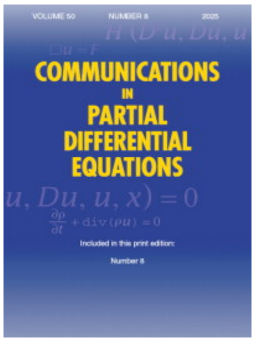

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An optimal design problem for a charge qubit

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ABSTRACT
In this article, we introduce a simple variational model describing the ground state of a superconducting charge qubit. The model gives rise to a shape optimization problem that aims at maximizing the number of qubit states at a given gating voltage. We show that for small values of the charge, optimal shapes exist and are $C^{2,\alpha}$-nearly spherical sets. In contrast, we prove that balls are not minimizers for large values of the charge and conjecture that optimal shapes do not exist, with the energy favoring disjoint collections of sets.

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Free boundary regularity; Hartree equation; nonlocal interactions; shape optimization

1. Introduction

This article studies the shape optimization problem associated with the energy functional

$$E_q(\Omega) := \inf_{u \in H_0^1(\Omega)} \left(\int_\Omega |\nabla u(x)|^2 \, dx + \frac{q}{2} \int_\Omega \int_\Omega \frac{u^2(x)\, u^2(y)}{|x-y|} \, dx \, dy \ : \ \int_\Omega u^2(x) dx = 1 \right), \quad (1)$$

for a fixed parameter $q > 0$ referred to as "charge" from now on, among all measurable sets $\Omega \subset \mathbb{R}^3$ with prescribed volume. With $q = 0$, minimizing $E_q(\Omega)$ among domains Ω with a fixed volume is the classical shape optimization problem for the first Dirichlet eigenvalue of the Laplacian, whose solution is known to be a ball (see [1] for an overview). The Dirichlet energy, together with a volume constraint on Ω, acts as a cohesive term that forces the perimeter of the set to be minimized. At the same time, the second term in the definition of the energy is the Coulombic repulsive energy for $q > 0$, favoring separation of charges and domain splitting. Thus, the considered problem falls into a general class of geometric variational problems with competing attractive and repulsive interactions that received a significant attention recently (for a broad overview, see [2]). It is the competition of these two interactions that makes the behavior of such problems highly non-trivial and interesting, and also makes their analysis challenging.

The choice of the particular energy functional in (1) is motivated by a basic model in quantum mechanics introduced by Hartree in 1927 [3]. We note that although this model was originally meant to describe the behavior of electrons in an atom, it suffers from a deficiency due to its lack of accounting for the fermionic nature of the electrons and in that context was superseded by a more appropriate Hartree-Fock model (see [4, 5] for a mathematical

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discussion of these models). Nevertheless, the so-called restricted Hartree equation

$$-\Delta u + Vu + \left(u^2 \star \frac{q}{|x|}\right)u = \varepsilon_q u \qquad \text{in } H^1(\mathbb{R}^3;\mathbb{C}),$$

where V is the external potential, ε_q is the lowest energy level and "$\star$" denotes a convolution, together with its associated energy functional naturally re-emerge in the context of superconductors, in which the variable u stands for the self-consistent ground state wave function of the Cooper pairs in the Bose-Einstein condensate at zero temperature. Thus, we interpret the energy in (1) as a simple model for the ground state energy of the Cooper pairs confined to a nanoscale superconducting island Ω that is embedded into an insulator ($V = 0$ in Ω and $V = +\infty$ in Ω^c) [6]. The obtained energy is a single-orbital Hartree functional for bosons, with the total number of condensate particles proportional to $q > 0$. By the well-known property of the single-orbital Schrödinger operator, the function u may be chosen to be real-valued and nonnegative [7].

A Cooper pair box, or a charge qubit, is an example of a quantum bit device that uses the charge states of the Cooper pairs in a superconducting nanoscale island to represent quantum information [8–11]. Ordinarily, a charge qubit is treated simply as a capacitor, with its quantum state corresponding to the discrete number of charges on the capacitor. Therefore, the shape of the island enters into the consideration solely through the value of the island's capacitance. At the same time, as the island's dimensions become smaller, as well as in the presence of a high dielectric constant matrix, an interplay between the kinetic and the potential energies of the Cooper pairs may become notable, making the dependence of the system's characteristics on the island shape less straightforward. A natural question one may then ask is whether one could take advantage of the superconducting island's shape to optimize some characteristics of the charge qubit. For example, one may ask what shape of the island at a given volume would maximize the number of stable charge states at a given gate voltage (i.e., for a given $V = V_0 < 0$ fixed in Ω), which is equivalent to minimizing the energy in (1) among all such domains. This is precisely the mathematical problem treated in this article.

As it is our interest to optimize $E_q(\Omega)$ with respect to Ω, we consider the optimal design problem

$$\inf_{\substack{\Omega\subset\mathbb{R}^3\\ |\Omega|=|B_1|}} E_q(\Omega) = \inf_{\substack{\Omega\subset\mathbb{R}^3\\ |\Omega|=|B_1|}} \inf_{u\in H^1_0(\Omega)} \left\{ E_q(u,\Omega) \ : \ \int_\Omega u^2\,dx = 1 \right\}, \tag{2}$$

where

$$E_q(u,\Omega) := \int_\Omega |\nabla u(x)|^2\,dx + \frac{q}{2}\int_\Omega\int_\Omega \frac{u^2(x)\,u^2(y)}{|x-y|}\,dx\,dy. \tag{3}$$

Notice that due to the scaling properties of the energy functional in (3) any choice of the volume of Ω may be reduced to that of $|\Omega| = |B_1| = \frac{4}{3}\pi$ in (2) by redefining q. By the same reason, all the physical constants in the problem may be absorbed in the value of the dimensionless constant q, which is thus the only non-trivial parameter of the model (see Appendix A for details).

Remark 1.1 (The shape optimization viewpoint). From a purely mathematical point of view, problem (2) is a shape optimization problem in which the functional to be optimized is a non-local, nonlinear perturbation of the first eigenvalue of the Dirichlet Laplacian. The nonlocal nature of the perturbation, even under some very strong regularity assumptions on the state variable u or the set Ω drastically prevents adapting standard shape optimization techniques. In particular, to our knowledge no second-order shape derivatives or symmetrization results are available in the literature for such functionals, implying that even local rigidity of critical points is in principle a highly nontrivial question.

1.1. Main results and detailed strategy of their proof

The main result of the article is the following:

Theorem 1.2. *For all $\varepsilon > 0$ there exists $q^* = q^*(\varepsilon) > 0$ and $\alpha \in (0,1)$ such that, for all $0 < q \le q^*$, there exists an optimal set for problem (2). Furthermore, every optimal set Ω is $C^{2,\alpha}$-nearly spherical, namely, there is a function $\varphi_\varepsilon : \partial B_1 \to \mathbb{R}$ of class $C^{2,\alpha}$ such that $\|\varphi_\varepsilon\|_{C^{2,\alpha}} \le \varepsilon$ and*

$$\partial\Omega = \Big\{(1+\varphi_\varepsilon(x))x : x \in \partial B_1\Big\}.$$

Remark 1.3. It is natural to conjecture that for sufficiently small q problem (2) admits a unique minimizer: the ball. A natural strategy to prove that conjecture would be to develop a quantified second-order shape derivative of the energy. To our knowledge, no second-order shape derivative for such a nonlocal energy is available at present, and providing one appears to be quite challenging.

The strategy of the proof of *Theorem 1.2* is quite involved, hence we offer an outline here. Its overall structure follows ideas developed in [12–18] for isoperimetric-type variational problems and developed later on in [19, 20] for spectral optimization problems (this list is not exhaustive). In our case, it can be summarized, broadly speaking, in three main steps.

(1) Prove existence of an optimal set among equibounded sets.
(2) Prove regularity of the optimal shapes: any minimizer previously found is a nearly spherical set.
(3) Remove the equiboundedness hypothesis.

Point (3) is almost independent and is dealt with in Section 5. It is in fact an improvement of spectral surgery techniques developed in [21, 19], allowing to reduce to the uniformly bounded setting. The technique nevertheless works on connected sets, hence its proof relies on some mild regularity that we have to show in the previous sections.

The core of the proof is contained in points (1) and (2). Their proof, performed in the (quite long) Section 3, is convoluted. Since its technical details may hide the ideas behind it, we describe it here.

- First we restrict ourselves to considering the problem for equibounded sets, that is, sets uniformly contained in a large enough ball B_R, $R \gg 1$. This greatly simplifies the ensuing compactness arguments, and does not lead to a loss of generality in view of point (3).
- Then we show the existence of a minimizer. In order to do that, we transform the energy E_q into a new equivalent energy $E_{q,M}$. The related ground state energy is such that

the L^2 constraint on the density function u is replaced by a Lagrange multiplier[1], see formula (12).

- Next we eliminate the volume constraint on the admissible sets: following ideas from [22, 20, 19] we consider a new energy $\Omega \mapsto E_{q,M,\eta}(\Omega)$ where the volume constraint on Ω is replaced by a Lagrange multiplier (more precisely we need to use a penalizing piecewise linear function $\eta \mapsto f_\eta(|\Omega|)$, because of the different scalings of the addends in $E_{q,M}$). We are now able to show that a minimizer for $E_{q,M,\eta}$ exists, and it has finite perimeter. Such a property plays a crucial role later on in the proof. A major complication arises now: at this stage, we are not able to show that the problems of minimizing $E_{q,M}$ under volume constraint and the unconstrained minimization of $E_{q,M,\eta}$ are equivalent. To establish this, we hence need to show some deeper regularity results on the minimizers of the latter energy. To that end:
- We consider an equivalent *free boundary* formulation of the problem, where the minimization in Ω is replaced by the minimization of an energy functional $\mathcal{E}_{q,M,\eta}(u)$ for $u \in H^1_0(B_R)$. Every optimal Ω happens to be the support of an optimal function for $\mathcal{E}_{q,M,\eta}$.
- We then adapt techniques from the free boundary regularity theory to show that any minimizer of $\mathcal{E}_{q,M,\eta}$ is Lipschitz continuous and nondegenerate, thus its positivity set is open and satisfies density estimates from below and above. Note that here a major technical, but crucial, point is rendering all the implicit constants independent of q.
- With such a regularity at hand, by exploiting a quantitative version of the Faber–Krahn inequality we can prove that for q small minimizers of $E_{M,q,\eta}$ are close to a ball in the L^1 and in the Hausdorff distance. This is just enough to show the equivalence of the (minimization of) $E_{q,M}$ and $E_{q,M,\eta}$.
- Eventually, thanks to the fact that minimizers have finite perimeter, by means of a shape variation analysis and an improvement of flatness, we show that minimizers are in fact $C^{2,\alpha}$−regular.

