Existence of higher degree minimizers in the magnetic skyrmion problem

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Abstract

We demonstrate existence of topologically nontrivial energy minimizing maps of a given positive degree from bounded domains in the plane to $\mathbb{S}^2$ in a variational model describing magnetizations in ultrathin ferromagnetic films with Dzyaloshinskii-Moriya interaction. Our strategy is to insert tiny truncated Belavin-Polyakov profiles in carefully chosen locations of lower degree objects such that the total energy increase lies strictly below the expected Dirichlet energy contribution, ruling out loss of degree in the limits of minimizing sequences. The argument requires that the domain be either sufficiently large or sufficiently slender to accommodate a prescribed degree. We also show that these higher degree minimizers concentrate on point-like skyrmionic configurations in a suitable parameter regime.

Keywords: topological solitons, skyrmions, concentration phenomena, nanomagnetism

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1 Introduction

Topological solitons are a central notion for a number of nonlinear field theories arising as mathematical models of systems of very different physical nature [53]. Broadly speaking, they are certain special solutions of nonlinear partial differential equations of a field theory in the whole space that, on one hand, are in a certain sense localized and, on the other hand, exhibit a certain degree of persistence among more general classes of solutions. Topological solitons constitute the backbone of *topological defects*, which, in turn, are stable localized nonlinear excitations of the topologically trivial background state in a nonlinear system. The stability of these defects is closely related to the topological character of topological solitons through the fact that they cannot be smoothly deformed to the background state due to topological obstruction.

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A prime example of topological defects are Abrikosov vortices in type-II superconductors, which may be described by the Ginzburg-Landau theory [76]. Mathematically the vortex solution can already be captured by the single Ginzburg-Landau equation for a complex-valued field in the plane, with the individual vortex solution in $\mathbb{R}^2$ providing an example of a topological soliton. Starting with the studies in the applied mathematics literature [32, 34, 63] (this and the subsequent lists of references are not intended to be exhaustive), existence and uniqueness of equivariant solutions ("radial" solutions with prescribed degree $d \in \mathbb{Z}$) was established in [19, 36]. Furthermore, these solutions with $d = \pm 1$ were shown to be the only non-trivial locally minimizing solutions in the sense of De Giorgi for the associated energy [56, 70].

As for the topological defects within the Ginzburg-Landau theory, the Dirichlet boundary data of non-trivial topological degree force the existence of minimizers (of the same degree) exhibiting point-like vortices in the domain interior, where the energy concentrates as a coherence length parameter goes to zero [9]. One can obtain a lot of information about minimizing solutions of the Ginzburg-Landau energy, including locations and degrees of vortices, expansion of the energy with respect to the coherence length parameter and fine properties of minimizers [9, 40, 55, 64]. The connection between the vortex solutions with degree $d = \pm 1$ in the whole plane and the blowup limits of topological defects was established in [73]. There is a vast literature on the subject and many questions are still unresolved, for further references and some open problems see [15, 64, 71].

We note that the whole space Ginzburg-Landau vortex solutions do not actually represent global energy minimizers with prescribed topological degree, as the Dirichlet energy contribution of these solutions diverges logarithmically at infinity (however, compare with [1]). In search of the genuine global energy minimizing topological solitons for field theories, various stabilization mechanisms have been considered, starting with the model proposed by Skyrme [74]. For that model, existence of $\mathbb{S}^3$ valued topologically nontrivial energy minimizers in $\mathbb{R}^3$, termed *skyrmions*, was established for degrees $d = \pm 1$ [24–27, 51]. Another variant of the Skyrme model in $\mathbb{R}^3$ was investigated in [3, 29, 51], where minimizers were found to exist for an infinite subset of degrees $d \in \mathbb{Z}$. In $\mathbb{R}^2$, Skyrme model with additional energy terms yields the so-called *baby skyrmions* as maps from the plane to $\mathbb{S}^2$ with degree $d = \pm 1$ [48, 52]. More recently, strong numerical evidence was provided for the existence of *hopfions* as locally energy minimizing maps from $\mathbb{R}^3$ to $\mathbb{S}^2$ [69], following the early work in [14], for the energy containing higher derivative penalty terms in addition to the Dirichlet energy.

Recently, a growing body of work has emerged with the studies of chiral magnetic skyrmions, or simply *magnetic skyrmions* for shorthand, motivated by the experimental discovery of these configurations in chiral magnets and ultrathin ferromagnetic heterostructures [35, 59, 66, 79]. In these systems, the skyrmion solutions are typically stabilized by a chiral energy term called the Dzyaloshinskii-Moriya interaction (DMI), which promotes rotations of the magnetization vector with values in $\mathbb{S}^2$. Studies in the physics literature identified skyrmion configurations as locally minimizing solutions of the micromagnetic energy [10–13, 47, 65], which makes them attractive candidates for information technology applications [30, 41, 62, 80].

Mathematical studies of chiral magnetic skyrmions go back to [54], which treated a model similar to the one in [52] and in which the original Skyrme term is replaced by a DMI term appropriate for non-centrosymmetric cubic materials.[1] This paper established the existence of degree $d = 1$ (in our convention) global energy minimizers in the whole plane. Further studies

[1]In bulk chiral materials, the simplest form of the DMI energy density is given by a term proportional to $m \cdot (\nabla \times m)$, where $m = (m_1, m_2, m_3)$ and $\nabla = (\partial_1, \partial_2, \partial_3)$ [62]. Although this is different from the form appropriate for ultra-

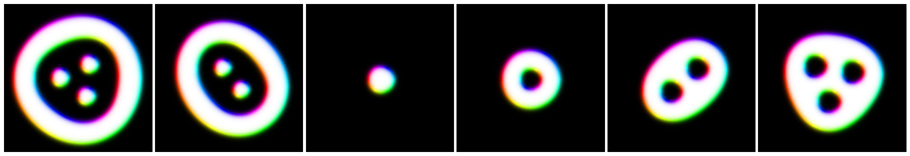

Figure 1: A series of numerical solutions of (1.2) with $d = 3, 2, 1, 0, -1, -2$, from left to right, obtained in [68]. Black and white regions show the domains where m is mostly down or up, respectively; the color indicates the direction of the m' component for intermediate values of m_3. For $d = 1$, the direction of m' is parallel to that of the gradient of m_3, which is the characteristic of the radial skyrmion solution.

of these and related minimizers can be found in [8, 22, 33, 42, 43, 49]. In a model that is appropriate for ultrathin multilayer materials with interfacial DMI existence results were obtained in [5–7], which also incorporated the non-local stabilizing effects of the stray field (see also the ansatz-based and numerical studies in [18]). Degree $d = 1$ minimizers were constructed in bounded domains in the plane under confinement in [57]. Also, precise asymptotic characterizations of the degree $d = 1$ energy minimizing solutions in the conformal limit, in which the energy is asymptotically dominated by the Dirichlet energy, have been obtained, showing that the energy minimizers approach some particular shrinking Belavin-Polyakov (BP) profiles [7, 22, 33, 57].

In its simplest form, the model describing the magnetization configurations $m : \mathbb{R}^2 \to \mathbb{S}^2$ in ultrathin ferromagnetic layers with perpendicular magnetic anisotropy and the interfacial DMI starts with the energy functional in the form of the sum of the exchange (Dirichlet), DMI and the anisotropy energy terms:

$$E(m) = \int_{\mathbb{R}^2} \left(|\nabla m|^2 + \kappa(m_3 \nabla \cdot m - m' \cdot \nabla m_3) + (Q-1)|m'|^2 \right) \, \mathrm{d}x, \tag{1.1}$$

where for $m = (m_1, m_2, m_3)$ we use the convention $m = (m', m_3)$, with $m' = (m_1, m_2)$. Here $\kappa \in \mathbb{R}$ and $Q \geq 1$ are dimensionless material parameters (the DMI constant and the material's quality factor respectively, see [6, 65] for the explanation). The associated Euler-Lagrange equation can be easily shown to be (see Proposition 2.2)

$$\begin{aligned} \Delta m + m|\nabla m|^2 + (Q-1)(m_3 e_3 - m_3^2 m) \\ - \kappa[(e_3 - m_3 m)\nabla \cdot m' - \nabla m_3 + (m' \cdot \nabla m_3)m] = 0, \end{aligned} \tag{1.2}$$

distributionally, where the pure gradient term is understood as $\nabla m_3 = (\partial_1 m_3, \partial_2 m_3, 0)$. As this equation is an $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ perturbation of the harmonic map equation from $\mathbb{R}^2$ to $\mathbb{S}^2$, its solutions are known to be smooth [58, Chapter 4]. We remark that in the simplest case of the parameters

thin ferromagnetic heterostructures in which the DMI is of interfacial origin and its energy density is proportional to $m_3 \nabla' \cdot m' - m' \cdot \nabla' m_3$, where $m = (m', m_3)$, $m' = (m_1, m_2)$ and $\nabla' = (\partial_1, \partial_2)$ [65], when $m = m(x_1, x_2)$ these two terms are equivalent up to a 90° rotation around the x_3-axis, since $m \cdot (\nabla \times m) = \tilde{m}_3 \nabla' \cdot \tilde{m}' - \tilde{m}' \cdot \nabla' \tilde{m}_3$ for $\tilde{m}_1 = m_2$, $\tilde{m}_2 = -m_1$ and $\tilde{m}_3 = m_3$. For this reason, we can interpret the results of the studies of bulk chiral materials in two dimensions in terms of models of ultrathin ferromagnetic heterostructures.

$Q = 1$ and $\kappa = 0$ equation (1.2) is just the harmonic map equation, whose solutions with bounded energy on the whole of $\mathbb{R}^2$ had been completely characterized [23, 46, 78]. They are minimizers of the energy in their respective homotopy classes determined by the topological degree [16, 17]

$$d = \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot (\partial_1 m \times \partial_2 m) \, \mathrm{d}x \in \mathbb{Z}, \tag{1.3}$$

which were first constructed in [4]. However, these solutions do not qualify as topological solitons due to the conformal invariance of the Dirichlet energy in $\mathbb{R}^2$ and hence the absence of a common characteristic length scale. We also note that for $\kappa = 0$ and $Q > 1$ equation (1.2) has no non-trivial solutions by the Derrick-Pohozaev argument [20], while for $Q \geq 1$ and $\kappa \neq 0$ sufficiently large (1.2) exhibits solutions in the form of spin spirals whose energy diverges to $-\infty$ [61, 65]. In contrast, for $Q > 1$ and $\kappa \neq 0$ sufficiently small there is always a solution with degree $d = 1$ that converges to $m = -e_3$ at infinity and for whose existence the DMI term is indispensable [6, 7, 33]. We remark, however, that at the same time the interplay between the DMI energy and the topological degree $d \neq 1$ appears to be far from straightforward for this type of profiles.

