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A multiple scale pattern formation cascade in reaction-diffusion systems of activator-inhibitor type

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A family of singular limits of reaction-diffusion systems of activator-inhibitor type in which stable stationary sharp-interface patterns may form is investigated. For concreteness, the analysis is performed for the FitzHugh-Nagumo model on a suitably rescaled bounded domain in $\mathbb{R}^N$, with $N \geqslant 2$. It is shown that when the system is sufficiently close to the limit the dynamics starting from the appropriate smooth initial data breaks down into five distinct stages on well-separated time scales, each of which can be approximated by a suitable reduced problem. The analysis allows to follow fully the progressive refinement of spatio-temporal patterns forming in the systems under consideration and provides a framework for understanding the pattern formation scenarios in a large class of physical, chemical, and biological systems modeled by the considered class of reaction-diffusion equations.

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1. Introduction

It is now well established that nonlinear systems of coupled reaction-diffusion equations may be capable of rich dynamical behaviors that give rise to the emergence of spatio-temporal patterns [3, 8, 12, 22]. Mathematical studies of patterns are complicated by the fact that even relatively "simple" systems of reaction-diffusion equations may possess solutions that can be extraordinarily complex [5, 19, 20, 25, 26]. At the same time, these complex solutions may arise generically in the situations that mimic physically relevant conditions and hence are important to the physical systems these equations model [8, 10, 19, 31].

Perhaps the most well-known class of pattern-forming systems exhibiting complex nonlinear behaviors are reaction-diffusion systems of activator-inhibitor type [8, 12]:

$$\alpha u_t = \varepsilon^2 \Delta u + f(u, v), \tag{1.1}$$

$$v_t = \Delta v + g(u, v). \tag{1.2}$$

Here, $u = u(y, t) \in \mathbb{R}$ is the activator variable, $v = v(y, t) \in \mathbb{R}$ is the inhibitor variable, f and g are the nonlinearities, ε and α are positive parameters denoting the ratios of the length and time scales of the activator and the inhibitor, respectively, $y \in \Omega_\varepsilon \subset \mathbb{R}^N$ is the spatial coordinate, and t is time. Equations (1.1) and (1.2) arise when modeling many applications in physics, chemistry, and biology, from combustion to autocatalytic chemical reactions and biological tissues undergoing morphogenesis [7, 8, 21].

The fact that u is the activator implies that there exists a positive feedback for u in (1.1), which in mathematical terms means that the nonlinearity f obeys the relation [8]

$$\frac{\partial f(u, v)}{\partial u} > 0 \tag{1.3}$$

in some range of values of u and v. Similarly, the fact that v is the inhibitor means that there is no positive feedback for v in (1.2), and that there is a negative feedback in the response of v to variations of u. Again, for (1.1) and (1.2) this can be expressed as [8]

$$\frac{\partial g(u, v)}{\partial v} < 0, \qquad \frac{\partial g(u, v)}{\partial u} \frac{\partial f(u, v)}{\partial v} < 0, \tag{1.4}$$

for all u and v. In particular, if (1.3) holds on an open interval of u for any fixed v, while $\partial f(u, v)/\partial u < 0$ outside the closure of this interval, then f is a cubic-like function. A canonical example is the FitzHugh-Nagumo system, a version of which has the following nonlinearities [19]:

$$f(u, v) = u - u^3 - v, \qquad g(u, v) = u - v - a, \tag{1.5}$$

where $a \in \mathbb{R}^+$ is a fixed parameter. For $v \in (-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9})$, f is bistable in the sense that the ordinary differential equation $u_t = f(u, v)$ has two stable solutions $h_-(v)$, $h_+(v)$ and one unstable solution $h_0(v)$, where $h_-(v) < h_0(v) < h_+(v)$ are the three solutions of the algebraic solution $f(u, v) = 0$. This is the kind of nonlinearity, which we will consider in this paper. A rich variety of patterns in this class of systems has been observed both numerically and analytically [14, 19, 25].

In view of the great complexity of the observed spatio-temporal dynamics, various types of reductions are usually employed to better understand these nonlinear phenomena. An especially fruitful approach which has been successfully used to study pattern formation, relies on the strong separation of spatial scales between the activator and the inhibitor. These studies are also motivated by the fact that strong time and length scale separation is routinely observed in applications [8]. In the case of (1.1) and (1.2) the length scale separation is expressed in the smallness of the parameter ε in (1.1) [4, 5, 13–15, 18, 25, 27]. One can get insights into the pattern formation scenarios by investigating the limit $\varepsilon \downarrow 0$ under various assumptions on the scaling of other parameters with ε [2, 30]. Nevertheless, the main difficulty in such an approach lies in the fact that the problem under consideration is intrinsically multiscale. Thus, it is not generally possible to analyze the events leading to the formation of a particular pattern using a single limit procedure. For reaction-diffusion systems of activator-inhibitor type with cubic-like nonlinearity f this point was already recognised in [14–16, 18, 19, 23, 24].

To our knowledge, the first rigorous attempt to analyze the sequence of pattern formation events arising at different time scales in ε in the class of systems (1.1)-(1.3) was made by Sakamoto [28]. More precisely, Sakamoto considered (1.1) and (1.2) under the assumptions that the domain Ω_ε is obtained from a fixed bounded domain Ω via rescaling by a factor of $\varepsilon^{1/3}$, consistent with the expected length scale of stable stationary sharp interface patterns [11, 14, 17, 18, 23, 24], as well as assuming that $\alpha = O(\varepsilon^{2/3})$. He was able to prove, under suitable assumptions on the nonlinearities, that the solutions of the initial-value problem for (1.1) and (1.2) with the initial data varying on the spatial scale of Ω_ε evolve in several stages on well-separated time scales when $\varepsilon \ll 1$. These stages can be summarized as follows:

1. The distribution of u approaches sharp interfaces on $O(\varepsilon^{2/3})$ time scale;
2. The interfaces move with normal velocity being a function of the average value of v, while the latter solves an ordinary differential equation, on the $O(1)$ time scale.

Note that in [28] the generation of interface result is proved only under the restrictive assumption that the initial data of v is a constant. It is also conjectured that after the completion of stage 2 above the interface will follow a different motion law on a slower $O(\varepsilon^{-2/3})$ time scale.

One question that naturally arises following the analysis of [28] is whether the formation and evolution of the spatio-temporal pattern can, in fact, be characterized across *all* time scales for a generic set of initial data, when ε is sufficiently small. Perhaps even more importantly, one should be interested in what are all possible phenomena that can be observed in the limit $\varepsilon \downarrow 0$ under different assumptions on the scaling of other quantities in the problem, such as α or the domain size. It is clear that the case studied in [28] is only one such scenario. What one needs to do is to systematically explore different scaling regimes to search for distinct reduced problems signifying qualitatively different pattern formation scenarios in the class of systems, which we consider. This paper provides a full study (across all time scales) of one family of scalings which leads to the same pattern formation scenario for $\varepsilon \ll 1$.

We are going to consider systems of reaction-diffusion equations of activator-inhibitor type under extra assumptions that $\Omega_\varepsilon = \varepsilon^{1/3}\Omega$, i.e. that Ω_ε is obtained by rescaling a fixed bounded domain Ω with $\varepsilon^{1/3}$, and

$$\alpha = O(\varepsilon^p), \qquad p \in (0, \tfrac{2}{3}). \tag{1.6}$$

The first scaling assumption is the same as in [28] and is motivated by the expected scale of stable interfacial patterns [11, 14, 17, 18, 23, 24]. The second scaling assumption is chosen so that it results in the *same* qualitative limit behavior as $\varepsilon \downarrow 0$ for all systems of activator-inhibitor type. Thus, the considered limit process generates a *universality class* of pattern-forming systems governed by (1.1) and (1.2) [3].

Let us briefly summarize here the conclusions of our analysis about the sequence of progressively longer evolution stages that will occur starting from the initial data varying on the scale of the domain Ω_ε for $\varepsilon \ll 1$ under a number of assumptions (for technical details, see the following sections):

1. u is frozen, v reaches its spatial average on the $O(\varepsilon^{2/3})$ time scale;
2. v is frozen, u forms sharp interfaces on the $O(\varepsilon^p)$ time scale; since $p < 2/3$ it follows that $\varepsilon^p \gg \varepsilon^{2/3}$;

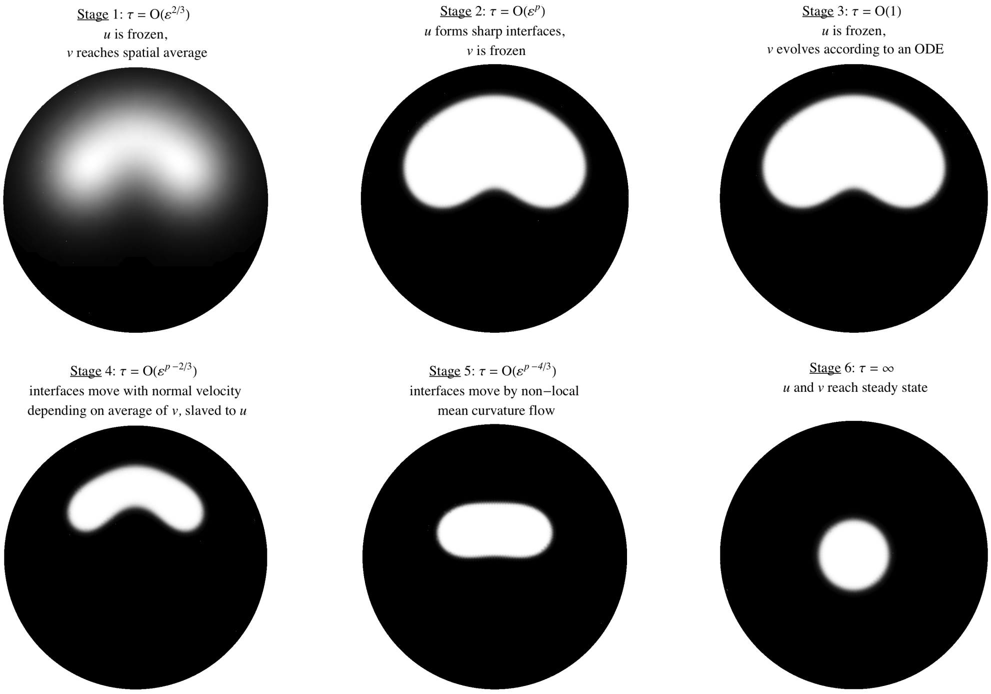

FIG. 1. An illustration of the dynamic stages at different time scales via schematic density plots of the activator variable

3. The interfaces do not move, the spatial average of v evolves by an ordinary differential equation on the $O(1)$ time scale, where $1 \gg \varepsilon^p$;
4. The interfaces move on the $O(\varepsilon^{p-2/3})$ time scale with normal velocity depending on the average of v which is slaved to the interface; note that $\varepsilon^{p-2/3} \gg 1$;
5. We also consider the $O(\varepsilon^{p-4/3})$ time scale and formally show that interfaces move by nonlocal mean-curvature, recalling that $\varepsilon^{p-4/3} \gg \varepsilon^{p-2/3}$.

This permits to characterize the evolution of patterns from the beginning to the end in the class of systems (1.1)–(1.3) with $\varepsilon \ll 1$. We illustrate this progression of stages in Fig. 1.

Our paper is organized as follows. In Section 2 we present the main results of this paper. In Section 3 we prove some preliminary estimates on u^ε and v^ε, which in particular imply that u^ε and v^ε are bounded. In Section 4 we deduce from the previous estimates that on the time interval $[0, \tau_1 \varepsilon^{2/3} |\ln \varepsilon|]$, u^ε is close to its initial condition u_0 and that v^ε is close to the spatial average of v_0. Following Xinfu Chen [2], we obtain in Section 5 that at the time $\tau_2^\varepsilon := \tau_2 \varepsilon^p |\ln \varepsilon|$, the solution u^ε develops an interface Γ and that v^ε stays close to the average of v_0. In Section 6 we prove that there exists a time τ_3^ε of order $|\ln \varepsilon|$ such that the interface Γ already formed does not move on the interval $[\tau_2^\varepsilon, \tau_2^\varepsilon + \tau_3^\varepsilon]$. Moreover, in each region, Ω^- and Ω^+, of Ω separated by Γ we deduce that v^ε is approximated by the solution, $t \mapsto \tilde{v}_3(t)$, of an ordinary differential equation. In Section 7, we prove that on the time interval $[\tau_2^\varepsilon + \tau_3^\varepsilon, \tau_4^\varepsilon]$, where τ_4^ε is of order $\varepsilon^{p-2/3}$, $(u^\varepsilon, v^\varepsilon)$ tends to the solution of a free boundary problem where the motion equation connects the velocity of Γ to the limit $\tilde{v}_4$ of v^ε and that $\tilde{v}_4$ is a solution of an algebraic equation. This leads us to consider in Section 8 a larger time interval, and we formally obtain that the interface moves by a nonlocal mean-curvature flow. Our proofs are based on the comparison principle associated with (1.1) and on a construction of non classical sub- and supersolutions of (1.1). For that purpose we use traveling wave solutions of a related one-dimensional parabolic system and a modified distance function to the interface. Then u^ε is squeezed between two functions which have the profile of the traveling wave and which converge to h_+ in Ω^+ and h_- in Ω^-. Thus we deduce the convergence of u^ε in each subdomain $\Omega^\pm$. Furthermore, the sub- and supersolutions depend on v^ε and hence on $h_\pm(v^\varepsilon)$ and $h_0(v^\varepsilon)$. This leads us to smoothly extend the functions $h_\pm(v)$, $h_0(v)$ to the whole of $\mathbb{R}$ and, as a consequence, the function $f(u, v) = -(u-h_+(v))(u-h_-(v))(u-h_0(v))$ to the whole of $\mathbb{R}^2$. We then introduce an extended problem for (1.1)–(1.2) and prove that its unique solution coincides with $(u^\varepsilon, v^\varepsilon)$.

2. Statement of results

We consider the following system

$$
(P^\varepsilon)\begin{cases}
\varepsilon^p u_t^\varepsilon = \varepsilon^{4/3}\Delta u^\varepsilon + f(u^\varepsilon, v^\varepsilon) & \text{in } \Omega\times(0,T) \qquad (2.1)\\
v_t^\varepsilon = \varepsilon^{-2/3}\Delta v^\varepsilon + g(u^\varepsilon, v^\varepsilon) & \text{in } \Omega\times(0,T) \qquad (2.2)\\
\dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T) \qquad (2.3)\\
u^\varepsilon(x,0) = u_0(x),\ \ v^\varepsilon(x,0) = v_0(x) \text{ for } x\in\Omega & \qquad (2.4)
\end{cases}
$$

and we suppose that
(H_1) $\Omega \subset R^N$ $(N \geqslant 2)$ a smooth (C^∞) bounded domain,
(H_2) $0 < p < 2/3$,
(H_3) $f(u, v) = u(1-u^2) - v$ and $g(u, v) = u - v - a$.
As it has been described in the introduction, Problem (P^ε) can be obtained from (1.1) and (1.2) by setting $\alpha = \varepsilon^p$ (cf. (1.6)) and $x = \varepsilon^{-1/3}y$, so that $y \mapsto x$ maps Ω_ε into Ω. Moreover in what follows we use the notation $\displaystyle⨍_\Omega h(x,t)dx = \frac{1}{|\Omega|}\int_\Omega h(x,t)dx$ for all functions h.

2.1 *First stage: v^ε becomes close to the spatial average of its initial condition in a time of order $\varepsilon^{2/3}|\ln\varepsilon|$*

We first prove the following result about the approximate attainment by the v^ε variable of its spatial average in the first stage.

Theorem 2.1 *Assume that u_0 and v_0 are in $C^2(\overline{\Omega})$ and satisfy the compatibility condition*

$$\frac{\partial u_0}{\partial n} = \frac{\partial v_0}{\partial n} = 0. \tag{2.5}$$

Let $(u^\varepsilon, v^\varepsilon)$ be the solution of (P^ε) then there exist positive constants τ_1, M_1 and $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$

$$|v^\varepsilon(x,\tau_1^\varepsilon) - ⨍_\Omega v_0(x)dx| \leqslant M_1\varepsilon^{\frac{4}{3(N+2)}},\ \textit{for all } x\in\Omega \tag{2.6}$$

and

$$u_0(x) - M_1\varepsilon^{2/3-p}|\ln\varepsilon| \leqslant u^\varepsilon(x,\tau_1^\varepsilon) \leqslant u_0(x) + M_1\varepsilon^{2/3-p}|\ln\varepsilon| \text{ for all } x \in \Omega \quad (2.7)$$

where $\tau_1^\varepsilon = \tau_1\varepsilon^{2/3}|\ln\varepsilon|$.

2.2 *Second stage: Generation of interface in a time of order* $\varepsilon^p|\ln\varepsilon|$

We prove in this section a generation of interface result at the time

$$\tau_2^\varepsilon = \tau_2\varepsilon^p|\ln\varepsilon| \text{ where } \tau_2 \text{ is a positive constant and } p \in (0, \tfrac{2}{3}) \quad (2.8)$$

Assuming that

$$\left|\fint_\Omega v_0(x)dx\right| \leqslant \frac{2\sqrt{3}}{9} - \sigma, \quad (2.9)$$

where $\sigma > 0$ is small enough such that Lemma B.1 is valid, then we have the following result

Theorem 2.2 *Let*

$$L^\varepsilon := \varepsilon^{\frac{p}{2}}|\ln\varepsilon| + \varepsilon^{2/3-p}|\ln\varepsilon| + \varepsilon^{\frac{4}{3(N+2)}} \quad (2.10)$$

then there exist positive constants ε_2*,* M_2 *and* τ_2 *such that for all* $\varepsilon \in [0,\varepsilon_2)$ *the solution* $(u^\varepsilon, v^\varepsilon)$ *of Problem* (P^ε) *satisfy that*

$$h_-\left(\fint v_0\right) - M_2L^\varepsilon \leqslant u^\varepsilon(x,\tau_2^\varepsilon) \leqslant h_+\left(\fint_\Omega v_0\right) + M_2L^\varepsilon, \forall x \in \Omega, \quad (2.11)$$

$$\left|u^\varepsilon(x,\tau_2^\varepsilon) - h_+\left(\fint_\Omega v_0\right)\right| \leqslant M_2L^\varepsilon, \forall x \in \Omega^{\varepsilon,+}, \quad (2.12)$$

$$\left|u^\varepsilon(x,\tau_2^\varepsilon) - h_-\left(\fint_\Omega v_0\right)\right| \leqslant M_2L^\varepsilon, \forall x \in \Omega^{\varepsilon,-}, \quad (2.13)$$

where

$$\Omega^{\varepsilon,+} := \left\{x \in \Omega,\ u_0(x) \geqslant h_0\left(\fint_\Omega v_0\right) + M_2L^\varepsilon\right\},$$
$$\Omega^{\varepsilon,-} := \left\{x \in \Omega,\ u_0(x) \leqslant h_0\left(\fint_\Omega v_0\right) - M_2L^\varepsilon\right\}.$$

Moreover there exists a positive constant $K > 0$ *such that*

$$|v^\varepsilon(x,\tau_2^\varepsilon) - \fint_\Omega v_0| \leqslant KL^\varepsilon. \quad (2.14)$$

2.3 *Third stage: time evolution with a fixed interface*

The goal of this section is the study of Problem (P^ε) on a time interval $[\tau_2^\varepsilon, \tau_3^\varepsilon]$, where

$$\tau_3^\varepsilon = \frac{c_{p,N}|\ln\varepsilon|}{m} \quad (2.15)$$

with

$$0 < c_{p,N} \leqslant \frac{1}{6}\min\left(\frac{p}{2}, \frac{4}{3(N+2)}, \frac{2}{3} - p\right), \tag{2.16}$$

and $m \geqslant 2$ is a constant to be chosen later. Setting

$$\Gamma_0 = \left\{x \in \Omega,\ u_0(x) = h_0\left(\fint_\Omega v_0\right)\right\}, \tag{2.17}$$

$$\Omega^{0,+} := \left\{x \in \Omega,\ u_0(x) > h_0\left(\fint_\Omega v_0\right)\right\} \quad \text{and} \quad \Omega^{0,-} := \left\{x \in \Omega,\ u_0(x) < h_0\left(\fint_\Omega v_0\right)\right\}, \tag{2.18}$$

we assume that Γ_0 is a smooth hypersurface. Moreover we also suppose that u_0 and $\fint_\Omega v_0$ satisfy

$$(H4)\begin{cases} u_0(x) - h_0\left(\fint_\Omega v_0\right) \geqslant \eta_0\, dist(x, \Gamma_0) & \text{if } x \in \Omega^{0,+} \\ u_0(x) - h_0\left(\fint_\Omega v_0\right) \leqslant -\eta_0 dist(x, \Gamma_0) & \text{if } x \in \Omega^{0,-} \end{cases}$$

where $dist(x, \Gamma_0)$ denotes the distance function from x to Γ_0. In this stage we prove that in the time interval $[\tau_2^\varepsilon, \tau_3^\varepsilon]$ the interface, already formed in stage 2, does not move and moreover that as $\varepsilon \downarrow 0$ v^ε tends to a function which only depends on t. Lastly, we define the set B

$$B := \left[-\frac{2\sqrt{3}}{9} + \frac{\sigma}{2}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{2}\right],$$

where σ is a small constant introduced in (2.9).

