: for δ > 0 independent of ε the energy is **not** bounded from below

=> need to choose $\delta_\varepsilon \to 0$ jointly with $\varepsilon \to 0$

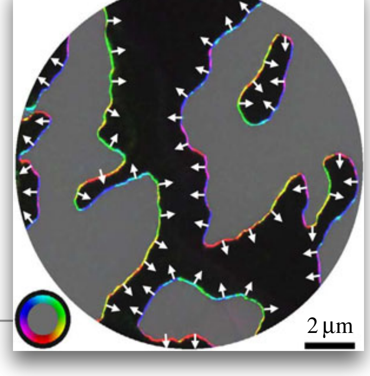
case δ = 0: $\mathbf{m}_\varepsilon \to \mathbf{m}_0 \in BV(\mathbb{R}^2; \{\pm\hat{\mathbf{z}}\})$ in $L^1(\mathbb{R}^2;\mathbb{R}^3)$

$$E_\varepsilon(\mathbf{m}_\varepsilon) \xrightarrow{\Gamma} E_0(\mathbf{m}_0) = (4\sqrt{Q-1} - \pi|\kappa|) P(\{m_0^\parallel = 1\}) - 4\int_{\{m_0^\parallel = 1\}} h\, d^2r$$

for $|\kappa| < \sqrt{Q-1}$ - technical assumption

M and Slastikov, 2016

Magnetic Modica-Mortola trick



$$E_\varepsilon^{MM}(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ \varepsilon|\nabla \mathbf{m}|^2 + \frac{Q-1}{\varepsilon}|\mathbf{m}^\perp|^2 - 2\kappa \mathbf{m}^\perp \cdot \nabla m^\parallel \right\} d^2r$$

$$\geq \int_{\mathbb{R}^2} \left\{ \frac{\varepsilon|\nabla m^\parallel|^2}{1-|m^\parallel|^2} + \frac{Q-1}{\varepsilon}(1-|m^\parallel|^2) - 2\kappa\sqrt{1-|m^\parallel|^2}\,|\nabla m^\parallel| \right\} d^2r$$

$$\geq 2\int_{\mathbb{R}^2} \left(\sqrt{Q-1} - \kappa\sqrt{1-|m^\parallel|^2} \right) |\nabla m^\parallel|\, d^2r$$

1st inequality is an equality for a Néel rotation

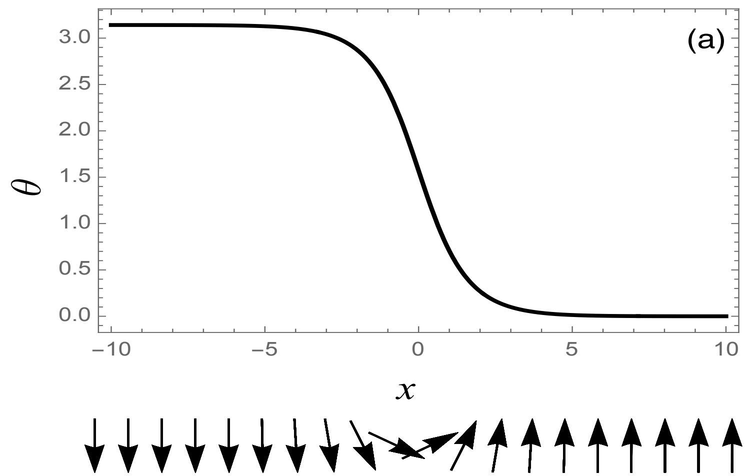
exp. image from G. Chen et al., 2013

2nd inequality is sharp => optimal 1D profile:

$$\mathbf{m} = (\operatorname{sech} x, 0, \tanh x)$$

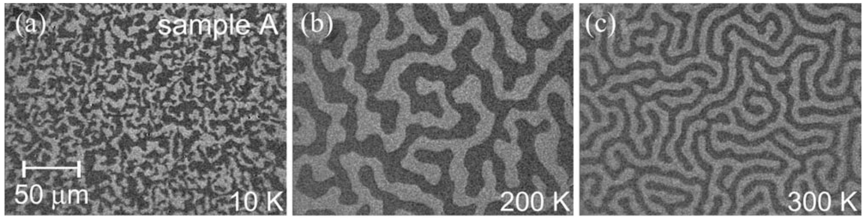
wall energy:

$$\sigma_{\text{wall}} = 4\sqrt{Q-1} - \pi\kappa$$



M and Slastikov, 2016

Sharp interface model



M. Yamanouchi et al., 2011

for $\mathbf{m} \in BV(\mathbb{R}^2; \{\pm\hat{\mathbf{z}}\})$ $\mathbf{m} = -\hat{\mathbf{z}}$ in Ω^c define

$$F_\varepsilon(\mathbf{m}) = (4\sqrt{Q-1} - \pi\kappa)P(\{m^\parallel = 1\}) - 4\int_{\{m^\parallel=1\}} h\, d^2r - \frac{\delta_\varepsilon}{8\pi}\int_{\mathbb{R}^2}\int_{\{|\mathbf{r}-\mathbf{r}'|>\varepsilon\}} \frac{(m^\parallel(\mathbf{r}) - m^\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2r\, d^2r'$$

subcritical case:

$$\sigma_{\text{wall}} = 4\sqrt{Q-1} - \pi\kappa - \frac{2\lambda}{\pi} > 0$$

$$\delta_\varepsilon = \frac{\lambda}{|\ln\varepsilon|}$$

Theorem 1. *For* $\mathbf{m}_\varepsilon \to \mathbf{m}_0$ *in* $L^1(\mathbb{R}^2; \mathbb{R}^3)$

M and Simon, 2019

$$F_\varepsilon(\mathbf{m}_\varepsilon) \xrightarrow{\Gamma} F_0(\mathbf{m}_0) = \sigma_{\text{wall}} P(\{m_0^\parallel = 1\}) - 4\int_{\{m_0^\parallel=1\}} h\, d^2r.$$

Microstructure beyond critical!

Knüpfer, M and Nolte, 2019
Bernand-Mantel, M and Slastikov, 2025

Back to micromagnetics

fix $\mathbf{m} = -\hat{\mathbf{z}}$ outside $\Omega_\varepsilon = \varepsilon^{-1}\Omega$ for fixed bdd domain Ω with smooth bdry

perform the Modica-Mortola rescaling, assume $\delta_\varepsilon = \lambda |\ln \varepsilon|^{-1}$

$$E_\varepsilon(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ \varepsilon |\nabla \mathbf{m}|^2 + \frac{Q-1}{\varepsilon} |\mathbf{m}^\perp|^2 - 2\kappa \mathbf{m}^\perp \cdot \nabla m^\parallel - 2h(m^\parallel + 1) \right\} d^2 r$$
$$+ \frac{\lambda}{4\pi |\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}^\perp(\mathbf{r}) \nabla \cdot \mathbf{m}^\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' - \frac{\lambda}{4\pi |\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla m^\parallel(\mathbf{r}) \cdot \nabla m^\parallel(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r'$$

assume $0 \le \kappa < \sqrt{Q-1}$ $\quad 0 \le \lambda < 2\pi(\sqrt{Q-1} - \kappa)$ $\quad Q > 1$

Theorem 2. *For* $\mathbf{m}_\varepsilon \to \mathbf{m}_0 \in BV(\mathbb{R}^2; \{\pm \hat{\mathbf{z}}\})$ *in* $L^1(\mathbb{R}^2; \mathbb{R}^3)$

$$E_\varepsilon(\mathbf{m}_\varepsilon) \xrightarrow{\Gamma} E_0(\mathbf{m}_0) = \sigma_{\text{wall}} P(\{m_0^\parallel = 1\}) - 4 \int_{\{m_0^\parallel = 1\}} h \, d^2 r.$$