Numerical studies of (1.2) reveal a wealth of locally energy minimizing solutions in $\mathbb{R}^2$ for various values of the topological degree [11, 31, 44, 45, 68]. For a sample of the observed numerical solutions, see Fig. 1. In particular, the problem turns out to be considerably richer than its Ginzburg-Landau counterpart, exhibiting a plethora of solutions beyond a simple equivariant "radial" form first studied in [10, 12]. Furthermore, much less is known about the topologically nontrivial globally minimizing configurations. For example, uniqueness and radial symmetry of the $d = 1$ minimizers for $Q > 1$ are not known and are only asymptotically obtained in the conformal limit $\kappa \to 0$ [7]. Furthermore, with the exception of some very special choices of models yielding explicit solutions via the Bogomolnyi trick [2] (see also [37, 39]), no existence for any other degree $d \neq 1$ has been known up to now, not even under confinement. This may be contrasted, for example, with the available results in [51] in which an infinite subset of degrees (possibly all of $\mathbb{Z}$) yields existence, and all degrees $d \in \mathbb{Z}$ yield minimizers under confinement. This is because in the Skyrme mechanisms the higher-order term in the energy prevents concentration and collapse of the minimizing sequences and a subsequent loss of the degree in the limit. In contrast, even under confinement the energy in (1.1) generally allows for concentration [50], and the question of existence of minimizers with prescribed degree is genuinely non-trivial.

In this paper, we investigate existence of energy minimizing solutions of (1.2) with prescribed degree $d \geq 1$ under confinement in a bounded domain $\Omega \subset \mathbb{R}^2$ subject to the Dirichlet boundary condition $m = -e_3$ on $\partial\Omega$. This formulation was used in our earlier work [57] to study degree $d = 1$ single skyrmion solutions and is relevant to ultrathin film ferromagnetic materials in suitable parameter regimes [21]. It is well suited for the study of multiple skyrmions, similarly to the Dirichlet problem for Ginzburg-Landau vortices [9]. In terms of the energy minimizing configurations for (1.1), our paper is the first to establish existence of higher degree magnetizations in the context of multiple magnetic skyrmions on large, bounded domains for $Q > 1$. We also establish existence of minimizers with higher degrees for $Q = 1$ for sufficiently slender domains characterized in terms of the domain's optimal Poincaré constant.

Our existence results open up the question of how skyrmions interact. At this point, it is not even clear whether multiple degree-one skyrmions or single high-degree skyrmions could develop. We do not address this issue here, which would require an analysis of the splitting alternative in the concentration compactness on the whole of $\mathbb{R}^2$. Our proof focuses instead on ruling out the vanishing alternative in the concentration compactness on bounded domains. It relies on a careful construction, inductively inserting a tiny, truncated BP profile in a location where the degree $d-1$ minimizer is almost constant and making sure that the energy increases by strictly less than the additional contribution in the exchange energy. This estimate then allows us to rule out loss of degree in weak limits of minimizing sequences.

A surprising amount of care needs to be taken in choosing the location for insertion as a result of the rigidity of the harmonic map problem: The error terms in the exchange energy when pasting together the lower degree minimizer and the BP profile need to be dominated by the gain in the DMI energy, which is of lower order. Ideally, one would thus choose a location where the local exchange contribution is small at lower order. We show this to be possible when our domain Ω is sufficiently large or sufficiently slender, using an appropriate covering argument to handle the problem. Our existence result is presented in Theorem 2.1. We note that numerical evidence suggests that a given domain can only support minimizers with the degree bounded above depending on the domain geometry.

As it stands, our existence result so far yields very little information on the structure of the obtained solutions. As was already noted, it would be natural to ask whether these minimizers may indeed be interpreted as topological defects consisting of multiple well-separated skyrmions. In analogy with the Ginzburg-Landau problem, we therefore consider the limit behavior of the minimizers in which the anisotropy term in the energy forces the magnetization to converge to $m = -e_3$ almost everywhere, which is achieved by sending the parameter Q to infinity. In this limit, we can prove that the Dirichlet energy density of the minimizers concentrates on a quantized atomic defect measure, see Theorem 2.5. However, our convergence result gives no further information on either the location of the limit measure support or its quantized amplitudes, which correspond to multiples of the degrees of the associated shrinking bubbles. At the heart of this issue lies the question of *interaction* between multiple skyrmions: Do they repel and settle in distinct locations, or do they coalesce into a genuinely higher degree object? Mathematically, this requires a better understanding of the rigidity properties of higher degree harmonic maps expected to arise as the blowup limits above and their interplay with the lower order terms in our energy, as the higher degree situation [67] is significantly more complex than in the degree one case [7, 38, 77].

The remainder of our paper is organized as follows. In Section 2, we give the precise mathematical formulation of the problem, state our main theorems and discuss how their conclusions depend on various ingredients of the problem. In this section we also give an outline of the arguments used in the proofs. In Section 3, we establish several technical results that provide key ingredients in the proofs of the main theorems. Finally, in Section 4 we conclude the proofs.

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2 Statement of results

We now give the precise mathematical statements of our results and outline our strategy of their proof.

2.1 Mathematical setup

Following the setup of our paper [57] on single skyrmions on a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, we wish to minimize the energy in (1.1) restricted to Ω under Dirichlet boundary condition $m = -e_3$ on $\partial\Omega$ [21], which after an integration by parts can be equivalently defined as [7]

$$\mathcal{E}(m) := \int_\Omega \left(|\nabla m|^2 - 2\kappa m' \cdot \nabla m_3 + (Q-1)|m'|^2 \right) \mathrm{d}x. \tag{2.1}$$

Passing from $m = (m', m_3)$ to $\tilde{m} := (-m', m_3)$ and noting that the energy remains unchanged when replacing m by $\tilde{m}$ and changing the sign of κ, throughout the rest of the paper we may assume without loss of generality that $\kappa > 0$.

For the energy in (2.1) and a given $d \in \mathbb{Z}$ we consider the set of admissible functions

$$\mathcal{A}_d := \left\{ m \in H^1(\Omega; \mathbb{S}^2) \ : \ m = -e_3 \text{ on } \partial\Omega,\ \mathcal{N}(m) = d \right\}, \tag{2.2}$$

which satisfy a specific Dirichlet boundary condition and whose topological degree $\mathcal{N}(m)$ is equal to d. The degree of a function $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$, where, as usual,

$$\mathring{H}^1(\mathbb{R}^2, \mathbb{S}^2) := \left\{ m \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^3) : \int_{\mathbb{R}^2} |\nabla m|^2 \, \mathrm{d}x < \infty,\ |m| = 1 \text{ a.e. in } \mathbb{R}^2 \right\}, \tag{2.3}$$

can be defined as [16, 17]

$$\mathcal{N}(m) = \frac{1}{4\pi} \int_\Omega m \cdot (\partial_1 m \times \partial_2 m) \, \mathrm{d}x. \tag{2.4}$$

It is well known that $\mathcal{N}(m) \in \mathbb{Z}$ for any $m \in \mathring{H}^1(\mathbb{R}^2; \mathbb{S}^2)$, see [16]. To apply this definition to our case of the bounded domain Ω, we extend $m \in \mathcal{A}_d$ to the whole of $\mathbb{R}^2$ by setting $m = -e_3$ outside Ω. Indeed, in the rest of this paper we will not distinguish between $m \in \mathcal{A}_d$ and its extension to the whole of $\mathbb{R}^2$.

We furthermore denote the smallest Dirichlet eigenvalue of the domain Ω associated with the optimal Poincaré constant of Ω by $\lambda_0 > 0$. Hence for all $f \in W^{1,2}_0(\Omega)$, we have

$$\int_\Omega |f|^2 \, \mathrm{d}x \le \lambda_0^{-1} \int_\Omega |\nabla f|^2 \, \mathrm{d}x. \tag{2.5}$$

Throughout the rest of the paper, $C > 0$ denotes a generic, universal constant which may change from line to line, unless specified otherwise. For simplicity, we also use the notation B_r to denote the open ball of radius r centered at the origin.

2.2 Main results

Our main result is formulated in the following theorem, giving a condition for existence of higher degree minimizers in terms of the quantity

$$\alpha(Q,\kappa) := \frac{2\kappa^2}{\sqrt{(Q-1)^2 + 4\lambda_0\kappa^2} + (Q-1)} \tag{2.6}$$

being small enough and the area of Ω being large enough compared to $\frac{d}{\kappa^2}$, see Theorem 2.1. Note that in both cases $Q = 1$ and $Q > 1$ we need κ to be small compared, respectively, to $\sqrt{\lambda_0}$ and $\sqrt{Q-1}$. This can be seen from the lower bounds on $\alpha(Q,\kappa)$ in Lemma 3.2. Notice also that the quantity $\alpha(Q,\kappa)$ implicitly depends on Ω via the value of λ_0.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. There exists a universal constant $\overline{C} > 0$ with the following property: Let $\kappa > 0$, $Q \geq 1$, and $d \in \mathbb{N}$ with*

$$\alpha(Q,\kappa) \leq \min\left\{\frac{2}{d+1}, \frac{1}{2}\right\}. \tag{2.7}$$

If

$$|\Omega| \geq \frac{\overline{C}d}{\kappa^2}, \tag{2.8}$$

then there exists a minimizer of $\mathcal{E}$ over $\mathcal{A}_d$.

A standard consequence of the minimality of $\mathcal{E}$ is that the minimizer solves the Euler-Lagrange equation in (1.2). More precisely, we have the following result.

Proposition 2.2. *Under the assumptions of Theorem 2.1, let m be a minimizer of $\mathcal{E}$ over $\mathcal{A}_d$. Then $m \in C^\infty(\Omega;\mathbb{R}^3)$, $|m| = 1$ in Ω, and m satisfies* (1.2) *classically in Ω. If, furthermore, Ω is a simply connected, bounded, open set with boundary of class $C^{1,\alpha}$ for some $\alpha > 0$, then $m \in C^\infty(\Omega;\mathbb{R}^3) \cap C(\overline{\Omega};\mathbb{R}^3)$.*

We also note that the above proposition applies to every critical point of $\mathcal{E}$ in $\mathcal{A}_d$ regardless of the choice of parameters. In fact, just stationarity with respect to smooth outer variations is sufficient, as in two-dimensional domains equation (1.2) has a good regularity theory [58, Chapter 4].