Theorem 2.3 *Assuming*

$$\frac{|\Omega^+(0)|}{|\Omega|} \in (\alpha_-, \alpha_+) \tag{2.19}$$

*where $\alpha_\pm$ are defined by (*B.1*) then the initial value problem*

$$(E)\begin{cases} (\tilde{v}_3)_t(t) &= -\tilde{v}_3(t) + h_+\big(\tilde{v}_3(t)\big)\dfrac{|\Omega^{0,+}|}{|\Omega|} + h_-\big(\tilde{v}_3(t)\big)\left(1 - \dfrac{|\Omega^{0,+}|}{|\Omega|}\right) - a \\ \tilde{v}_3(0) &= \fint_\Omega v_0(x)\, dx \end{cases}$$

possesses a unique solution $t \mapsto \tilde{v}_3(t)$ defined on $[0, +\infty]$ *such that*

$$\tilde{v}_3(t) \in B, \tag{2.20}$$

for all $t \in$ $[0, +\infty)$. *Let $\tilde{v}_{3,\infty} := \lim_{t\to\infty} \tilde{v}_3(t)$ then we have*

$$-\tilde{v}_{3,\infty} + h_+(\tilde{v}_{3,\infty})\frac{|\Omega^{0,+}|}{|\Omega|} + h_-(\tilde{v}_{3,\infty})\left(1 - \frac{|\Omega^{0,+}|}{|\Omega|}\right) - a = 0, \tag{2.21}$$

and moreover there exists a positive constant C such that

$$|\tilde{v}_3(\tau_3^\varepsilon) - \tilde{v}_{3,\infty}| \leqslant C\varepsilon^{\frac{c_{p,N}}{m}}. \tag{2.22}$$

Setting

$$u_3(x,t) = \begin{cases} h_+\big(\tilde v_3(t)\big) \text{ if } x \in \Omega^{0,+} \cup \Gamma_0 \text{ and } t \in [0,\tau_3^\varepsilon] \\ h_-\big(\tilde v_3(t)\big) \text{ if } x \in \Omega^{0,-} \text{ and } t \in [0,\tau_3^\varepsilon] \end{cases}$$

and denoting by $\tilde d(x,\Gamma_0)$ the signed distance to Γ_0 such that

$$\begin{cases} \tilde d(x,\Gamma_0) = dist(x,\Gamma_0) & \text{if } x \in \Omega^{0,+} \\ \tilde d(x,\Gamma_0) = -dist(x,\Gamma_0) & \text{if } x \in \Omega^{0,-} \end{cases}$$

we then obtain the convergence theorem

Theorem 2.4 *Assuming (2.19) and $(H4)$ there exist positive constants M_3 and $\varepsilon_3 > 0$ such that for all $\varepsilon \in (0,\varepsilon_3]$ the solution $(u^\varepsilon, v^\varepsilon)$ of Problem (P^ε) satisfies that*

$$|u^\varepsilon(x,t+\tau_2^\varepsilon) - u_3(x,t)| \leqslant M_3 \varepsilon^{\frac{c_{p,N}}{2}}, \tag{2.23}$$

for all $x \in \{x \in \Omega,\ |\tilde d(x,\Gamma_0)| \geqslant \varepsilon^{c_{p,N}}\}$ and $t \in [0,\tau_3^\varepsilon]$. Moreover we also have

$$|v^\varepsilon(x,t+\tau_2^\varepsilon) - \tilde v_3(t)| \leqslant M_3 \varepsilon^{\frac{c_{p,N}}{2}}, \tag{2.24}$$

for all $x \in \Omega$ and $t \in [0,\tau_3^\varepsilon]$.

2.4 *Fourth stage: Propagation of interface for large time*

The goal of this stage is to study Problem (P^ε) for $t \geqslant \tau_2^\varepsilon + \tau_3^\varepsilon$. We first consider the limit problem,

$$(Q_4)\begin{cases} V_{n,4} = \dfrac{3}{\sqrt 2} h_0(\tilde v_4) & \text{on } \Gamma_4(s),\ \ s \in (0,\tilde T_4), \quad (2.25) \\ h_+(\tilde v_4)\dfrac{|\Omega_4^+|}{|\Omega|} + h_-(\tilde v_4)\left(1 - \dfrac{|\Omega_4^+|}{|\Omega|}\right) - \tilde v_4 - a = 0 & \text{in } (0,\tilde T_4), \quad (2.26) \\ \Gamma_4|_{|t=0} = \Gamma_0, \end{cases}$$

where $\tilde v_4 = \tilde v_4(s)$, $\Omega_4^+(s)$ the interior of $\Gamma_4(s)$, $\Omega_4^-(s) = \Omega \setminus \overline{\Omega_4^+(s)}$ and $V_{n,4}$ is the normal velocity of $\Gamma_4(s)$ in the direction of $\Omega_4^+(s)$. We note that $\tilde v_4(0) = \tilde v_{3,\infty}$ and that the velocity $V_{n,4}$ only depends on s and we state that (Q_4) is well posed locally in time.

Theorem 2.5 *Assume that (2.19) holds. There exists $\tilde T_4 > 0$ such that the free boundary Problem (Q_4) has a unique smooth solution $(\tilde v_4(s), \Gamma_4(s))$ for $s \in [0,\tilde T_4]$.*

Let

$$u_4(x,s) = \begin{cases} h_+\big(\tilde v_4(s)\big) \text{ if } x \in \Omega_4^+(s) \cup \Gamma_4(s) \text{ and } s \in [0,\tilde T_4] \\ h_-\big(\tilde v_4(s)\big) \text{ if } x \in \Omega_4^-(s) \text{ and } s \in [0,\tilde T_4], \end{cases}$$

we then obtain the following convergence result

Theorem 2.6 *Assume that (2.19) and $(H4)$ hold. There exists a positive constant T_4 such that for all $\varepsilon^* > 0$ we have*

$$\left|u^\varepsilon(x,\varepsilon^{p-2/3}s + \tau_2^\varepsilon + \tau_3^\varepsilon) - u_4(x,s)\right| \leqslant \varepsilon^*, \tag{2.27}$$

for all $(x, s) \in \{(x, s) \in \Omega \times [0, T_4],\ |\tilde{d}(x, \Gamma_4(s))| \geqslant \varepsilon^*\}$ *and all* ε *small enough. Moreover*

$$\left|v^\varepsilon(x, \varepsilon^{p-2/3}s + \tau_2^\varepsilon + \tau_3^\varepsilon) - \tilde{v}_4(s)\right| \leqslant \varepsilon^*, \tag{2.28}$$

for all $(x, s) \in \Omega \times [0, T_4]$ *and* ε *small enough.*

We note that in general the solution of (Q_4) may not exist for all $s > 0$ because of the possibility of Γ_4 vanishing in finite time. Nevertheless, if the solution of (Q_4) exists globally in time, it must necessarily reach a steady state as $s \to \infty$. Indeed, if $P(\Omega_4^+)$ is the perimeter of Ω_4^+, which in view of the regularity of Γ_4 coincides with the $(N-1)$-dimensional Hausdorff measure of Γ_4, from (2.25) and our sign convention on $V_{n,4}$ we obtain

$$\frac{d|\Omega_4^+(s)|}{ds} = -\frac{3P\big(\Omega_4^+(s)\big)}{\sqrt{2}} h_0\big(\tilde{v}_4(s)\big). \tag{2.29}$$

At the same time, (2.26) is equivalent to

$$|\Omega_4^+| = \frac{a + \tilde{v}_4 - h_-(\tilde{v}_4)}{h_+(\tilde{v}_4) - h_+(\tilde{v}_4)}\, |\Omega|. \tag{2.30}$$

It is a calculus exercise to show that the right-hand side of (2.30) is a strictly monotone increasing smooth function of $\tilde{v}_4 \in (-2\sqrt{3}/9, 2\sqrt{3}/9)$ for all $a \in (-1, 1)$. Therefore, for any given $|\Omega_4^+| \in (0, |\Omega|)$ and $a \in (-1, 1)$ there is at most one value of $\tilde{v}_4$ that satisfies (2.30), and this value of $\tilde{v}_4$ is increasing as $|\Omega_4^+|$ increases. In particular, introducing

$$|\Omega_{4,\infty}^+| = \frac{a+1}{2}|\Omega|, \tag{2.31}$$

it follows that $\tilde{v}_4 > 0$ whenever $|\Omega_4^+| > |\Omega_{4,\infty}^+|$, and vice versa. At the same time, recalling that the sign of $h_0(\tilde{v}_4)$ coincides with that of $\tilde{v}_4$, from (2.29) we get that $|\Omega_4^+(s)| \to |\Omega_{4,\infty}^+|$ and $\tilde{v}_4(s) \to 0$ as $s \to \infty$, if the solution of (Q_4) exists for all $s > 0$. In fact, this convergence is exponential, as can be easily seen from the linearization of (2.29). Thus, in the limit $s \to \infty$ the interface $\Gamma_4(s)$ solving (Q_4) must converge (in Hausdorff sense) to some limiting interface $\Gamma_{4,\infty}$ enclosing a set $\Omega_{4,\infty}^+$ whose measure satisfies (2.31), while $\tilde{v}_4$ vanishes asymptotically. We note that the coupling between $\Gamma_4(s)$ and $\tilde{v}_4(s)$ is such that small deviations of $|\Omega_4(s)|$ from $\frac{1}{2}(1+a)|\Omega|$ are restored via the solution of (Q_4) on the time scale of the Fourth stage. This property should naturally be inherited by the solutions of (P^ε) for small enough ε. Therefore, on longer time scales one should expect that the limiting problem preserves the volume of the zero super-level set of $u^\varepsilon(\cdot, t)$ for all t larger than the Fourth stage time scale. Noting that $\lim_{\varepsilon \to 0} u^\varepsilon(x, \varepsilon^{p-2/3}s) = u_4(x, s) = h_\pm(\tilde{v}_4(s)) \to \pm 1$ as $s \to \infty$ for all $x \in \Omega_{4,\infty}^\pm$, respectively, with $\Omega_{4,\infty}^- := \Omega \backslash \overline{\Omega_{4,\infty}^+}$, we should then expect

$$\lim_{\varepsilon \to 0} \fint_\Omega u^\varepsilon(x, t)\, dx = a \qquad\qquad t \gg \varepsilon^{p-2/3}. \tag{2.32}$$

In other words, beyond the Fourth stage the evolution governed by (P^ε) should become asymptotically mass-preserving, which is an interesting feature of the considered problem, since a priori problem (P^ε) does not have such a property.

2.5 *Fifth stage: Propagation with non local mean curvature*

In this stage we assume that there exists a solution of the free boundary problem

$$\begin{cases} V_n = K + C'(0)\hat{w}_5 - \frac{1}{|\Gamma_5|}\left(\int_{\Gamma_5} K + C'(0)\int_{\Gamma_5}\hat{w}_5\right) \text{ for all } \tau \in (0, T_5), \\ \Gamma_5(0) = \Gamma_{5,0}, \end{cases}$$

where $\hat{w}_5$ satisfies

$$(Q_5)\begin{cases} -\Delta\hat{w}_5 = u_5 - a & \text{in } \Omega \times (0, T_5), \\ \dfrac{\partial \hat{w}_5}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T_5), \\ \displaystyle\int_\Omega \hat{w}_5 dx = 0 & \text{for all } \tau \in (0, T_5), \end{cases}$$

with

$$u_5(x,t) = \begin{cases} 1 & \text{if } x \in \Omega_5^+(\tau), \\ -1 & \text{if } x \in \Omega_5^-(\tau), \end{cases}$$

where V_n, K, $\Omega_5^+(\tau)$ and $\Omega_5^-(\tau)$ denote respectively the normal velocity in the direction of $\Omega_5^+(\tau)$, the sum of principal curvatures (positive if $\Omega_5^+(\tau)$ is convex), the interior and the exterior of $\Gamma_5(\tau)$, respectively. We remark that (Q_5) only makes sense if $⨍_\Omega u_5(s,\tau)\,dx = a \in (-1,1)$. We formally show that $u^\varepsilon(\cdot, \varepsilon^{p-4/3}s)$ solving (P^ε) converges to $u_5(\cdot,\tau)$ as ε tends to zero when $\tau \in (0, T_5)$ and $\Gamma_{5,0} = \Gamma_{4,\infty}$, where $\Gamma_{4,\infty}$ is the asymptotic limit of $\Gamma_4(s)$ solving (Q_4) as $s \to \infty$, provided it exists. Note that the free boundary problem above preserves $|\Omega_5(\tau)|$, which in view of (2.31) satisfies

$$|\Omega_5(\tau)| = \frac{1+a}{2}|\Omega| \qquad \text{for all } \tau \in (0, T_5). \tag{2.33}$$

This is due, as was already mentioned, to the strong restoring effect on $|\Omega_5(\tau)|$ from the spatial average of v inherited from the Fourth stage. We also note that the resulting volume-preserving nonlocal mean curvature flow is a gradient flow. Hence one expects that the interface $\Gamma_5(\tau)$ ultimately reaches a steady state as $\tau \to \infty$, if the solution is global in time.

3. Preliminary estimates

Lemma 3.1 *Assume that u_0 and v_0 are in $C^2(\overline{\Omega})$ and satisfy the homogeneous Neumann boundary conditions* (2.5)*; then there exists a unique solution of the system (P^ε) for all $0 < T \leqslant \infty$. Moreover there exists a positive constant C_0 such that for all $\varepsilon > 0$*

$$|u^\varepsilon(x,t)| + |v^\varepsilon(x,t)| \leqslant C_0 \textit{ for all } x \in \Omega \textit{ and } t \geqslant 0. \tag{3.1}$$

Proof. From standard theory for parabolic systems we deduce the existence of a unique solution $(u^\varepsilon, v^\varepsilon)$ of (P^ε). Moreover applying the Corollary 14.8 of [29] we obtain the estimate (3.1).

Next we state some estimates, which will be useful in what follows, namely

Lemma 3.2 *We set $\overline{v}^\varepsilon(t) := ⨍_\Omega v^\varepsilon(x,t)dx$; then there exist positive constants $\tilde{C}_0$ and τ_1 such that*

$$\left|\overline{v}^\varepsilon(t) - ⨍_\Omega v_0(x)dx\right| \leqslant \tilde{C}_0 t, \textit{ for all } t \geqslant 0, \tag{3.2}$$

and

$$\max_{x\in\overline{\Omega}} |v^\varepsilon(x,t) - \overline{v}^\varepsilon(t)| \leqslant \tilde{C}_0 \varepsilon^{\frac{4}{3(N+2)}},\ \textit{for all } t \geqslant \tau_1^\varepsilon := \tau_1 \varepsilon^{2/3} |\ln \varepsilon|. \tag{3.3}$$

Proof. Integrating (2.2) on Ω and on $[0,t]$ for all $t \geqslant 0$ and using (3.1) we obtain that

$$\left| \int_0^t (\overline{v}^\varepsilon)_s(s) ds \right| = |\overline{v}^\varepsilon(t) - \overline{v}^\varepsilon(0)| \leqslant C_1 t,$$

which coincides with (3.2). Next we prove a preliminary estimate, which will be useful to obtain (3.3), namely

$$\|\nabla v^\varepsilon(.,t)\|^2_{L^2(\Omega)} + \|v^\varepsilon(.,t) - \overline{v}^\varepsilon(t)\|^2_{L^2(\Omega)} \leqslant C_2 \|\nabla v_0\|^2_{L^2(\Omega)} \exp\Big(-\frac{\lambda}{\varepsilon^{2/3}} t \Big) + C_2 \varepsilon^{4/3}. \tag{3.4}$$

Multiplying (2.2) by Δv^ε and integrating the result on Ω we obtain

$$\int_\Omega \nabla v^\varepsilon \nabla v^\varepsilon_t = -\varepsilon^{-2/3} \int_\Omega |\Delta v^\varepsilon|^2 - \int_\Omega \Delta v^\varepsilon g(u^\varepsilon, v^\varepsilon),$$

so that using (3.1)

$$\frac{d}{dt} \|\nabla v^\varepsilon\|^2_{L^2(\Omega)} \leqslant -\varepsilon^{-2/3} \int_\Omega |\Delta v^\varepsilon|^2 + \varepsilon^{2/3} C_3.$$

This together with the inequality

$$\|\Delta v^\varepsilon\|^2_{L^2(\Omega)} \geqslant \lambda \|\nabla v^\varepsilon\|^2_{L^2(\Omega)},$$

gives

$$\frac{d}{dt} \|\nabla v^\varepsilon\|^2_{L^2(\Omega)} \leqslant -\varepsilon^{-2/3} \lambda \|\nabla v^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{2/3} C_3,$$

which by Gronwall Lemma implies

$$\|\nabla v^\varepsilon(.,t)\|^2_{L^2(\Omega)} \leqslant \|\nabla v^\varepsilon(.,0)\|^2_{L^2(\Omega)} \exp\left(-\frac{\lambda}{\varepsilon^{2/3}} t \right) + \frac{C_3}{\lambda} \varepsilon^{4/3}.$$

This together with the Poincaré inequality gives (3.4). We now prove that there exists a constant K_0 independent of ε such that

$$|v^\varepsilon(x,t) - v^\varepsilon(x',t)| \leqslant K_0 |x - x'|,\ \text{for } t \geqslant 0,\ \ x, x' \in \overline{\Omega} \tag{3.5}$$

$$|\nabla v^\varepsilon(x,t) - \nabla v^\varepsilon(x',t)| \leqslant K_0 |x - x'|^{3/4},\ \text{for } t \geqslant 0,\ \ x, x' \in \overline{\Omega}. \tag{3.6}$$

Setting $s := \frac{t}{\varepsilon^{2/3}}$ we deduce from (2.2) that v^ε is the solution of

$$v_s + Av = v + \varepsilon^{2/3} g(u, v) =: G(v, x, s),$$

where $Av = -\Delta v + v$. As it is done in [28], one can check that there exists a constant $C_{\alpha,p}$ such that

$$\|A^\alpha v(s)\|_{L^p} \leqslant C_0 \|A^\alpha v(0)\|_{L^p} + C_{\alpha,p},$$

with $\alpha \in (0,1)$, $p > 1$. Further since

$$D(A^{\alpha}) \subset C^{1+\nu}(\overline{\Omega}),\ \text{for } 2\alpha - N/p > 1 + \nu \text{ and } \nu \in (0,1),$$

we obtain choosing $\alpha = \frac{15}{16}$ and $p > 8N$ that

$$v^{\varepsilon}(.,t) \in C^{1+\nu}(\overline{\Omega}) \text{ and } |v^{\varepsilon}(.,t)|_{C^{1+\nu}(\overline{\Omega})} \leqslant K_0,\ \text{for } \nu = 3/4,$$

which gives (3.5) and (3.6). Applying (3.4) for all $\tau \geqslant \tau_1^{\varepsilon} := \tau_1 \varepsilon^{2/3} |\ln \varepsilon|$ where $\tau_1 = \dfrac{4}{3\lambda}$ we obtain that

$$\|\nabla v^{\varepsilon}(.,t)\|^2_{L^2(\Omega)} + \|v^{\varepsilon}(.,t) - \overline{v}^{\varepsilon}(t)\|^2_{L^2(\Omega)} \leqslant C_2 \|\nabla v^{\varepsilon}(.,0)\|^2_{L^2(\Omega)} \exp\left(-\frac{4}{3}|\ln \varepsilon|\right) + C_2 \varepsilon^{4/3},$$

so that

$$\|v^{\varepsilon}(.,t) - \overline{v}^{\varepsilon}(t)\|^2_{L^2(\Omega)} = O(\varepsilon^{4/3}). \tag{3.7}$$

Furthermore in view of (3.5)-(3.7), we deduce from Lemma 3.2 of [28] that for all $\tau \geqslant \tau_1^{\varepsilon} := \tau_1 \varepsilon^{2/3} |\ln \varepsilon|$

$$\max_{x \in \overline{\Omega}} \left| v^{\varepsilon}(x,t) - \overline{v}^{\varepsilon}(t) \right| \leqslant C \|v^{\varepsilon}(.,t) - \overline{v}^{\varepsilon}(t)\|_{L^2(\Omega)}^{\frac{2\tilde{\nu}}{N+2\tilde{\nu}}} \tag{3.8}$$

where $\tilde{\nu} \in (0,1]$ and C is a positive constant. Thus (3.7) and (3.8) with $\tilde{\nu} = 1$ imply (3.3), which concludes the proof of Lemma 3.2.