Remark 1.4. It is quite natural to suppose that the closeness of minimizers to a ball should directly imply that the equiboundedness constraint might be removed. This would simplify the above general strategy. Nevertheless, we are not able to do that as all the regularity estimates (in particular the crucial density estimates) do depend on R, with constants diverging as $R \to +\infty$.

The second result of the article provides some information about the solutions when q is large. We show that in this regime the ball cannot be optimal and that any optimal set must have diameter at least of order $q^{1/2} \gg 1$.

Theorem 1.5. *There exists a universal $\overline{q} > 0$ such that if $q > \overline{q}$ then B_1 is not optimal for problem (2). More precisely, any minimizer must have* $\operatorname{diam}(\Omega) \geq Cq^{1/2}$*, for a universal constant $C > 0$.*

The proof of this result follows by estimating the energy on a suitable union of small balls at very large mutual distance, as the repulsive term becomes dominant. We expect that this

[1]We will transform the energy two more times, for a total of four (equivalent) energies! This appears quite cumbersome, yet we are not able to reduce these complications.

should lead to nonexistence of minimizers in the q large regime, in view also of the result by Lu and Otto [23] about the closely related Thomas-Fermi-Dirac-Von Weizsäcker energy. The proof of such a conjecture seems quite challenging, though, due to the lack of a uniform bound with respect to q on the L^∞ norm of u.

Plan of the article. In Section 2, we show some preliminary results on the nature of the energy $E_q(\cdot,\Omega)$ and in particular we study its Euler-Lagrange equation. In doing so, we investigate the nonlinear and nonlocal eigenvalue problem associated to the Euler-Lagrange equation of the energy. Then we recall some notions about quasi-open sets and the quantitative Faber-Krahn inequality. Section 3 is devoted to the proof of Theorem 1.2 in the equibounded setting (namely Theorem 3.1). More precisely, we introduce a new functional without the measure constraint, then prove existence of minimizers and their mild regularity properties (using a free boundary formulation), and finally we show the equivalence with problem (2) in an equibounded setting. In Section 4, we first show what is the optimality condition at the free boundary and then prove that minimizers are $C^{2,\alpha}$-nearly spherical, employing free boundary regularity techniques. Section 5 is devoted to a surgery argument which allows to conclude the proof of Theorem 1.2. Finally, in Section 6 we prove Theorem 1.5 concerning the regime when q is large.

2. Preliminaries

We present here some basic properties of the functionals that will be used in the proofs. Throughout the article, we adopt the following notations: for $\Omega \subset \mathbb{R}^3$ a bounded open set and $u \in H^1_0(\Omega)$ denoting with $\star$ the usual convolution, we let

$$v_u(x) = u^2(x) \star \frac{1}{|x|} = \int_\Omega \frac{u^2(y)}{|x-y|}\, dy,$$

with $v_u \in W^{2,3}_{\mathrm{loc}}(\mathbb{R}^3)$ by [24, Theorem 9.9] and Sobolev embedding. We begin by establishing a uniform bound on v_u.

Lemma 2.1. *Let $\overline{x} \in \mathbb{R}^3$, let $\Omega \subset \mathbb{R}^3$ be a bounded open set, and let $\varphi \in H^1_0(\Omega)$. Then*

$$\int_\Omega \frac{\varphi^2(x)}{|x-\overline{x}|}\, dx \le 2\|\varphi\|_{L^2(\Omega)}\, \|\nabla\varphi\|_{L^2(\Omega)}.$$

Proof. Up to extending with the value $\varphi(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$, we can suppose $\varphi \in H^1_0(\mathbb{R}^3)$. By Hölder inequality, we obtain

$$\int \frac{\varphi^2(x)}{|x-\overline{x}|}\, dx \le \left\| \frac{\varphi(\cdot)}{|\cdot - \overline{x}|} \right\|_{L^2(\mathbb{R}^3)} \|\varphi\|_{L^2(\mathbb{R}^3)}$$

Then we can apply the classical Hardy-Sobolev inequality, see for instance [25, *Corollary 2* of Section 2.1.7]

$$\left\| \frac{\varphi}{|\cdot - \overline{x}|} \right\|_{L^2(\mathbb{R}^3)} \le 2\|\nabla\varphi\|_{L^2(\mathbb{R}^3)},$$

to obtain the result. □

We recall now that for $\phi, \psi : \Omega \to \mathbb{R}$ the Coulomb energy denoted by

$$D(\phi,\psi) = \int_\Omega \int_\Omega \frac{\phi(x)\psi(y)}{|x-y|}\,dx\,dy$$

is a well-defined positive definite bilinear form, provided $D(|\phi|,|\psi|) < \infty$ [7, Theorem 9.8]. We note that in the definition of $D(\cdot,\cdot)$ the dependence on Ω is implicit, as it is the domain of the functions ϕ, ψ.

2.1. Minimization of $E_q(u,\Omega)$ in u.

Existence of u achieving the infimum in the definition of $E_q(\Omega)$ is an application of the Direct Method in the Calculus of Variations.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set and let $q > 0$. Then the minimization problem*

$$E_q(\Omega) = \inf\left\{E_q(v,\Omega) : v \in H_0^1(\Omega),\ \int_\Omega v^2\,dx = 1\right\}. \tag{4}$$

admits a solution with constant sign (nonnegative, without loss of generality).

Proof. Let $(u_n)_n$ be a minimizing sequence for the energy. As all the addends in the definition of $E_q(u,\Omega)$ are positive, we infer that $(u_n)_n$ is bounded in $H_0^1(\Omega)$.

Then up to passing to a subsequence, $(u_n)_n$ converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $u \in H_0^1(\Omega)$. In particular the L^2−convergence implies that $\|u\|_{L^2(\Omega)} = 1$. By lower semicontinuity with respect to the weak convergence, we have that

$$\int |\nabla u|^2\,dx \le \liminf_{n\to+\infty} \int |\nabla u_n|^2\,dx$$

and, by Fatou *Lemma*

$$D(u^2,u^2) = \iint_{\Omega\times\Omega} \frac{u^2(x)u^2(y)}{|x-y|}\,dxdy \le \liminf_{n\to+\infty} D(u_n^2,u_n^2).$$

Hence, the energy is lower semicontinuous

$$E_q(u,\Omega) \le \liminf_{n\to+\infty} E_q(u_n,\Omega),$$

and u is a minimizer. The fact that u can be chosen of constant sign follows since

$$E_q(|u|,\Omega) = E_q(u,\Omega),$$

and since $|u|$ is an admissible competitor. □

Remark 2.3. It is not difficult to check that the scale invariant functional

$$\widetilde{E}_q(v,\Omega) := \frac{\int_\Omega |\nabla v|^2\,dx}{\|v\|_{L^2}^2} + \frac{q}{2}\frac{\int_\Omega\int_\Omega \frac{v^2(x)v^2(y)}{|x-y|}\,dx\,dy}{\|v\|_{L^2}^4}, \qquad v \in H_0^1(\Omega),$$

leads to an equivalent, yet unconstrained, minimization problem

$$\min\left\{E_q(v,\Omega) : v \in H_0^1(\Omega),\ \int_\Omega v^2\,dx = 1\right\} = \min\left\{\widetilde{E}_q(v,\Omega) : v \in H_0^1(\Omega), v \not\equiv 0\right\}.$$

In particular, we can always choose an optimal function $u \in H_0^1(\Omega)$ for the unconstrained problem which satisfies, a posteriori, the constraint $\|u\|_{L^2} = 1$.

Let us introduce the semilinear operator on $H_0^1(\Omega)$,

$$L_{\Omega,q}(u) = -\Delta u + q\Big(u^2 \star \frac{1}{|\cdot|}\Big)u.$$

One easily shows that $L_{\Omega,q}$ is a positive operator, that is, $(L_{\Omega,q}(u), u)_{L^2} \geq 0$. We say that a number $\lambda > 0$ is a (nonlinear) eigenvalue for $L_{\Omega,q}$ if there exists a non-null function v such that

$$L_{\Omega,q}(v) = \lambda v, \qquad \int_\Omega v^2\,dx = 1, \tag{5}$$

in distributional sense. Such a function is then called an eigenfunction corresponding to λ.

By means of a first variation, it is possible to check that (as it is observed in [4]) the optimal function u attaining $E_q(\Omega)$ is a (nonnegative) normalized eigenfunction for the operator $L_{\Omega,q}$ associated to the eigenvalue

$$\lambda_q = \int_\Omega |\nabla u|^2\,dx + qD(u^2, u^2).$$

We note that for $q \in (0, 1]$ and $|\Omega| = |B_1|$ the quantity λ_q is uniformly bounded from below and above:

$$\lambda_0(B_1) \leq \lambda_0(\Omega) \leq \lambda_q \leq 2E_q(\Omega) \leq 2E_1(B_1), \tag{6}$$

where $\lambda_0(A)$ denotes the first eigenvalue of the Dirichlet Laplacian of an open set $A \subset \mathbb{R}^3$, and we used the Faber-Krahn inequality in the first inequality of (6). Here we are limiting our analysis to the sets satisfying $E_q(\Omega) \leq E_q(B_1)$. This is not restrictive since otherwise the ball would be optimal for E_q. In other words, the Euler-Lagrange equation for $E_q(\Omega)$ is

$$L_{\Omega,q}(u) = \lambda_q u, \qquad u \in H_0^1(\Omega). \tag{7}$$

By classical elliptic theory, we deduce now a useful uniform bound on the L^∞ norm of solutions to (7). Precisely, we have the following result.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set, let $q \in (0, 1]$ and let u be a solution of*

$$L_{\Omega,q}(u) = \lambda_q u \qquad \text{in } H_0^1(\Omega), \qquad \int_\Omega u^2\,dx = 1. \tag{8}$$

Then $u \in C^\infty(\Omega)$ and, hence, is a classical solution of (7). Furthermore, if $\lambda_q \leq M$ then

$$\|u\|_{L^\infty(\Omega)} \leq C,$$

for a constant $C = C(M, |\Omega|) > 0$ depending only on M and $|\Omega|$.