The statement of Theorem 2.1 may be simplified in two extreme cases to yield a more explicit dependence on the domain Ω. The first case corresponds to the value of $Q > 1$ fixed and the domain being sufficiently large, so that the value of λ_0 is negligible in the definition of $\alpha(Q,\kappa)$. Since

$$\alpha(Q,\kappa) \leq \frac{\kappa^2}{Q-1} \tag{2.9}$$

for all $Q > 1$ and $\kappa > 0$, Theorem 2.1 immediately yields the following result.

Corollary 2.3. *Let $d \in \mathbb{N}$, $Q > 1$ and $0 < \kappa < \sqrt{(Q-1)\min\left\{\frac{2}{d+1}, \frac{1}{2}\right\}}$. Then there exists a minimizer of $\mathcal{E}$ over $\mathcal{A}_d$ for all $|\Omega| \geq \frac{\overline{C}d}{\kappa^2}$, for some $\overline{C} > 0$ universal.*

This corollary implies, in particular, that given $d \in \mathbb{N}$ and $Q > 1$ fixed, given $\kappa > 0$ sufficiently small depending on d and Q, and given a bounded domain $\Omega_0 \subset \mathbb{R}^2$, there exists a scale factor $s_0 > 0$ sufficiently large depending on d, κ and Ω_0 such that for all $s > s_0$ and all $\Omega = s\Omega_0$ the energy $\mathcal{E}$ admits a minimizer in the admissible class $\mathcal{A}_d$. In particular, this is true for any $d \in \mathbb{N}$ in the simplest case of Ω_0 being a disk.

On the other hand, for $d \gg 1$ the corollary gives existence only for $\kappa < C\sqrt{(Q-1)/d}$, implying that we must have $|\Omega| \geq C' d^2/(Q-1)$, for some $C, C' > 0$ universal. Notice that this scaling of $|\Omega|$ with d is consistent with configurations in which the skyrmions line the domain boundary, one skyrmion per natural length scale $l = 1/\sqrt{Q-1}$ of a single skyrmion. Nevertheless, one would rather expect that the skyrmions fill the interior of the domain Ω uniformly, which would yield a different scaling $|\Omega| \geq Cd/(Q-1)$, for some $C > 0$ universal. Our methods are too coarse to discriminate between these two possibilities, which would require to rule out preferential placement of skyrmions close to the domain boundary.

In the opposite extreme of $Q = 1$ we have instead the following corollary of Theorem 2.1.

Corollary 2.4. *Let $d \in \mathbb{N}$, $Q = 1$ and $0 < \kappa < \sqrt{\lambda_0} \min\left\{\frac{2}{d+1}, \frac{1}{2}\right\}$. Then there exists a minimizer of $\mathcal{E}$ over $\mathcal{A}_d$ for all $|\Omega| \geq \frac{\overline{C}d}{\kappa^2}$, for some $\overline{C} > 0$ universal.*

We note that in the case $Q = 1$ we cannot ensure existence of minimizers by rescaling $s\Omega$ with some large scale factor $s > 0$ since doing so decreases λ_0, which feeds back into the smallness condition for κ. In particular, existence of minimizers depends on the *shape* of Ω, not only on its area as in Corollary 2.3. This point may be illustrated by considering Ω to be a strip of width 1 and varying length $L \geq 1$, i.e. $\Omega = (0, L) \times (0, 1)$. Clearly in this case the value of λ_0 may be bounded below by a universal constant uniformly in L. Thus, we have existence for all $0 < \kappa < C/d$ for some $C > 0$ universal and $L \geq L_0$ with $L_0 = \overline{C}d/\kappa^2$. However, our methods do not allow to conclude whether there is a minimizer of $\mathcal{E}$ over $\mathcal{A}_d$ for any $d > 1$ in the case $Q = 1$ and $L = 1$, no matter what the value of κ is, as there is currently no reasonable quantitative information on the value of the universal constant $\overline{C}$ in the statement of Theorem 2.1. Similarly, contrary to the case $Q > 1$ we do not know if Corollary 2.4 ever applies with $d > 1$ and $Q = 1$ when Ω is a disk of any given radius. This should be contrasted with the result in [57] for $d = 1$, which yields existence of minimizers in this case for an arbitrary domain Ω, provided κ is sufficiently small. Also notice that for $d \gg 1$ and Ω in the form of the strip as above, Corollary 2.4 only yields existence for $L \geq Cd^3$ with some $C > 0$ universal, while we expect that existence may hold as soon as $L \geq Cd$.

We now let $d \in \mathbb{N}$, $\kappa > 0$ and the domain Ω satisfying condition (2.8) be fixed and consider the limit $Q \to \infty$, in which minimizers of $\mathcal{E}$ over $\mathcal{A}_d$ are expected to concentrate. In view of (2.9), we have existence for any $Q > 1$ sufficiently big. We may thus analyze the asymptotic behaviour of the minimizers in this regime. For technical reasons, we additionally assume Ω to be simply connected and have a $C^{1,\alpha}$ boundary.

Theorem 2.5. *Let $d \in \mathbb{N}$ and $\kappa > 0$, and let $\Omega \subset \mathbb{R}^2$ be a bounded, open, simply connected domain with a $C^{1,\alpha}$ boundary for some $\alpha > 0$, satisfying*

$$|\Omega| \geq \frac{\overline{C}d}{\kappa^2}, \tag{2.10}$$

where $\overline{C}$ is as in Theorem 2.1. Then for each $Q > 1$ large enough there exists a minimizer of $\mathcal{E}$ over $\mathcal{A}_d$. Furthermore, if (Q_n) is a sequence such that $Q_n \to \infty$ as $n \to \infty$, and m_n is a *minimizer of $\mathcal{E}$ with $Q = Q_n$ over $\mathcal{A}_d$, there exist $k \in \mathbb{N}$ with $k \leq d$, $x_1, \dots, x_k \in \overline{\Omega}$ distinct and $d_1, \dots, d_k \in \mathbb{N}$ with $\sum_{j=1}^{k} d_j = d$ such that as $n \to \infty$ we have*

$$m_n \rightharpoonup -e_3 \qquad \text{in } W^{1,2}(\Omega; \mathbb{R}^3), \tag{2.11}$$

and for $d\mu_n := \left(|\nabla m_n|^2 - 2\kappa m' \cdot \nabla m_3 + (Q-1)|m'|^2\right) \mathrm{d}x$ there holds

$$\mu_n \overset{*}{\rightharpoonup} \sum_{j=1}^{k} 8\pi d_j \delta_{x_j} \tag{2.12}$$

in the sense of measures, possibly up to extraction of a subsequence.

In other words, this theorem says that for $d \in \mathbb{N}$, $\kappa > 0$ and the domain Ω all fixed, as $Q \to \infty$ the energy density of minimizers over $\mathcal{A}_d$ concentrates on a sum of $k \in \mathbb{N}$ quantized delta-measures supported at points x_j in the closure of Ω. Their amplitudes $8\pi d_j$ correspond to the energies of the harmonic maps with degrees $1 \leq d_j \leq d$ in the whole of $\mathbb{R}^2$, indicating a bubbling phenomenon. In that sense one can interpret the minimizers for $Q \gg 1$ as collections of k well-separated skyrmionic configurations. However, unless $d = 1$ we cannot conclude that $k = d$ or, equivalently, that all $d_j = 1$ for all $1 \leq j \leq k$, as is suggested by the results of numerical simulations in this regime. In other words, based on the above result we cannot conclude that minimizers with degree $d > 1$ and $Q \gg 1$ resemble a collection of d well-separated skyrmions (i.e., minimizers with degree $d = 1$ in $\mathbb{R}^2$). The latter would require a much finer analysis of the interaction of skyrmions that goes well beyond the scope of the present paper.

2.3 Strategy of the proof

We apply the direct method of calculus of variations to establish existence of minimizers of $\mathcal{E}$ over $\mathcal{A}_d$ for a given value of $d \in \mathbb{N}$ in Theorem 2.1. As we work on a bounded domain with Dirichlet boundary conditions, lower-semicontinuity and coercivity of the energy functional easily follow by the Sobolev embedding and the Cauchy-Schwarz inequality, respectively. Therefore, the only issue in the proof of existence of minimizers is to ensure that the degree d is preserved when passing to the weak limit of minimizing sequences. This can be achieved by establishing a sort of strict subadditivity of the energy with respect to the degree, which is the subject of the lemmas in Section 3.

More precisely, the main point in our existence argument is proving that for each incremental increase in degree the infimal energy is increased by strictly less than 8π, which is the energy contribution of a shrinking BP profile (a harmonic bubble of degree 1). This is the content of Lemma 3.4. In analogy to the existence of a single skyrmion [54], we can then rule out loss of degree in weak limits of minimizing sequences. For κ sufficiently small depending on d, an increase in degree is prevented by the coercivity properties of the energy presented in Lemma 3.1. We note that in the considered regime the energy is qualitatively dominated by the exchange energy term.

We prove the required upper bound on the infimal energies by constructing a competitor, in which we insert a tiny, truncated BP profile at a point around which the lower degree minimizer is very close to $-e_3$ in the H^1-topology. As we already mentioned, this is a surprisingly delicate issue as the exchange energy incurred by pasting in the BP profile needs to be dominated by the gain in the DMI energy, which is of order κ^2 (at least for $Q = 1$, this is sharp). Therefore, we need to insert the skyrmion in a location where the exchange energy is small at order κ^2. Surprisingly, this only seems to be possible with further assumptions on Ω and the parameters appearing in the energy, including the value of the degree. Lemma 3.5 finally guarantees the existence of such a location in sufficiently large domains by a standard covering argument using the Hardy-Littlewood maximal operator.

Lastly, to prove our concentration result in Theorem 2.5 we use a characterization of weak limits of minimizing sequences of the Dirichlet energy due to Lin [50]: Such limits are harmonic maps which are smooth away from finitely many points at which a suitable defect measure indicates concentration of Dirichlet energy. Due to our boundary data, the harmonic map component is constant, leaving concentration as the only non-trivial effect.