<u>gap</u> with the supercritical regime $\qquad$ already present for λ = 0

Key estimate

Lemma 3. *Let* $f \in \mathring{H}^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *be such that* $f + c \in L^2(\mathbb{R}^2)$, *for some* $c \in \mathbb{R}$. *Then, for any* $0 < r < R$, *it holds*

$$\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(f(x)-f(y))^2}{|x-y|^3}\,dy\,dx \leq \frac{2}{\pi}\ln\left(\frac{R}{r}\right)\|f\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}|\nabla f|\,dx + \frac{r}{4}\int_{\mathbb{R}^2}|\nabla f|^2\,dx + \frac{2}{R}\int_{\mathbb{R}^2}(f+c)^2\,dx.$$

DeSimone, Knüpfer and Otto, 2006; Knüpfer, M and Nolte, 2019; M, 2019

=> compactness:

$$\begin{aligned} E_\varepsilon(\mathbf{m}) \;&\geq\; \left(1 - \frac{\lambda}{|\ln\varepsilon|}\right)E_\varepsilon^{MM}(\mathbf{m}) - \frac{\lambda}{\pi}\int_{\mathbb{R}^2}|\nabla m^\parallel|\,d^2r - \frac{2\lambda}{|\ln\varepsilon|}\int_{\mathbb{R}^2}(m^\parallel+1)\,d^2r \\ &\quad -2\int_{\mathbb{R}^2}h(m^\parallel+1)\,d^2r \\ &\geq\; 2\left(\sqrt{Q-1} - \kappa - \frac{\lambda}{2\pi} - \frac{\lambda\sqrt{Q-1}}{|\ln\varepsilon|}\right)\int_{\mathbb{R}^2}|\nabla m^\parallel|\,d^2r - C \end{aligned}$$

Γ-convergence

$$E_\varepsilon(\mathbf{m}_\varepsilon) \geq \int_{\mathbb{R}^2} |\nabla \Phi_\lambda(m_\varepsilon^\|)|\, d^2r - 2\int_{\mathbb{R}^2} h(m_\varepsilon^\| + 1)\, d^2r - \frac{c}{|\ln \varepsilon|}$$

where

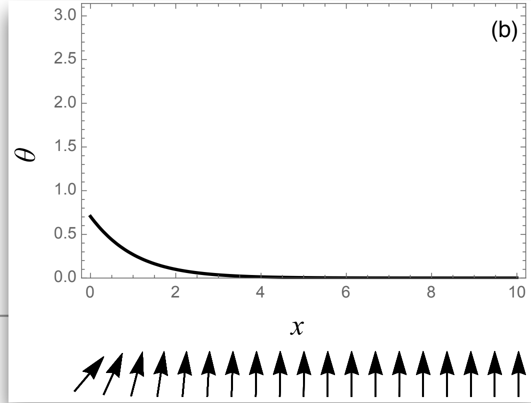
$$\begin{aligned} \Phi_\lambda(s) &= 2\int_0^s \left(\sqrt{Q-1} - \kappa\sqrt{1-t^2} - \frac{\lambda}{2\pi}\right) dt \\ &= 2s\sqrt{Q-1} - \kappa(s\sqrt{1-s^2} + \arcsin s) - \frac{\lambda s}{\pi} \end{aligned}$$

defines a one-to-one, strictly increasing, odd and continuously differentiable map from [-1,1] to $I = \left[-2\sqrt{Q-1} + \frac{\pi\kappa}{2} + \frac{\lambda}{\pi},\, 2\sqrt{Q-1} - \frac{\pi\kappa}{2} - \frac{\lambda}{\pi}\right]$

hence the lower bound follows:

$$\liminf_{\varepsilon \to 0} E_\varepsilon(\mathbf{m}_\varepsilon) \geq \int_{\mathbb{R}^2} |\nabla \Phi_\lambda(m_0^\|)|\, d^2r - 2\int_{\mathbb{R}^2} h(m_0^\| + 1)\, d^2r = \frac{\sigma_{\text{wall}}}{2}\int_{\mathbb{R}^2} |\nabla m_0^\|| - 2\int_{\mathbb{R}^2} h(m_0^\| + 1)\, d^2r$$

upper bound by approximation and a painstaking computation for smooth sets



Bounded domain: edge walls

now consider $\mathbf{m}_\varepsilon : \Omega \to \mathbb{S}^2$, minimal model (λ = 0):

$$E_\varepsilon(\mathbf{m}) = \int_\Omega \left\{ \varepsilon |\nabla \mathbf{m}|^2 + \frac{Q-1}{\varepsilon} |\mathbf{m}^\perp|^2 + \kappa (m^\parallel \nabla \cdot \mathbf{m}^\perp - \mathbf{m}^\perp \cdot \nabla m^\parallel) - 2h(m^\parallel + 1) \right\} d^2 r$$