Proof. Thanks to Lemma 2.1, and by the Young inequality, we deduce that

$$\|v_u\|_{L^\infty(\Omega)} \leq 1 + \|\nabla u\|^2_{L^2(\Omega)} \leq 1 + \lambda_q.$$

Hence the function

$$c(x) = \lambda_q - qv_u(x)$$

is equibounded in Ω by a constant depending only on λ_q. As u solves

$$\Delta u + c(x)u = 0, \qquad \text{in } H_0^1(\Omega),$$

by [24, Theorem 8.8] we can assert that $u \in H^2_{\mathrm{loc}}(\Omega)$ and by a simple bootstrap argument, $u \in C^\infty(\Omega)$ so that it is a classical solution of (7). Eventually, by [24, Theorem 8.15] we conclude that

$$\|u\|_{L^\infty(\Omega)} \le C(M, |\Omega|)\|u\|_{L^2(\Omega)} = C(M, |\Omega|),$$

where $C(M, |\Omega|) > 0$ is a constant depending only on M and $|\Omega|$. □

Corollary 2.5. *There exists $q_0 \in (0, 1]$ such that for every $q \in (0, q_0]$ and every bounded open set $\Omega \subset \mathbb{R}^3$ with $|\Omega| = |B_1|$ every nonnegative minimizer u of $E_q(u, \Omega)$ solves*

$$-\Delta u = c(x)u \quad \text{in } H^1_0(\Omega), \qquad \int_\Omega u^2\,dx = 1,$$

for some $c \in C^\infty(\Omega; \mathbb{R}^+)$. In particular, u is superharmonic.

Proof. We set $c(x) = \lambda_q - qv_u(x)$. The positivity of c for all q sufficiently small universal follows by the uniform L^∞ bound proven in Lemma 2.4 and by the uniform bounds on λ_q from (6). Since $u \ge 0$, then it is superharmonic for q below a certain threshold q_0. □

From now on, $q_0 > 0$ always refers to the universal constant in Corollary 2.5.

Remark 2.6. We note that in general one cannot expect that a minimizer u of $E_q(u, \Omega)$ has the entire set Ω as its support. This can can be already seen in the case $q = 0$ with Ω consisting of two disjoint sets whose first Dirichlet eigenvalues are distinct. It is not difficult to show that this situation persists to the case of $q > 0$ sufficiently small depending on Ω.

On the other hand, we can prove that the energy E_q is the same on Ω and on the positivity set of u and give some further characterizations.

Lemma 2.7. *Let $q > 0$, let Ω be a bounded open set and let $u \in H^1_0(\Omega)$ be a nonnegative minimizer of $E_q(u, \Omega)$. Then $E_q(\Omega) = E_q(\{u > 0\})$. Furthermore, if Ω_i is a connected component of Ω then either $u > 0$ or $u \equiv 0$ on Ω_i.*

Proof. It was already observed that we can choose $u \ge 0$, moreover, as u is continuous, $\{u > 0\}$ is an open set. By the optimality of u, $E_q(\Omega) \ge E_q(\{u > 0\})$. On the other hand, $\{u > 0\} \subset \Omega$ and by monotonicity of the functional $E_q(\Omega)$ in Ω, we infer $E_q(\Omega) = E_q(\{u > 0\})$. The last part of the statement follows from Lemma 2.4 and [7, Theorem 9.10]. □

Remark 2.8. We highlight again that if Ω is not connected, then we only know up to this point that the set $\{u > 0\}$ coincides with the union of some of the connected components of Ω and has the same energy as Ω.

We show now that $E_q(u, \Omega)$ has a unique (up to sign) minimizer u if Ω is a *connected* set (or, more generally, if $u > 0$ on all connected components of Ω). To do so, we follow an approach proposed in [26] (later revisited in [27]), based on a *hidden convexity property* of the functional.

Proposition 2.9. *Let $q > 0$ and let Ω be a connected bounded open set. Let $u, v \in H_0^1(\Omega)$ be such that $\|u\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega)} = 1$. Let $\sigma_t = \sigma_t(u, v)$ be defined as*

$$\sigma_t = ((1-t)u^2 + tv^2)^{1/2}, \qquad \text{for } t \in (0,1).$$

Then the map $g : t \mapsto E_q(\sigma_t, \Omega)$ is strictly convex. In particular, the function u attaining the minimum of $E_q(u, \Omega)$ is unique.

Proof. Thanks to Lemma 2.7 and the connectedness of Ω, we deduce that the optimal function u for $E_q(\Omega)$ is strictly positive in Ω. Then the claim follows directly from [26, Lemma 4], which assures that the map $t \mapsto E_q(\sigma, \Omega)$ is strictly convex. Eventually, if u, v are two distinct (strictly positive) minimizers, since $\|\sigma_t(u, v)\|_{L^2(\Omega)} = 1$ by strict convexity it follows immediately that $E_q(\sigma_t, \Omega) < E_q(u, \Omega) = E_q(v, \Omega)$, a contradiction. □

As a side result of *Proposition* 2.9, we have also the radiality of u, when Ω is a ball.

Corollary 2.10. *Let $\Omega \subset \mathbb{R}^3$ be a ball. Then the unique solution to*

$$\min \left\{ E_q(u, \Omega) : u \in H_0^1(\Omega), \int_\Omega u^2\, dx = 1 \right\},$$

is radial.

Proof. The uniqueness of the solution follows from Proposition 2.9. Concerning the radiality, it follows from the rotational invariance of the energy $E_q(u, \Omega)$. □

2.2. Fraenkel asymmetry and quantitative Faber-Krahn inequality

Here we recall the sharp quantitative version of the Faber–Krahn inequality. We first remind the notion of *Fraenkel asymmetry*: for a set $\Omega \subset \mathbb{R}^3$ with finite measure we define

$$\mathcal{A}(\Omega) = \inf_{x \in \mathbb{R}^3} \frac{|\Omega \Delta (B + x)|}{|\Omega|}, \tag{9}$$

where B denotes the ball of measure $|\Omega|$ centered at the origin. We also recall that we denote by $\lambda_0(A)$ the first eigenvalue of the Dirichlet Laplacian of a set $A \subset \mathbb{R}^3$.

Theorem 2.11. *[20] There exists a universal positive constant $\widehat{\sigma} > 0$ such that for all open sets $\Omega \subset \mathbb{R}^3$ with finite measure we have*

$$|\Omega|^{2/3}\lambda_0(\Omega) - |B_1|^{2/3}\lambda_0(B_1) \geq \widehat{\sigma}\mathcal{A}(\Omega)^2,$$

where $\mathcal{A}$ denotes the Fraenkel asymmetry.

2.3. Some facts about quasi-open sets

Finally, we recall some definitions and facts about quasi-open sets which will be needed in the following.

Definition 2.12. A *quasi-open* set is a measurable set $\Omega \subset \mathbb{R}^3$ such that for all $\varepsilon > 0$ there exists K_ε compact such that its Newtonian capacity $\mathrm{cap}(K_\varepsilon) < \varepsilon$ and $\Omega \setminus K_\varepsilon$ is open. Similarly, a function $u : \Omega \to \mathbb{R}$ is *quasi-continuous* if for all $\varepsilon > 0$ there exists a compact set K_ε such that $\mathrm{cap}(K_\varepsilon) < \varepsilon$ and the restriction of u to $\Omega \setminus K_\varepsilon$ is continuous. Eventually, we say that a property holds *quasi-everywhere* on a set if it holds up to a set of null-capacity.

It is well-known that every $u \in H^1(B_R)$ admits a quasi-continuous representative $\widetilde{u}$. Moreover, if $\widetilde{u}$ and $\widehat{u}$ are two quasi-continuous representatives of u, then they are equal quasi-everywhere. Therefore, in this article, for every $u \in H^1(B_R)$ we identify it with its quasi-continuous representative. A quasi-open set is then simply a superlevel set of (the quasi-continuous representative of) a function $u \in H^1(B_R)$. For more details on quasi-open sets and quasi-continuous functions, and the definition of capacity we refer to [1, Chapter 2], [28, Chapter 3] or [29, Chapter 2 and 4].

Let us also stress that it is standard to define the Sobolev space H^1_0 on a quasi-open set $\Omega \subset \mathbb{R}^3$ as

$$H^1_0(\Omega) = \{u \in H^1(\mathbb{R}^3) : u = 0 \text{ quasi- everywhere in } \mathbb{R}^3 \setminus \Omega\}.$$

If Ω is an open set, this definition coincides with the usual one, see [28, Section 3.3.5] for more details.

Remark 2.13. If $\Omega \subset \mathbb{R}^3$ is a quasi-open set and $u \in H^1_0(\Omega)$ is a nonnegative solution to (8), then $u \in L^\infty(\Omega)$ and its L^∞ norm is bounded by a constant as in Lemma 2.4. This can be checked with an approximation argument directly from the definition of a quasi-open set.

3. An existence result for an auxiliary problem

In this section, we begin the proof of the following result, which can be seen as an equi-bounded version of Theorem 1.2.