3 Auxiliary results

We start by formulating several key technical results used in the proof of Theorem 2.1. We recall that the quantity $\alpha(Q,\kappa)$ appearing in all the lemmas below was defined in (2.6).

At the core of the proof of existence of minimizers of $\mathcal{E}$ are simple lower bounds for the energy that control the L^2-norm of ∇m for sufficiently small κ, together with a construction showing that the infimum energy is strictly below the topological lower bound

$$\int_\Omega |\nabla m|^2 \,\mathrm{d}x \geq 8\pi d \qquad \forall m \in \mathcal{A}_d, \tag{3.1}$$

for the case of the pure Dirichlet energy and $d \in \mathbb{N}$ (see, e.g., [54, (3.3)] or [7, Lemma A.3]).

Lemma 3.1. *Let $\kappa > 0$ and $Q \geq 1$. For $m \in H^1(\Omega;\mathbb{S}^2)$ satisfying $m = -e_3$ on $\partial\Omega$ we have*

i) the following lower bounds on the energy:

$$\mathcal{E}(m) \geq (1-\alpha(Q,\kappa)) \int_\Omega |\nabla m|^2 \,\mathrm{d}x, \tag{3.2}$$

$$\mathcal{E}(m) \geq (1-2\alpha(Q,\kappa)) \int_\Omega |\nabla m|^2 \,\mathrm{d}x + \frac{1}{2}(Q-1)\int_\Omega \left(1-m_3^2\right) \mathrm{d}x, \tag{3.3}$$

ii) the following upper bound on the energy:

$$\inf_{\mathcal{A}_d} \mathcal{E} < 8\pi d. \tag{3.4}$$

As can be seen from estimates (3.2) and (3.3), the energy $\mathcal{E}$ may be used to control simultaneously the exchange and the anisotropy energy when

$$\alpha(Q,\kappa) \leq \frac{1}{2}. \tag{3.5}$$

We also need some basic properties of the function $\alpha(Q,\kappa)$.

Lemma 3.2. *Let $\kappa > 0$ and $Q \geq 1$ satisfy inequality* (3.5)*. If $\lambda_0 \geq Q - 1$, we have*

$$\frac{2\kappa^2}{3\lambda_0} \leq \alpha(Q, \kappa), \tag{3.6}$$

while for $\lambda_0 < Q - 1$, we have

$$\frac{2\kappa^2}{3(Q-1)} \leq \alpha(Q, \kappa). \tag{3.7}$$

Note that the first estimate (3.6) is non-optimal if $Q = 1$, as then one would have $\alpha(Q, \kappa) = \lambda_0^{-\frac{1}{2}}\kappa$, but for our purposes this estimate is sufficient.

We now come to the result that is at the heart of our existence proof. We begin with a basic observation that the minimal energy cannot go up by more that 8π, the Dirichlet energy of the degree 1 harmonic map from $\mathbb{R}^2$ to $\mathbb{S}^2$, when the value of the degree in the admissible class is increased by 1. This fact is completely independent of the parameters of the model and simply reflects the leading order role of the Dirichlet energy.

Proposition 3.3. *Let $\kappa \in \mathbb{R}$, $Q \in \mathbb{R}$, and $d \in \mathbb{N}$. Then $\inf_{\mathcal{A}_{d+1}} \mathcal{E} \leq \inf_{\mathcal{A}_d} \mathcal{E} + 8\pi$.*

The conclusion of this proposition reflects a possible bubbling phenomenon: assuming a minimizer over $\mathcal{A}_d$ exists, a minimizing sequence from $\mathcal{A}_{d+1}$ could converge to that minimizer everywhere except at one point, around which the profile approaches a sequence of vanishing BP profiles that disappear in the limit. In this situation the degree of the minimizing sequence would not be preserved in the limit, failing to yield existence of a minimizer over $\mathcal{A}_{d+1}$. Furthermore, in this case we would have $\inf_{\mathcal{A}_{d+1}} \mathcal{E} = \inf_{\mathcal{A}_d} \mathcal{E} + 8\pi$.

Lemma 3.4 below, which will allow us to rule out loss of degree in weak limits of minimizing sequences, expresses the following deeper result: Given enough control on the energy density, it is possible to insert a carefully chosen, truncated BP profile in such a way that the energy increases by *strictly* less than 8π. The proof requires a careful construction ensuring that the energy gain in the DMI term of the inserted BP profile wins out over the error terms incurred in the insertion procedure.

Lemma 3.4. *There exists a universal constant $\varepsilon > 0$ with the following property: Let $\kappa > 0$ and $Q \geq 1$ satisfy inequality* (3.5)*, and let $d \in \mathbb{N} \cup \{0\}$. If $m \in \mathcal{A}_d$ satisfies*

$$\frac{1}{\pi r^2} \int_{B_r(x)} \left(|\nabla m|^2 + \max\{\lambda_0, Q-1\} |m + e_3|^2 \right) \, \mathrm{d}y \leq \varepsilon \kappa^2 \tag{3.8}$$

for some $x \in \Omega$ and all $r > 0$ such that $B_r(x) \subset \Omega$, then there exists $\bar{m} \in \mathcal{A}_{d+1}$ with

$$\mathcal{E}(\bar{m}) < \mathcal{E}(m) + 8\pi. \tag{3.9}$$

We observe that the above lemma works under an assumption of smallness of the energy density in a subdomain of Ω. In the following lemma we show that this assumption is true in sufficiently large or sufficiently slender domains.

Lemma 3.5. *There exists a universal constant $C_0 > 0$ such that for all $\varepsilon > 0$, $\Omega \subset \mathbb{R}^2$ open, bounded, and Lipschitz, $d \in \mathbb{N}$, and $\kappa > 0$ and $Q \geq 1$ satisfying inequality* (3.5) *the following* *holds: Let*

$$\beta(\kappa, Q, d) := \begin{cases} 1 & \text{if } \lambda_0 \geq Q-1, \\ \frac{d\kappa^2}{Q-1} & \text{if } \lambda_0 < Q-1. \end{cases} \tag{3.10}$$

If $|\Omega| \geq C_0\left(\beta(\kappa, Q, d)+1\right)\frac{d}{\varepsilon\kappa^2}$, $m \in \mathcal{A}_d$ *and* $\mathcal{E}(m) \leq 8\pi d$*, then there exists* $x \in \Omega$ *such that for all* $r > 0$ *with* $B_r(x) \subset \Omega$*, we have*

$$\frac{1}{\pi r^2}\int_{B_r(x)} \left(|\nabla m|^2 + \max\{\lambda_0, Q-1\}|m+e_3|^2\right)\,\mathrm{d}y \leq \varepsilon\kappa^2. \tag{3.11}$$

We now proceed to the proofs of the above lemmas.

Proof of Lemma 3.1. The construction in part ii) of the statement can be done in the same way as in [57, Lemma 3.2], inserting d small, truncated BP profiles into the domain.

In order to prove the statement in part i), we consider $\alpha > 0$ to be determined and estimate using the Poincaré, Cauchy-Schwarz, and Young inequalities

$$\begin{aligned} &\int_\Omega \left(\alpha|\nabla m|^2 - 2\kappa m' \cdot \nabla m_3 + (Q-1)|m'|^2\right)\,\mathrm{d}x \\ &\geq \alpha\|\nabla m_3\|_2^2 + \left(\alpha\lambda_0 + Q - 1\right)\|m'\|_2^2 - 2\kappa\|m'\|_2\|\nabla m_3\|_2 \\ &\geq 2\left(\sqrt{\alpha\left(\alpha\lambda_0+Q-1\right)} - \kappa\right)\|m'\|_2\|\nabla m_3\|. \end{aligned} \tag{3.12}$$

The smallest $\alpha > 0$ ensuring that the right hand side is non-negative is given by the positive solution of the quadratic equation

$$\lambda_0\alpha^2 + (Q-1)\alpha - \kappa^2 = 0, \tag{3.13}$$

which is given by $\alpha = \alpha(Q,\kappa)$, where $\alpha(Q,\kappa)$ is defined in (2.6). This proves the estimate (3.2). For $\tilde{\alpha} = 2\alpha(Q,\kappa)$ we furthermore have

$$\lambda_0\tilde{\alpha}^2 + \frac{Q-1}{2}\tilde{\alpha} - \kappa^2 > \lambda_0\alpha^2 + (Q-1)\alpha - \kappa^2 = 0, \tag{3.14}$$

similarly giving estimate (3.3). □

Proof of Lemma 3.2. If $\lambda_0 \geq Q-1$, we use definition (2.6) to note that

$$\alpha(Q,\kappa) \geq \frac{2\frac{\kappa^2}{\lambda_0}}{\sqrt{1+4\frac{\kappa^2}{\lambda_0}}+1} = \frac{1}{2}\left(-1+\sqrt{1+\frac{4\kappa^2}{\lambda_0}}\right), \tag{3.15}$$

which implies that from $\alpha(Q,\kappa) \leq \frac{1}{2}$ it follows that $\frac{\kappa^2}{\lambda_0} \leq \frac{3}{4}$ and hence $\frac{2\kappa^2}{3\lambda_0} \leq \alpha(Q,\kappa)$. If on the other hand we have $\lambda_0 \leq Q-1$, we note that the same type of algebra as above for $\alpha(Q,\kappa) \leq \frac{1}{2}$ gives $\frac{\kappa^2}{Q-1} \leq \frac{3}{4}$ and $\frac{2\kappa^2}{3(Q-1)} \leq \alpha(Q,\kappa)$, concluding the proof. □

We defer the proof of Proposition 3.3 to the end of this section, as it is independent of the proofs of the remaining lemmas, and we can also take advantage of the construction in the proof of Lemma 3.4.

Proof of Lemma 3.4. The strategy is to insert a truncated BP profile in some ball $B_\delta(x)$ in order to increase the degree while controlling the energy.