4. Proof of the first stage: v^{ε} becomes close to $\fint_{\Omega} v_0$ in a time of order $\varepsilon^{2/3}|\ln \varepsilon|$

Proof of Theorem 2.1. By (3.2) with $\tau = \tau_1^{\varepsilon} = \tau_1 \varepsilon^{2/3} |\ln \varepsilon|$ we obtain that

$$\left| \overline{v}^{\varepsilon}(\tau_1^{\varepsilon}) - \fint_{\Omega} v_0(x)dx \right| \leqslant \tilde{C}_0 \tau_1 \varepsilon^{2/3} |\ln \varepsilon|.$$

This together with (3.3) gives

$$\max_{x \in \overline{\Omega}} \left| v^{\varepsilon}(x,\tau_1^{\varepsilon}) - \fint_{\Omega} v_0(x)dx \right| \leqslant C_1 \varepsilon^{\frac{4}{3(N+2)}},$$

which implies (2.6). Next we prove (2.7). We set $U^{\pm}(x,t) := u_0(x) \pm \frac{C_1}{\varepsilon^p} t$ where C_1 is a constant such that $C_1 \geqslant 2C_0 + 2 + (C_0+1)^3$ with C_0 defined in lemma 3. Denoting by L^{ε} the parabolic operators associated to (2.1) one can check that

$$L^{\varepsilon}(U^+, v^{\varepsilon}) \geqslant 0 \text{ and } L^{\varepsilon}(U^-, v^{\varepsilon}) \leqslant 0, \tag{4.1}$$

on $\Omega \times [0, \tau_1^{\varepsilon}]$ and then deduce from the comparison principle that

$$U^-(x,t) \leqslant u^{\varepsilon}(x,t) \leqslant U^+(x,t),\ \text{for all } (x,t) \in \Omega \times [0,\tau_1^{\varepsilon}].$$

Applying this with $t = \tau_1^{\varepsilon}$ we obtain (2.7), which completes the proof of Theorem 2.1. □

5. Proof of the second stage: Generation of interface in a time of order $\varepsilon^p|\ln\varepsilon|$

Proof of Theorem 2.2. Since the details of the computations are given in [2] we only give the main steps of the proof. First we remark that for all $v \in \mathbb{R}$ such that $|v| \leqslant 2\frac{\sqrt{3}}{9} - \sigma$ where $\sigma > 0$ is a positive constant, the algebraic equation $f(.,v) = 0$ has three solutions $h_-(v) < h_0(v) < h_+(v)$. As it is done in [2], we now introduce an approximation of the function f. To begin with let $s \mapsto \rho(s) \in C^\infty(\mathbb{R})$ be a cut-off function satisfying

$$\begin{cases} \rho(s) = 1, & \text{if } |s| \leqslant 1, \\ \rho(s) = 0, & \text{if } |s| \geqslant 2, \\ 0 < \rho(s) < 1, & \text{if } 1 < |s| < 2, \\ -2 < s\rho'(s) \leqslant 0, & \text{if } s \in \mathbb{R}, \\ |\rho''(s)| \leqslant 4, & \text{if } s \in \mathbb{R}, \end{cases}$$

then we set $\rho_0 = \rho\left(\dfrac{u - h_0(v)}{\varepsilon^{\frac{p}{2}}|\ln\varepsilon|}\right)$, $\rho_+ = \rho\left(\dfrac{u - h_+(v)}{\varepsilon^{\frac{p}{2}}|\ln\varepsilon|}\right)$ and $\rho_- = \rho\left(\dfrac{u - h_-(v)}{\varepsilon^{\frac{p}{2}}|\ln\varepsilon|}\right)$ and

$$\tilde{f}(u,v) := \rho_0\frac{u - h_0(v)}{|\ln\varepsilon|} + \rho_+\frac{h_+(v) - u}{|\ln\varepsilon|} + \rho_-\frac{h_-(v) - u}{|\ln\varepsilon|} + (1 - \rho_0 - \rho_- - \rho_+)f(u,v). \quad (5.1)$$

Thus following the proof of estimate (3.8) in [2] one can check that

$$|\tilde{f}(u,v) - f(u,v)| \leqslant C_f\varepsilon^{\frac{p}{2}}|\ln\varepsilon|, \text{ for all } u \in [C_0, C_0]. \quad (5.2)$$

We next show that the solution u^ε can be approximated by the solution of the following ordinary differential equation

$$(ODE)\begin{cases} \omega_s(\zeta,s,w) = \tilde{f}(\omega,w), \text{ for all } s > 0, \\ \omega(\zeta,0,w) = \zeta, \end{cases}$$

where $\zeta \in [-C_0, C_0]$ and $w \in B$. Replacing ε by $\varepsilon^{\frac{p}{2}}$ in the proof of Lemma 3.2 in [2], one can obtain the following properties of ω

Lemma 5.1 *Assume that $\zeta \in [-C_0, C_0]$ and $w \in B$ and let $\omega(\zeta,s,w)$ be the solution of (ODE). Then $\omega \in C^2(\mathbb{R} \times \mathbb{R}^+ \times B)$ and*

$$\omega_\zeta(\zeta,s,w) > 0. \quad (5.3)$$

There exist positive constants τ_2 and ε_0 such that for all $\varepsilon \in (0,\varepsilon_0]$ and $s \geqslant \tau_2$, we have

$$\omega(\zeta,s,w) \geqslant h_+(w) - 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon|, \forall\zeta \in [h_0(w) + 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon|, \infty), \quad (5.4)$$

$$\omega(\zeta,s,w) \leqslant h_-(w) + 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon|, \forall\zeta \in (-\infty, h_0(w) - 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon|], \quad (5.5)$$

and

$$h_-(w) - 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon| \leqslant \omega(\zeta,s,w) \leqslant h_+(w) + 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon|, \forall\zeta \in [-C_0, C_0]. \quad (5.6)$$

Moreover, there exists a positive constant C_1 such that for all $\varepsilon \in (0,\varepsilon_0]$ and $s \in [0, \tau_0|\ln\varepsilon|]$, we have

$$|\omega_{\zeta\zeta}| \leqslant C_1\frac{\omega_\zeta}{\varepsilon^{\frac{p}{2}}}. \quad (5.7)$$

We are now in a position to prove the generation interface. Let $\tau_2^\varepsilon = \tau_2\varepsilon^p|\ln\varepsilon|$, where τ_2 is defined in Lemma 5.1. Using (3.2) we have

$$\left|\overline{v}^\varepsilon(t) - \fint_\Omega v_0(x)dx\right| \leqslant \tilde{C}_0\tau_2\varepsilon^p|\ln\varepsilon| \tag{5.8}$$

for all $t \in [0, \tau_2^\varepsilon]$ and then by (3.3) and the definition of L^ε,(2.10), we obtain

$$\left|v^\varepsilon(x,t) - \fint_\Omega v_0(x)dx\right| \leqslant \left|v^\varepsilon(x,t) - \overline{v}^\varepsilon(t)\right| + \left|\overline{v}^\varepsilon(t) - \fint_\Omega v_0(x)dx\right| \leqslant \tilde{C}_0 L^\varepsilon, \tag{5.9}$$

for all $(x,t) \in \Omega \times [\tau_1^\varepsilon, \tau_2^\varepsilon]$, which coincides with (2.14). Setting

$$u_2^\varepsilon(x,t) = u^\varepsilon(x, \tau_1^\varepsilon + t) \text{ and } v_2^\varepsilon(x,t) = v^\varepsilon(x, \tau_1^\varepsilon + t), \text{ for all } (x,t) \in \Omega \times [0, \tau_2^\varepsilon - \tau_1^\varepsilon], \tag{5.10}$$

and

$$u^\pm(x,t) = \omega\left(u_0 \pm l\left(\frac{t}{\varepsilon^{\frac{p}{2}}} + \varepsilon^{2/3-p}|\ln\varepsilon|\right), \frac{t}{\varepsilon^p}, \fint_\Omega v_0 \mp lL^\varepsilon\right), \text{ for all } (x,t) \in \Omega \times [0, \tau_2^\varepsilon - \tau_1^\varepsilon], \tag{5.11}$$

where l is a constant to be chosen later. We next prove

$$u^-(x,t) \leqslant u_2^\varepsilon(x,t) \leqslant u^+(x,t), \text{ for all } (x,t) \in \Omega \times [0, \tau_2^\varepsilon - \tau_1^\varepsilon]. \tag{5.12}$$

To that purpose, we first compute the derivatives of u^-, namely

$$\varepsilon^p u_t^- = -l\varepsilon^{\frac{p}{2}}\omega_\zeta + \omega_s = -l\varepsilon^{\frac{p}{2}}\omega_\zeta + \tilde{f}\left(\omega, , \fint_\Omega v_0 + lL^\varepsilon\right), \tag{5.13}$$

and by (5.7)

$$|\Delta u^-| = |\Delta u_0 \omega_\zeta + |\nabla u_0|^2 \omega_{\zeta\zeta}| \leqslant \tilde{A}_0 \frac{\omega_\zeta}{\varepsilon^{\frac{p}{2}}}, \tag{5.14}$$

where $\tilde{A}_0$ is a positive constant. Thus by (5.13), (5.14) and (5.10), (5.9), (5.2), (5.3) we have that

$$L^\varepsilon(u^-, v_2^\varepsilon) \leqslant \left(C_f\varepsilon^{\frac{p}{2}}|\ln\varepsilon| + \tilde{C}_0 L^\varepsilon - lL^\varepsilon\right) + \omega_\zeta\varepsilon^{\frac{p}{2}}\left(-l + \tilde{A}_0\right),$$

for all $(x,t) \in \Omega\times[0, \tau_2^\varepsilon - \tau_1^\varepsilon]$. Thus $L^\varepsilon(u^-, v_2^\varepsilon) \leqslant 0$ for $l > \tilde{A}_0 + C_f + \tilde{C}_0$. Similarly, one can check that $L^\varepsilon(u^+, v_2^\varepsilon) \geqslant 0$. Further noting that $u^\pm(x,0) = \omega(u_0 \pm l\varepsilon^{2/3-p}|\ln\varepsilon|, 0, \fint_\Omega v_0 \mp lL^\varepsilon) = u_0(x) \pm l\varepsilon^{2/3-p}$ we deduce from (2.7) and (5.10) that

$$u^-(x,0) \leqslant u_2^\varepsilon(x,0) = u^\varepsilon(x,\tau_1^\varepsilon) \leqslant u^+(x,0)$$

for $l > M_1$. By

$$\frac{\partial u^\pm}{\partial n} = \omega_\zeta \frac{\partial u_0}{\partial n} = 0 = \frac{\partial u^\varepsilon}{\partial n}$$

and the comparison principle we obtain (5.12). Let us apply (5.12) at $t = \tau_2^\varepsilon - \tau_1^\varepsilon$; then we deduce from (5.6), (A.11) and (2.9) that

$$u^\varepsilon(x,\tau_2^\varepsilon) = u_2^\varepsilon(x, \tau_2^\varepsilon - \tau_1^\varepsilon) \geqslant h_-\left(\fint_\Omega v_0 + lL^\varepsilon\right) - 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon| \geqslant h_-\left(\fint_\Omega v_0\right) - (lK_4 + 2)L^\varepsilon.$$

Similarly one can check that

$$u^\varepsilon(x,\tau_2^\varepsilon) = u_2^\varepsilon(x,\tau_2^\varepsilon - \tau_1^\varepsilon) \leqslant h_+\left(\fint_\Omega v_0\right) + (lK_4+2)L^\varepsilon, \tag{5.15}$$

so that (2.11) is obtained. Further by (A.11) and (5.4) we have

$$u^\varepsilon(x,\tau_2^\varepsilon) = u_2^\varepsilon(x,\tau_2^\varepsilon - \tau_1^\varepsilon) \geqslant h_+(\fint_\Omega v_0 + lL^\varepsilon) - 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon| \geqslant h_-(\fint_\Omega v_0) - (lK_4+2)L^\varepsilon$$

provided that

$$u_0 - l\left(\frac{\tau_2^\varepsilon - \tau_1^\varepsilon}{\varepsilon^{\frac{p}{2}}}\right) - l\varepsilon^{2/3-p}|\ln\varepsilon| \geqslant h_0(\fint_\Omega v_0 + lL^\varepsilon) + 2\varepsilon^{\frac{p}{2}}|\ln\varepsilon|.$$

This last condition is satisfied if

$$u_0 \geqslant h_0(\fint_\Omega v_0) + CL^\varepsilon,$$

for $C > (K_4 l + l\tau_2 + l + 2)$. This together with (5.15) implies (2.12). In the same way, one can prove (2.13) and this concludes the proof of Theorem 2.2. □

6. Proof of the third stage: Time evolution with a fixed interface

Proof of Theorem 2.3. We set

$$K(\alpha, v) := \alpha h_+(v) + (1-\alpha)h_-(v) - v - a \tag{6.1}$$

and $\overline{K}(v) := K(\frac{|\Omega^{0,+}|}{|\Omega|}, v)$, so that the ODE of the Problem (E) coincides with

$$(\tilde{v}_3)_t = \overline{K}(\tilde{v}_3).$$

By the Cauchy theorem (E) admits a unique solution on a maximal time interval $I = [0, \tilde{T}_3)$. Since $v \mapsto \overline{K}(v)$ is strictly decreasing and since by (2.19)

$$\overline{K}(\frac{2\sqrt{3}}{9}) = \sqrt{3}\left(\frac{|\Omega^{0,+}|}{|\Omega|} - \frac{8}{9}\right) - a < 0 \text{ and } \overline{K}(-\frac{2\sqrt{3}}{9}) = \sqrt{3}\left(\frac{|\Omega^{0,+}|}{|\Omega|} - \frac{1}{9}\right) - a > 0$$

we deduce that there exists a unique $\omega \in (-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9})$ such that $\overline{K}(\omega) = 0$. Moreover we suppose that σ is small enough to ensure that $\omega \in (-\frac{2\sqrt{3}}{9} + \frac{\sigma}{2}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{2})$.

1. If $\omega = \fint_\Omega v_0$, then $\tilde{v}_3(t) = \fint_\Omega v_0$, for all $t \in [0, \tilde{T}_3)$ with $\tilde{T}_3 = \infty$. In this case (2.20), (2.21) and (2.22) are satisfied.

2. If $\displaystyle\fint_\Omega v_0 > \omega$ then since $\tilde{v}_3(t)$ and ω are two different solutions of (E) we have

$$\tilde{v}_3(t) > \omega, \text{ for all } t \in I$$

and then $\overline{K}(\tilde{v}_3(t)) < \overline{K}(\omega) = 0$, for all $t \in I$. Thus by the ODE $\tilde{v}_3$ is non increasing and we have

$$\omega < \tilde{v}_3(t) < \fint_\Omega v_0, \text{ for all } t \in I. \tag{6.2}$$

Further by classical argument, one can prove that $\tilde{T}_3 = +\infty$, $\lim_{t\to\infty} \tilde{v}_3(t) = \omega$ and that $\lim_{t\to\infty}(\tilde{v}_3)_t = 0$. Thus

$$\omega = \tilde{v}_{3,\infty} \tag{6.3}$$

and (2.20), (2.21) are satisfied. We now prove (2.22). Let $w(t) = e^t(\tilde{v}_3(t) - \omega)$ then using (6.2) and the fact that $h_\pm$ are non increasing we have

$$\begin{aligned} w_t &= e^t(\tilde{v}_3 - \omega) + e^t\big(\overline{K}(\tilde{v}_3) - \overline{K}(\omega)\big) \\ &= e^t \frac{|\Omega^{0,+}|}{|\Omega|}\Big(h_+(\tilde{v}_3) - h_+(\omega)\Big) + e^t\left(1 - \frac{|\Omega^{0,+}|}{|\Omega|}\right)\Big(h_+(\tilde{v}_3) - h_+(\omega)\Big) \leqslant 0, \end{aligned}$$

so that $w(t) \leqslant w(0)$ for all $t \in [0, \infty)$. This gives

$$0 \leqslant \tilde{v}_3(t) - \omega \leqslant e^{-t}\left(\fint_\Omega v_0 - \omega\right), \text{ for all } t \in [0, \infty).$$

Thus for $t = \tau_3^\varepsilon = \frac{c_{p,N}|\ln\varepsilon|}{m}$ we obtain $0 \leqslant \tilde{v}_3(\tau_2^\varepsilon + \tau_3^\varepsilon) - \omega \leqslant C\varepsilon^{\frac{c_{p,N}}{m}}$, which in view of (6.3) coincides with (2.22).

3. One can give similar arguments in the case $\displaystyle\fint_\Omega v_0 < \omega$ and conclude the proof of Theorem 2.3. □

In what follows we introduce preliminary notations, which will be useful in the sequel. Let $\hat{h}_\pm$ and $\hat{h}_0$ be the perturbations of $h_\pm$ and h_0 defined in Appendix A. Setting

$$\hat{f}(s, v) := -\big(s - \hat{h}_0(v)\big)(s - \hat{h}_+(v))(s - \hat{h}_-(v)) \tag{6.4}$$

we denote by $(\alpha^\varepsilon, \beta^\varepsilon)$ the solution of the following system

$$(\hat{P}^\varepsilon)\begin{cases} \varepsilon^p \alpha_t^\varepsilon = \varepsilon^{4/3}\Delta\alpha^\varepsilon + \hat{f}(\alpha^\varepsilon, \beta^\varepsilon) \text{ in } \Omega \times (0, T) & (6.5) \\ \beta_t^\varepsilon = \varepsilon^{-2/3}\Delta\beta^\varepsilon + g(\alpha^\varepsilon, \beta^\varepsilon) \quad \text{in } \Omega \times (0, T) & (6.6) \\ \dfrac{\partial\alpha^\varepsilon}{\partial n} = \dfrac{\partial\beta^\varepsilon}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0, T) & (6.7) \end{cases}$$

with the initial conditions

$$\alpha^\varepsilon(x, 0) = u_0(x), \;\; \beta^\varepsilon(x, 0) = v_0(x), \text{ for } x \in \Omega. \tag{6.8}$$

By standard theory for parabolic systems there exists a unique solution $(\alpha^\varepsilon, \beta^\varepsilon)$ of $(\hat{P}^\varepsilon)$ such that

$$|\alpha^\varepsilon(x, t)| + |\beta^\varepsilon(x, t)| \leqslant \hat{C}_0, \text{ for all } x \in \Omega \text{ and } t \geqslant 0. \tag{6.9}$$

We now claim that there exists a positive constant $\tilde{C}_0$ such that

$$\left|\overline{\beta}^{\varepsilon}(t) - \overline{\beta}^{\varepsilon}(0)\right| \leqslant \tilde{C}_0 t, \text{ for all } t \geqslant 0 \tag{6.10}$$

and

$$\max_{x\in\overline{\Omega}} \left|\beta^{\varepsilon}(x,t) - \overline{\beta}^{\varepsilon}(t)\right| \leqslant \tilde{C}_0 \varepsilon^{\frac{4}{3(N+2)}}, \text{ for all } t \geqslant \tau_1^{\varepsilon} := \tau_1 \varepsilon^{2/3} |\ln \varepsilon|. \tag{6.11}$$

Since (P^{ε}) and $(\hat{P}^{\varepsilon})$ have the same second parabolic equation the proof of (6.10) and (6.11) are exactly the same as the one of (3.2) and (3.3) respectively. Moreover we denote by $(\alpha_3^{\varepsilon}, \beta_3^{\varepsilon})$ the solution of system $(\hat{P}^{\varepsilon})$ with the initial condition

$$\alpha_3^{\varepsilon}(x,0) = u^{\varepsilon}(x, \tau_2^{\varepsilon}) \text{ and } \beta_3^{\varepsilon}(x,0) = v^{\varepsilon}(x,\tau_2^{\varepsilon}), \text{ for } x \in \Omega. \tag{6.12}$$

We now consider the interface Γ_3^{ε} defined by the interface motion equation

$$V_{n,3}^{\varepsilon} = \varepsilon^{2/3-p} \frac{3}{\sqrt{2}} \hat{h}_0(\overline{\beta}_3^{\varepsilon}), \quad \Gamma_3^{\varepsilon}|_{t=0} = \Gamma_0, \tag{6.13}$$

where $\overline{\beta}_3^{\varepsilon} = \int_{\Omega} \beta_3^{\varepsilon}(x,t)dx$ and $V_{n,3}^{\varepsilon}$ denotes the normal velocity of Γ_3^{ε}. One can show that Problem (6.13) possesses a unique classical solution Γ_3^{ε} on a maximal time interval $[0, \tilde{t}_3^{\varepsilon}]$. We then deduce from (6.13) and the definition of $\hat{h}_0$ that $|V_{n,3}^{\varepsilon}| \leqslant C\varepsilon^{2/3-p}$, where C is a constant independent on ε and $\tilde{t}_3^{\varepsilon}$. We now set

$$\phi \;:\; \Gamma_0 \times \mathbb{R} \to \mathbb{R}^N \quad \phi(s,z) = s + zn(s);$$

there exists δ_0 such that

$$\text{for all } 0 < \delta \leqslant \delta_0, \phi \text{ is a } C^{\infty} \text{ diffeomorphism from } \Gamma_0 \times (-\delta, \delta) \text{ to its range, } \Gamma_0(\delta). \tag{6.14}$$

This yields setting

$$l_3^{\varepsilon}(t) := \varepsilon^{2/3-p} \frac{3}{\sqrt{2}} \int_0^t \hat{h}_0\big(\overline{\beta}_3^{\varepsilon}(r)\big)dr, \tag{6.15}$$

that $|l_3^{\varepsilon}(t)| \leqslant C\varepsilon^{2/3-p}t$, for all $t \in [0, \tilde{t}_3^{\varepsilon}]$. Thus the interface Γ_3^{ε} is well defined on $[0, \tau_3^{\varepsilon}]$, where $\tau_3^{\varepsilon} = \frac{c_{p,N}|\ln \varepsilon|}{m}$. This gives that (6.13) admits a unique classical solution on the time interval $[0, \tau_3^{\varepsilon}]$. Note that for $t \in [0, \tau_3^{\varepsilon}]$ Γ_3^{ε} divides Ω into two subdomains $\Omega_3^{\varepsilon,\pm}(t)$, with $\Omega_3^{\varepsilon,\pm}(0) = \Omega^{0,\pm}$.