Theorem 3.1. *There exists $R_0 > 1$ such that, for all $R \geq R_0$ there exists $\overline{q} = \overline{q}(R) > 0$ such that for all $q \leq \overline{q}$ there exists a minimizer for the problem*

$$\min\Big\{E_q(\Omega) : \Omega \subset B_R,\ |\Omega| = |B_1|\Big\}. \tag{10}$$

Moreover, for all $\varepsilon > 0$ there exists $q_\varepsilon = q_\varepsilon(R) > 0$ such that if $q \leq q_\varepsilon$ then any minimizer is an $(\varepsilon, C^{2,\alpha})$−nearly spherical set, with $\alpha = \alpha(\varepsilon) \in (0, 1)$.

The proof of this result is carried out through several steps, divided in the next (sub)sections, as outlined in the introduction.

3.1. Removing the L^2 norm constraint in E_q

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded quasi-open set of measure $|B_1|$ and let $q \leq q_0$. There exists $\overline{M} > 0$ (uniform for all $q \leq q_0$) such that for all $M \geq \overline{M}$, the minimizers of the problem (4) are the same as those of*

$$\min\Big\{E_{q,M}(v, \Omega) : v \in H^1_0(\Omega)\Big\}, \tag{11}$$

where

$$E_{q,M}(v, \Omega) = E_q(v, \Omega) + M\Big|\int v^2\, dx - 1\Big|.$$

Proof. Problem (11) admits a minimizer, as it can be seen as in the proof of Lemma 2.2. We set

$$0 < \alpha = \min \left\{ E_q(v,\Omega) : v \in H_0^1(\Omega), \int v^2 \, dx = 1 \right\} = E_q(u,\Omega).$$

Since $\int u^2 = 1$, clearly for all $M > 0$, $E_{q,M}(u,\Omega) = E_q(u,\Omega)$, hence

$$\alpha \geq \min \left\{ E_{q,M}(v,\Omega) : v \in H_0^1(\Omega) \right\}.$$

Let us assume by contradiction that there is a sequence $M_k \to +\infty$ such that

$$\min \left\{ E_{q,M_k}(v,\Omega) : v \in H_0^1(\Omega) \right\} = E_{q,M_k}(\widehat{u}_k,\Omega) < \alpha.$$

We call $\sigma_k = \|\widehat{u}_k\|_{L^2} - 1$ and define $u_k = \frac{\widehat{u}_k}{1+\sigma_k}$, so that $\|u_k\|_{L^2} = 1$. This definition makes sense for all large enough k, since $\sigma_k \to 0$ as $k \to +\infty$. We can then compute

$$\begin{aligned} \alpha > E_{q,M_k}(\widehat{u}_k,\Omega) &= (1+\sigma_k)^2 \int |\nabla u_k|^2 \, dx + \frac{q}{2}(1+\sigma_k)^4 D(u_k^2,u_k^2) + M_k \left| (1+\sigma_k)^2 - 1 \right| \\ &= E_q(u_k,\Omega) + (2\sigma_k + o(\sigma_k)) \int |\nabla u_k|^2 \, dx + \frac{q}{2}(4\sigma_k + o(\sigma_k)) D(u_k^2,u_k^2) \\ &\quad + 2M_k(|\sigma_k| + o(\sigma_k)). \end{aligned}$$

Noting that $\int |\nabla u_k|^2 \, dx$ and $D(u_k^2,u_k^2)$ are uniformly bounded by 2α and $q \leq q_0$, for $M_k \geq 10\alpha$ and k large enough, we conclude that

$$\alpha > E_{q,M_k}(\widehat{u}_k,\Omega) \geq E_q(u_k,\Omega) \geq \alpha,$$

a contradiction. We note that the choice of M_k for reaching a contradiction is independent of q and Ω. □

In accordance with the notation of the previous theorem, we set

$$E_{q,M}(\Omega) := \min \left\{ E_q(v,\Omega) + M \left| \int v^2 \, dx - 1 \right| : v \in H_0^1(\Omega) \right\} \tag{12}$$

and note that, as a consequence of Proposition 3.2, if u_Ω is a minimizer of $E_{q,M}(\Omega)$ for $M \geq \overline{M}$, then $\int u_\Omega^2 \, dx = 1$, thus, a posteriori, $E_q(\Omega) = E_{q,M}(\Omega)$.

Remark 3.3. From now on, we fix once and for all a constant $M > \overline{M}$, and we stress that this constant does not depend on $q \leq q_0$ and will not be changed later in the article.

3.2. Another auxiliary problem: removing the volume constraint

In order to get rid of the measure constraint, we follow an approach first proposed by Aguilera, Alt and Caffarelli [22]. Let $\eta \in (0,1)$ and consider the piecewise linear function

$$f_\eta \colon \mathbb{R}^+ \to \mathbb{R}, \qquad f_\eta(s) = \begin{cases} \eta(s - |B_1|), & \text{if } s \leq |B_1|, \\ \frac{1}{\eta}(s - |B_1|), & \text{if } s \geq |B_1|. \end{cases}$$

It is easy to check that, for all $0 \le s_2 \le s_1$, there holds

$$\eta(s_1 - s_2) \le f_\eta(s_1) - f_\eta(s_2) \le \frac{1}{\eta}(s_1 - s_2). \tag{13}$$

We introduce then the functional (recalling that M is fixed, see *Remark* 3.3 and Proposition 3.2),

$$E_{q,M,\eta}(\Omega) := E_{q,M}(\Omega) + f_\eta(|\Omega|), \tag{14}$$

and, for $R > 1$, the minimization problem

$$\min\left\{E_{q,M,\eta}(\Omega) : \Omega \subset B_R, \Omega \text{ open}\right\}. \tag{15}$$

To prove the existence of a minimizer for problem (15), we need first to work in the setting of quasi-open sets (see Section 2.3) and then recover the regularity.

Thus, we first focus on the problem:

$$\min\left\{E_{q,M,\eta}(\Omega) : \Omega \subset B_R, \Omega \text{ quasi-open}\right\}. \tag{16}$$

We aim to prove that problem (16) is equivalent to the constrained problem (10), at least for q and η small enough. To do that we first have to prove existence and some mild regularity of minimizers of $E_{q,M,\eta}$. We begin by showing a lower bound for $E_{q,M,\eta}$ on equibounded sets.

Lemma 3.4. *Let $R > 1$, $q \in (0, q_0]$ and $\eta \in (0,1)$. Then, for all quasi-open $\Omega \subset B_R$, we have*

$$E_{q,M,\eta}(\Omega) \ge \lambda_0(B_1)R^{-2} - |B_1|.$$

Proof. Let $v \in H^1_0(\Omega)$, with $\int_\Omega v^2\,dx = 1$ be a function attaining the infimum in the definition of $E_{q,M}(\Omega) = E_q(\Omega)$, so that

$$E_{q,M,\eta}(\Omega) = \int |\nabla v|^2\,dx + \frac{q}{2}D(v^2, v^2) + f_\eta(|\Omega|).$$

By the monotonicity of Dirichlet eigenvalues, the inclusion $\Omega \subset B_R$, the positivity of $D(\cdot,\cdot)$, the scaling properties of λ_0, and the definition of f_η, we obtain

$$\int |\nabla v|^2\,dx + \frac{q}{2}D(v^2, v^2) + f_\eta(|\Omega|) \ge \lambda_0(\Omega) - \eta\max\{|B_1| - |\Omega|, 0\} \ge \lambda_0(B_1)R^{-2} - |B_1|,$$

so the claim is proved. □

The following existence result is mostly an adaptation to our situation of [20, Lemma 4.6] and [19, Lemma 3.2], which are in turn inspired by [30, Theorem 2.2 and Lemma 2.3].

Lemma 3.5. *Let $\eta \in (0,1)$, $q \in (0, q_0]$ and let $R > 1$. There exists a minimizer for problem (16). Moreover, all minimizers have perimeter uniformly bounded by a constant depending on R, η.*

Proof. Let $(\Omega_n)_n \subset B_R$ be a sequence of smooth sets such that

$$E_{q,M,\eta}(\Omega_n) \le \inf\left\{E_{q,M,\eta}(\Omega) : \Omega \subset B_R,\ \text{quasi-open}\right\} + \frac{1}{n}.$$

Let u_n be an optimal function with unit L^2 norm for the minimization of $E_q(\Omega_n)$ or equivalently $E_{q,M}(\Omega_n)$, so that by Lemma 2.7, either $\Omega_n = \{u_n > 0\}$, or $\{u_n > 0\}$ is a union of some connected components[2] of Ω_n, with $E_{q,M,\eta}(\{u_n > 0\}) \le E_{q,M,\eta}(\Omega_n)$, so that $(\{u_n > 0\})_n$ still forms a minimizing sequence and we can replace Ω_n with $\{u_n > 0\}$. Let $t_n = 1/\sqrt{n}$. We define

$$\widetilde{\Omega}_n := \{u_n > t_n\}.$$

We have

$$E_{q,M,\eta}(\Omega_n) \le E_{q,M,\eta}(\widetilde{\Omega}_n) + \frac{1}{n},$$

and since we can take $\widetilde{u}_n := (u_n - t_n)_+$ as a competitor in the minimization problem defining $E_M(\widetilde{\Omega}_n)$, we obtain

$$\begin{aligned}
&\int_{\{u_n>0\}} |\nabla u_n|^2\,dx + \frac{q}{2}D(u_n^2, u_n^2) + M\left|\int u_n^2\,dx - 1\right| + f_\eta(|\{u_n > 0\}|) \\
&\le \int_{\{u_n>t_n\}} |\nabla u_n|^2\,dx + \frac{q}{2}D(\widetilde{u}_n^2, \widetilde{u}_n^2) + M\left|\int (\widetilde{u}_n)^2\,dx - 1\right| + f_\eta(|\{u_n > t_n\}|) + \frac{1}{n}.
\end{aligned} \tag{17}$$