Step 1. Construction of a cutoff. Let $m \in \mathcal{A}_d$ satisfy (3.8). Shifting domain if necessary, we may assume $x = 0$, so that for all $\delta > 0$ with $B_\delta \subset \Omega$, we have

$$\int_{B_\delta} \left(|\nabla m|^2 + \max\{\lambda_0, Q-1\} |m + e_3|^2 \right) \, \mathrm{d}x \leq C \varepsilon \kappa^2 \delta^2. \tag{3.16}$$

By [28, Theorems 4.19 and 4.21], up to a redefinition on a set of measure zero we have $m|_{\partial B_r} \in H^1(\partial B_r; \mathbb{S}^2)$ with $\nabla_\tau (m|_{\partial B_r}) = (\nabla_\tau m)|_{\partial B_r}$, where ∇_τ denotes the tangential derivative on the circle, for almost all $0 < r \leq \delta$. From estimate (3.16) and an averaging argument we can therefore conclude that there exists a circle of radius $\frac{3}{4}\delta \leq r_0 \leq \delta$ such that

$$\int_{\partial B_{r_0}} \left(|\nabla_\tau m|^2 + \max\{\lambda_0, Q-1\} |m + e_3|^2 \right) \, \mathrm{d}\mathcal{H}^1 \leq C \varepsilon \kappa^2 \delta. \tag{3.17}$$

In particular, we immediately obtain

$$\int_{\partial B_{r_0}} \left(|\nabla_\tau m'|^2 + \max\{\lambda_0, Q-1\} |m'|^2 \right) \, \mathrm{d}\mathcal{H}^1 \leq C \varepsilon \kappa^2 \delta. \tag{3.18}$$

Let now $(m')_{r_0} := \frac{1}{2\pi r_0} \int_{\partial B_{r_0}} m' \, \mathrm{d}\mathcal{H}^1$. Up to a redefinition on a set of $\mathcal{H}^1$ measure zero, we have $m'|_{\partial B_{r_0}} \in C(\partial B_{r_0}; \mathbb{R}^2)$, and for all $x, y \in \partial B_{r_0}$ we have by the fundamental theorem of calculus and the Cauchy-Schwarz inequality:

$$|m'(x) - m'(y)|^2 \leq C |x - y| \int_{\partial B_{r_0}} |\nabla_\tau m'|^2 \, \mathrm{d}\mathcal{H}^1. \tag{3.19}$$

Choosing y such that $m'(y) = (m')_{r_0}$ and using estimate (3.18), for all $x \in \partial B_{r_0}$ we obtain

$$|m'(x) - (m')_{r_0}|^2 \leq C \varepsilon \kappa^2 \delta^2, \tag{3.20}$$

from which for all $x \in \partial B_{r_0}$ we get by Jensen's inequality and estimate (3.18) that

$$|m'(x)|^2 \leq C \varepsilon \kappa^2 \left(\frac{1}{\max\{\lambda_0, Q-1\}} + \delta^2 \right). \tag{3.21}$$

Choosing

$$\delta \leq \frac{1}{\sqrt{\max\{\lambda_0, Q-1\}}} \tag{3.22}$$

and using inequality (3.5) and Lemma 3.2, for all $x \in \partial B_{r_0}$ we thus have

$$|m'(x)|^2 \le \frac{C\varepsilon\kappa^2}{\max\{\lambda_0, Q-1\}} \le C\varepsilon. \tag{3.23}$$

Now we want to show that m_3 is close to -1 on ∂B_{r_0}. Using the estimate (3.17), we have

$$\int_{\partial B_{r_0}} \left(|\nabla_\tau(m_3+1)|^2 + \max\{\lambda_0, Q-1\}(1+m_3)^2\right) \,\mathrm{d}\mathcal{H}^1 \le C\varepsilon\kappa^2\delta. \tag{3.24}$$

As in the preceding argument, for all $x \in \partial B_{r_0}$ it follows that

$$|1 + m_3(x)|^2 \le C\varepsilon. \tag{3.25}$$

Therefore, assuming ε to be sufficiently small universal and using $1 + m_3 = \frac{|m'|^2}{1-m_3}$, for all $x \in \partial B_{r_0}$ we obtain

$$|1 + m_3(x)| \le C\varepsilon. \tag{3.26}$$

Inside B_{r_0} we only have integral estimates on m, but we want to ensure that $|m'|$ is small and m_3 is close to -1 pointwise. In order to achieve this, we will extend m' from ∂B_{r_0} inside B_{r_0} in a controlled way, keeping it small pointwise and not appreciably increasing its Dirichlet energy. We then recover m_3 from the length constraint.

For $x \in D_{r_0} := B_{r_0} \setminus B_{r_0/2}$, let

$$v(x) := \left(\frac{2}{r_0}|x| - 1\right)\left(m'\left(r_0\frac{x}{|x|}\right) - (m')_{r_0}\right). \tag{3.27}$$

An explicit calculation together with estimate (3.20) and assumption (3.22) yields

$$\int_{D_{r_0}} |v|^2 \,\mathrm{d}x \le Cr_0 \int_{\partial B_{r_0}} \left|m' - (m')_{r_0}\right|^2 \,\mathrm{d}\mathcal{H}^1 \le \frac{C\varepsilon\kappa^2\delta^2}{\max\{\lambda_0, Q-1\}}. \tag{3.28}$$

Similarly, with the help of estimate (3.18) a decomposition into tangential and normal derivatives together with the Poincaré inequality on ∂B_{r_0} gives

$$\begin{aligned} \int_{D_{r_0}} |\nabla v|^2 \,\mathrm{d}x &\le Cr_0 \int_{\partial B_{r_0}} \left(\left|\nabla_\tau m'\right|^2 + \frac{1}{r_0^2}\left|m' - (m')_{r_0}\right|^2\right) \mathrm{d}\mathcal{H}^1 \\ &\le Cr_0 \int_{\partial B_{r_0}} \left|\nabla_\tau m'\right|^2 \,\mathrm{d}\mathcal{H}^1 \le C\varepsilon\kappa^2\delta^2. \end{aligned} \tag{3.29}$$

We may now define $\tilde{m}'(x) := v(x) + (m')_{r_0}$, which by the first part of estimate (3.23), as well as estimates (3.28) and (3.29) satisfies

$$\int_{D_{r_0}} |\tilde{m}'|^2 \,\mathrm{d}x \le \frac{C\varepsilon\kappa^2\delta^2}{\max\{\lambda_0, Q-1\}}, \tag{3.30}$$

$$\int_{D_{r_0}} |\nabla\tilde{m}'|^2 \,\mathrm{d}x \le C\varepsilon\kappa^2\delta^2. \tag{3.31}$$

Using the second part of estimate (3.23) and the definition of v, we have $|v(x)|^2 \le C\varepsilon$ for any $x \in D_{r_0}$, as well as $|(m')_{r_0}|^2 \le C\varepsilon$. Therefore, it follows for all $x \in D_{r_0}$ that

$$|\tilde{m}'(x)|^2 \le C\varepsilon. \tag{3.32}$$

Defining $\tilde{m}_3 = -\sqrt{1 - |\tilde{m}'|^2}$, for all $x \in D_{r_0}$ we also get that

$$|1 + \tilde{m}_3(x)| \le C\varepsilon. \tag{3.33}$$

Step 2: Energy estimates for the cutoff. We now want to estimate the energy terms inside $D_{r_0} = B_{r_0} \setminus B_{r_0/2}$. By the estimates (3.31) and (3.32), as well as $|\nabla \tilde{m}_3(x)|^2 \le \frac{|\tilde{m}'(x)|^2 |\nabla \tilde{m}'(x)|^2}{1 - |\tilde{m}'(x)|^2}$ for a.e. $x \in D_{r_0}$, we obtain for all $\varepsilon > 0$ universally small that

$$\int_{D_{r_0}} |\nabla \tilde{m}_3|^2 \, \mathrm{d}x \le 2 \int_{D_{r_0}} |\nabla \tilde{m}'|^2 |\tilde{m}'|^2 \, \mathrm{d}x \le C\varepsilon^2 \kappa^2 \delta^2. \tag{3.34}$$

Therefore, again by estimate (3.31) we have for $\varepsilon < 1$:

$$\int_{D_{r_0}} |\nabla \tilde{m}|^2 \, \mathrm{d}x \le C\varepsilon \kappa^2 \delta^2. \tag{3.35}$$

The anisotropy term is already controlled by estimate (3.30). In order to estimate the DMI term, we note that by the Cauchy-Schwarz inequality, together with estimates (3.32) and (3.34) we have

$$\int_{D_{r_0}} |\tilde{m}' \cdot \nabla \tilde{m}_3| \, \mathrm{d}x \le \left(\int_{D_{r_0}} |\nabla \tilde{m}_3|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{D_{r_0}} |\tilde{m}'|^2 \, \mathrm{d}x \right)^{1/2} \le C\varepsilon^{\frac{3}{2}} \kappa \delta^2. \tag{3.36}$$

Step 3: Inserting a Belavin-Polyakov profile. Into the hole $B_{r_0/2}$, we aim to insert a suitably truncated, rotated Belavin-Polyakov profile ϕ of radius ρ satisfying $\phi(x) = -e_3$ for $x \in \partial B_{r_0/2}$.

Let us recall the construction of an appropriate Belavin-Polyakov profile [7]. To this end, for $L \ge 2$ and $r > 0$ we truncate the function

$$f(r) := \frac{2r}{1 + r^2} \tag{3.37}$$

via defining

$$f_L(r) := \begin{cases} f(r) & \text{if } r \le \frac{L}{2}, \\ 2f\left(\frac{L}{2}\right)\left(1 - L^{-1} r\right) & \text{if } \frac{L}{2} < r \le L, \\ 0 & \text{if } L < r. \end{cases} \tag{3.38}$$

For $x \in \mathbb{R}^2$ this translates to a truncated Belavin-Polyakov profile

$$\Phi_L(x) := \left(-f_L(|x|) \frac{x}{|x|}, \operatorname{sign}(1 - |x|) \sqrt{1 - f_L^2(|x|)} \right), \tag{3.39}$$

with the original Belavin-Polyakov profile being

$$\Phi(x) := \left(-\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right) = \lim_{L\to\infty} \Phi_L(x). \tag{3.40}$$

In the spirit of estimates [7, (A.66) to (A.71)], we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla \Phi_L|^2 \, \mathrm{d}x &\le \frac{8\pi L^2}{4+L^2} + 2\pi \int_{\frac{L}{2}}^{L} r \left(\frac{4 f^2\left(\frac{L}{2}\right) L^{-2}}{1 - f^2\left(\frac{L}{2}\right)} + \frac{f_L^2(r)}{r^2} \right) \mathrm{d}r \\ &\le 8\pi + C L^{-2}, \end{aligned} \tag{3.41}$$

where $C > 0$ is some universal constant. In addition, we have $\mathcal{N}(\Phi_L) = 1$ for all $L \ge 2$, since $\nabla\Phi_L \to \nabla\Phi$ in $L^2(\mathbb{R}^2)$ and $\Phi_L \to \Phi$ pointwise a.e. as $L \to \infty$, and $\mathcal{N}(\Phi_L)$ is a continuous function of $L \ge 2$ with values in $\mathbb{N}$.