DMI has an effect near the sample boundary

M and Slastikov, 2016

Ansatz:

$$\mathbf{m} = (\sin\theta, 0, \cos\theta),$$

$$\theta(x) = 2 \arctan e^{(x_0 - x)\sqrt{Q-1}}, \qquad x_0 = \frac{\ln \tan\left(\frac{\theta_0}{2}\right)}{\sqrt{Q-1}}, \qquad \theta_0 \in (0, \pi)$$

Optimal?

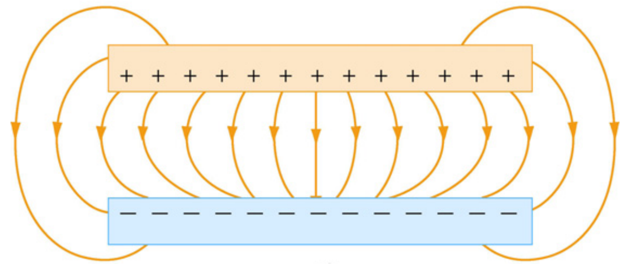
$$E(\mathbf{m}) = 2\sqrt{Q-1} \int_0^\infty |\sin\theta| \, |\theta'| \, dx + \int_0^\infty \left(|\theta'| - \sqrt{Q-1} \, |\sin\theta| \right)^2 dx - \kappa \theta_0$$

$$\geq - \int_0^\infty \left(2\sqrt{Q-1} \, |\sin\theta| - \kappa \right) \theta' dx = \int_0^{\theta_0} \left(2\sqrt{Q-1} \, |\sin\theta| - \kappa \right) d\theta.$$

Yes, provided θ_0 minimizes $F(\theta_0) = 2\sqrt{Q-1}\,(1 - \cos\theta_0) - \kappa\theta_0$. => wall energy:

$$\sigma_{\mathrm{edge}} = 2\sqrt{Q-1} \left(1 - \sqrt{1 - \frac{\kappa^2}{4(Q-1)}} \right) - \kappa \arcsin\left(\frac{\kappa}{2\sqrt{(Q-1)}} \right)$$

What about the stray field?



the original 2D energy accounting for the dipolar interactions in $\mathbb{R}^2$:

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}_\perp|^2 + \kappa (m^\parallel \nabla \cdot \mathbf{m}^\perp - \mathbf{m}_\perp \cdot \nabla m_\parallel) \right\} d^2 r$$
$$+ \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot \mathbf{m}_\perp(\mathbf{r}) \nabla \cdot \mathbf{m}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r'$$
$$- \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(\mathbf{r}) - m_\parallel(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r \, d^2 r' - 2 \int_{\mathbb{R}^2} h(m^\parallel + 1) \, d^2 r$$

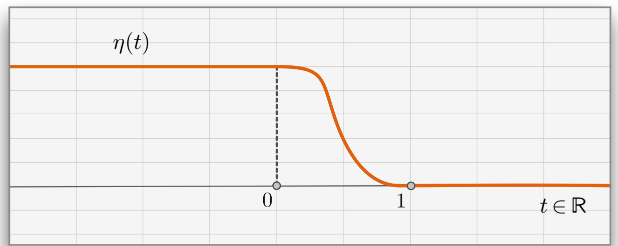
restricting the integrals to Ω would neglect the *fringe fields*

<u>solution</u>: extend $\mathbf{m}$ by zero outside Ω ?