Using the uniform bound on the L^∞ norm of u_n, see Lemma 2.4, we observe that

$$\begin{aligned}
\int_{\{u_n>t_n\}} u_n^2 - (u_n - t_n)^2\,dx &= \int_{\{u_n>t_n\}} 2t_n u_n - t_n^2\,dx \\
&\le 2\|u_n\|_{L^\infty} t_n |\{u_n > t_n\}| \le C(R) t_n |\{u_n > t_n\}|,
\end{aligned}$$

$$\text{and} \quad \int_{\{0\le u_n\le t_n\}} u_n^2\,dx \le t_n^2 |\{0 < u_n \le t_n\}|,$$

and obtain the estimate (possibly increasing the value of $C(R)$)

$$\int_{\{u_n>0\}} u_n^2\,dx - \int_{\{u_n>t_n\}} (u_n - t_n)^2\,dx \le C(R) t_n |\{u_n > 0\}|. \tag{18}$$

Noting that $D(\widetilde{u}_n^2, \widetilde{u}_n^2) - D(u_n^2, u_n^2) \le 0$, recalling the property (13) of f_η and (18), we can rewrite (17) as

$$\int_{\{0<u_n<t_n\}} |\nabla u_n|^2\,dx + \eta|\{0 < u_n < t_n\}| \le C(R) M t_n |\{u_n > 0\}| + \frac{1}{n} \le C(R, M) t_n + \frac{1}{n}. \tag{19}$$

On the other hand, since $\eta < 1$, using coarea formula, the arithmetic-geometric-mean inequality and (19), we obtain

$$\begin{aligned}
2\eta \int_0^{t_n} P(\{u_n > s\})\,ds &= 2\eta \int_{\{0<u_n<t_n\}} |\nabla u_n|\,dx \\
&\le \eta \int_{\{0<u_n<t_n\}} |\nabla u_n|^2\,dx + \eta|\{0 < u_n < t_n\}| \le C(R, M) t_n + \frac{1}{n}.
\end{aligned}$$

As $t_n = 1/\sqrt{n}$ we can find a level $0 < s_n < 1/\sqrt{n}$ such that the sets $W_n := \{u_n > s_n\}$ satisfy

$$P(W_n) \le \frac{2}{t_n} \int_0^{t_n} P(\{u_n > s\})\,ds \le \frac{C(R, M)}{\eta} + \frac{1}{\eta t_n n} \le C(R, M, \eta) + \frac{1}{\eta\sqrt{n}}.$$

[2]Notice that since Ω_n are smooth then u_n are regular functions and there are no measure-theoretical issues in defining connected components.

It is easy to check that $(W_n)_n$ is still a minimizing sequence for problem (16). In fact, with arguments similar to the ones used above, we obtain:

$$
\begin{aligned}
E_{q,M,\eta}(W_n) = &\int_{\{u_n>s_n\}} |\nabla u_n|^2\,dx + \frac{q}{2}D\Big((u_n-s_n)_+^2,(u_n-s_n)_+^2)\Big)\\
&+ M\Big|\int_{\{u_n>s_n\}}(u_n-s_n)^2\,dx - 1\Big| + f_\eta(|\{u_n>s_n\}|)\\
&\le E_{q,M,\eta}(\Omega_n) + C(R,M)s_n + f_\eta(|\{u_n>s_n\}|) - f_\eta(|\{u_n>0\}|)\\
&\le E_{q,M,\eta}(\Omega_n) + \frac{C(R,M)}{\sqrt{n}} - \eta|\{0<u_n<s_n\}| \le E_{q,M,\eta}(\Omega_n) + \frac{C(R,M)}{\sqrt{n}},
\end{aligned}
\tag{20}
$$

where we used that $D\Big((u_n-s_n)_+^2,(u_n-s_n)_+^2\Big) - D(u_n^2,u_n^2) \le 0$ and property (18) with s_n in place of t_n. Moreover, since the sets of the sequence $(W_n)_n$ have equibounded perimeter, there exists a Borel set W_∞ such that (up to passing to subsequences)

$$
W_n \to W_\infty, \text{ in } L^1(B_R), \qquad P(W_\infty) \le C(R,M,\eta). \tag{21}
$$

On the other hand, an optimal function (normalized in L^2) w_n attaining $E_{q,M,\eta}(W_n)$, is equibounded in $H^1(B_R)$. In fact, being $(W_n)_n$ a minimizing sequence for $E_{q,M,\eta}$, we have that

$$
\int |\nabla w_n|^2\,dx + \int w_n^2\,dx \le 1 + E_{q,M,\eta}(W_n) \le 1 + C.
$$

Hence, up to passing to subsequences, there is $w \in H_0^1(B_R)$ such that

$$
w_n \to w \quad \text{strongly in } L^2(B_R), \text{ weakly in } H_0^1(B_R) \text{ and pointwise a.e.} \tag{22}
$$

Let $W := \{w > 0\}$, and recall that we are identifying w with its quasi-continuous representative. Then, thanks to (21) and (22), we deduce

$$
\chi_W(x) \le \liminf_{n\to+\infty} \chi_{W_n}(x) = \chi_{W_\infty}(x), \qquad \text{for a.e. } x \in B_R,
$$

hence $|W \setminus W_\infty| = 0$, that is $W \subset W_\infty$ up to a negligible set. We now observe that $\Omega \mapsto f_\eta(|\Omega|)$ is continuous with respect to the L^1 convergence of sets, the Dirichlet energy is lower semicontinuous with respect to the weak H^1 convergence and the functional $D(\cdot,\cdot)$ is lower semicontinuous with respect to the strong L^2 convergence by Fatou lemma. We can therefore pass to the limit in (20) and obtain

$$
\begin{aligned}
&\int_W |\nabla w|^2\,dx + \frac{q}{2}D(w^2,w^2) + f_\eta(|W_\infty|)\\
&\quad\le \liminf_{n\to+\infty}\int_{W_n} |\nabla w_n|^2\,dx + \frac{q}{2}D(w_n^2,w_n^2) + f_\eta(|W_\infty|)\\
&\quad\le \liminf_{n\to+\infty} E_{q,M,\eta}(W_n) = \inf_{\Omega\subset B_R} E_{q,M,\eta}(\Omega) \le E_q(W) + f_\eta(|W|).
\end{aligned}
\tag{23}
$$

In conclusion, using also (23), we have

$$
\eta|W_\infty \setminus W| = \eta(|W_\infty| - |W|) \le f_\eta(|W_\infty|) - f_\eta(|W|) \le 0,
$$

thus $|W_\infty \setminus W| = 0$, which entails $W = W_\infty$ a.e. and this is the desired minimizer for problem (16). □

3.3. Free boundary formulation

In the previous section, we introduced the functional $E_{q,M,\eta}$, see (14), we proved existence and mild regularity properties of minimizers (namely: they are sets of finite perimeter). In this section we improve the regularity for such sets. This will be needed in the sequel but allows us also to show the equivalence between unconstrained minimizers of $E_{q,M,\eta}$ and volume constrained minimizers of E_q and $E_{q,M}$. The crucial remark is that one may consider, in place of the shape functional $E_{q,M,\eta}$, a functional defined on the larger space $H^1_0(B_R)$ and take a *free boundary* approach. Let us define, for $u \in H^1_0(B_R)$,

$$\mathcal{E}_{q,M,\eta}(u) = \int_{\{u>0\}} |\nabla u|^2\, dx + \frac{q}{2}\int_{\{u>0\}}\int_{\{u>0\}} \frac{u^2(x)u^2(y)}{|x-y|}\, dxdy \\ + M\left|\int_{\{u>0\}} u^2\, dx - 1\right| + f_\eta(|\{u>0\}|),$$

and we note that one could have equivalently integrated over B_R in all the integrals above.

Lemma 3.6. *Let $\eta \in (0,1)$, $q \in (0, q_0]$ and Ω be a minimizer for problem (16). Then every minimal function of $E_q(\Omega)$ is a minimizer of $\mathcal{E}_{q,M,\eta}$. Viceversa, if w minimizes $\mathcal{E}_{q,M,\eta}$, then $\Omega = \{w > 0\}$ is a minimizer of $E_{q,M,\eta}$. Furthermore, it is possible to select a minimizer Ω for $E_{q,M,\eta}$ which coincides with the support of an optimal function for $E_q(\Omega)$.*

Proof. Concerning the first claim, let Ω be an optimal set for $E_{q,M,\eta}$ and u an optimal function for $E_q(\Omega)$. We immediately note that $\{u > 0\} = \Omega$ up to sets of zero measure, otherwise $\{u > 0\} \subset \Omega$, thus $f_\eta(|\{u>0\}|) < f_\eta(|\Omega|)$ so that $E_{q,M,\eta}(\Omega) > E_{q,M,\eta}(\{u>0\})$, a contradiction with the optimality of Ω. As a consequence, $\mathcal{E}_{q,M,\eta}(u) = E_{q,M,\eta}(\{u>0\}) = E_{q,M,\eta}(\Omega)$. Therefore, for all $v \in H^1_0(B_R)$, we have

$$\mathcal{E}_{q,M,\eta}(u) = E_{q,M,\eta}(\Omega) \le E_{q,M,\eta}(\{v>0\}) = \mathcal{E}_{q,M,\eta}(v),$$

namely u is a minimizer for $\mathcal{E}_{q,M,\eta}$.

Let us focus on the second claim: let u be an optimal function for $\mathcal{E}_{q,M,\eta}$, and we call $\Omega = \{u > 0\}$. For all $\widetilde{\Omega} \subset B_R$, calling $\widetilde{u}$ any optimal function attaining $E_q(\widetilde{\Omega})$, we have

$$E_{q,M,\eta}(\Omega) = \mathcal{E}_{q,M,\eta}(u) \le \mathcal{E}_{q,M,\eta}(\widetilde{u}) \le E_{q,M,\eta}(\widetilde{\Omega}),$$

as requested. Notice that the last inequality is not *a priori* an equality, as in principle $\{\widetilde{u} > 0\} \subset \widetilde{\Omega}$.

Concerning the last part of the statement, it is enough to notice that, given an optimal function u for $E_q(\Omega)$, $\{u>0\}$ is always a minimizer for $E_{q,M,\eta}$. □

In the rest of this section, we focus on the new (equivalent) formulation of problem (16)

$$\min\left\{\mathcal{E}_{q,M,\eta}(u) \,:\, u \in H^1_0(B_R)\right\}. \tag{24}$$

As a consequence of Lemma 3.6 and Remark 2.8, working on problem (24) means selecting an optimal set $\{u > 0\}$ for the original problem (2) or for (16), namely the union of the connected components of Ω where u is nonzero.