Now let us modify this truncated profile by a suitable rotation to match the boundary conditions $\phi_\rho = \left((m')_{r_0}, -\sqrt{1-(m')^2_{r_0}}\right)$ on $\partial B_{r_0/2}$. Let $R \in SO(3)$ be such that

$$-Re_3 = \left((m')_{r_0}, -\sqrt{1-(m')^2_{r_0}}\right), \tag{3.42}$$

which due to the estimate (3.23) for m' we may choose to satisfy

$$|R - \mathrm{id}\,|^2 \le \frac{C\varepsilon\kappa^2}{\max\{\lambda_0, Q-1\}} \le C\varepsilon, \tag{3.43}$$

where $|\cdot|$ denotes the usual Frobenius norm. Then for $0 < \rho \ll \delta$ and $\phi_\rho(x) := R\Phi_{\frac{r_0}{2\rho}}(\rho^{-1}x)$ we have $\mathcal{N}(\phi_\rho) = 1$ and $\phi_\rho(x) = \left((m')_{r_0}, -\sqrt{1-(m')^2_{r_0}}\right)$ for all $x \in \partial B_{r_0/2}$. Due to the estimate (3.41), the rotation R not affecting the Dirichlet energy and recalling that $\frac{3}{4}\delta \le r_0 \le \delta$, we also get

$$\int_{\mathbb{R}^2} |\nabla\phi_\rho|^2 \, \mathrm{d}x \le 8\pi + \frac{C\rho^2}{\delta^2}. \tag{3.44}$$

To estimate the DMI energy of ϕ_ρ, we use the argument in the proof of estimate [7, (A.53)] to obtain

$$\int_{\mathbb{R}^2} \Phi_L' \cdot \nabla\Phi_{L,3} \, \mathrm{d}x > \frac{1}{C}, \tag{3.45}$$

for all L sufficiently large universal. Since $\Phi_L(x) + e_3$ and $\Phi(x) + e_3$ decay as $\frac{1}{|x|}$ and $\nabla\Phi_L(x)$ and $\nabla\Phi(x)$ decay as $\frac{1}{|x|^2}$ as $|x| \to \infty$, with the help of estimate (3.43) for L sufficiently large and ε sufficiently small universal we also observe that

$$\int_{\mathbb{R}^2} (R\Phi_L)' \cdot \nabla(R\Phi_L)_3 \, \mathrm{d}x > \frac{1}{C}. \tag{3.46}$$

In total, this gives

$$\int_{B_{\frac{r_0}{2}}} \phi'_\rho \cdot \nabla\phi_{\rho,3}\,\mathrm{d}x \geq \frac{\rho}{C}, \tag{3.47}$$

for all ε and ρ/δ sufficiently small universal.

Additionally, due to $|\phi'_\rho(x)|^2 \leq C\left(\frac{\rho^2|x|^2}{(\rho^2+|x|^2)^2} + |R - \mathrm{id}\,|^2\right)$ for $x \in B_{r_0/2}$ and estimate (3.43) we have

$$\int_{B_{\frac{r_0}{2}}} |\phi'_\rho|^2\,\mathrm{d}x \leq C\left(\rho^2 \log\frac{\delta}{\rho} + \frac{\varepsilon\kappa^2\delta^2}{\max\{\lambda_0, Q-1\}}\right), \tag{3.48}$$

again for all ε and ρ/δ sufficiently small universal.

Putting together estimates (3.44), (3.47) and (3.48), we arrive at

$$\mathcal{E}(\phi_\rho) \leq 8\pi + C\left(\frac{\rho^2}{\delta^2} + (Q-1)\rho^2\log\frac{\delta}{\rho}\right) - \frac{\kappa\rho}{C} + C\varepsilon\kappa^2\delta^2. \tag{3.49}$$

Step 4. Construction of a competitor. We now construct a degree $d+1$ competitor $\bar{m}$ by inserting ϕ_ρ into $B_{r_0/2}$ via

$$\bar{m}(x) = \begin{cases} \phi_\rho(x) & |x| \leq \frac{r_0}{2}, \\ \tilde{m}\,(x) & \frac{r_0}{2} < |x| < r_0, \\ m(x) & |x| \geq r_0, \end{cases} \tag{3.50}$$

for $x \in \Omega$. As on $\partial B_{r_0/2}$ we have

$$\phi_\rho = \left((m')_{r_0}, -\sqrt{1-(m')^2_{r_0}}\right) = \tilde{m}, \tag{3.51}$$

the map $\bar{m}$ is well defined in $H^1(\Omega;\mathbb{S}^2)$.

Step 5: Prove $\bar{m} \in \mathcal{A}_{d+1}$. It is clear due to construction that we have

$$\mathcal{N}(\bar{m})|_{B_{r_0/2}} := \frac{1}{4\pi}\int_{B_{r_0/2}} \bar{m}\cdot(\partial_1\bar{m}\times\partial_2\bar{m})\,\mathrm{d}x = 1. \tag{3.52}$$

Additionally, by estimates (3.16) and (3.35) we have

$$\begin{aligned} \left|\mathcal{N}(\bar{m}) - \mathcal{N}(m) - \mathcal{N}(\bar{m})|_{B_{r_0/2}}\right| &= \left|\int_{D_{r_0}} \bar{m}\cdot(\partial_1\bar{m}\times\partial_2\bar{m})\,\mathrm{d}x - \int_{B_{r_0}} m\cdot(\partial_1 m\times\partial_2 m)\,\mathrm{d}x\right| \\ &\leq C\left(\int_{D_{r_0}} |\nabla\bar{m}|^2\,\mathrm{d}x + \int_{B_{r_0}} |\nabla m|^2\,\mathrm{d}x\right) \\ &\leq C\varepsilon\kappa^2\delta^2. \end{aligned} \tag{3.53}$$

As we know that $\mathcal{N}(m) = d$, in view of discreteness of the degree and the assumption (3.5) we deduce that $\mathcal{N}(\bar{m}) = d + 1$ for any choice of δ satisfying (3.22), with $\varepsilon > 0$ small universal.

Step 6: Conclusion. We know that outside of B_{r_0} the maps m and $\bar{m}$ coincide. Therefore, we just need to show that when restricted to B_{r_0} we have

$$\mathcal{E}(\bar{m})|_{B_{r_0}} < \mathcal{E}(m)|_{B_{r_0}} + 8\pi. \tag{3.54}$$

Using estimates (3.16), (3.5) and Lemma 3.2, we know that inside B_{r_0} we have $\int_{B_{r_0}} |m'|^2 \, \mathrm{d}x \leq C\varepsilon\delta^2$ and hence by the Cauchy-Schwarz and Young inequalities

$$\mathcal{E}(m)|_{B_{r_0}} \geq \int_{B_{r_0}} |\nabla m|^2 \, \mathrm{d}x - 2\kappa \int_{B_{r_0}} \nabla m_3 \cdot m' \, \mathrm{d}x \geq -C\kappa^2 \int_{B_{r_0}} |m'|^2 \, \mathrm{d}x \geq -C\varepsilon\kappa^2\delta^2. \tag{3.55}$$

Moreover, using estimates (3.30), (3.35), (3.36), and (3.49) we obtain

$$\mathcal{E}(\bar{m})|_{B_{r_0}} \leq 8\pi + C\left(\frac{\rho^2}{\delta^2} + (Q-1)\rho^2 \log\frac{\delta}{\rho}\right) - \frac{\kappa\rho}{C} + C\varepsilon\kappa^2\delta^2. \tag{3.56}$$

Choosing $\rho = \kappa\delta^2/(2C^2)$, we arrive at

$$\mathcal{E}|_{B_{r_0}}(\bar{m}) \leq 8\pi - \frac{\kappa^2\delta^2}{4C^3} + C(Q-1)\kappa^2\delta^4 \ln\left(\frac{1}{\kappa\delta}\right) + C\varepsilon\kappa^2\delta^2. \tag{3.57}$$

Taking $\varepsilon > 0$ small enough universal, $\delta > 0$ small enough depending on λ_0, Q and κ (it is enough to ensure that in addition to (3.22) the quantities $\kappa\delta$, $(Q-1)\delta$ and $\delta\ln(1/(\kappa\delta))$ are all sufficiently small universal), we arrive at

$$\mathcal{E}|_{B_{r_0}}(\bar{m}) - \mathcal{E}(m)|_{B_{r_0}} \leq 8\pi - \frac{\kappa^2\delta^2}{5C^3} + C\varepsilon\kappa^2\delta^2 < 8\pi, \tag{3.58}$$

which concludes the proof. □

Proof of Lemma 3.5. Under our assumptions on κ, Q, and the energy, estimate (3.2) of Lemma 3.1 implies that

$$\int_\Omega |\nabla m|^2 \, \mathrm{d}x \leq 16\pi d, \tag{3.59}$$

so that by the Poincaré inequality, we have

$$\lambda_0 \int_\Omega |m + e_3|^2 \, \mathrm{d}x \leq 16\pi d. \tag{3.60}$$

On the other hand, estimate (3.3) of Lemma 3.1 and the topological lower bound (3.1) give

$$8\pi d\,(1 - 2\alpha(Q,\kappa)) + \frac{1}{2}(Q-1)\int_\Omega \left(1 - m_3^2\right) \mathrm{d}x \leq 8\pi d, \tag{3.61}$$

which by (2.9) then implies

$$\int_\Omega \left(1-m_3^2\right)\,\mathrm{d}x \le \frac{Cd\alpha(Q,\kappa)}{(Q-1)} \le \frac{Cd\kappa^2}{(Q-1)^2}. \tag{3.62}$$

Thus, in view of the fact that the extension of m satisfies $m = -e_3$ outside Ω, by [7, Lemma 5.1] we obtain

$$\int_\Omega |m_3+1|^2\,\mathrm{d}x \le \frac{1}{4\pi}\int_\Omega |\nabla m|^2\,\mathrm{d}x \int_\Omega \left(1-m_3^2\right)\,\mathrm{d}x \le \frac{Cd^2\kappa^2}{(Q-1)^2}. \tag{3.63}$$