Next we introduce a smooth truncated approximation of the signed distance function to the interface Γ_3^{ε}; more precisely let $0 < \frac{r_0}{2} < \delta_0$ we define $d_3^{\varepsilon}(x,t) \in C^2(\Omega \times (0, \tau_3^{\varepsilon}))$ as

$$d_3^{\varepsilon}(x,t) = \begin{cases} r_0 & \text{if } x \in \Omega_3^{\varepsilon,+}(t) \text{ and } dist\big(x, \Gamma_3^{\varepsilon}(t)\big) \geqslant r_0 \\ -r_0 & \text{if } x \in \Omega_3^{\varepsilon,-}(t) \text{ and } dist\big(x, \Gamma_3^{\varepsilon}(t)\big) \geqslant r_0 \\ dist\big(x, \Gamma_3^{\varepsilon}(t)\big) & \text{if } x \in \Omega_3^{\varepsilon,+}(t) \text{ and } dist\big(x, \Gamma_3^{\varepsilon}(t)\big) \leqslant \frac{r_0}{2} \\ -dist\big(x, \Gamma_3^{\varepsilon}(t)\big) & \text{if } x \in \Omega_3^{\varepsilon,-}(t) \text{ and } dist\big(x, \Gamma_3^{\varepsilon}(t)\big) \leqslant \frac{r_0}{2}, \end{cases}$$

and extend it smoothly for $x \in \{r_0/2 < dist(x, \Gamma_3^\varepsilon(t)) < r_0\}$. Moreover we assume that ε is small enough so that

$$\frac{\partial d_3^\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega \times (0, \tau_3^\varepsilon). \tag{6.16}$$

Further let $(U(z, V, \delta), C(V, \delta))$ be the solution of the following system

$$(TW) \begin{cases} U_{zz}(z, V, \delta) + C(V, \delta)U_z + \hat{f}(U(z, V, \delta), V) = \delta, \forall z \in \mathbb{R}, \\ \lim_{z\to+\infty} U(z, V, \delta) = \hat{h}_+(V, \delta), \quad \lim_{z\to-\infty} U(z, V, \delta) = \hat{h}_-(V, \delta), \\ U(0, V) = \hat{h}_0(V, \delta). \end{cases}$$

The basic properties of $(U(z, V, \delta), C(V, \delta))$ are recalled in Lemma A.1. We now define for all $x \in \Omega$ and $t \in [0, \tau_3^\varepsilon]$

$$U_3^\pm(x, t) = U\left(\frac{d_3^\varepsilon(x, t) \pm S_1 S^\varepsilon e^{mt}}{\varepsilon^{2/3}}, \overline{\beta}_3^\varepsilon(t), \mp S_2 S^\varepsilon\right) \tag{6.17}$$

where

$$S^\varepsilon := \varepsilon^{3c_{p,N}} |\ln \varepsilon| \tag{6.18}$$

and S_1, S_2 are positive constants to be determined later. Note that by the definitions of L^ε and $c_{p,N}$ (see (2.10) and (2.16)) we have

$$S^\varepsilon \geqslant L^\varepsilon \text{ and } \lim_{\varepsilon\downarrow 0} \frac{S^\varepsilon}{\varepsilon^{2/3}} = \infty. \tag{6.19}$$

We next state four lemmas, which will be useful to prove Theorem 2.4.

Lemma 6.1 *The functions $U_3^\pm(x, t)$ satisfy that*

$$\hat{L}^\varepsilon(U_3^+, \beta_3^\varepsilon) := \varepsilon^p (U_3^+)_t - \varepsilon^{4/3} \Delta U_3^+ - \hat{f}(U_3^+, \beta_3^\varepsilon) \geqslant 0, \text{ on } \Omega \times [0, \tau_3^\varepsilon] \tag{6.20}$$

$$\hat{L}^\varepsilon(U_3^-, \beta_3^\varepsilon) := \varepsilon^p (U_3^-)_t - \varepsilon^{4/3} \Delta U_3^- - \hat{f}(U_3^-, \beta_3^\varepsilon) \leqslant 0, \text{ on } \Omega \times [0, \tau_3^\varepsilon] \tag{6.21}$$

and $\dfrac{\partial U_3^\pm}{\partial n} = 0$ *on* $\partial\Omega \times [0, \tau_3^\varepsilon]$.

Proof. It follows from (6.16) that $\dfrac{\partial U_3^\pm}{\partial n} = 0$. Furthermore we have that

$$\hat{L}^\varepsilon(U_3^+, \overline{\beta}_3^\varepsilon) = I_1 + I_2 + I_3 + S_2 S^\varepsilon \tag{6.22}$$

with

$$I_1 := \varepsilon^p U_v (\overline{\beta}_3^\varepsilon)_t - \varepsilon^{2/3} U_z \Delta d_3^\varepsilon + U_{zz}(1 - |\nabla d_3^\varepsilon|^2) \tag{6.23}$$

$$I_2 := U_z\left(\varepsilon^{p-2/3}(d_3^\varepsilon)_t + \frac{3}{\sqrt{2}} \hat{h}_0(\overline{\beta}_3^\varepsilon, -S_2 S^\varepsilon) + \varepsilon^{p-2/3} e^{mt} m S_1 S^\varepsilon\right) \tag{6.24}$$

and

$$I_3 := -\hat{f}(U, \beta_3^\varepsilon) + \hat{f}(U, \overline{\beta}_3^\varepsilon), \tag{6.25}$$

where the derivatives of U are evaluated at the point $\big(\frac{d_3^\varepsilon(x,t)+S_1S^\varepsilon e^{mt}}{\varepsilon^{2/3}}, \overline{\beta}_3^\varepsilon(t)-S_2S^\varepsilon\big)$. Since the estimates of I_1, I_2 and I_3 are standard (see [2], [6], [28], [11]) we only give the main steps of the computations. We have by the definition of d_3^ε and the property (A.6) of U that

$$\Big|U_{zz}(1-|\nabla d_3^\varepsilon|^2)\Big| \leqslant C \sup_{|d^\varepsilon|\geqslant \frac{r_0}{2}} \Big|U_{zz}\big(\frac{d_3^\varepsilon(x,t)+S_1S^\varepsilon e^{mt}}{\varepsilon^{2/3}}, \overline{\beta}_3^\varepsilon(t)-S_2S^\varepsilon\big)\Big|$$
$$\leqslant CK_1 \sup_{|d_3^\varepsilon|\geqslant \frac{r_0}{2}} e^{-K_2|\frac{d_3^\varepsilon+S_1S^\varepsilon e^{mt}}{\varepsilon^{2/3}}|}.$$

Moreover since $\tau_3^\varepsilon = \frac{c_{p,N}|\ln\varepsilon|}{m}$ we have for $t\in[0,\tau_3^\varepsilon]$

$$|d_3^\varepsilon + S_1S^\varepsilon e^{mt}| \geqslant \frac{r_0}{2} - S_1S^\varepsilon e^{m\tau_3^\varepsilon} = \frac{r_0}{2} - S_1S^\varepsilon\varepsilon^{-c_{p,N}} \quad \text{for } |d_3^\varepsilon| \geqslant \frac{r_0}{2},$$

so that

$$\Big|U_{zz}(1-|\nabla d_3^\varepsilon|^2)\Big| \leqslant CK_1 e^{\frac{K_2}{\varepsilon^{2/3}}[-r_0/2+S_1S^\varepsilon\varepsilon^{-c_{p,N}}]} \leqslant C_1\varepsilon^p, \tag{6.26}$$

where we have used the fact that $S^\varepsilon\varepsilon^{-c_{p,N}}$ tends to 0, as $\varepsilon\downarrow 0$. Noting that Δd_3^ε is bounded and also using (A.5), (6.26) and the fact that $|(\overline{\beta}^\varepsilon)_\tau| \leqslant C$ (see(6.6) and (6.9)) we deduce from (6.23) that

$$I_1 \geqslant -D_1\varepsilon^p. \tag{6.27}$$

To estimate I_2, we first note that the motion equation (6.13) together with the mean value theorem and the smoothness of the function d_3^ε implies

$$\Big|(d_3^\varepsilon)_t + \varepsilon^{2/3-p}\frac{3}{\sqrt{2}}\hat{h}_0(\overline{\beta}_3^\varepsilon)\Big| \leqslant D|d_3^\varepsilon|, \quad \text{in } \overline{\Omega}\times[0,\tau_3^\varepsilon].$$

Substituting this into (6.25) and using (A.10) we obtain

$$I_2 \geqslant U_z\Big(-\varepsilon^{p-2/3}D|d_3^\varepsilon| - K_4S_2\frac{3}{\sqrt{2}}S^\varepsilon + \varepsilon^{p-2/3}me^{mt}S_1S^\varepsilon\Big). \tag{6.28}$$

Moreover we have that

$$me^{mt}S_1S^\varepsilon - D|d_3^\varepsilon| \geqslant -D|d_3^\varepsilon + e^{mt}S_1S^\varepsilon| + e^{mt}S_1S^\varepsilon(m-D).$$

Substituting this into (6.28) we obtain

$$I_2 \geqslant -\varepsilon^p DU_z\frac{|d_3^\varepsilon + e^{mt}S_1S^\varepsilon|}{\varepsilon^{2/3}} + U_zS^\varepsilon\Big[-K_4S_2\frac{3}{\sqrt{2}} + \varepsilon^{p-2/3}S_1(m-D)e^{mt}\Big].$$

Noting that the second term is positive for $m > D$ and using (A.6) combinated with the fact that $\sup_{z\in R}|ze^{-K_2z}|$ is bounded we obtain

$$I_2 \geqslant -D_2\varepsilon^p, \tag{6.29}$$

for ε small enough. Next we estimate I_3. Using (A.13) and (6.11) with $t \geqslant \tau_2^\varepsilon$ we obtain that

$$I_3 \geqslant -c_{\hat{f}} \left| \beta_3^\varepsilon(x,t) - \overline{\beta}_3^\varepsilon(t) \right| \geqslant -D_3 \varepsilon^{\frac{4}{3(N+2)}}. \tag{6.30}$$

Substituting (6.27),(6.29) and (6.30) into (6.22) we deduce that

$$\hat{L}^\varepsilon\big(U_3^+, \beta_3^\varepsilon\big) \geqslant S_2 S^\varepsilon - \varepsilon^p (D_1 + D_2) - D_3 \varepsilon^{\frac{4}{3(N+2)}}.$$

Thus for $S_2 \geqslant D_1 + D_2 + D_3$ we obtain $\hat{L}^\varepsilon(U_3^+, \beta_3^\varepsilon) \geqslant 0$, which coincides with (6.20). One can prove the inequality (6.21) in a similar way.

Next we prove the following inequalities on the initial functions, namely

Lemma 6.2

$$U_3^-(x,0) \leqslant \alpha_3^\varepsilon(x,0) \leqslant U_3^+(x,0), \textit{ for all } x \in \Omega. \tag{6.31}$$

Proof. In view of (6.12) and (3.2) we first note that

$$\left| \overline{\beta}_3^\varepsilon(0) - \fint_\Omega v_0 \right| = \left| \overline{v}^\varepsilon(\tau_2^\varepsilon) - \fint_\Omega v_0 \right| \leqslant \tilde{C}_0 \tau_2 \varepsilon^p |\ln \varepsilon|. \tag{6.32}$$

Further if $d_3^\varepsilon(x,0) \leqslant -\frac{M_2}{\eta_0} L^\varepsilon$ then $x \in \Omega^{0,-}$ and thus by $(H4)$

$$u_0(x) - h_0\left(\fint_\Omega v_0\right) \leqslant -\eta_0 dist(x, \Gamma_0) \leqslant -M_2 L^\varepsilon,$$

so that $x \in \Omega^{\varepsilon,-}$. Thus we deduce from (6.12), (6.19) and (2.13) that

$$\alpha_3^\varepsilon(x,0) = u^\varepsilon(x, \tau_2^\varepsilon) \leqslant h_-\left(\fint_\Omega v_0\right) + M_2 L^\varepsilon \leqslant h_-\left(\fint_\Omega v_0\right) + M_2 S^\varepsilon. \tag{6.33}$$

Moreover since U is strictly increasing we have

$$U_3^+(x,0) = U\Big(\frac{d_3^\varepsilon(x,0) + S_1 S^\varepsilon}{\varepsilon^{2/3}}, \overline{\beta}_3^\varepsilon(0), -S_2 S^\varepsilon\Big) \geqslant \hat{h}_-\big(\overline{\beta}_3^\varepsilon(0), -S_2 S^\varepsilon\big)$$

Further since by (6.12) $\overline{\beta}_3^\varepsilon(0) - S_2 S^\varepsilon = \overline{v}^\varepsilon(\tau_2^\varepsilon) - S_2 S^\varepsilon$ we deduce from (6.32) and (2.9) that $\overline{\beta}_3^\varepsilon(0) - S_2 S^\varepsilon \in (-\frac{2\sqrt{3}}{9} + \frac{\sigma}{4}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{4})$ for ε small enough, so that by (A.2)

$$\hat{h}_-\Big(\overline{\beta}_3^\varepsilon(0), -S_2 S^\varepsilon\Big) = h_-\Big(\overline{\beta}_3^\varepsilon(0) - S_2 S^\varepsilon\Big).$$

Then by (A.12), (A.10) and (6.32) we obtain

$$U_3^+(x,0) \geqslant h_-\Big(\overline{\beta}_3^\varepsilon(0)\Big) + K_3 S_2 S^\varepsilon \geqslant h_-\left(\fint_\Omega v_0\right) - K_4 \tilde{C}_0 \tau_2 \varepsilon^p |\ln \varepsilon| + K_3 S_2 S^\varepsilon.$$

Thus for $S_2 \geqslant \dfrac{M_2 + K_4 \tilde{C}_0 \tau_2}{K_3}$ and ε small enough we obtain by (6.33) that

$$\alpha_3^\varepsilon(x,0) \leqslant U_3^+(x,0) \text{ for all } x \in \Omega \text{ such that } d_3^\varepsilon(x,0) \leqslant -\frac{M_2}{\eta_0} L^\varepsilon. \tag{6.34}$$

If $d_3^\varepsilon(x,0) \geqslant -\frac{M_2}{\eta_0}L^\varepsilon$ then choosing $S_1 \geqslant 2\frac{M_2}{\eta_0}$ we have by (A.5) and (A.7) successively that

$$U_3^+(x,0) \geqslant U\left(\frac{M_2 L^\varepsilon}{\varepsilon^{2/3}\eta_0}, \overline{\beta}_3^\varepsilon(0), -S_2 S^\varepsilon\right) \geqslant \hat{h}_+(\overline{\beta}_3^\varepsilon(0), -S_2 S^\varepsilon) - K_1 e^{-K_2 \frac{M_2 L^\varepsilon}{\varepsilon^{2/3}\eta_0}}.$$

By (A.12), (A.10) and (6.32) we obtain as previously that

$$U_3^+(x,0) \geqslant h_+\left(\fint_\Omega v_0\right) + S^\varepsilon K_3 S_2 - K_4 \tilde{C}_0 \tau_2 \varepsilon^p |\ln\varepsilon| - K_1 e^{-K_2 \frac{M_2 L^\varepsilon}{\varepsilon^{2/3}\eta_0}},$$

and then by (2.11) one has for $S_2 \geqslant \dfrac{M_2 + K_4\tilde{C}_0\tau_2 + K_1}{K_3}$ and ε small enough that

$$\alpha_3^\varepsilon(x,0) \leqslant U_3^+(x,0) \text{ for all } x \in \Omega \text{ such that } d_3^\varepsilon(x,0) \geqslant -\frac{M_2}{\eta_0}L^\varepsilon.$$

This together with (6.34) implies that $\alpha_3^\varepsilon(x,0) \leqslant U_3^+(x,0)$, for all $x \in \Omega$. Similarly one can show that $\alpha_3^\varepsilon(x,0) \geqslant U_3^-(x,0)$, for all $x \in \Omega$. This completes the proof of Lemma 6.2. Next we prove the following result

Lemma 6.3 *There exists a positive constant L_1 such that*

$$\left|\alpha_3^\varepsilon(x,t) - \hat{h}_+\big(\overline{\beta}_3^\varepsilon(t)\big)\right| \leqslant L_1 \varepsilon^{3c_{p,N}} |\ln\varepsilon| \text{ if } d_3^\varepsilon(x,t) \geqslant 2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon| \tag{6.35}$$

and

$$\left|\alpha_3^\varepsilon(x,t) - \hat{h}_-\big(\overline{\beta}_3^\varepsilon(t)\big)\right| \leqslant L_1 \varepsilon^{3c_{p,N}} |\ln\varepsilon| \text{ if } d_3^\varepsilon(x,t) \leqslant -2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|, \tag{6.36}$$

for all $(x,t) \in \Omega \times [0,\tau_3^\varepsilon]$.

Proof. Applying the comparison principle to the equation (2.1) with the functions $U_3^\pm$ we deduce from (6.16), the lemmas 6.1 and 6.2 that

$$U_3^-(x,t) \leqslant \alpha_3^\varepsilon(x,t) \leqslant U_3^+(x,t) \text{ for all } (x,t) \in \Omega \times [0,\tau_3^\varepsilon]. \tag{6.37}$$

For $d_3^\varepsilon(x,t) \geqslant 2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|$ we have by (6.18) that

$$d_3^\varepsilon(x,t) - S_1 e^{mt} S^\varepsilon \geqslant 2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon| - S_1 e^{m\tau_3^\varepsilon} S^\varepsilon = S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|.$$

This in view of (A.5), (A.7), (A.10) and the fact that $\lim_{\varepsilon\downarrow 0} \frac{\varepsilon^{2c_{p,N}}}{\varepsilon^{2/3}} = +\infty$ implies

$$\begin{aligned}
\alpha_3^\varepsilon(x,t) &\geqslant U_3^-(x,t) \geqslant U_3^-\left(\frac{S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|}{\varepsilon^{2/3}}, \overline{\beta}_3^\varepsilon(t), S_2 S^\varepsilon\right) \\
&\geqslant \hat{h}_+(\overline{\beta}_3^\varepsilon(t), S_2 S^\varepsilon) - K_1 e^{-K_2 \frac{S_1 \varepsilon^{2c_{p,N}}|\ln\varepsilon|}{\varepsilon^{2/3}}} \\
&\geqslant \hat{h}_+(\overline{\beta}_3^\varepsilon(t)) - K_4 S_2 S^\varepsilon - K_1 S^\varepsilon \geqslant \hat{h}_+(\overline{\beta}_3^\varepsilon(t)) - L_1 S^\varepsilon,
\end{aligned} \tag{6.38}$$

for $L_1 \geqslant K_4 S_2 + K_1$ and ε small enough. Moreover we have by (A.10) that

$$\alpha_3^\varepsilon(x,t) \leqslant U_3^+(x,t) \leqslant \hat{h}_+(\overline{\beta}_3^\varepsilon(t), -S_2 S^\varepsilon) \leqslant \hat{h}_+\big(\overline{\beta}_3^\varepsilon(t)\big) + K_4 S_2 S^\varepsilon \leqslant \hat{h}_+\big(\overline{\beta}_3^\varepsilon(t)\big) + L_1 S^\varepsilon.$$

This together with (6.38) implies (6.35). Similarly one can check (6.36).