In the next results, we work with a minimizer of the form $\Omega = \{u > 0\}$. This is not restrictive, as next lemma shows.

Lemma 3.7. *For all $R \geq 1$, $\eta \in (0,1)$ and $q \leq q_0$, if Ω is an optimal set for problem (16) and u is an associated (nonnegative) optimal function attaining $E_q(\Omega)$, then $\{u > 0\} = \Omega$ (up to a negligible set). Moreover, Ω is connected.*

Proof. Let us argue for the sake of contradiction and assume that $\{u > 0\}$ is strictly contained in Ω. We already know from Remark 2.8 that the above assumptions entail that $u \equiv 0$ on a connected component ω, with positive measure. Therefore, $f_\eta(|\{u > 0\}|) < f_\eta(|\Omega|)$ and we have contradicted the minimality of Ω for $E_{q,M,\eta}$. Concerning the last part of the statement, if $\Omega = \{u > 0\}$ is the disjoint union of two components Ω_1 and Ω_2, by increasing the distance between the two components we are strictly decreasing the Coulomb energy term, while the other terms of $E_{q,M,\eta}$ are unchanged. Thus we constradict again the optimality of Ω. □

Remark 3.8. We stress that if we could prove that an optimal function for (24) is a quasi-minimizer for the functional

$$J(u) = \int_{\{u>0\}} |\nabla u|^2\,dx + |\{u > 0\}|,$$

i.e. letting $J_{x,r}(u) = \int_{\{u>0\}\cap B_r(x)} |\nabla u|^2 + |\{u > 0\} \cap B_r(x)|$, we have, for some $\beta > 0$,

$$J_{x,r}(u) \leq (1 + Kr^\beta)J_{x,r}(v), \qquad \text{for all admissible } v \text{ and for all } x, r,$$

this would strongly simplify the regularity proof, see for example [31]. Unfortunately, this does not seem to be the case in our setting, due to the presence of the nonlocal double integral term $D(u^2, u^2)$. Therefore, we use a careful modification of the standard free boundary regularity techniques developed starting from [32].

Lemma 3.9. *Let $R > 1$, $\eta \in (0,1)$, $q \in (0, q_0]$, let Ω be an optimal set for problem (16), and let $u \in H^1_0(\Omega)$ be any (nonnegative) function attaining $E_q(\Omega) = E_{q,M}(\Omega)$. Then for every $\kappa \in (0,1)$ there are positive constants K_0, ρ_0 depending only on κ, η, R such that the following assertion holds: if $\rho \leq \rho_0$ and $x_0 \in \overline{B_R}$, then*

$$\fint_{\partial B_\rho(x_0)\cap B_R} u\,d\mathcal{H}^2 \leq K_0\rho \implies u \equiv 0 \text{ in } B_{\kappa\rho}(x_0) \cap B_R. \tag{25}$$

Proof. By Lemma 3.7, we know that $\Omega = \{u > 0\}$, that u is optimal for problem (24) and has unitary L^2 norm. We extend u to zero outside B_R, so that by Lemma 2.4, u solves distributionally $-\Delta u \leq \gamma_1$ in $\mathbb{R}^3$ where (since $q \leq q_0$)

$$\gamma_1 := 2 \sup_x \left(\lambda_q - qv_u(x)\right) \|u\|_{L^\infty(\Omega)} > 0.$$

The positivity of γ_1 follows by the bound (uniform in q) on $\|u\|_{L^\infty(\Omega)}$ (see Lemma 2.4) and, consequently, on $\|v_u\|_{L^\infty(\Omega)}$, while $\lambda_q \geq \lambda_0(\Omega) \geq \lambda_0(B_1) > 0$. Then the function

$$x \mapsto u(x) + \gamma_1 \frac{|x - x_0|^2 - \rho^2}{6}$$

is subharmonic in B_ρ (recalling that we are in three dimensional setting). Thus, for every $\kappa \in (0,1)$, there exists $c = c(\kappa)$ such that

$$\delta_\rho := \sup_{B_{\sqrt{\kappa}\rho}(x_0)} u \leq c\left(\fint_{\partial B_\rho(x_0)\cap B_R} u\,d\mathcal{H}^2 + \gamma_1\rho^2\right) \leq c(K_0\rho + \rho^2). \tag{26}$$

Let us show now that there exists $\rho > 0$ small enough so that there exists a positive solution w of

$$\begin{cases} -\Delta w = \frac{M}{2}(u+w), & \text{in } B_{\sqrt{\kappa\rho}}(x_0) \setminus B_{\kappa\rho}(x_0), \\ w = \delta_\rho, & \text{on } \partial B_{\sqrt{\kappa\rho}}(x_0), \\ w = 0, & \text{on } B_{\kappa\rho}(x_0), \end{cases} \tag{27}$$

where M is fixed large enough so that the statement of Proposition 3.2 holds. To show existence of a solution, let $\rho > 0$ be such that

$$\alpha(\rho) := \lambda_0(B_{\sqrt{\kappa\rho}} \setminus B_{\kappa\rho})^{-1} \left(\frac{M}{2} \|u\|_{L^2(B_{\sqrt{\kappa\rho}}(x_0)\setminus B_{\kappa\rho}(x_0))} + \frac{M}{2} \right) \le \frac{1}{4}.$$

This can be easily obtained as $\lambda_0(B_{\sqrt{\kappa\rho}} \setminus B_{\kappa\rho}) \to +\infty$ as $\rho \to 0$ and since $u \in L^\infty(\Omega)$. Then any minimizing sequence for the energy

$$\varphi \mapsto \frac{1}{2} \int_{B_{\sqrt{\kappa\rho}}(x_0)\setminus B_{\kappa\rho}(x_0)} |\nabla\varphi|^2 \, dx - \frac{M}{2} \int_{B_{\sqrt{\kappa\rho}}(x_0)\setminus B_{\kappa\rho}(x_0)} \left(\frac{\varphi}{2} + u \right) \varphi \, dx \tag{28}$$

with boundary conditions as in (27) can be chosen, after standard computations, made of nonnegative functions (as passing to the modulus decreases the energy) and such that

$$\int_{B_{\sqrt{\kappa\rho}}(x_0)\setminus B_{\kappa\rho}(x_0)} |\varphi_n|^2 \, dx \le \alpha(\rho) \int_{B_{\sqrt{\kappa\rho}}(x_0)\setminus B_{\kappa\rho}(x_0)} |\nabla\varphi_n|^2 \, dx \le \frac{1}{4} \int_{B_{\sqrt{\kappa\rho}}(x_0)\setminus B_{\kappa\rho}(x_0)} |\nabla\varphi_n|^2 \, dx.$$

Hence the sequence $(\|\varphi_n\|_{H^1})$ is uniformly bounded and a positive minimizer for the energy (28) exists and solves its Euler-Lagrange equation, namely (27). By standard elliptic regularity, we obtain that the L^∞ norm of any positive solution w is bounded by a constant $\gamma_2 > 0$ depending only on κ and M. By definition, $w \ge u$ on $\partial B_{\sqrt{\kappa\rho}}(x_0)$; therefore, the function

$$v = \begin{cases} u, & \text{in } \mathbb{R}^3 \setminus B_{\sqrt{\kappa\rho}}(x_0), \\ \min\{u, w\}, & \text{in } B_{\sqrt{\kappa\rho}}(x_0), \end{cases}$$

satisfies

$$v \le u, \qquad \{v > 0\} \subset \{u > 0\}, \qquad \{v > 0\} \setminus B_{\sqrt{\kappa\rho}}(x_0) = \{u > 0\} \setminus B_{\sqrt{\kappa\rho}}(x_0),$$

thus $D(v^2, v^2) \le D(u^2, u^2)$ (and we can neglect these contributions in the following computations). Since $v \in H_0^1(B_R)$, inequality (24) gives

$$\begin{aligned} &\int_{B_{\sqrt{\kappa\rho}}(x_0)} |\nabla u|^2 \, dx + f_\eta(|\{u > 0\}|) \\ &\le \int_{B_{\sqrt{\kappa\rho}}(x_0)} |\nabla v|^2 \, dx + M \left| \int_{B_{\sqrt{\kappa\rho}}(x_0)} (v^2 - u^2) \, dx \right| + f_\eta(|\{v > 0\}|). \end{aligned}$$

As $v = 0$ in $B_{\kappa\rho}$, using also (13), we get

$$\begin{aligned} \eta|\{u > 0\} \cap B_{\kappa\rho}(x_0)| &\le \eta|(\{u > 0\} \setminus \{v > 0\}) \cap B_{\sqrt{\kappa\rho}}(x_0)| \\ &\le f_\eta(|\{u > 0\}|) - f_\eta(|\{v > 0\}|). \end{aligned}$$

On the other hand, we can rewrite the term involving the L^2 norm of the functions as

$$\Big|\int_{B_{\sqrt{\kappa}\rho}(x_0)} (v^2 - u^2)\,dx\Big| = \int_{B_{\sqrt{k}\rho}(x_0)} (u^2 - v^2)\,dx = \int_{B_{\kappa\rho}(x_0)} u^2\,dx + \int_{(B_{\sqrt{\kappa}\rho}(x_0)\setminus B_{\kappa\rho}(x_0))\cap\{u>w\}} (u^2 - w^2)\,dx.$$