In total, we get

$$\max\{\lambda_0, Q-1\}\int_\Omega |m_3+1|^2\,\mathrm{d}x \le C\beta(\kappa,Q,d)d, \tag{3.64}$$

where we recall that

$$\beta(\kappa,Q,d) = \begin{cases} 1 & \text{if } \lambda_0 \ge Q-1, \\ \frac{d\kappa^2}{Q-1} & \text{if } \lambda_0 < Q-1. \end{cases} \tag{3.65}$$

For $x\in\mathbb{R}^2$ we consider the Hardy-Littlewood maximal functions

$$M_1(x) := \sup_{r>0}\frac{1}{\pi r^2}\int_{B_r(x)} |m+e_3|^2\,\mathrm{d}y, \tag{3.66}$$

$$M_2(x) := \sup_{r>0}\frac{1}{\pi r^2}\int_{B_r(x)} |\nabla m|^2\,\mathrm{d}y, \tag{3.67}$$

which are well known to be bounded in weak L^1 if the original functions are bounded in L^1, see [75, Chapter 3, Theorem 1.1]. Consequently, for all $t>0$, there exists a universal constant $C_1>0$ such that

$$\left|\left\{M_1(x)>t\right\}\right| \le \frac{C_1\beta(\kappa,Q,d)d}{t\max\{\lambda_0,Q-1\}}, \tag{3.68}$$

$$\left|\left\{M_2(x)>t\right\}\right| \le \frac{C_1 d}{t}. \tag{3.69}$$

Choosing $t_1 := 3C_1\frac{\beta(\kappa,Q,d)d}{|\Omega|\max\{\lambda_0,Q-1\}}$ and $t_2 := 3C_1\frac{d}{|\Omega|}$ we get that

$$\left|\Omega\cap\left\{M_1(x)\le t_1\right\}\right| \ge \frac{2}{3}|\Omega|, \tag{3.70}$$

$$\left|\Omega\cap\left\{M_2(x)\le t_2\right\}\right| \ge \frac{2}{3}|\Omega|. \tag{3.71}$$

Thus, there exists $x\in\Omega$ such that

$$M_1(x) \le 3C_1\frac{\beta(\kappa,Q,d)d}{|\Omega|\max\{\lambda_0,Q-1\}} \tag{3.72}$$

and

$$M_2(x) \leq 3C_1 \frac{d}{|\Omega|}. \tag{3.73}$$

The statement follows provided $3C_1 \frac{\beta(\kappa,Q,d)d}{|\Omega|} \leq \varepsilon\kappa^2$ and $3C_1 \frac{d}{|\Omega|} \leq \varepsilon\kappa^2$, which is the case under our assumption on $|\Omega|$ for $C_0 = 3C_1$. □

Proof of Proposition 3.3. Letting $m \in \mathcal{A}_d$ and passing to the precise representative if necessary [28, Theorem 4.19], pick $x \in \Omega$ to be a point of continuity of m that is also a Lebesgue point of ∇m. Without loss of generality, we may assume that $x = 0$ and, hence,

$$\int_{B_\delta} |\nabla m|^2 \, dx \leq C\delta^2, \qquad m(0) = \lim_{|y|\to 0} m(y), \qquad |m(0)| = 1. \tag{3.74}$$

for some $C > 0$ independent of $\delta \ll 1$.

Up to a rigid rotation of m, we may assume for the moment that $m(0) = -e_3$. Arguing as in the proof of Lemma 3.4, we can find $r_0 \in \left(\frac{3}{4}\delta, \delta\right)$ such that $m|_{\partial B_{r_0}} \in H^1(\partial B_{r_0}; \mathbb{S}^2)$ and $|m' - (m')_{r_0}| \leq C\delta$ for some constant $C > 0$ independent of $\delta \ll 1$. Furthermore, by continuity of m at the origin we have $m_3(x) < 0$ for all $x \in \partial B_{r_0}$ and $\delta \ll 1$. Therefore, as in the proof of Lemma 3.4 we can define a cutoff $\tilde{m}_\delta$ in D_{r_0} for every $\delta \ll 1$ such that $\tilde{m}_\delta = m$ on ∂B_{r_0}, $\tilde{m}_\delta = \left((m')_{r_0}, -\sqrt{1 - (m')^2_{r_0}}\right)$ on $\partial B_{r_0/2}$, and

$$\int_{D_{r_0}} |\nabla \tilde{m}_\delta|^2 \, dx \leq C\delta^2, \tag{3.75}$$

for some $C > 0$ independent of $\delta \ll 1$. By the Cauchy-Schwarz inequality, we also have

$$\int_{D_{r_0}} |\tilde{m}'_\delta \cdot \nabla \tilde{m}_{\delta,3}| \, dx \leq C\delta^2, \tag{3.76}$$

for some $C > 0$ independent of $\delta \ll 1$. As the estimates in (3.75) and (3.76) are rotation-invariant, existence of an $\tilde{m}_\delta$ satisfying these estimates and interpolating from m on ∂B_{r_0} to a constant unit vector on $\partial B_{r_0/2}$ follows for an arbitrary value of $m(0)$.

We now set $\rho_\delta = \delta^2$, and for $\delta \ll 1$ and $x \in B_{r_0/2}$ we define a truncated Belavin-Polyakov profile $\phi_{\rho_\delta}(x) = R\Phi_{\frac{\rho}{2r_0}}(\rho_\delta^{-1} x)$, where $R \in SO(3)$ is a rotation satisfying $Re_3 = -\tilde{m}_\delta|_{\partial B_{r_0/2}}$. By an explicit calculation we get

$$\int_{B_{r_0/2}} |\nabla \phi_{\rho_\delta}|^2 \, dx \leq 8\pi + C\delta^2, \qquad \int_{B_{r_0/2}} |\phi'_{\rho_\delta} \cdot \nabla \phi_{\rho_\delta,3}| \, dx \leq C\delta^2, \tag{3.77}$$

for some $C > 0$ independent of $\delta \ll 1$.

Finally, we construct a competitor $\bar{m}_\delta \in H^1(\Omega; \mathbb{S}^2)$ exactly as in Lemma 3.4. Clearly, by (3.74),(3.75) and the definition of ϕ_{ρ_δ} we have $\bar{m}_\delta \in \mathcal{A}_{d+1}$ for all $\delta \ll 1$. Furthermore, by (3.74)–(3.77) and again the Cauchy-Schwarz inequality for the DMI term in $\mathcal{E}(m)$ we have

$$\mathcal{E}(\bar{m}_\delta) \leq \mathcal{E}(m) + 8\pi + C\delta^2, \tag{3.78}$$

for some $C > 0$ independent of $\delta \ll 1$. We then conclude by applying (3.78) to a minimizing sequence $(m_n) \in \mathcal{A}_d$ and using a diagonal argument. □

Remark 3.6. *We note that contrary to Lemma 3.4, the construction in the proof of Proposition 3.3 does not allow one to conclude a strict inequality, since the rotation R is a priori not close to identity and, hence, we may not pick a negative contribution from the DMI term. Furthermore, even if we were able to show that the competitor produces a net negative contribution to the DMI energy, we would not be able to conclude a priori that it beats the possible positive gain in the exchange energy due to the cutoff.*

4 Proofs of the main theorems

Now we conclude the proofs of Theorem 2.1, Proposition 2.2, and Theorem 2.5.

Proof of Theorem 2.1. Step 1. Ensuring the applicability of Lemma 3.5. We first show that under assumption (2.7) it is possible to find $\overline{C} > 0$ universal such that if assumption (2.8) holds, then the condition on $|\Omega|$ of Lemma 3.5 with $\varepsilon > 0$ from Lemma 3.4 is satisfied. That is, we need to ensure that $|\Omega| \geq C_0\,(\beta(\kappa, Q, d) + 1)\,\frac{d}{\varepsilon\kappa^2}$ with $\varepsilon > 0$ universal from Lemma 3.4, $\beta(\kappa, Q, d)$ defined in (3.10), and $C_0 > 0$ being a universal constant from Lemma 3.5.

If $\lambda_0 \geq Q - 1$, then $\beta(\kappa, Q, d) = 1$, and using inequality (2.8) we have

$$|\Omega| \geq \frac{\overline{C} d}{\kappa^2} = C_0\,(\beta(\kappa, Q, d) + 1)\,\frac{d}{\varepsilon\kappa^2}, \tag{4.1}$$

if $\overline{C} = 2C_0/\varepsilon$. Otherwise, we have $\lambda_0 < Q - 1$. Using condition (2.7) and Lemma 3.2, we have

$$\frac{2}{d} \geq \alpha(Q, \kappa) \geq \frac{2\kappa^2}{3(Q-1)}, \tag{4.2}$$

and therefore $\beta(\kappa, Q, d) = \frac{d\kappa^2}{Q-1} \leq 3$. Consequently, invoking the inequality (2.8), we have

$$|\Omega| \geq \frac{\overline{C} d}{\kappa^2} \geq C_0\,(\beta(\kappa, Q, d) + 1)\,\frac{d}{\varepsilon\kappa^2}, \tag{4.3}$$

if $\overline{C} = 4C_0/\varepsilon$.

Step 2. Convergence of minimizing sequences and lower semicontinuity. Let $(m_n) \in \mathcal{A}_d$ be a minimizing sequence. By Lemma 3.1, since inequality (3.5) holds by assumption, we get that (m_n) is uniformly bounded in $H^1(\Omega; \mathbb{S}^2)$. Consequently, there exists a subsequence (not relabeled) and $m_\infty \in H^1(\Omega; \mathbb{S}^2)$ such that $m_n \to m_\infty$ in L^2 and $\nabla m_n \rightharpoonup \nabla m_\infty$ in L^2 as $n \to \infty$. Furthermore, by a weak-times-strong argument, we get $\int_\Omega m_n' \cdot \nabla m_{n,3}\,\mathrm{d}x \to \int_\Omega m_\infty' \cdot \nabla m_{\infty,3}\,\mathrm{d}x$ and, therefore, we have

$$\mathcal{E}(m_\infty) \leq \liminf_{n\to\infty} \mathcal{E}(m_n) = \inf_{\mathcal{A}_d} \mathcal{E}. \tag{4.4}$$

Thus, it remains to prove that $m_\infty \in \mathcal{A}_d$, i.e., that $\mathcal{N}(m_\infty) = d$.