Lemma 6.4 *There exists $L_2 > 0$ such that*

$$\left|l_3^\varepsilon(t)\right| \leqslant L_2\varepsilon^{2/3-p}|\ln\varepsilon| \tag{6.39}$$

and

$$\left|\overline{\beta}_3^\varepsilon(t) - \tilde{v}_3(t)\right| \leqslant L_2\varepsilon^{c_{p,N}}|\ln\varepsilon|^2, \tag{6.40}$$

for all $t \in [0, \tau_3^\varepsilon]$.

Proof. By (6.15) we have $|l_3^\varepsilon(t)| \leqslant \varepsilon^{2/3-p} C\frac{c_{p,N}|\ln\varepsilon|}{m}$ for all $t \in [0, \tau_3^\varepsilon]$, which implies (6.39). We now show (6.40). Indeed, integrating (6.6) on Ω we first note that

$$\begin{aligned}
(\overline{\beta}_3^\varepsilon)_t(t) &= -\overline{\beta}_3^\varepsilon(t) + \frac{1}{|\Omega|}\left(\int_{\Omega_3^{\varepsilon,+}} \alpha_3^\varepsilon(x,t)dx + \int_{\Omega_3^{\varepsilon,-}} \alpha_3^\varepsilon(x,t)dx\right) - a \\
&= -\overline{\beta}_3^\varepsilon(t) + \frac{\hat{h}_+(\overline{\beta}_3^\varepsilon(t))|\Omega_3^{\varepsilon,+}(t)| + \hat{h}_-(\overline{\beta}_3^\varepsilon(t))|\Omega_3^{\varepsilon,-}(t)|}{|\Omega|} - a \\
&\quad + \frac{1}{|\Omega|}\int_{\Omega_3^{\varepsilon,+}}\left[\alpha_3^\varepsilon(x,t) - \hat{h}_+(\overline{\beta}_3^\varepsilon(t))\right]dx + \frac{1}{|\Omega|}\int_{\Omega_3^{\varepsilon,-}}\left[\alpha_3^\varepsilon(x,t) - \hat{h}_-(\overline{\beta}_3^\varepsilon(t))\right]dx \\
&= -\overline{\beta}^\varepsilon(t) + \frac{\hat{h}_+(\overline{\beta}_3^\varepsilon(t))|\Omega_3^{\varepsilon,+}(t)| + \hat{h}_-(\overline{\beta}_3^\varepsilon(t))|\Omega_3^{\varepsilon,-}(t)|}{|\Omega|} - a \\
&\quad + \frac{1}{|\Omega|}\left(J_1^{\varepsilon,+} + J_2^{\varepsilon,+} + J_3^{\varepsilon,-} + J_4^{\varepsilon,-}\right)
\end{aligned} \tag{6.41}$$

where

$$\begin{aligned}
J_1^{\varepsilon,+} &:= \int_{\{x\in\Omega,\ d_3^\varepsilon(x,t)\geqslant 2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|\}}\left[\alpha_3^\varepsilon(x,t) - \hat{h}_+\big(\overline{\beta}_3^\varepsilon(t)\big)\right]dx \\
J_2^{\varepsilon,+} &:= \int_{\{x\in\Omega,\ 0\leqslant d_3^\varepsilon(x,t)< 2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|\}}\left[\alpha_3^\varepsilon(x,t) - \hat{h}_+\big(\overline{\beta}_3^\varepsilon(t)\big)\right]dx \\
J_3^{\varepsilon,-} &:= \int_{\{x\in\Omega,\ d_3^\varepsilon(x,t)\leqslant -2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|\}}\left[\alpha_3^\varepsilon(x,t) - \hat{h}_-\big(\overline{\beta}_3^\varepsilon(t)\big)\right]dx \\
J_4^{\varepsilon,-} &:= \int_{\{x\in\Omega,\ 0\geqslant d_3^\varepsilon(x,t)> -2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|\}}\left[\alpha_3^\varepsilon(x,t) - \hat{h}_-\big(\overline{\beta}_3^\varepsilon(t)\big)\right]dx.
\end{aligned}$$

By (6.35) we have that

$$\left|J_1^{\varepsilon,+}\right| \leqslant L_1\varepsilon^{3c_{p,N}}|\ln\varepsilon||\Omega|. \tag{6.42}$$

Since $\hat{h}_+(\overline{\beta}_3^\varepsilon)$ and α_3^ε are bounded, we obtain

$$\left|J_2^{\varepsilon,+}\right| \leqslant C2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|. \tag{6.43}$$

Similarly, one can prove that $|J_3^{\varepsilon,-}| + |J_4^{\varepsilon,-}| \leqslant \tilde{C}2S_1\varepsilon^{2c_{p,N}}|\ln\varepsilon|$, which we substitute into (6.41) to deduce also in view of (6.42), (6.43) that $\overline{\beta}_3^\varepsilon$ satisfies

$$(\overline{\beta}_3^\varepsilon)_t(t) = -\overline{\beta}_3^\varepsilon(t) + \frac{\hat{h}_+\big(\overline{\beta}_3^\varepsilon(t)\big)|\Omega_3^{\varepsilon,+}(t)| + \hat{h}_-\big(\overline{\beta}_3^\varepsilon(t)\big)|\Omega_3^{\varepsilon,-}(t)|}{|\Omega|} - a + \omega_1(t)$$

where $|\omega_1(t)| \leqslant C\varepsilon^{2c_{p,N}}|\ln\varepsilon|$, for all $t \in [0, \tau_3^\varepsilon]$. Thus setting

$$\hat{K}(\alpha, v) := \alpha\hat{h}_+(v) + (1-\alpha)\hat{h}_-(v) - v - a \tag{6.44}$$

we obtain

$$(\overline{\beta}_3^\varepsilon)_t(t) = \hat{K}\left(\frac{|\Omega_3^{\varepsilon,+}(t)|}{|\Omega|}, \overline{\beta}_3^\varepsilon(t)\right) + \omega_1(t). \tag{6.45}$$

Since $\tilde{v}_3 \in (-\frac{2\sqrt{3}}{9} + \frac{\sigma}{2}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{2})$ we have in view of Theorem 2.3

$$(\tilde{v}_3)_t = \hat{K}\left(\frac{|\Omega^{0,+}|}{|\Omega|}, \tilde{v}_3\right).$$

Thus also using (6.45) we deduce for all $t \in [0, \tau_3^\varepsilon]$ that

$$\begin{aligned}\left|(\tilde{v}_3 - \overline{\beta}_3^\varepsilon)(t)\right| &= \left|\int_0^t \hat{K}\left(\frac{|\Omega^{0,+}|}{|\Omega|}, \tilde{v}_3(s)\right) - \hat{K}\left(\frac{|\Omega_3^{\varepsilon,+}(s)|}{|\Omega|}, \overline{\beta}_3^\varepsilon(s)\right)ds + \left(\tilde{v}_3(0) - \overline{\beta}_3^\varepsilon(0)\right)\right| \\ &\leqslant \int_0^t \left(|\tilde{v}_3(s) - \overline{\beta}_3^\varepsilon(s)| + I(s) + II(s)\right)ds + C\varepsilon^{2c_{p,N}}|\ln\varepsilon|\tau_3^\varepsilon + |\tilde{v}_3(0) - \overline{\beta}_3^\varepsilon(0)|, \end{aligned} \tag{6.46}$$

where

$$\begin{aligned} I(s) &= \left|\frac{\hat{h}_+(\tilde{v}_3)|\Omega^{0,+}| + \hat{h}_-(\tilde{v}_3)|\Omega^{0,-}|}{|\Omega|} - \frac{\hat{h}_+(\overline{\beta}_3^\varepsilon)|\Omega^{0,+}| + \hat{h}_-(\overline{\beta}_3^\varepsilon)|\Omega^{0,-}|}{|\Omega|}\right|, \\ II(s) &= \left|\frac{\hat{h}_+(\overline{\beta}_3^\varepsilon)|\Omega^{0,+}| + \hat{h}_-(\overline{\beta}_3^\varepsilon)|\Omega^{0,-}|}{|\Omega|} - \frac{\hat{h}_+(\overline{\beta}_3^\varepsilon)|\Omega_3^{\varepsilon,+}| + \hat{h}_-(\overline{\beta}_3^\varepsilon)|\Omega_3^{\varepsilon,-}|}{|\Omega|}\right|. \end{aligned}$$

By (A.9) we have that

$$I(s) \leqslant K_4\left|\tilde{v}_3(s) - \overline{\beta}_3^\varepsilon(s)\right|. \tag{6.47}$$

Moreover since $|\Omega^{0,-}| - |\Omega_3^{\varepsilon,-}| = |\Omega_3^{\varepsilon,+}| - |\Omega^{0,+}|$ and since $\hat{h}_\pm$ are uniformly bounded one gets

$$II(s) \leqslant C(\Omega)\left||\Omega^{0,+}| - |\Omega_3^{\varepsilon,+}|\right|. \tag{6.48}$$

Further by (6.39) one gets $\left||\Omega^{0,+}| - |\Omega_3^{\varepsilon,+}|\right| \leqslant C(\Gamma_0)\varepsilon^{2/3-p}|\ln\varepsilon|$, which in view of (6.48) implies

$$II(s) \leqslant C(\Omega, \Gamma_0)\varepsilon^{2/3-p}|\ln\varepsilon|.$$

This with (6.32), (6.46) and (6.47) gives that

$$\begin{aligned}\left|(\tilde{v}_3 - \overline{\beta}_3^\varepsilon)(t)\right| &\leqslant \int_0^t (1+K_4)|\tilde{v}_3(s) - \overline{\beta}_3^\varepsilon(s)|ds + C(\Omega,\Gamma_0)\varepsilon^{2/3-p}|\ln\varepsilon|\tau_3^\varepsilon + C\varepsilon^{2c_{p,N}}|\ln\varepsilon|\tau_3^\varepsilon \\ &\quad + \tilde{C}_0\tau_2\varepsilon^p|\ln\varepsilon| \\ &\leqslant (1+K_4)\int_0^t |\tilde{v}_3(s) - \overline{\beta}_3^\varepsilon(s)|ds + C\varepsilon^{2c_{p,N}}|\ln\varepsilon|\tau_3^\varepsilon,\end{aligned}$$

for all $t \in [0, \tau_3^\varepsilon]$. Thus by Gronwall's Lemma we deduce

$$\left|(\tilde{v}_3 - \overline{\beta}^\varepsilon)(t)\right| \leqslant C\varepsilon^{2c_{p,N}} |\ln\varepsilon| \tau_3^\varepsilon e^{(1+K_4)t},$$

for all $t \in [0, \tau_3^\varepsilon]$. Now recalling that $\tau_3^\varepsilon = \frac{c_{p,N}}{m} |\ln\varepsilon|$ and choosing $m \geqslant 1 + K_4$ we obtain (6.40), which ends the proof of Lemma 6.4. □

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. Let $M_3 := \max(\tilde{C}_0 + L_2, L_1 + L_2 K_4)$, we have by (6.11) and (6.40) that

$$\left|\beta_3^\varepsilon(x,t) - \tilde{v}_3(t)\right| \leqslant \left|\beta_3^\varepsilon(x,t) - \overline{\beta}_3^\varepsilon(t)\right| + \left|\overline{\beta}_3^\varepsilon(t) - \tilde{v}_3(t)\right| \leqslant M_3 \varepsilon^{c_{p,N}} |\ln\varepsilon|^2, \quad \text{for } (x,t) \in \overline{\Omega} \times [0, \tau_3^\varepsilon]. \tag{6.49}$$

Further since $\tilde{v}_3 \in (-\frac{2\sqrt{3}}{9} + \frac{\sigma}{2}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{2})$ we deduce that

$$\beta_3^\varepsilon \in \Big(-\frac{2\sqrt{3}}{9} + \frac{\sigma}{3}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{3}\Big), \text{ for all } (x,t) \in \overline{\Omega} \times [0, \tau_3^\varepsilon] \tag{6.50}$$

and ε small enough. Thus by (A.2) $\hat{h}_\pm(\beta_3^\varepsilon) = h_\pm(\beta_3^\varepsilon)$ and $\hat{h}_0(\beta_3^\varepsilon) = h_0(\beta_3^\varepsilon)$, so that both problems $(\hat{P}^\varepsilon)$ and (P^ε) coincides. This gives in view of (6.8) that

$$\alpha_3^\varepsilon(x,t) = \alpha^\varepsilon(x, t+\tau_2^\varepsilon) = u^\varepsilon(x, t+\tau_2^\varepsilon) \quad \text{and} \quad \beta_3^\varepsilon(x,t) = \beta^\varepsilon(x, t+\tau_2^\varepsilon) = v^\varepsilon(x, t+\tau_2^\varepsilon), \tag{6.51}$$

for all $(x,t) \in \overline{\Omega} \times [0, \tau_3^\varepsilon]$. Then (2.24) follows directly from (6.49) and we also deduce from (6.40) and (6.51) that

$$\left|\overline{v}^\varepsilon(x, \tau_2^\varepsilon + \tau_3^\varepsilon) - \tilde{v}_3(\tau_3^\varepsilon)\right| \leqslant L_2 \varepsilon^{c_{p,N}} |\ln\varepsilon|^2. \tag{6.52}$$

Let $x \in \Omega$ such that $|\tilde{d}(x, \Gamma_0)| \geqslant \varepsilon^{c_{p,N}}$ and let $s \in [0, T_4]$. We first assume that $\tilde{d}(x, \Gamma_0) \geqslant \varepsilon^{c_{p,N}}$ then we obtain from (6.39) that

$$\left|d_3^\varepsilon(x,t)\right| \geqslant r_0/2 \geqslant 2S_1 \varepsilon^{2c_{p,N}} |\ln\varepsilon|,$$

or

$$d_3^\varepsilon(x,t) \geqslant dist(x, \Gamma_0) - L_2 \varepsilon^{2/3-p} |\ln\varepsilon| \geqslant 2S_1 \varepsilon^{2c_{p,N}} |\ln\varepsilon|,$$

for ε small enough. Thus by Lemma 6.3, (A.11) and (6.40) we deduce

$$\begin{aligned}
\left|\alpha_3^\varepsilon(x,t) - h_+\big(\tilde{v}_3(t)\big)\right| &\leqslant \left|\alpha_3^\varepsilon(x,t) - h_+\big(\overline{\beta}_3^\varepsilon(t)\big)\right| + \left|h_+(\overline{\beta}_3^\varepsilon(t)) - h_+\big(\tilde{v}_3(t)\big)\right| \\
&\leqslant \left|\alpha_3^\varepsilon(x,t) - h_+\big(\overline{\beta}_3^\varepsilon(t)\big)\right| + K_4 \left|\overline{\beta}_3^\varepsilon(t) - \tilde{v}_3(t)\right| \\
&\leqslant L_1 \varepsilon^{3c_{p,N}} |\ln\varepsilon| + L_2 K_4 \varepsilon^{c_{p,N}} |\ln\varepsilon|^2 \leqslant M_3 \varepsilon^{\frac{c_{p,N}}{2}}.
\end{aligned}$$

This together with (6.51) gives (2.23) in the case $\tilde{d}(x, \Gamma_0) \geqslant \varepsilon^{c_{p,N}}$. Similarly one can prove (2.23) in the case $\tilde{d}(x, \Gamma_0) \leqslant -\varepsilon^{c_{p,N}}$, which completes the proof of Theorem 2.4. □

7. Proof of the fourth stage: Propagation of interface for large time

Proof of Theorem 2.5. Let δ_0 and ϕ be defined by (6.14), we set for $r \in \mathbb{R}$

$$\Gamma_0^*(r) := \begin{cases} \{s + zn(s), 0 \leqslant z < r\}, & \text{if } r > 0, \\ \{s + zn(s), r < z \leqslant 0\}, & \text{if } r < 0, \end{cases}$$

then by [1] we have, since V_n only depend on s, that

$$vol\big(\Gamma_0^*(r)\big) = P(r), \tag{7.1}$$

where P is a polynomial function with coefficients which only depend on Γ_0. We now assume that $(\tilde{v}_4(s), \Gamma_4(s))$ has a unique smooth solution of (Q_4) on a time interval $[0, \tilde{T}_4]$ and we set

$$l(s) := \int_0^s V_{n,4}(\tau)d\tau = \frac{3}{\sqrt{2}}\int_0^s h_0\big(\tilde{v}_4(\tau)\big)d\tau, \tag{7.2}$$

so that $\Gamma_4(s) = \{Z + l(s)n(Z),\ Z \in \Gamma_0\}$ where $n(Z)$ is the normal of Γ_0 at Z. Then we have $|l(s)| \leqslant \delta_0$ on a time interval, which we denote again by $[0, \tilde{T}_4]$ and thus by (7.1)

$$|\Omega_4^+(s)| = |\Omega_4^+(0)| + P\big(l(s)\big). \tag{7.3}$$

This yields in view of the assumption (2.19) that

$$\frac{|\Omega_4^+(s)|}{|\Omega|} \in (\alpha_-, \alpha_+),$$

for all s in a time interval, still denoted by $[0, \tilde{T}_4]$. Thus by (2.26) and Lemma B.1 below, we obtain

$$\tilde{v}_4(s) = W\left(\frac{|\Omega_4^+(s)|}{|\Omega|}\right), \text{ for all } s \in [0, \tilde{T}_4],$$

so that in view of (7.3)

$$\tilde{v}_4(s) = W\left(\frac{\big|\Omega_4^+(0)\big| + P\big(l(s)\big)}{|\Omega|}\right) \quad \text{and} \quad l_s = \frac{3}{\sqrt{2}}h_0\left(W\left(\frac{\big|\Omega_4^+(0)\big| + P\big(l(s)\big)}{|\Omega|}\right)\right), \tag{7.4}$$

for all $t \in [0, \tilde{T}_4]$.