Thanks to the two inequalities above and the definition of v, we can infer

$$\begin{aligned}
\int_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx + \eta|\{u>0\}\cap B_{\kappa\rho}| &\le \int_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx + f_\eta(|\{u>0\}|)\\
&\quad - f_\eta(|\{v>0\}|)\\
&\le \int_{B_{\sqrt{\kappa}\rho}(x_0)\setminus B_{\kappa\rho}(x_0)} (|\nabla v|^2 - |\nabla u|^2)\,dx + M\int_{B_{\kappa\rho}(x_0)} u^2\,dx\\
&\quad + M\int_{(B_{\sqrt{\kappa}\rho(x_0)}\setminus B_{\kappa\rho}(x_0))\cap\{u>w\}} (u^2 - w^2)\,dx\\
&\le 2\int_{(B_{\sqrt{\kappa}\rho(x_0)}\setminus B_{\kappa\rho}(x_0))\cap\{u>w\}} (|\nabla w|^2 - \nabla u\cdot\nabla w)\,dx + M\int_{B_{\kappa\rho}(x_0)} u^2\\
&\quad + M\int_{(B_{\sqrt{\kappa}\rho}(x_0)\setminus B_{\kappa\rho}(x_0))\cap\{u>w\}} (u^2 - w^2)\,dx.
\end{aligned} \tag{29}$$

On the other hand testing (27) with $(u-w)_+$ and integrating over $B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}$, we obtain

$$\int_{(B_{\sqrt{\kappa}\rho}(x_0)\setminus B_{\kappa\rho}(x_0))\cap\{u>w\}} (|\nabla w|^2 - \nabla u\cdot\nabla w)\,dx + \frac{M}{2}\int_{(B_{\sqrt{\kappa}\rho}(x_0)\setminus B_{\kappa\rho}(x_0))\cap\{u>w\}} (u^2 - w^2)\,dx = \int_{\partial B_{\kappa\rho}(x_0)} \frac{\partial w}{\partial \nu} u\,d\mathcal{H}^2, \tag{30}$$

where ν denotes the outer unit normal exiting from $B_{\kappa\rho}$ and thanks to the fact that $w=0$ on $\partial B_{\kappa\rho}(x_0)$ and $w \ge u$ on $\partial B_{\sqrt{\kappa}\rho}(x_0)$. Recalling that $\|u\|_{L^\infty} \le \gamma_1$ and $\|w\|_{L^\infty} \le \gamma_2$, we now fix $\gamma_3 = \frac{M}{2}(\gamma_1 + \gamma_2)$ and consider the solution to the problem

$$\begin{cases} -\Delta\widetilde{w} = \gamma_3, & \text{in } B_{\sqrt{\kappa}\rho}(x_0)\setminus B_{\kappa\rho}(x_0),\\ \widetilde{w} = \delta_\rho, & \text{on } \partial B_{\sqrt{\kappa}\rho}(x_0),\\ \widetilde{w} = 0, & \text{on } B_{\kappa\rho}(x_0), \end{cases}$$

since the torsion function on an annulus is explicit (see [20]), with a direct computation one obtains

$$\left|\frac{\partial\widetilde{w}}{\partial\nu}\right| \le \beta_1\frac{\delta_\rho + \rho^2}{\rho}, \qquad \text{on } \partial B_{\kappa\rho}(x_0),$$

for some $\beta_1 = \beta_1(\kappa, M)$. By comparison, since $\widetilde{w} - w$ is superharmonic (by (27)) it follows

$$\left|\frac{\partial w}{\partial\nu}\right| \le \left|\frac{\partial\widetilde{w}}{\partial\nu}\right| \le \beta_1\frac{\delta_\rho + \rho^2}{\rho}, \qquad \text{on } \partial B_{\kappa\rho}(x_0).$$

We can now combine (29) and (30) to obtain

$$\int_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx + \eta|\{u>0\}\cap B_{\kappa\rho}(x_0)| \le \beta_1(\kappa)\frac{\delta_\rho+\rho^2}{\rho}\int_{\partial B_{\kappa\rho}(x_0)} u\,d\mathcal{H}^2 + M\int_{B_{\kappa\rho}(x_0)} u^2\,dx.$$

By the uniform bound on the L^∞ norm of u, we have the estimate

$$M\int_{B_{\kappa\rho}(x_0)} u^2\,dx \le \beta_2\delta_\rho^2|\{u>0\}\cap B_{\kappa\rho}(x_0)|,$$

for some $\beta_2(\kappa, M)$.

Then, using the definition of δ_ρ, the trace inequality in $W^{1,1}$ and the arithmetic geometric mean inequality we obtain

$$\begin{aligned}\int_{\partial B_{\kappa\rho}(x_0)} u\,d\mathcal{H}^2 &\le C(\kappa)\left(\frac{1}{\rho}\int_{B_{\kappa\rho}(x_0)} u\,dx + \int_{B_{\kappa\rho}(x_0)} |\nabla u|\,dx\right)\\ &\le \beta_3\left(\left(\frac{\delta_\rho}{\rho}+\frac{1}{2}\right)|\{u>0\}\cap B_{\kappa\rho}(x_0)| + \frac{1}{2}\int_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx\right),\end{aligned}$$

for some $\beta_3 = \beta_3(\kappa) > 0$. By collecting the above estimates, recalling again (26) we have, for all $\rho \le \rho_0$

$$\begin{aligned}&\eta\int_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx + \eta|\{u>0\}\cap B_{\kappa\rho}(x_0)|\\ &\le \beta_1\frac{\delta_\rho+\rho^2}{\rho}\int_{\partial B_{\kappa\rho}(x_0)} u\,d\mathcal{H}^2 + \delta_\rho(1+\delta_\rho\beta_2)|\{u>0\}\cap B_{\kappa\rho}(x_0)|\\ &\le \beta_1(c(K_0+\rho)+\rho)\int_{\partial B_{\kappa\rho}(x_0)} u\,d\mathcal{H}^2\\ &\quad + c(K_0\rho+\rho^2)(1+c(K_0\rho+\rho^2)\beta_3)|\{u>0\}\cap B_{\kappa\rho}(x_0)|\\ &\le \beta_1\beta_3(c(K_0+\rho)+\rho)\left[\left(\frac{\delta_\rho}{\rho}+\frac{1}{2}\right)|\{u>0\}\cap B_{\kappa\rho}(x_0)| + \frac{1}{2}\int_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx\right]\\ &\quad + c(K_0\rho+\rho^2)(1+c(K_0\rho+\rho^2)\beta_2)|\{u>0\}\cap B_{\kappa\rho}(x_0)|\\ &\le \beta_1\beta_3(c(K_0+\rho)+\rho)\left(2c(K_0+\rho)+\frac{1}{2}\right)\left[\int_{B_{\kappa\rho}(x_0)} |\nabla u|^2 + |\{u>0\}\cap B_{\kappa\rho}(x_0)|\right].\end{aligned}$$

Eventually, by choosing $K_0, \rho_0 < \overline{\rho}$ small enough so that

$$\beta_1\beta_3(c(K_0+\rho_0)+\rho_0)\left(2c(K_0+\rho_0)+\frac{1}{2}\right) \le \eta/4,$$

we conclude that $u \equiv 0$ in $B_{\kappa\rho}$, for all $\rho \le \rho_0$. □

Remark 3.10. The statement of Lemma 3.9 and in particular (25) can be also equivalently stated as

$$\|u\|_{L^\infty(B_\rho(x_0))} \le K_0\rho \quad \Longrightarrow \quad u \equiv 0 \text{ in } \ B_{\kappa\rho}(x_0) \cap B_R,$$

see for example [19, Remark 4.3].

In other words, Lemma 3.9 implies that if $x_0 \in \overline{\Omega}$, then there is a constant $C = C(R, \eta) > 0$, which can be taken independent of x_0, such that

$$\sup_{B_\rho(x_0)\cap B_R} u \ge C\rho, \qquad \text{and} \qquad \fint_{\partial B_\rho(x_0)\cap B_R} u\, d\mathcal{H}^2 \ge C\rho.$$

Lemma 3.11. *Let R, η, q, Ω and u be as in Lemma 3.9. The function u can be extended to a Lipschitz continuous function defined in the whole B_R, with Lipschitz constant $L = L(R, \eta)$. In particular, $\Omega = \{u > 0\} \subset B_R$ is an open set.*

Proof. We follow the approach of [31, Section 3.2], first proposed in [33].

Step 1. We prove an estimate on the nonnegative Radon measure $|\Delta u|$, namely

$$|\Delta u|(B_r(x_0)) \le Cr^2, \qquad \text{for all } x_0 \in B_R \text{ and } 0 < r < 1 \text{ such that } B_{2r}(x_0) \subset B_R$$

for a universal constant $C > 0$. Let $\psi \in C_c^\infty(B_{2r}(x_0))$ for some $B_{2r}(x_0) \subset B_R$, with $\|\psi\|_{L^\infty} \le c$, and we test the optimality of u against $u + \psi$, obtaining:

$$\begin{aligned}
&\int_{\{u>0\}} |\nabla u|^2\, dx + \frac{q}{2} D(u^2, u^2) + f_\eta(|\{u > 0\}|) \\
&\quad \le \int_{\{u+\psi>0\}} |\nabla(u+\psi)|^2\, dx + \frac{q}{2} D\Big((u+\psi)^2, (u+\psi)^2\Big) + f_\eta(|\{u + \psi > 0\}|),
\end{aligned}$$

which implies

$$\begin{aligned}
-2\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\psi\, dx \le \int_{B_{2r}(x_0)} &|\nabla\psi|^2\, dx + C_\eta |\{u = 0\} \cap B_{2r}(x_0)| \\
&+ \frac{q}{2}\int_{B_{2r}(x_0)}\int_{B_R(0)} P(x,y)\, dxdy
\end{aligned}$$

where

$$P(x,y) = \frac{4u(x)\psi(x)u^2(y) + 4u(x)\psi(x)\psi^2(y) + 2\psi^2(x)u^2(y) + 4u(x)\psi(x)u(y)\psi(y) + \psi^2(x)\psi^2(y)}{|x-y|}.$$

Recalling that $\|\psi\|_{L^\infty} \le c$ and using Lemma 2.4, we can control the nonlocal term as

$$\int_{B_{2r}(x_0)}\int_{B_R(0)} P(x,y)\, dxdy \le C_1 \int_{B_{2r}(x_0)}\int_{B_R(0)} \frac{1}{|x-y|}\, dxdy \le C_2 R^2 |B_{2r}(x_0)| \le C_3 r^3.$$