Arguing as in [16, 54] and [7, Lemma A.3], we complete the squares to get for all $m \in H^1(\Omega; \mathbb{S}^2)$ that

$$\int_\Omega |\nabla m|^2\,\mathrm{d}x \pm 8\pi\mathcal{N}(m) = \int_\Omega |\partial_1 m \mp m \times \partial_2 m|^2\,\mathrm{d}x. \tag{4.5}$$

As a result, by the lower semicontinuity of the right-hand side in (4.5) and the continuity of the DMI and anisotropy terms we have

$$\mathcal{E}(m_\infty) \pm 8\pi \mathcal{N}(m_\infty) \le \liminf_{n\to\infty} \mathcal{E}(m_n) \pm 8\pi d. \tag{4.6}$$

Step 3. Proving $1 \le \mathcal{N}(m_\infty) \le d$. With the help of Lemma 3.1 we know that $\mathcal{E}(m_\infty) \ge 0$ and $\inf_{\mathcal{A}_d} \mathcal{E} < 8\pi d$. Therefore, using (4.6) with the "–" sign", we obtain $-8\pi\mathcal{N}(m_\infty) < 0$, implying $\mathcal{N}(m_\infty) \ge 1$.

The inequality (4.6) with the "+" sign, together with Lemma 3.1 and the topological bound (3.1) yield

$$8\pi(2 - \alpha(Q,\kappa))\mathcal{N}(m_\infty) < 16\pi d. \tag{4.7}$$

Therefore, under assumption (2.7) we have

$$\mathcal{N}(m_\infty) < \frac{2d}{2 - \alpha(Q,\kappa)} \le d + 1. \tag{4.8}$$

Thus, by discreteness of the degree we have $\mathcal{N}(m_\infty) \le d$. Note that this is the only place in our argument where we crucially need $\alpha(Q,\kappa)$ to be bounded proportionally to d^{-1}.

Step 4. Proving $\mathcal{N}(m_\infty) = d$. If $d = 1$, we are done now, producing a minimizer of $\mathcal{E}$ over $\mathcal{A}_1$ (compare with [57, Theorem 2.1] in the case $Q = 1$). So in the following we may assume $d \ge 2$.

We argue by contradiction and assume $N := \mathcal{N}(m_\infty) < d$. Using estimate (4.6), we have

$$\inf_{\mathcal{A}_N} \mathcal{E} + 8\pi\,(d - N) \le \inf_{\mathcal{A}_d} \mathcal{E}. \tag{4.9}$$

Since $1 \le N < d$ and we already have existence for $d = 1$, we can use induction and assume that the minimum of $\mathcal{E}$ over $\mathcal{A}_{d'}$ is attained for all $1 \le d' < d$, with $\min_{\mathcal{A}_{d'}} \mathcal{E} < 8\pi d'$ by Lemma 3.1. Then applying first Lemma 3.5, then Lemma 3.4 repeatedly to the minimizers of $\mathcal{E}$ in $\mathcal{A}_{d'}$ for $N \le d' < d$, we obtain

$$\inf_{\mathcal{A}_d} \mathcal{E} < \min_{\mathcal{A}_N} \mathcal{E} + 8\pi\,(d - N)\,. \tag{4.10}$$

However, this contradicts estimate (4.9), and, therefore, the assumption $N < d$ is false, proving the assertion of the theorem. □

Proof of Proposition 2.2. Arguing as in [72], let $\xi \in C_c^\infty(\Omega;\mathbb{R}^3)$. Then, for $|t| < t_0$ with $t_0 > 0$ small enough, we have

$$m^t := \frac{m + t\xi}{|m + t\xi|} \in \mathcal{A}_d. \tag{4.11}$$

Indeed, for $i = 1, 2$, we get $|m^t| = 1$ a.e. in Ω, $m^t = -e_3$ on $\partial\Omega$ and

$$m^t = m + t\,(\xi - (\xi\cdot m)m) + O(t^2), \tag{4.12}$$

$$\partial_i m^t = \partial_i m + t\left(\partial_i\xi - \left(m\cdot\partial_i\xi + \xi\cdot\partial_i m\right)m - (\xi\cdot m)\partial_i m\right) + O(t^2(1 + |\nabla m|)), \tag{4.13}$$

for a.e. $x \in \Omega$, where the constants in the O-notation depend on ξ, but not on m. In particular, we have $m^t \in H^1(\Omega; \mathbb{S}^2)$ for all $|t| < t_0$ and m^t is continuous in $H^1(\Omega; \mathbb{R}^3)$ at $t = 0$. Therefore, by continuity of the degree in $H^1(\Omega; \mathbb{R}^3)$ we have $\mathcal{N}(m^t) = d$ in a sufficiently small neighborhood of $t = 0$.

We now compute the derivative of $\mathcal{E}(m^t)$ at $t = 0$. Rewriting the anisotropy term as $(Q - 1)(1 - m_3^2)$ and interpreting $\nabla m_3 = (\partial_1 m_3, \partial_2 m_3, 0)$, by minimality of m we have

$$
\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(m^t)\bigg|_{t=0} = 2\int_\Omega \left(\nabla m : \nabla \xi - |\nabla m|^2 m \cdot \xi\right) \mathrm{d}x \\
+ 2\kappa \int_\Omega \left((\nabla \cdot m')(e_3 - m_3 m) - (\nabla m_3 - (m \cdot \nabla m_3) m)\right) \cdot \xi \, \mathrm{d}x \qquad (4.14) \\
- 2(Q-1)\int_\Omega \left(m_3 e_3 - m_3^2 m\right) \cdot \xi \, \mathrm{d}x.
\end{aligned}
$$

This gives the distributional version of equation (1.2).

As the lower order terms in this equation are all L^2-integrable, standard regularity theory implies that $m \in C^\infty(\Omega; \mathbb{S}^2)$, see [58, Chapter 4]. Thus, the Euler-Lagrange equation is satisfied classically. If Ω is additionally simply connected with a $C^{1,\alpha}$ boundary, then we also have continuity up to the boundary, see [60, Theorem 1.1, Remark 1.4, and Corollary 1.6]. □

Proof of Theorem 2.5. By inequality (2.9) the condition (2.7) is satisfied for $Q_n \gg 1$ and, therefore, existence of a minimizer $m_n \in \mathcal{A}_d(\Omega)$ follows from Theorem 2.1 for all n large enough.

By Lemma 3.1 and the Poincaré inequality, the sequence $(m_n + e_3)$ is bounded in $W^{1,2}(\mathbb{R}^2; \mathbb{R}^3)$ after constant extension by $-e_3$ outside of Ω. Indeed, using Lemma 3.1 we have

$$
\lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla m_n|^2 \, \mathrm{d}x = 8\pi d, \qquad (4.15)
$$

since $\alpha(Q_n, \kappa) \to 0$ by inequality (2.9). For $x \in \mathbb{R}^2$, let

$$
\Phi(x) := \left(-\tfrac{2x}{1+|x|^2} \quad \tfrac{1-|x|^2}{1+|x|^2}\right). \qquad (4.16)
$$

Then by conformal invariance of the Dirichlet energy in two dimensions, see for example [7, Lemma A.2], also $\hat{m}_n : \mathbb{S}^2 \to \mathbb{S}^2$ defined as $\hat{m}_n := m_n \circ \Phi^{-1}$ satisfies

$$
\lim_{n\to 0} \int_{\mathbb{S}^2} |\nabla \hat{m}_n|^2 \, \mathrm{d}\mathcal{H}^2 = 8\pi d. \qquad (4.17)
$$

Consequently, in view of the topological lower bound (3.1) it is a minimizing sequence of the Dirichlet energy among maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ of degree d.

Additionally, by Proposition 2.2 we have $m_n \in C(\mathbb{R}^2; \mathbb{S}^2)$ and thus $\hat{m}_n \in C(\mathbb{S}^2; \mathbb{S}^2)$. By [50, Theorem 1"], there exists a subsequence, a weakly harmonic map $\hat{m}_\infty \in W^{1,2}(\mathbb{S}^2; \mathbb{S}^2)$ and a non-negative Radon measure ν on $\mathbb{S}^2$ such that

$$
\hat{m}_n \rightharpoonup \hat{m}_\infty \qquad (4.18)
$$

in $W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$, and for $d\hat{\mu}_n := |\nabla \hat{m}_n|^2 \, \mathrm{d}\mathcal{H}^2$ and $d\hat{\mu}_\infty := |\nabla \hat{m}_\infty|^2 \, \mathrm{d}\mathcal{H}^2$ we have

$$\hat{\mu}_n \overset{*}{\rightharpoonup} \hat{\mu}_\infty + \nu, \tag{4.19}$$

as measures when $n \to \infty$. By [50, Theorem 5.8], there exist $k \in \mathbb{N} \cup \{0\}$, $z_1, \dots, z_k \in \mathbb{S}^2$, and $d_1, \dots, d_k \in \mathbb{N}$ such that

$$\nu = \sum_{j=1}^{k} 8\pi d_j \delta_{z_j}, \tag{4.20}$$

with the convention that $\nu = 0$ if $k = 0$.

Turning our attention to $\hat{m}_\infty$, we observe that because it is a stationary point of the Dirichlet energy on the sphere, it must be an energy-minimizing map in its own homotopy class, a fact that was first independently proved by Lemaire [46] and Wood [78], see also [23, (10.5)]. However, since $\hat{m}_\infty(x) = -e_3$ for $x \in \mathbb{S}^2 \setminus \Phi(\Omega)$, we must have that $\hat{m}_\infty = -e_3$ on the entirety of $\mathbb{S}^2$, which can be proved by, for example, using [16, Lemma A.1].

We thus have

$$\hat{\mu}_n \overset{*}{\rightharpoonup} \sum_{j=1}^{k} 8\pi d_j \delta_{z_j}, \tag{4.21}$$

as measures when $n \to \infty$. In particular, from convergence (4.17) it follows that $k \neq 0$ and $\sum_{j=1}^{k} d_j = d$. Since $|\nabla \hat{m}_n|^2(x) = 0$ for all $x \notin \Phi(\overline{\Omega})$, we must have $z_1, \dots, z_k \in \Phi\left(\overline{\Omega}\right)$.

Pulling back these statements to the plane by precomposing with Φ, we obtain the convergence of the exchange energy density to the sum of delta measures as in (2.12). At the same time, by estimate (3.62) that applies to m_n we get that the rest of the terms in the energy go to zero in the sense of measures as $n \to \infty$. This gives the desired result. □

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