We now consider the following ODE

$$(S_4)\begin{cases} Y_s = H(Y), \text{ for } s \in [0, \tilde{T}_4] \\ Y(0) = 0, \end{cases}$$

with $H(.) = \frac{3}{\sqrt{2}}h_0 \circ W\left(\frac{|\Omega_4^+(0)| + P(.)}{|\Omega|}\right)$. By the Cauchy–Lipschitz theorem we have that (S_4) admits a unique solution on a maximal time interval (s_1, s_2) with $s_1 < 0 < s_2$. Thus choosing $\tilde{T}_4 \in (0, s_2)$ such that $|l(ts)| \leqslant \delta_0$ and $\frac{|\Omega_4^+(0)| + P(l(s))}{|\Omega|} \in (\alpha_-, \alpha_+)$ and setting

$$\tilde{v}_4(s) := W\left(\frac{\big|\Omega_4^+(0)\big| + P\big(l(s)\big)}{|\Omega|}\right) \text{ and } \Gamma_4(s) := \{Z + l(s)n(Z),\ Z \in \Gamma_0\}$$

we conclude that $(\tilde{v}_4, \Gamma_4)$ is the unique solution of (Q_4) on $[0, \tilde{T}_4]$, which ends the proof of Theorem 2.5. $\square$

The goal of this stage is to study Problem (P^ε) on the $O(\varepsilon^{p-2/3})$ time scale. This leads us to introduce the corresponding change of variable, namely

$$s := \varepsilon^{2/3-p}(t - \tau_2^\varepsilon - \tau_3^\varepsilon). \tag{7.5}$$

and to deduce from Problem $(\hat{P}^\varepsilon)$ the following system

$$(\hat{P}_4^\varepsilon)\begin{cases} (\alpha_4^\varepsilon)_s = \varepsilon^{2/3}\Delta\alpha_4^\varepsilon + \dfrac{1}{\varepsilon^{2/3}}\hat{f}(\alpha_4^\varepsilon,\beta_4^\varepsilon) & \text{in } \Omega\times(0,T), \quad (7.6)\\ (\beta_4^\varepsilon)_s = \varepsilon^{p-4/3}\Delta\beta_4^\varepsilon + \varepsilon^{p-2/3}g(\alpha_4^\varepsilon,\beta_4^\varepsilon) & \text{in } \Omega\times(0,T), \quad (7.7)\\ \dfrac{\partial\alpha_4^\varepsilon}{\partial n} = \dfrac{\partial\beta_4^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T), \quad (7.8)\end{cases}$$

with the initial conditions

$$\alpha_4^\varepsilon(x,0) = \alpha^\varepsilon(x,\tau_2^\varepsilon+\tau_3^\varepsilon) = u^\varepsilon(x,\tau_2^\varepsilon+\tau_3^\varepsilon) \text{ for } x\in\Omega, \tag{7.9}$$
$$\beta_4^\varepsilon(x,0) = \beta^\varepsilon(x,\tau_2^\varepsilon+\tau_3^\varepsilon) = v^\varepsilon(x,\tau_2^\varepsilon+\tau_3^\varepsilon) \text{ for } x\in\Omega. \tag{7.10}$$

By (A.1) we have for $\beta_4^\varepsilon \in (-\frac{2\sqrt{3}}{9}+\frac{\sigma}{4}, \frac{2\sqrt{3}}{9}-\frac{\sigma}{4})$ that

$$\alpha_4^\varepsilon(x,s) = \alpha^\varepsilon(x,\varepsilon^{p-2/3}s+\tau_2^\varepsilon+\tau_3^\varepsilon) = u^\varepsilon(x,\varepsilon^{p-2/3}s+\tau_2^\varepsilon+\tau_3^\varepsilon) \tag{7.11}$$
$$\beta_4^\varepsilon(x,s) = \beta^\varepsilon(x,\varepsilon^{p-2/3}s+\tau_2^\varepsilon+\tau_3^\varepsilon) = v^\varepsilon(x,\varepsilon^{p-2/3}s+\tau_2^\varepsilon+\tau_3^\varepsilon). \tag{7.12}$$

Moreover let Γ_4^ε be the interface defined by the motion equation

$$V_{n,4}^\varepsilon = \frac{3}{\sqrt{2}}\hat{h}_0\big(\overline{\beta}_4^\varepsilon(s)\big), \quad \Gamma_4^\varepsilon|_{s=0} = \Gamma_0, \tag{7.13}$$

where $V_{n,4}^\varepsilon$ is the velocity of Γ_4^ε. One can prove that Problem (7.13) admits a unique classical solution on a time interval $[0,\tilde{s}_4^\varepsilon]$, for some positive constant $\tilde{s}_4^\varepsilon$. We then deduce from (7.13) and the construction of $\hat{h}_0$ that $|V_{n,4}^\varepsilon| \leqslant C$, where C is a constant independent on ε and $\tilde{s}_4^\varepsilon$. This yields setting

$$l_4^\varepsilon(s) := \frac{3}{\sqrt{2}}\int_0^s \hat{h}_0\big(\overline{\beta}_4^\varepsilon(z)\big)dz, \tag{7.14}$$

that $|l_4^\varepsilon(s)| \leqslant C\tilde{s}_4^\varepsilon$, for all $s\in[0,\tilde{s}_4^\varepsilon]$. Thus the interface Γ_4^ε is well defined on $[0,\frac{\delta_0}{C})$, where δ_0 is defined by (6.14). This gives that (7.13) admits a unique classical solution on the time interval $[0,\tilde{s}_4]$, with $0<\tilde{s}_4<\frac{\delta_0}{C}$. Further for $s\in[0,\tilde{s}_4]$, Γ_4^ε divides Ω into two subdomains, $\Omega_4^{\varepsilon,\pm}(s)$. Let $0<\frac{r_0}{2}<\delta_0$ we introduce as in the stage 6 a smooth truncated approximation of the signed distance function to the interface Γ_4^ε, namely

$$d_4^\varepsilon(x,s) = \begin{cases} r_0 & \text{if } x\in\Omega_4^{\varepsilon,+}(s) \text{ and } dist\big(x,\Gamma_4^\varepsilon(s)\big)\geqslant r_0\\ -r_0 & \text{if } x\in\Omega_4^{\varepsilon,-}(s) \text{ and } dist\big(x,\Gamma_4^\varepsilon(s)\big)\geqslant r_0\\ dist\big(x,\Gamma_4^\varepsilon(s)\big) & \text{if } x\in\Omega^{\varepsilon,+}(s) \text{ and } dist\big(x,\Gamma_4^\varepsilon(s)\big)\leqslant \frac{r_0}{2}\\ -dist\big(x,\Gamma_4^\varepsilon(s)\big) & \text{if } x\in\Omega_4^{\varepsilon,-}(s) \text{ and } dist\big(x,\Gamma_4^\varepsilon(s)\big)\leqslant \frac{r_0}{2},\end{cases}$$

and extended smoothly for $x \in \{r_0/2 < dist(x, \Gamma_4^\varepsilon(s)) < r_0\}$. Moreover we also assume that

$$\frac{\partial d_4^\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega \times (0, \tilde{s}_4). \tag{7.15}$$

We set

$$T_4 := \min(\tilde{T}_4, \tilde{s}_4). \tag{7.16}$$

and we define for $s \in [0, T_4]$

$$U_4^\pm(x,s) = U\left(\frac{d_4^\varepsilon(x,s) \pm R_1 R^\varepsilon e^{\tilde{m}s}}{\varepsilon^{2/3}}, \overline{\beta}_4^\varepsilon(s), \mp R_2 R^\varepsilon\right), \tag{7.17}$$

where

$$R^\varepsilon = \varepsilon^{\frac{c_{p,N}}{3}} |\ln \varepsilon| \tag{7.18}$$

and R_1, R_2 and $\tilde{m}$ are positive constants to be determined later. As in Section 2.3 we now prove that $U_4^\pm$ are sub and super-solution of (7.6).

Lemma 7.1 *The functions $U_4^\pm(x,s)$ satisfy that*

$$\hat{L}_4^\varepsilon(U_4^+, \beta_4^\varepsilon) := (U_4^+)_s - \varepsilon^{2/3}\Delta U_4^+ - \frac{1}{\varepsilon^{2/3}}\hat{f}(U_4^+, \beta_4^\varepsilon) \geqslant 0, \ \text{on } \Omega \times [0, T_4] \tag{7.19}$$

$$\hat{L}_4^\varepsilon(U_4^-, \beta_4^\varepsilon) := (U_4^-)_s - \varepsilon^{2/3}\Delta U_4^- - \frac{1}{\varepsilon^{2/3}}\hat{f}(U_4^-, \beta_4^\varepsilon) \leqslant 0, \ \text{on } \Omega \times [0, T_4] \tag{7.20}$$

and $\dfrac{\partial U_4^\pm}{\partial n} = 0$ *on* $\partial\Omega \times [0, T_4]$.

Proof. It follows from (7.15) that $\dfrac{\partial U_4^\pm}{\partial n} = 0$. Moreover we have

$$L_4^\varepsilon(U_4^+, \beta^\varepsilon) = I_1 + I_2 + I_3 + R_2\frac{R^\varepsilon}{\varepsilon^{2/3}} \tag{7.21}$$

with

$$I_1 := U_v(\overline{\beta}_4^\varepsilon)_s - U_z\Delta d_4^\varepsilon + \frac{1}{\varepsilon^{2/3}}U_{zz}(1 - |\nabla d_4^\varepsilon|^2) \tag{7.22}$$

$$I_2 := \frac{1}{\varepsilon^{2/3}}U_z((d_4^\varepsilon)_s + C\frac{3}{\sqrt{2}}\hat{h}_0(\overline{\beta}_4^\varepsilon, -R_2R^\varepsilon) + e^{\tilde{m}s}\tilde{m}R_1R^\varepsilon) \tag{7.23}$$

and

$$I_3 := \frac{1}{\varepsilon^{2/3}}\left(\hat{f}(U_4^+, \overline{\beta}_4^\varepsilon) - \hat{f}(U_4^+, \beta_4^\varepsilon)\right),$$

where the derivatives of U are evaluated at the point $\left(\frac{d_4^\varepsilon(x,s) + R_1R^\varepsilon e^{\tilde{m}s}}{\varepsilon^{2/3}}, \overline{\beta}_4^\varepsilon(s), -R_2R^\varepsilon\right)$. Since the computations are similar to those done in Lemma 6.1 we only give the main estimates. We have by the definition of d_4^ε and (A.6) that

$$\left|U_{zz}(1 - |\nabla d_4^\varepsilon|^2)\right| \leqslant CK_1 \sup_{|d_4^\varepsilon| \geqslant \frac{r_0}{2}} e^{-K_2\left|\frac{d_4^\varepsilon + R_1R^\varepsilon e^{\tilde{m}s}}{\varepsilon^{2/3}}\right|}$$

and for all $s \in [0, T_4]$

$$\left|d_4^\varepsilon + R_1 e^{\tilde{m}s} R^\varepsilon\right| \geqslant \frac{r_0}{2} - R_1 R^\varepsilon e^{\tilde{m}T_4}, \text{ for } |d_4^\varepsilon| \geqslant \frac{r_0}{2},$$

so that, since $\lim_{\varepsilon\downarrow 0} R^\varepsilon = 0$, $|U_{zz}(1-|\nabla d_4^\varepsilon|^2)| \leqslant CK_1$. Substituting this into (7.22) and also using the fact that U_ν, Δd_4^ε and U_z are bounded and that by (7.7), $|(\overline{\beta}_4^\varepsilon)_t| \leqslant \varepsilon^{p-2/3}C$ we deduce that

$$I_1 \geqslant -F_1\varepsilon^{p-2/3}. \tag{7.24}$$

To estimate I_2 we remark that d_4^ε satisfies

$$\left|(d_4^\varepsilon)_s + \frac{3}{\sqrt{2}}\hat{h}_0(\overline{\beta}_4^\varepsilon)\right| \leqslant \tilde{D}|d_4^\varepsilon|, \quad \text{in } \overline{\Omega}\times[0,T_4],$$

which we substitute into (7.23) to obtain in view of (A.4), (A.5), (A.10) that

$$I_2 \geqslant \frac{1}{\varepsilon^{2/3}} U_z\big(-\tilde{D}|d_4^\varepsilon| - K_4R_2R^\varepsilon + \tilde{m}e^{\tilde{m}s}R_1R^\varepsilon\big).$$

Moreover since

$$\tilde{m}e^{\tilde{m}s}R_1R^\varepsilon - \tilde{D}|d_4^\varepsilon| \geqslant -\tilde{D}|d_4^\varepsilon + e^{\tilde{m}s}R_1R^\varepsilon| + e^{\tilde{m}s}R_1R^\varepsilon(\tilde{m}-\tilde{D}),$$

we deduce as in the proof of (6.29) that

$$I_2 \geqslant -F_2, \tag{7.25}$$

for ε small enough. As it is done in the proof of (6.30) we obtain from (A.13) and (6.11) that

$$I_3 \geqslant -\frac{c_{\hat{f}}}{\varepsilon^{2/3}}\left|\beta^\varepsilon(x, \varepsilon^{p-2/3}s + \tau_2^\varepsilon + \tau_3^\varepsilon) - \overline{\beta}^\varepsilon(\varepsilon^{p-2/3}t + \tau_2^\varepsilon + \tau_3^\varepsilon)\right| \geqslant -\frac{F_3}{\varepsilon^{2/3}}\varepsilon^{\frac{4}{3(N+2)}}. \tag{7.26}$$

Substituting (7.24), (7.25) and (7.26) into (7.21) we deduce

$$\hat{L}_4^\varepsilon(U_4^+, \beta_4^\varepsilon) \geqslant \frac{1}{\varepsilon^{2/3}}R_2R^\varepsilon - F_1\varepsilon^{p-2/3} - F_2 - F_3\frac{1}{\varepsilon^{2/3}}\varepsilon^{\frac{4}{3(N+2)}},$$

for all $(x,s) \in \Omega\times[0,T_4]$. Thus for $R_2 \geqslant F_1 + F_2 + F_3$ we obtain $\hat{L}_4^\varepsilon(U_4^+, \beta_4^\varepsilon) \geqslant 0$, which coincides with (7.19). One can prove the inequality (7.20) in a similar way.

Next we state the following estimates on the initial condition, namely

Lemma 7.2

$$U_4^-(x,0) \leqslant \alpha_4^\varepsilon(x,0) \leqslant U_4^+(x,0), \textit{ for all } x \in \Omega.$$

Proof. We first recall that by (6.50) and (6.51)

$$\overline{v}^\varepsilon(\tau_2^\varepsilon + \tau_3^\varepsilon) \in \Big(-\frac{2\sqrt{3}}{9} + \frac{\sigma}{3}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{3}\Big). \tag{7.27}$$

First case: $d_4^\varepsilon(x,0) \leqslant -\varepsilon^{c_{p,N}}$. Then by definition of d_4^ε we have $\tilde{d}(x,\Gamma_0) = -dist(x,\Gamma_0) \leqslant -\varepsilon^{c_{p,N}}$ and thus using (7.9) and (2.23)

$$\alpha_4^\varepsilon(x,0) = u^\varepsilon(x, \tau_2^\varepsilon + \tau_3^\varepsilon) \leqslant h_+(\tilde{v}_3(\tau_3^\varepsilon)) + M_3\varepsilon^{c_{p,N}/3}. \tag{7.28}$$

Further by (A.5), (7.10), (7.27) (A.2) and (A.12) we obtain

$$\begin{aligned}U_4^+(x,0) &= U\Big(\frac{d_4^\varepsilon(x,0)+R_1R^\varepsilon}{\varepsilon^{2/3}},\overline{\beta}_4^\varepsilon(0),-R_2R^\varepsilon\Big)\\ &\geqslant \hat{h}_-(\overline{\beta}_4^\varepsilon(0),-R_2R^\varepsilon)=\hat{h}_-(\overline{v}^\varepsilon(\tau_2^\varepsilon+\tau_3^\varepsilon),-R_2R^\varepsilon)=h_-(\overline{v}^\varepsilon(\tau_2^\varepsilon+\tau_3^\varepsilon)-R_2R^\varepsilon)\\ &\geqslant h_-(\overline{v}^\varepsilon(\tau_2^\varepsilon+\tau_3^\varepsilon))+K_3R_2R^\varepsilon.\end{aligned}$$

This in view of (6.52) and (A.11) gives

$$U_4^+(x,0)\geqslant h_-\big(\tilde{v}_3(\tau_3^\varepsilon)\big)-K_4L_2\varepsilon^{c_{p,N}}|\ln\varepsilon|^2+K_3R_2R^\varepsilon,$$

which together with (7.28) implies $\alpha_4^\varepsilon(x,0)\leqslant U_4^+(x,0)$ for $R_2\geqslant \dfrac{M_3+K_4L_2}{K_3}$.

Second case: $d_4^\varepsilon(x,0)\geqslant -\varepsilon^{c_{p,N}}$. We note that $d_4^\varepsilon(x,0)+R_1R^\varepsilon\geqslant \dfrac{R_1R^\varepsilon}{2}$ and then as previously using (A.7), (A.12), (7.27), (7.10), (2.24) and (A.11) we obtain

$$\begin{aligned}U_4^+(x,0) &\geqslant U\left(\frac{R_1R^\varepsilon}{2\varepsilon^{2/3}},\overline{\beta}_4^\varepsilon(0),-R_2R^\varepsilon\right)\geqslant \hat{h}_+\big(\overline{\beta}_4^\varepsilon(0),-R_2R^\varepsilon\big)-K_1e^{-K_2\frac{R_1R^\varepsilon}{2\varepsilon^{2/3}}}\\ &\geqslant h_+\big(\tilde{v}_3(\tau_3^\varepsilon)\big)-K_4L_2\varepsilon^{c_{p,N}}|\ln\varepsilon|^2+K_3R_2R^\varepsilon-K_1R^\varepsilon,\end{aligned}$$

for ε small enough. Thus for $R_2\geqslant \frac{M_3+K_4L_2+K_1}{K_3}$ we deduce in view of (2.23) and (7.9) that $\alpha_4^\varepsilon(x,0)=u^\varepsilon(x,\tau_2^\varepsilon+\tau_3^\varepsilon)\leqslant U_4^+(x,0)$ for all $x\in\Omega$ such that $d_4^\varepsilon(x,0)\geqslant -\varepsilon^{c_{p,N}}$.

Finally we have obtained that $\alpha_4^\varepsilon(x,0)\leqslant U_4^+(x,0)$ for all $x\in\Omega$. Similarly, one can check that $\alpha_4^\varepsilon(x,0)\geqslant U_4^-(x,0)$ for all $x\in\Omega$, which ends the proof of Lemma 7.2. □

Next we prove the following result

Lemma 7.3 *There exist two positive constants, R_3 and R_4, such that*

$$\left|\alpha_4^\varepsilon(x,s)-\hat{h}_+equation(\overline{\beta}_4^\varepsilon(s))\right|\leqslant R_3R^\varepsilon \text{ if } d_4^\varepsilon(x,s)\geqslant R_4R^\varepsilon \tag{7.29}$$

and

$$\left|\alpha_4^\varepsilon(x,s)-\hat{h}_-\big(\overline{\beta}_4^\varepsilon(s)\big)\right|\leqslant R_3R^\varepsilon \text{ if } d_4^\varepsilon(x,s)\leqslant -R_4R^\varepsilon, \tag{7.30}$$

for all $s\in[0,T_4]$.

Proof. Using the comparison principle we deduce from lemmas 7.1 and 7.2 that

$$U_4^-(x,s)\leqslant u_4^\varepsilon(x,s)\leqslant U_4^+(x,s). \tag{7.31}$$

For $d_4^\varepsilon(x,s)\geqslant R_4R^\varepsilon$, where $R_4\geqslant R_1+R_1e^{\tilde{m}T_4}$ we obtain that

$$d_4^\varepsilon(x,s)-R_1e^{\tilde{m}s}R^\varepsilon\geqslant R_4R^\varepsilon-R_1\varepsilon^{\tilde{m}T_4}R^\varepsilon\geqslant R_1R^\varepsilon.$$

This in view of (7.31), (A.7) and (A.10) implies that

$$\begin{aligned}\alpha_4^\varepsilon(x,s) &\geqslant U_4^-(x,s)\geqslant U^-\Big(\frac{R_1R^\varepsilon}{\varepsilon^{2/3}},\overline{\beta}_4^\varepsilon(s),R_2R^\varepsilon\Big)\\ &\geqslant \hat{h}_+\big(\overline{\beta}_4^\varepsilon(s),R_2R^\varepsilon\big)-K_1e^{-\frac{K_2R_1R^\varepsilon}{\varepsilon^{2/3}}}\\ &\geqslant \hat{h}_+\big(\overline{\beta}_4^\varepsilon(s)\big)-K_4R_2R^\varepsilon-K_1R^\varepsilon\geqslant h_+\big(\overline{\beta}_4^\varepsilon(s)\big)-R_3R^\varepsilon,\end{aligned} \tag{7.32}$$

for $R_3 \geqslant K_4 R_2 + K_1$ and ε small enough. Moreover we have by (A.12) that

$$\alpha_4^\varepsilon(x,s) \leqslant U_4^+(x,s) \leqslant \hat{h}_+\big(\overline{\beta}_4^\varepsilon(s), -R_2R^\varepsilon\big) \leqslant \hat{h}_+\big(\overline{\beta}_4^\varepsilon(s)\big) + K_4R_2R^\varepsilon \leqslant \hat{h}_+\big(\overline{\beta}_4^\varepsilon(s)\big) + R_3R^\varepsilon.$$

This combined with (7.32) implies (7.29). Similarly one can check (7.30).