No! - the dipolar energy is generically *undefined* (e.g., for $\mathbf{m} = -\hat{\mathbf{z}}$)

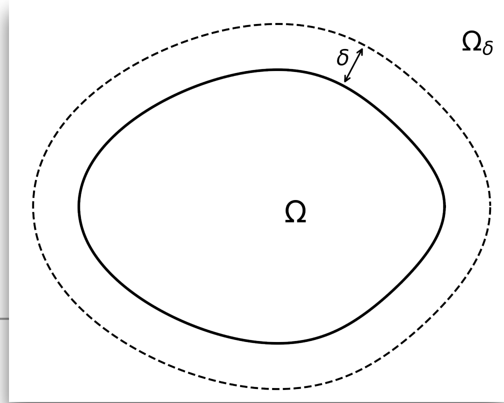
=> regularize the edge with a smooth cutoff: $|\mathbf{m}| = \eta_\delta$

$$\eta_\delta(\mathbf{r}) = \eta(\delta^{-1} \mathrm{dist}(\mathbf{r}, \Omega)) \qquad \delta \ll 1$$



Chaves-O'Flynn and M, 2013; M, Osipov and Vanden-Eijnden, 2015; Lund, M and Slastikov, 2018; Morini et al., 2023; Di Fratta, M and Slastikov, 2024

Micromagnetics of the film edge



two-dimensional energy with a regularized edge:

$$E(\mathbf{m}) = \int_{\mathbb{R}^2} \eta_\delta^2 \left\{ |\nabla \mathbf{m}|^2 + (Q-1)|\mathbf{m}^\perp|^2 + \kappa(m^\parallel \nabla \cdot \mathbf{m}^\perp - \mathbf{m}^\perp \cdot \nabla m^\parallel) \right\} d^2 r$$
$$+ \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot (\eta_\delta \mathbf{m}^\perp)(\mathbf{r}) \nabla \cdot (\eta_\delta \mathbf{m}^\perp)(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^2 r \, d^2 r'$$
$$- \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\eta_\delta(\mathbf{r}) m^\parallel(\mathbf{r}) - \eta_\delta(\mathbf{r}') m^\parallel(\mathbf{r}'))^2}{|\mathbf{r}-\mathbf{r}'|^3} d^2 r \, d^2 r' - 2 \int_{\mathbb{R}^2} \eta_\delta m^\parallel h \, d^2 r$$

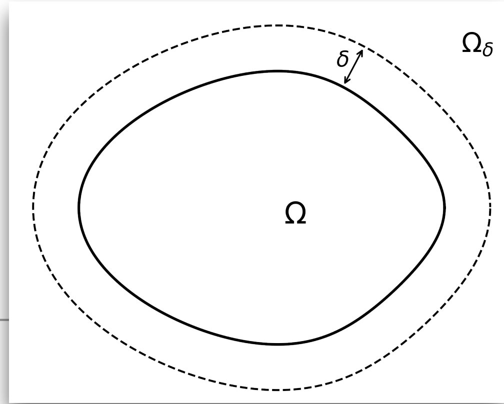
rescale lengths by ε, set $\delta = \delta_\varepsilon$, define $\Omega_{\varepsilon\delta_\varepsilon} = \Omega + B_{\varepsilon\delta_\varepsilon}$, take $\mathbf{m} \in H^1(\Omega_{\varepsilon\delta_\varepsilon}; \mathbb{S}^2)$

with $\delta_\varepsilon = \lambda |\ln \varepsilon|^{-1}$ and integrating by parts in the last term we obtain

assume $\quad Q > 1 \qquad 0 \leq \kappa < \sqrt{Q-1} \qquad 0 \leq \lambda < 2\pi(\sqrt{Q-1} - \kappa)$