Lemma 7.4 *There exists a function $L_4 \in C([0,T_4])$ such that l_4^ε tends to L_4 uniformly on* $[0,T_4]$*, as ε tends to 0. Moreover*

$$L_4(s) = \int_0^s \lim_{\varepsilon\downarrow 0} \hat{h}_0\big(\overline{\beta}_4^\varepsilon(z)\big)dz \text{ for all } s \in [0,T_4]. \tag{7.33}$$

Further L_4 is differentiable almost everywhere on $[0,T_4]$ *and there exists a positive constant $\overline{L}_4$ such that*

$$|(L_4)_s(s)| \leqslant \overline{L}_4 \text{ almost everywhere on } [0,T_4]. \tag{7.34}$$

Proof. We deduce from (7.14) that l_4^ε and $(l_4^\varepsilon)_s$ are bounded uniformly with respect to ε on a time interval $[0,T_4]$. Thus there exist a function L_4 and a subsequence of ε, which we denote again by ε such that l_4^ε tends to L_4 uniformly on $[0,T_4]$, as ε tends to 0. This together with (7.14) gives (7.33). Further since $\hat{h}_0$ is smooth and $\overline{\beta}_4^\varepsilon$ is bounded on $[0,T_4]$ we have that L_4 is a Lipschitz function. Thus L_4 is differentiable almost everywhere on $[0,T_4]$ and

$$(L_4)_s(s) = \lim_{\varepsilon\downarrow 0} \hat{h}_0\big(\overline{\beta}_4^\varepsilon(s)\big), \text{ for almost } s \in [0,T_4].$$

This with the fact that $\hat{h}_0$ is smooth and $\overline{\beta}_4^\varepsilon$ is bounded implies that $(L_4)_s$ is bounded for almost $s \in [0,T_4]$, which coincides with (7.34). □

Lemma 7.5

$$\overline{\beta}_4^\varepsilon \text{ tends uniformly to } \tilde{v}_4 \text{ on } [0,T_4]. \tag{7.35}$$

$$L_4(s) = \frac{3}{\sqrt{2}} \int_0^s h_0\big(\tilde{v}_4(z)\big)dz, \text{ for all } s \in [0,T_4]. \tag{7.36}$$

Proof. In what follows we check that $\overline{\beta}_4^\varepsilon$ satisfies

$$\varepsilon^{2/3-p}(\overline{\beta}_4^\varepsilon)_s = -\overline{\beta}_4^\varepsilon + \hat{h}_+(\overline{\beta}_4^\varepsilon)\frac{|\Omega_4^{\varepsilon,+}|}{|\Omega|} + \hat{h}_-(\overline{\beta}_4^\varepsilon)\left(1 - \frac{|\Omega_4^{\varepsilon,+}|}{|\Omega|}\right) - a + \omega_2(s), \tag{7.37}$$

$$\overline{\beta}_4^\varepsilon(0) = \tilde{v}_{3,\infty} + \omega_3(s), \tag{7.38}$$

where $|\omega_2(s)| \leqslant CR^\varepsilon$ and $\omega_3(s) \leqslant C\varepsilon^{\frac{c_{p,N}}{m}}$, for all $s \in [0,T_4]$. We first prove (7.38). We have by (7.10) that

$$\overline{\beta}_4^\varepsilon(0) = \overline{v}^\varepsilon\big(\tau_2^\varepsilon + \tau_3^\varepsilon\big) = \tilde{v}_\infty^3 + \big[-\tilde{v}_\infty^3 + \tilde{v}_3(\tau_2^\varepsilon + \tau_3^\varepsilon)\big] + \big[-\tilde{v}_3(\tau_2^\varepsilon + \tau_3^\varepsilon) + \overline{v}^\varepsilon(\tau_2^\varepsilon + \tau_3^\varepsilon)\big].$$

Thus (7.38) follows directly from (2.22) and (2.24). Since the proof of (7.37) is very similar to the proof of (6.45), we omit the details of the computation. Integrating (7.7) on Ω we obtain

$$\begin{aligned}\varepsilon^{2/3-p}(\overline{\beta}_4^{\varepsilon})_s(s) &= -\overline{\beta}_4^{\varepsilon}(s) + \frac{1}{|\Omega|}\biggl(\int_{\Omega_4^{\varepsilon,+}} \alpha_4^{\varepsilon}(x,s)dx + \int_{\Omega_4^{\varepsilon,-}} \alpha_4^{\varepsilon}(x,s)dx\biggr) - a \\ &= -\overline{\beta}_4^{\varepsilon}(s) + \frac{\hat{h}_+(\overline{\beta}_4^{\varepsilon}(s))|\Omega_4^{\varepsilon,+}(s)| + \hat{h}_-(\overline{\beta}_4^{\varepsilon}(s))|\Omega_4^{\varepsilon,-}(s)|}{|\Omega|} - a \\ &\quad + \frac{1}{|\Omega|}\biggl(\tilde{J}_1^{\varepsilon,+} + \tilde{J}_2^{\varepsilon,+} + \tilde{J}_3^{\varepsilon,-} + \tilde{J}_4^{\varepsilon,-}\biggr), \qquad (7.39)\end{aligned}$$

where

$$\begin{aligned}\tilde{J}_1^{\varepsilon,+} &:= \int_{\{x\in\Omega,\ d_4^{\varepsilon}(x,s)\geqslant R_4R^{\varepsilon}\}} \Bigl[\alpha_4^{\varepsilon}(x,s) - \hat{h}_+\bigl(\overline{\beta}_4^{\varepsilon}(s)\bigr)\Bigr]dx \\ \tilde{J}_2^{\varepsilon,+} &:= \int_{\{x\in\Omega,\ 0\leqslant d_4^{\varepsilon}(x,s)<R_4R^{\varepsilon}\}} \Bigl[\alpha_4^{\varepsilon}(x,s) - \hat{h}_+\bigl(\overline{\beta}_4^{\varepsilon}(s)\bigr)\Bigr]dx \\ \tilde{J}_3^{\varepsilon,-} &:= \int_{\{x\in\Omega,\ d_4^{\varepsilon}(x,s)\leqslant -R_4R^{\varepsilon}} \Bigl[\alpha_4^{\varepsilon}(x,s) - \hat{h}_-\bigl(\overline{\beta}_4^{\varepsilon}(s)\bigr)\Bigr]dx \\ \tilde{J}_4^{\varepsilon,-} &:= \int_{\{x\in\Omega,\ 0\geqslant d_4^{\varepsilon}(x,s)>-R_4R^{\varepsilon}\}} \Bigl[\alpha_4^{\varepsilon}(x,s) - \hat{h}_-\bigl(\overline{\beta}_4^{\varepsilon}(s)\bigr)\Bigr]dx.\end{aligned}$$

As it is done in the proof of Lemma 6.4, we deduce from Lemma 7.3 and the fact that $h_{\pm}(\overline{\beta}_4^{\varepsilon})$ and α_4^{ε} are bounded that $|\tilde{J}_i^{\varepsilon,+}| \leqslant CR^{\varepsilon}$ for $i = 1,2$ and $|\tilde{J}_i^{\varepsilon,-}| \leqslant CR^{\varepsilon}$ for $i = 3,4$ which we substitute into (7.39) to deduce that $\overline{\beta}_4^{\varepsilon}$ satisfies (7.37).

Moreover by the motion equation (7.13) we obtain as it is done in the proof of Theorem 2.5 that

$$|\Omega_4^{\varepsilon,+}(s)| = |\Omega^{0,+}| + P\bigl(l_4^{\varepsilon}(s)\bigr),$$

where P is the polynomial function introduced in (7.1). This together with Lemma 7.4 yields that

$$|\Omega_4^{\varepsilon,+}| \text{ tends to a function } \zeta \text{ uniformly on } [0,T_4], \qquad (7.40)$$

which satisfies

$$\zeta(s) = |\Omega^{0,+}| + P\bigl(L_4(s)\bigr), \text{ for all } s \in [0,T_4]. \qquad (7.41)$$

We set

$$\Theta(\alpha,v) := K(\alpha,v) + v = h_+(v)\alpha + h_-(v)(1-\alpha) - a \qquad (7.42)$$

and

$$\hat{\Theta}(\alpha,v) := \hat{h}_+(v)\alpha + \hat{h}_-(v)(1-\alpha) - a. \qquad (7.43)$$

By (2.19), $\frac{\zeta(0)}{|\Omega|} \in (\alpha_-,\alpha_+)$ thus there exists a time, which we denote again by T_4 such that $\frac{\zeta(t)}{|\Omega|} \in (\alpha_-,\alpha_+)$, for all $t \in [0,T_4]$. Thus setting $\chi_4(s) = W(\frac{\zeta(s)}{|\Omega|})$ we deduce from Lemma B.1 that $\chi_4 \in (-2\frac{\sqrt{3}}{9}+\sigma, 2\frac{\sqrt{3}}{9}-\sigma)$ and

$$\Theta\Bigl(\frac{\zeta}{|\Omega|},\chi_4\Bigr)(s) - \chi_4(s) = 0. \qquad (7.44)$$

Further by (A.2) we have $\Theta(\frac{\zeta}{|\Omega|},\chi_4) = \hat{\Theta}(\frac{\zeta}{|\Omega|},\chi_4) = \chi_4$. Since $\chi_4 = W(\frac{\zeta}{|\Omega|})$ we obtain from (7.41), Lemma 7.4 and the smoothness of W that χ_4 is differentiable almost everywhere on $[0,T_4]$ and moreover using (7.34) and (B.3) we also obtain that $|(\chi_4)_s(s)| \leqslant \overline{\chi}_4$, for almost s in $[0,T_4]$. This together with (7.44) gives that χ_4 satisfies a similar ODE to the one satisfied by $\overline{\beta}_4^\varepsilon$, namely

$$\varepsilon^{2/3-p}(\chi_4)_s + \chi_4 = \hat{\Theta}\Big(\frac{\zeta}{|\Omega|},\chi_4\Big) + \omega_4 \tag{7.45}$$

where $|\omega_4(s)| \leqslant \overline{\chi}_4\varepsilon^{2/3-p}$, for almost $s \in [0,T_4]$. Further since $\zeta(0) = \frac{|\Omega^{0,+}|}{|\Omega|}$ we have by (2.21) and Lemma B.1 that $\tilde{v}_{3,\infty} = W(\frac{\zeta(0)}{|\Omega|})$, so that in view of the definition of χ_4

$$\chi_4(0) = \tilde{v}_{3,\infty}. \tag{7.46}$$

We now prove that

$$\overline{\beta}_4^\varepsilon \text{ tends uniformly to } \chi_4 \text{ on } [0,T_4]. \tag{7.47}$$

Setting $\phi^\varepsilon = \overline{\beta}_4^\varepsilon - \chi_4$, subtracting (7.45) from (7.37) we obtain

$$\varepsilon^{2/3-p}(\phi^\varepsilon)_s + \phi^\varepsilon(s) = \hat{\Theta}\Big(\frac{|\Omega_4^{\varepsilon,+}|}{|\Omega|},\overline{\beta}_4^\varepsilon\Big)(s) - \hat{\Theta}\Big(\frac{\zeta}{|\Omega|},\chi_4\Big)(s) + \omega_2(s) - \omega_4(s) \tag{7.48}$$

Furthermore one can check that for all functions $I \in C^1([0,1]\times[-C,C])$

$$\begin{aligned} I(\alpha_2,z_2) - I(\alpha_1,z_1) &= \int_0^1 \frac{\partial I}{\partial u}\big((1-u)\alpha_1 + u\alpha_2, (1-u)z_1 + uz_2\big)du \\ &= (\alpha_2-\alpha_1)\int_0^1 \frac{\partial I}{\partial \alpha}\big((1-u)\alpha_1 + u\alpha_2, (1-u)z_1 + uz_2\big)du \\ &\quad + (z_2-z_1)\int_0^1 \frac{\partial I}{\partial z}\big((1-u)\alpha_1 + u\alpha_2, (1-u)z_1 + uz_2\big)du. \end{aligned}$$

Applying this with $I = \hat{\Theta}$, $\alpha_1 = \frac{\zeta}{|\Omega|}$, $\alpha_2 = \frac{|\Omega_4^{\varepsilon,+}|}{|\Omega|}$, $z_1 = \chi_4$, $z_2 = \overline{\beta}_4^\varepsilon$ and the constant $C = \hat{C}_1 := \max(2\frac{\sqrt{3}}{9} - \sigma, \hat{C}_0)$ so that $(\chi_4,\overline{\beta}_4^\varepsilon) \in [-\hat{C}_1,\hat{C}_1]^2$ we deduce

$$\varepsilon^{2/3-p}(\phi^\varepsilon)_s + \phi^\varepsilon(s) = \Big(\frac{|\Omega^{\varepsilon,+}|}{|\Omega|} - \frac{\zeta}{|\Omega|}\Big)\Psi_1^\varepsilon(s) + \phi^\varepsilon(s)\Psi_2^\varepsilon(s) + \omega_2(s) - \omega_4(s), \tag{7.49}$$

where

$$\Psi_1^\varepsilon(s) = \int_0^1 \frac{\partial\hat{\Theta}}{\partial\alpha}\big(\alpha(u,s),z(u,s)\big)du = \int_0^1 \hat{h}_+\big(z(u,s)\big) - \hat{h}_-\big(z(u,s)\big)du$$

$$\Psi_2^\varepsilon(s) = \int_0^1 \frac{\partial\hat{\Theta}}{\partial z}\big(\alpha(u,s),z(u,s)\big)du = \int_0^1 \alpha(u,s)\hat{h}'_+\big(z(u,s)\big) + \big(1-\alpha(u,s)\big)\hat{h}'_-\big(z(u,s)\big)du$$

and with

$$\alpha(u,s) = (1-u)\frac{\zeta(s)}{|\Omega|} + u\frac{|\Omega_4^{\varepsilon,+}(s)|}{|\Omega|} \in [0,1] \text{ and } z(u,s) = (1-u)\chi_4(s) + u\overline{\beta}_4^\varepsilon(s) \in [-\hat{C}_1,\hat{C}_1],$$

for all $(u,s) \in [0,1] \times [0,T_4]$. Since $\hat{h}_+$ and $\hat{h}_-$ are strictly decreasing function on $\mathbb{R}$, there exist a constant $H > 0$ such that

$$\hat{h}'_+(v) \leqslant -H \text{ and } \hat{h}'_-(v) \leqslant -H, \text{ for all } v \in [-\hat{C}_1, \hat{C}_1]$$

and thus

$$\Psi_2^\varepsilon(s) = \int_0^1 \alpha(u,s)\hat{h}'_+\big(z(u,s)\big) + \big(1-\alpha(u,s)\big)\hat{h}'_-\big(z(u,s)\big)du \leqslant -H, \text{ for all } s \in [0,T_4]. \tag{7.50}$$

Now setting $\varphi^\varepsilon(s) := e^{\varepsilon^{p-2/3}s}\phi^\varepsilon(s)$ we obtain from (7.49) that φ^ε satisfies

$$(\varphi^\varepsilon)_s = \varepsilon^{p-2/3}\varphi^\varepsilon\Psi_2^\varepsilon + \varepsilon^{p-2/3}e^{\varepsilon^{p-2/3}s}\Psi_3^\varepsilon \tag{7.51}$$

where $\Psi_3^\varepsilon(s) = \big(\frac{|\Omega_4^{\varepsilon,+}|}{|\Omega|} - \frac{\zeta}{|\Omega|}\big)\Psi_1^\varepsilon(s) + \omega_2(s) - \omega_4(s)$. Setting $N_2(s) := \displaystyle\int_0^s \Psi_2^\varepsilon(\tau)d\tau$ and solving the ordinary differential equation (7.51) we then have

$$\varphi^\varepsilon(s) = \varphi^\varepsilon(0)e^{\varepsilon^{p-2/3}N_2(s)} + \varepsilon^{p-2/3}e^{\varepsilon^{p-2/3}N_2(s)}\int_0^s e^{-\varepsilon^{p-2/3}N_2(\tau)}e^{\varepsilon^{p-2/3}\tau}\Psi_3^\varepsilon(\tau)d\tau,$$

so that since $\phi^\varepsilon(s) = e^{-\varepsilon^{p-2/3}s}\varphi^\varepsilon(s)$

$$\phi^\varepsilon(s) = \phi^\varepsilon(0)e^{\varepsilon^{p-2/3}(N_2(s)-s)} + \varepsilon^{p-2/3}\int_0^s e^{\varepsilon^{p-2/3}(N_2(s)-N_2(\tau)+\tau-s)}\Psi_3^\varepsilon(\tau)d\tau.$$

Further let $\tilde{\varepsilon}_0 > 0$ then by (7.40) we have for all ε small enough that

$$\big|\Psi_3^\varepsilon(s)\big| \leqslant \tilde{\varepsilon}_0, \text{ for all } s \in [0,T_4].$$

Noting that by (7.50) N_2 is non-increasing we then deduce that

$$\big|\phi^\varepsilon(s)\big| \leqslant \big|\phi^\varepsilon(0)\big|e^{\varepsilon^{p-2/3}(N_2(s)-s)} + \tilde{\varepsilon}_0\varepsilon^{p-2/3}\int_0^s e^{\varepsilon^{p-2/3}(\tau-s)}d\tau.$$

This together with (7.50) implies

$$\big|\phi^\varepsilon(s)\big| \leqslant \big|\phi^\varepsilon(0)\big|e^{-\varepsilon^{p-2/3}s(H+1)} + \tilde{\varepsilon}_0 \leqslant \big|\phi^\varepsilon(0)\big| + \tilde{\varepsilon}_0,$$

so that in view of (7.38), (7.46) gives (7.47). Moreover since $\chi_4 \in (-2\frac{\sqrt{3}}{9}+\sigma, 2\frac{\sqrt{3}}{9}-\sigma)$ we deduce from (7.47) that

$$\beta_4^\varepsilon \in \Big(-2\frac{\sqrt{3}}{9} + \frac{\sigma}{4}, 2\frac{\sqrt{3}}{9} - \frac{\sigma}{4}\Big). \tag{7.52}$$

Furthermore letting $\varepsilon \downarrow 0$ into (7.14) and also using (7.47) we deduce that

$$L_4(s) = \frac{3}{\sqrt{2}}\int_0^s h_0\big(\chi_4(z)\big)dz, \text{ for all } s \in [0,T_4]. \tag{7.53}$$

Setting

$$G_4(s) := \big\{Z + L_4(s)n(Z),\ Z \in \Gamma_0\big\}, \text{ for all } s \in [0,T_4],$$

where $n(Z)$ is the normal of Γ_0 at Z we deduce from (7.41), (7.44) and (7.53) that (G_4, χ_4) satisfies Problem (Q_4). Thus by the uniqueness of the solution of Problem (Q_4) we deduce that

$$G_4(s) = \Gamma_4(s) \text{ and } \chi_4(s) = \tilde{v}_4(s), \text{ for all } s \in [0, T_4]. \tag{7.54}$$

Finally (7.35) follows directly from (7.47) and (7.54) while (7.36) follows from (7.53) and (7.54), which concludes the proof of Lemma 7.5. □

We are now in a position to prove Theorem 2.6.

Proof of Theorem 2.6. To begin with we note that (7.52), (7.11) and (7.12) imply

$$\alpha_4^\varepsilon(x, s) = u^\varepsilon\big(x, \varepsilon^{p-2/3}s + \tau_2^\varepsilon + \tau_3^\varepsilon\big) \quad \text{and} \quad \beta_4^\varepsilon(x, s) = v^\varepsilon\big(x, \varepsilon^{p-2/3}s + \tau_2^\varepsilon + \tau_3^\varepsilon\big). \tag{7.55}$$

for all $(x, s) \in \overline{\Omega} \times [0, T_4]$.

Let $\varepsilon^* \in (0, \frac{\sigma}{4})$ then by (7.35) we have

$$\sup_{[0,T_4]} \big|\overline{\beta}_4^\varepsilon(s) - \tilde{v}_4(s)\big| \leqslant \frac{\varepsilon^*}{2}, \tag{7.56}$$

for all ε small enough. Thus (2.28) follows directly from (3.3), (7.56) and (7.55).

We next show (2.27). Let $\varepsilon^* > 0$ and $(x, s) \in \Omega \times [0, T_4]$ such that $|\tilde{d}(x, \Gamma_4(s))| > \varepsilon^*$. We first assume that $\tilde{d}(x, \Gamma_4(s)) > \varepsilon^*$ then we obtain from Lemma 7.4 and (7.36) that

$$|d_4^\varepsilon(x, t)| \geqslant r_0/2 \text{ or } d_4^\varepsilon(x, t) \geqslant \tilde{d}(x, \Gamma_4(s)) - \varepsilon^*/2 \geqslant \varepsilon^*/2,$$

for ε small enough. Thus in both cases we obtain $d_4^\varepsilon(x, t) \geqslant R_4 R^\varepsilon$, for ε small enough. This yields using (A.11) and (7.29) that

$$\begin{aligned}\big|\alpha_4^\varepsilon(x, s) - h_+\big(\tilde{v}_4(s)\big)\big| &\leqslant \big|\alpha_4^\varepsilon(x, s) - h_+\big(\overline{\beta}_4^\varepsilon(s)\big)\big| + \big|h_+\big(\overline{\beta}_4^\varepsilon(s)\big) - h_+\big(\tilde{v}_4(s)\big)\big| \\ &\leqslant R_3 R^\varepsilon + K_4\big|\overline{\beta}_4^\varepsilon(s) - \tilde{v}_4(s)\big|,\end{aligned}$$

which in view of (7.55) and (7.56) implies (2.27) in the case $\tilde{d}(x, \Gamma_4(s)) > \varepsilon^*$. Similarly one can check (2.27) in the case $\tilde{d}(x, \Gamma_4(s)) < -\varepsilon^*$, which achieves the proof of Theorem 2.6. □

8. Formal derivation of the fifth stage: Propagation with mean curvature

Setting $\tau := \varepsilon^{4/3-p}t$ and $w^\varepsilon := \varepsilon^{-2/3}v^\varepsilon$ we obtain from (P^ε) the system

$$(P_5^\varepsilon)\begin{cases} (u_5^\varepsilon)_\tau = \Delta u_5^\varepsilon + \dfrac{1}{\varepsilon^{4/3}} f(u_5^\varepsilon, \varepsilon^{2/3}w_5^\varepsilon) & \text{in } \Omega \times (0, T_5) \quad (8.1) \\ \varepsilon^{2-p}(w_5^\varepsilon)_\tau = \Delta w_5^\varepsilon + u_5^\varepsilon - a - \varepsilon^{2/3}w_5^\varepsilon & \text{in } \Omega \times (0, T_5) \quad (8.2) \\ \dfrac{\partial u_5^\varepsilon}{\partial n} = \dfrac{\partial w_5^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T_5) \quad (8.3) \\ u_5^\varepsilon(x, 0) = u_{5,0}^\varepsilon(x), \;\; w_5^\varepsilon(x, 0) = w_{5,0}^\varepsilon(x) \text{ for } x \in \Omega. & \quad (8.4) \end{cases}$$

We present in this section a formal derivation of the singular limit of Problem (P_5^ε) as $\varepsilon \downarrow 0$. More precisely we show heuristically how to obtain the motion equation

$$V_n = K + C'(0)\hat{w}_5 - \frac{1}{|\Gamma_5|}\left(\int_{\Gamma_5} K + C'(0)\int_{\Gamma_5}\hat{w}_5\right) \text{ on } \Gamma_5(\tau),\ \tau \in (0, T_5) \tag{8.5}$$

where $\hat{w}_5$ satisfies

$$(Q_5)\begin{cases} -\Delta\hat{w}_5 = u_5 - a & \text{in } \Omega \times (0, T_5) \\ \dfrac{\partial\hat{w}_5}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T_5) \\ \displaystyle\int_\Omega \hat{w}_5 dx = 0 & \text{for all } t \in (0, T_5), \end{cases}$$

with

$$u_5(x, \tau) = \begin{cases} 1 & \text{if } x \in \Omega_5^+(\tau) \cup \Gamma_5(\tau) \\ -1 & \text{if } x \in \Omega_5^-(\tau), \end{cases}$$

and where V_n, K, $\Omega_5^+(\tau)$ and $\Omega_5^-(\tau)$ denote respectively the normal velocity, the the sum of principal curvatures (positive if $\Omega_5^+(\tau)$ is convex), the interior and the exterior of $\Gamma_5(\tau)$, respectively. We remark that (Q_5) only makes sense if $\displaystyle⨍_\Omega u_5(x, \tau)\, dx = a$, for all $\tau \in (0, T_5)$. We define the operator

$$L^5(\psi) = \psi_\tau - \Delta\psi + \frac{1}{\varepsilon^{4/3}} f(\psi, \varepsilon^{2/3} w_5^\varepsilon)$$

and we set $\overline{w_5^\varepsilon}(\tau) := ⨍_\Omega w_5^\varepsilon(x, \tau)dx$. Then $\overline{w_5^\varepsilon}$ satisfies the initial value problem

$$\begin{cases} \varepsilon^{2-p}(\overline{w_5^\varepsilon})_\tau = ⨍_\Omega u_5^\varepsilon - a - \varepsilon^{2/3}\overline{w_5^\varepsilon} \text{ for } \tau \in (0, T_5) \\ \overline{w_5^\varepsilon}(0) = ⨍_\Omega w_5^\varepsilon(x, 0)dx. \end{cases} \tag{8.6}$$

The function $\hat{w}_5^\varepsilon(x, \tau) := w_5^\varepsilon(x, \tau) - ⨍_\Omega w_5^\varepsilon(x, \tau)dx$ satisfies

$$\begin{cases} \varepsilon^{2-p}(\hat{w}_5^\varepsilon)_\tau = \Delta\hat{w}_5^\varepsilon + (u_5^\varepsilon - ⨍_\Omega u_5^\varepsilon) - \varepsilon^{2/3}\hat{w}_5^\varepsilon \text{ in } \Omega \times (0, T_5) & (8.7) \\ \dfrac{\partial\hat{w}_5^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T_5) & (8.8) \\ \hat{w}_5^\varepsilon(x, 0) = w_5^\varepsilon(x, 0) - ⨍_\Omega w_5^\varepsilon(x, 0)dx \quad \text{for } x \in \Omega. \end{cases}$$

By definition we have that

$$\int_\Omega \hat{w}_5^\varepsilon(x, \tau)dx = 0, \text{ for } \tau \in (0, T_5). \tag{8.9}$$

Next we make the assumption that for ε small enough the function u_5^ε can be approximated by

$$U\left(\frac{d(x, \tau)}{\varepsilon^{2/3}}, \varepsilon^{2/3} w_5^\varepsilon(x, \tau)\right),$$

where d is the signed distance to the interface $\Gamma_5(\tau)$, strictly positive in $\Omega_5^+(\tau)$ and strictly negative in $\Omega_5^-(\tau)$. A classical computation gives that

$$L^5(U) = U_{zz}\frac{1}{\varepsilon^{4/3}}\big(1-|\nabla d|^2\big) + U_z\frac{1}{\varepsilon^{2/3}}\Big(d_\tau - \Delta d + \lim_{\varepsilon\downarrow 0}\frac{C(\varepsilon^{2/3}w_5^\varepsilon)}{\varepsilon^{2/3}}\Big) + O(1),$$

where the functions U_{zz} and U_z are evaluated at the point $(\frac{d(x,\tau)}{\varepsilon^{2/3}}, \varepsilon^{2/3}w_5^\varepsilon(x,\tau))$. Denoting by w_5 the limit of w_5^ε as $\varepsilon\downarrow 0$, we deduce that

$$d_\tau = \Delta d - C'(0)w_5 \quad \text{on } \Gamma_5(\tau). \tag{8.10}$$

Since $-d_\tau = V_n$ and $\Delta d = -K$ we rewrite (8.10) as

$$V_n = K + C'(0)w_5 \quad \text{on } \Gamma_5(\tau). \tag{8.11}$$

We write again $w_5(x,\tau) = \overline{w}_5(\tau) + \hat{w}_5(x,\tau)$, where $\overline{w}_5(\tau) = ⨍_\Omega w_5(x,\tau)dx$ and $\hat{w}_5(x,\tau) = w_5(x,\tau) - \overline{w}_5(\tau)$. Finally, we formally deduce from (8.6) that

$$⨍_\Omega u_5 = a.$$

Indeed, if $⨍_\Omega u_5^\varepsilon > a$ by a small $O(\varepsilon^{2/3})$ quantity, then on the short $O(\varepsilon^{4/3-p})$ time scale the value of $\overline{w_5^\varepsilon}$ would reach an $O(1)$ positive value, see (8.6). In turn, by (8.11) this would result in an $O(1)$ positive contribution to V_n (recall that $C'(0) = \frac{3}{\sqrt{2}}h_0'(0) > 0$), which would result in returning the value of $⨍_\Omega u_5^\varepsilon$ to a on an $O(\varepsilon^{2/3})$ time scale. The argument works the same if $⨍_\Omega u_5^\varepsilon < a$ by a small $O(\varepsilon^{2/3})$ quantity. With this information, from (8.7)–(8.9) we then formally deduce that

$$\begin{cases} -\Delta\hat{w}_5 = u_5 - a \\ \dfrac{\partial\hat{w}_5}{\partial n} = 0 \\ \displaystyle\int_\Omega \hat{w}_5 = 0, \end{cases}$$

which coincides with Problem (Q_5).

Finally, differentiating the mass constraint with respect to time, we deduce that

$$0 = \frac{d}{d\tau}\Big(\int_\Omega u_5\Big) = 2\int_{\Omega_5^+}(u_5)_\tau + 2\int_{\Gamma_5} V_n,$$

which yields $\int_{\Gamma_5} V_n = 0$. This together with (8.11) gives

$$0 = \int_{\Gamma_5} V_n = \int_{\Gamma_5} K + C'(0)\Big(\int_{\Gamma_5}\overline{w}_5 + \int_{\Gamma_5}\hat{w}_5\Big),$$

so that $C'(0)\overline{w}_5 = -⨍_{\Gamma_5} K - C'(0)⨍_{\Gamma_5}\hat{w}_5$, which we substitute into (8.11) to conclude that

$$V_n = K + C'(0)\hat{w}_5 - \frac{1}{|\Gamma_5|}\Big(\int_{\Gamma_5} K + C'(0)\int_{\Gamma_5}\hat{w}_5\Big) \quad \text{on } \Gamma_5(\tau). \tag{8.12}$$

This completes the formal derivation of the singular limit Problem (8.5) $-$ (Q_5).

A. Appendix. Travelling wave solutions

Let $|v| < \frac{2\sqrt{3}}{9}$ then the equation $f(s,v) = 0$ has three zeros such that $h_-(v) < h_0(v) < h_+(v)$. We define $\hat{h}_\pm(v)$ by

$$\hat{h}_\pm(v) = h_\pm(v) \text{ for } v \in [0, \frac{2\sqrt{3}}{9} - \frac{\sigma}{4})$$

and

$$\hat{h}_\pm(v) = -\hat{h}_\mp(-v) \text{ for } v \in (-\frac{2\sqrt{3}}{9} + \frac{\sigma}{4}, 0]$$

with $\lim_{v\to\pm\infty} \hat{h}_\pm(v) = \pm\frac{1}{\sqrt{3}}$, $\lim_{v\to\mp\infty} \hat{h}_\pm(v) = \pm\frac{2}{\sqrt{3}}$ and smoothly extend in $(-\infty, -\frac{2\sqrt{3}}{9} + \frac{\sigma}{4}) \cup (\frac{2\sqrt{3}}{9} + \frac{\sigma}{4}, +\infty)$ in such way that $\hat{h}_\pm$ are non-increasing functions. Finally we set $\hat{h}_0 = -(\hat{h}_+ + \hat{h}_-)$. Note that

$$\hat{h}_\pm(v) = h_\pm(v) \text{ and } \hat{h}_0(v) = h_0(v) \text{ for all } v \in \Big[-\frac{2\sqrt{3}}{9} + \frac{\sigma}{4}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{4}\Big] \tag{A.1}$$

and that $\hat{h}_-(v) < \hat{h}_0(v) < \hat{h}_+(v)$. Besides $h_\pm$ (respectively $\hat{h}_\pm$) are strictly decreasing functions on $(-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9})$ (respectively on $\mathbb{R}$). Moreover we set

$$\hat{f}(s,v) := -\big(s - \hat{h}_-(v)\big)\big(s - \hat{h}_0(v)\big)\big(s - \hat{h}_+(v)\big), \text{ for all } (s,v) \in \mathbb{R}^2.$$

Further we also introduce $\hat{h}_-(v,\delta) < \hat{h}_0(v,\delta) < \hat{h}_+(v,\delta)$ the three solutions of $\hat{f}(s,v) = \delta$. One can check that this functions are well defined for $\delta \in [0, \delta_0)$ and δ_0 small enough. We also will use the notations

$$\hat{h}_\pm(v, 0) = \hat{h}_\pm(v).$$

By definition we then have

$$\hat{h}(v,\delta) = h_\pm(v+\delta), \text{ for all } (v, v+\delta) \in \Big[-\frac{2\sqrt{3}}{9} + \frac{\sigma}{4}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{4}\Big] \tag{A.2}$$

We next recall the properties of $h_\pm$ and $\hat{h}_\pm$ and of the travelling wave solutions of the equation $u_t = u_{zz} + \hat{f}(u,v) - \delta$.

Lemma A.1 *Let $(U(z,v,\delta), C(v,\delta))$ be the solution of the problem*

$$(TW)\begin{cases} U_{zz} + C(v,\delta)U_z + \hat{f}(U,v) = \delta \\ \lim_{z\to-\infty} U(z,v,\delta) = \hat{h}_-(v,\delta), \quad \lim_{z\to+\infty} U(z,v,\delta) = \hat{h}_+(v,\delta), \\ U(0,v,\delta) = \hat{h}_0(v,\delta). \end{cases}$$

Then we have that

$$U(z,v,\delta) = \hat{h}_+(v,\delta) - \frac{\hat{h}_+(v,\delta) - \hat{h}_-(v,\delta)}{1 + \frac{\hat{h}_0(v,\delta)-\hat{h}_-(v,\delta)}{\hat{h}_+(v,\delta)-\hat{h}_0(v,\delta)} e^{\frac{\hat{h}_+(v,\delta)-\hat{h}_-(v,\delta)}{\sqrt{2}} z}}, \tag{A.3}$$

$$C(v,\delta) = \frac{1}{\sqrt{2}}(2\hat{h}_0(v,\delta) - \hat{h}_-(v,\delta) - \hat{h}_+(v,\delta)) = \frac{3}{\sqrt{2}}\hat{h}_0(v,\delta). \tag{A.4}$$

Moreover there exist K, K_1, K_2 positive constants such that, for all $z \in R$ and v in a compact subset of $\mathbb{R}$ we have that

$$K \geqslant U_z(z, v, \delta) > 0 \text{ and } |U_\nu| \leqslant K \tag{A.5}$$

$$|U_z(z, v, \delta)| + |U_{zz}(z, v, \delta)| \leqslant K_1 e^{-K_2|z|}, \tag{A.6}$$

$$|U(z, v, \delta) - \hat{h}_+(v, \delta)| \leqslant K_1 e^{-K_2 z} \text{ if } z \geqslant 0, \tag{A.7}$$

$$|U(z, v, \delta) - \hat{h}_-(v, \delta)| \leqslant K_1 e^{K_2 z} \text{ if } z \leqslant 0. \tag{A.8}$$

Further there exist K_3 and K_4 positive constants such that

$$|\hat{h}_\pm(v) - \hat{h}_\pm(w)| \leqslant K_4|v - w|, \text{ for all } (v, w) \in \mathbb{R}^2 \tag{A.9}$$

$$|\hat{h}_i(v, \delta) - \hat{h}_i(v, 0)| \leqslant K_4|\delta|, \text{ for } v \in \mathbb{R} \text{ and } |\delta| \in [0, \delta_0) \tag{A.10}$$

$$|h_i(v) - h_i(w)| \leqslant K_4|v - w|, \tag{A.11}$$

for $(v, w) \in [-\frac{2\sqrt{3}}{9} + \frac{\sigma}{8}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{8}]^2$ with $i = +, -, 0$ and

$$h_\pm(v - \delta) \geqslant h_\pm(v) + K_3\delta, \text{ for } \delta \in [0, \delta_0),\ (v - \delta, v) \in \Big[-\frac{2\sqrt{3}}{9} + \frac{\sigma}{8}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{8}\Big]^2. \tag{A.12}$$

Besides there exists a constant $c_{\hat{f}} \geqslant 0$ such that

$$\big|\hat{f}(s, v) - \hat{f}(s, w)\big| \leqslant c_{\hat{f}}|v - w|, \tag{A.13}$$

for $s \in [-\hat{C}_0, \hat{C}_0]$ and $(v, w) \in \mathbb{R}^2$.

Proof. We refer to [2, 6] for the proof of (A.3)–(A.12). Since the derivatives of $\hat{h}_0$ and $\hat{h}_\pm$ are bounded on $\mathbb{R}$ the estimate (A.13) is obvious. □

B. Appendix. The equation $K(\alpha, v) = 0$

We set

$$\alpha_- := \frac{a - \frac{2\sqrt{3}}{9} + \sigma - h_-\Big(-\frac{2\sqrt{3}}{9} + \sigma\Big)}{h_+\Big(-\frac{2\sqrt{3}}{9} + \sigma\Big) - h_-\Big(-\frac{2\sqrt{3}}{9} + \sigma\Big)} \quad \text{and} \quad \alpha_+ := \frac{a + \frac{2\sqrt{3}}{9} - \sigma - h_-\Big(\frac{2\sqrt{3}}{9} - \sigma\Big)}{h_+\Big(\frac{2\sqrt{3}}{9} - \sigma\Big) - h_-\Big(\frac{2\sqrt{3}}{9} - \sigma\Big)}, \tag{B.1}$$

then one can check that

$$\lim_{\sigma\to 0} \alpha_- = \frac{a}{\sqrt{3}} + \frac{1}{9} \quad \text{and} \quad \lim_{\sigma\to 0} \alpha_+ = \frac{a}{\sqrt{3}} + \frac{8}{9} \tag{B.2}$$

and we next study the equation $K(\alpha, v) = 0$, for $\alpha \in [\alpha_-, \alpha_+]$ and $v \in [-2\frac{\sqrt{3}}{9} + \sigma, 2\frac{\sqrt{3}}{9} - \sigma]$.

Lemma B.1 *Let σ small enough, then there exists a function $W \in C^\infty([\alpha_-, \alpha_+], [-2\frac{\sqrt{3}}{9} + \sigma, 2\frac{\sqrt{3}}{9} - \sigma)]$ such that for all $\alpha \in [\alpha_-, \alpha_+]$ we have*

$$\begin{cases} K(\alpha, v) = 0 \\ -2\frac{\sqrt{3}}{9} + \sigma < v < 2\frac{\sqrt{3}}{9} - \sigma \end{cases} \Leftrightarrow v = W(\alpha),$$

where K is defined by (6.1). Moreover

$$\left|\frac{\partial W(\alpha)}{\partial \alpha}\right| \leqslant 2\sqrt{3}, \text{ for all } \alpha \in [\alpha_-, \alpha_+]. \tag{B.3}$$

Proof. We first remark that $\alpha \mapsto K(\alpha, v)$ is a nondecreasing function on [0, 1] for all $v \in [-2\frac{\sqrt{3}}{9} + \sigma, 2\frac{\sqrt{3}}{9} - \sigma]$ while $v \to K(\alpha, v)$ is a strictly increasing function on $[-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9}]$, for all $\alpha \in [0, 1]$. Let $\sigma > 0$, $v_+ = \frac{2\sqrt{3}}{9} - \sigma$ and $v_- = -\frac{2\sqrt{3}}{9} + \sigma$, where σ is small enough so that $v_- < 0 < v_+$ and let $\alpha_\pm$ defined as previously then $v \mapsto K(\alpha, v)$ takes its values between $K(\alpha, v_+)$ and $K(\alpha, v_-)$. Moreover since $K(\alpha, v_+) < K(\alpha_+, v_+) = 0$ and similarly $K(\alpha, v_-) > K(\alpha_-, v_-) = 0$ we claim that for $\alpha \in [\alpha_-, \alpha_+]$ the equation $K(\alpha, v) = 0$ has a unique solution $v = W(\alpha) \in (v_-, v_+)$. Since $\frac{\partial K}{\partial v} \neq 0$ we now deduce from the implicit function Theorem applied to the function $(\alpha, v) \mapsto K(\alpha, v)$ that the function $\alpha \mapsto W(\alpha) \in C^\infty((\alpha_-, \alpha_+), (-2\frac{\sqrt{3}}{9} + \sigma, 2\frac{\sqrt{3}}{9} - \sigma))$ and that

$$\frac{\partial W}{\partial \alpha}(\alpha) = -\frac{\frac{\partial K}{\partial \alpha}}{\frac{\partial K}{\partial v}} = -\frac{h_+(v) - h_-(v)}{\alpha h'_+(v) + (1-\alpha) h'_-(v) - 1} > 0.$$

Thus $\alpha \mapsto W(\alpha)$ is a nondecreasing function on (α_-, α_+). Further we have $0 \leqslant h_+(v) - h_-(v) \leqslant 2\sqrt{3}$ and $|\frac{\partial K}{\partial v}| = |\alpha h'_+(v) + (1-\alpha)h'_-(v) - 1| \geqslant 1$, so that (B.3) is satisfied. Finally taking $\sigma/2$ instead of σ in this proof we obtain Lemma B.1. □

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