Ω is a bounded domain with C^2 boundary

Micromagnetics of the film edge



rescaled two-dimensional energy with a regularized edge:

$$
\begin{aligned}
E_\varepsilon(\mathbf{m}) \quad = \quad & \int_{\Omega_{\varepsilon\delta_\varepsilon}} \eta_{\varepsilon\delta_\varepsilon}^2 \left(\varepsilon |\nabla \mathbf{m}|^2 + \frac{Q-1}{\varepsilon} |\mathbf{m}^\perp|^2 \right) d^2 r - 2 \int_{\Omega_{\varepsilon\delta_\varepsilon}} \eta_{\varepsilon\delta_\varepsilon} m^\| h \, d^2 r \\
& + \kappa \int_{\Omega_{\varepsilon\delta_\varepsilon}} \eta_{\varepsilon\delta_\varepsilon}^2 \left(m^\| \nabla \cdot \mathbf{m}^\perp - \mathbf{m}^\perp \cdot \nabla m^\| \right) d^2 r \\
& + \frac{\lambda}{4\pi |\ln \varepsilon|} \int_{\Omega_{\varepsilon\delta_\varepsilon}} \int_{\Omega_{\varepsilon\delta_\varepsilon}} \frac{\nabla \cdot (\eta_{\varepsilon\delta_\varepsilon} \mathbf{m}^\perp)(\mathbf{r}) \nabla \cdot (\eta_{\varepsilon\delta_\varepsilon} \mathbf{m}^\perp)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r' \\
& - \frac{\lambda}{4\pi |\ln \varepsilon|} \int_{\Omega_{\varepsilon\delta_\varepsilon}} \int_{\Omega_{\varepsilon\delta_\varepsilon}} \frac{\nabla (\eta_{\varepsilon\delta_\varepsilon} m^\|)(\mathbf{r}) \cdot \nabla (\eta_{\varepsilon\delta_\varepsilon} m^\|)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^2 r \, d^2 r'
\end{aligned}
$$

define $\sigma_{\text{wall}} = 4\sqrt{Q-1} - \pi\kappa - \dfrac{2\lambda}{\pi}$

$$
\sigma_{\text{edge}} = 2\sqrt{Q-1} \left(1 - \sqrt{1 - \frac{\kappa^2}{4(Q-1)}} \right) - \kappa \arcsin \left(\frac{\kappa}{2\sqrt{(Q-1)}} \right) - \frac{\lambda}{2\pi}
$$

Main result

$$\sigma_{\text{wall}} = 4\sqrt{Q-1} - \pi\kappa - \frac{2\lambda}{\pi}$$

Theorem 4. *For* $\mathbf{m}_\varepsilon \to \mathbf{m}_0 \in BV(\Omega; \{\pm\hat{\mathbf{z}}\})$ *in* $L^1(\Omega; \mathbb{R}^3)$

$$E_\varepsilon(\mathbf{m}_\varepsilon) \xrightarrow{\Gamma} E_0(\mathbf{m}_0) = \sigma_{\text{wall}} P(\{m_0^\parallel = 1\}) + \sigma_{\text{edge}} \mathcal{H}^1(\partial\Omega) - 2\int_\Omega m_0^\parallel h \, d^2 r.$$

- existence of skyrmionic bubbles, e.g. in the presence of an MFM tip

- non-eistence of skyrmionic bubbles without an applied field

but skyrmionic bubbles are observed *experimentally* under zero field!

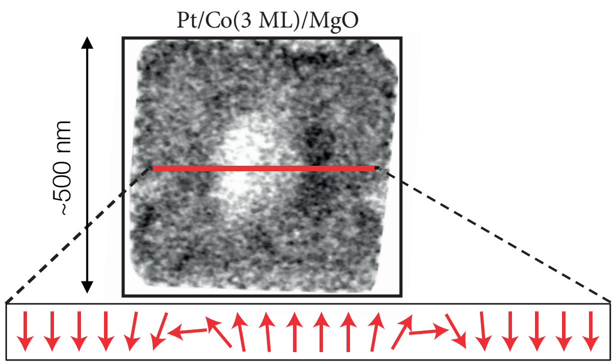
need to consider the regime of DMI strength/film thickness close to critical

case $\lambda = 0$ and $0 < 4\sqrt{Q-1} - \pi\kappa \ll 1$

- existence of radial Néel skyrmion solution

- minimizers have the form of a bubble of large radius

M, Simon and Slastikov, 2026+



Boulle et al., 2016