Thus we obtain

$$-2\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\psi\, dx \le \int_{B_{2r}(x_0)} |\nabla\psi|^2\, dx + C_\eta |\{u = 0\} \cap B_{2r}(x_0)| + Cqr^3. \tag{31}$$

We now set, for all $\varphi \in C_c^\infty(B_{2r}(x_0))$, $\psi = \pm r^{3/2}\|\nabla\varphi\|_{L^2}^{-1}\varphi$ and from (31) we deduce, for some $\widetilde{C} > 0$

$$\left|\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\varphi\, dx\right| \le \widetilde{C} r^{3/2} \|\nabla\varphi\|_{L^2(B_{2r}(x_0))}$$

It is then enough to choose $\varphi \in C_c^\infty(B_{2r}(x_0))$ with $\varphi \geq 0$ and $\varphi = 1$ in $B_r(x_0)$ and with $\|\nabla\varphi\|_{L^\infty(B_{2r})} \leq \frac{2}{r}$ (notice that this is compatible with the requirement $\|\psi\|_{L^\infty} \leq c$ indepdendently of r) to obtain, for some constant $C > 0$:

$$|\Delta u|(B_r(x_0)) \leq |\Delta u|(\varphi) = \left|\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\varphi \, dx\right| \leq Cr^2. \tag{32}$$

Step 2. We prove that the Laplacian estimate (32) of Step 1 entails (recall that $\mathcal{H}^2(\partial B_r) = 4\pi r^2$)

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_0)} u \, d\mathcal{H}^2 \leq u(x_0) + Cr \qquad \text{for all } x_0 \in B_R, \tag{33}$$

for some constant $C > 0$. This follows from [33, Lemma 3.6], which assures that, for all $x_0 \in B_R$, it holds

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_0)} u \, d\mathcal{H}^2 - u(x_0) = \int_0^r \frac{1}{4\pi s^2}\Delta u(B_s(x_0)) \, ds. \tag{34}$$

It is then enough to put together (34) and (32) to obtain (33).

Now, let us take $x_0 \in \partial\{u > 0\} \cap B_R$ and a sequence of $x_n \to x_0$ such that $u(x_n) = 0$ for all n and with $x_n \in B_{r_1}(x_0) \subset B_R$. For those points (33) reads as

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_n)} u \, d\mathcal{H}^2 \leq u(x_n) + Cr = Cr, \qquad \text{for all } r < r_1, \tag{35}$$

and the constant C does not depend on n. Since $u \in H^1(B_R)$, the map $x \mapsto \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u \, d\mathcal{H}^2$ is continuous, see [31, Remark 3.6]. We can then pass to the limit as $n \to \infty$ in (35) to deduce

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_0)} u \, d\mathcal{H}^2 \leq Cr, \qquad \text{for all } r < r_1.$$

Finally, passing to the limit as $r \to 0$, we obtain that $u(x_0) = 0$ (recalling that we are considering the quasi continuous representative of the Sobolev function u), thus $\Omega \cap \partial\Omega = \{u > 0\} \cap \partial\{u > 0\} = \emptyset$, hence $\Omega = \{u > 0\}$ is an open set.

Step 3. We conclude, using Step 2, that u is Lipschitz continuous in B_R, as in [31, Lemma 3.5], see also [33, Theorem 3.1 and 4.1]. □

An immediate and fundamental consequence of Lemmas 3.9 and 3.11 is the following density estimate on Ω.

Lemma 3.12. *Let $R > 1$ and $\eta \in (0, 1)$. There exists $q_1 \in (0, q_0]$ such that for all $q \in (0, q_1]$, calling Ω an optimal set for problem (15) and u a positive normalized function attaining $E_{q,M,\eta}(\Omega)$, there exist positive constants $\theta = \theta(R, \eta)$ and $\rho_0 = \rho_0(R, \eta) < 1$ such that for every $x_0 \in \partial\Omega$ and every $\rho \leq \rho_0$, we have*

$$\theta \leq \frac{|\Omega \cap B_\rho(x_0)|}{|B_\rho|} \leq (1 - \theta).$$

Proof. Let us start from the lower bound. We can assume that $x_0 = 0 \in \partial\Omega = \partial\{u > 0\}$. Thus, the nondegeneracy condition of Remark 3.10 implies that

$$\|u\|_{L^\infty(B_{\rho/2})} \geq C\frac{\rho}{2}.$$

Thus, there is a point $y \in B_{\rho/2}$ such that $u(y) \geq C\frac{\rho}{2}$. On the other hand, the Lipschitz continuity of u, with constant $L = L(R, \eta)$, implies that $u > 0$ on a ball with radius $\frac{\rho}{2}\min\{1, \frac{C}{L}\}$, and so we conclude.

The upper bound can be obtained as in [32], see also [31, Section 5.1], with a few modifications. Precisely, let $x_0 = 0 \in \partial\Omega$ and consider the function h which is equal to u outside B_ρ, and inside B_ρ it is the solution of

$$\begin{cases} -\Delta h = \gamma_1 & \text{in } B_\rho, \\ h = u & \text{on } \partial B_\rho, \end{cases}$$

where $\gamma_1 = \gamma_1(\rho) := 2\sup_x \left(\lambda_q - qv_u(x)\right)\|u\|_{L^\infty(B_\rho)} > 0$. As a consequence, we obtain that $-\Delta(h-u) = \gamma_1 - (\lambda_q u - qv_u)u \geq 0$ in B_ρ. In particular, we have that $u \leq h$ and $\{u > 0\} \subset \{h > 0\}$ in B_ρ. Moreover, since h is the torsion function multiplied by γ_1 with boundary datum u, recalling that $u(0) = 0$ and that u is Lipschitz continuous with constant $L = L(R, \eta)$, see Lemma 3.11, we deduce by a simple comparison argument that

$$\|h\|_{L^\infty(B_\rho)} \leq C_h\rho, \tag{36}$$

with a constant C_h depending only on R, η. Thus, testing the optimality of u with h, using also (13) and an integration by parts, we have

$$\begin{aligned} \frac{1}{\eta}|B_\rho \cap \{u = 0\}| &+ \frac{q}{2}\int_{B_\rho}\int_{B_\rho}\frac{h^2(x)h^2(y)}{|x-y|}\,dxdy + M\left|\int_{B_\rho}(h^2-u^2)\,dx\right| \\ &\geq \int_{B_\rho}|\nabla u|^2\,dx - \int_{B_\rho}|\nabla h|^2\,dx \\ &= \int_{B_\rho}|\nabla(u-h)|^2\,dx + 2\int_{B_\rho}\Big(\nabla h\cdot\nabla(u-h)\Big)\,dx \\ &= \int_{B_\rho}|\nabla(u-h)|^2\,dx + 2\int_{B_\rho}(-\Delta h)(u-h)\Big)\,dx + 2\int_{B_\rho}(u-h)\frac{\partial h}{\partial\nu}\,d\mathcal{H}^2 \\ &= \int_{B_\rho}|\nabla(u-h)|^2\,dx - 2\gamma_1\int_{B_\rho}(h-u)\,dx. \end{aligned}$$

Let us first treat the terms not involving the gradient: thanks to (36), we can bound the term (using also the scaling of the Riesz energy and the fact that $\rho \leq 1$)

$$\frac{q}{2}\int_{B_\rho}\int_{B_\rho}\frac{h^2(x)h^2(y)}{|x-y|}\,dxdy \leq C_h^4\rho^4\rho^5 D(\chi_{B_1}, \chi_{B_1})\frac{q}{2} \leq C_a|B_\rho|q, \tag{37}$$

for a constant $C_a > 0$, depending only on R, η. On the other hand, using again the bound L^∞ on h and u, we have

$$\int_{B_\rho}(h^2-u^2)\,dx + \int_{B_\rho}(h-u)\,dx \leq (C_h+1)\int_{B_\rho}h\,dx \leq (C_h+1)C_h\rho|B_\rho| = C_h'\rho|B_\rho|, \tag{38}$$

for a constant $C_h' > 0$ depending only on R, η.

Let us now focus on the gradient term. By the Poincaré and Cauchy-Schwarz inequalities, we have

$$\int_{B_\rho} |\nabla(u-h)|^2\,dx \geq \frac{C_d}{|B_\rho|}\left(\frac{1}{\rho}\int_{B_\rho}(h-u)\,dx\right)^2,$$

so in order to prove the upper bound in the claim, we first need to show that $\frac{1}{\rho|B_\rho|}\int_{B_\rho}(h-u)\,dx$ is bounded from below by a positive constant. Notice that, by the non-degeneracy of u (see Remark 3.10), we have

$$C\rho \leq \sup_{B_{\rho/2}} u \leq \sup_{B_{\rho/2}} h\,.$$

On the other hand, since $h(x)+\frac{\gamma_1}{6}|x|^2$ is harmonic in B_ρ, the Harnack inequality in B_ρ implies

$$C\rho \leq \sup_{B_{\rho/2}} h \leq C_d\big(h(x)+\gamma_1\rho^2\big) \quad \text{for every} \quad x\in B_{\frac{\rho}{2}}\,.$$

Thus, by taking ρ_0 such that $2C_d\rho_0\gamma_1 \leq C$, we obtain that $h \geq C_dC\rho = \overline{C}\rho$ in $B_{\frac{\rho}{2}}$. On the other hand, if $L = L(R,\eta)$ is the Lipschitz constant of u (by Lemma 3.11), then for any $\varepsilon\in(0,1)$, $u \leq L\varepsilon\rho$ in $B_{\varepsilon\rho}$. Then

$$\int_{B_\rho}(h-u)\,dx \geq \int_{B_{\varepsilon\rho}}(h-u)\,dx \geq (\overline{C}\rho - L\varepsilon\rho)|B_{\varepsilon\rho}|,$$

which, after choosing $\varepsilon \leq \frac{1}{2}$ small enough, shows that

$$\frac{1}{\rho}\int_{B_\rho}(h-u)\,dx \geq C_0|B_\rho|,$$

for some constant $C_0 > 0$ depending only on R,η and thus

$$\int_{B_\rho}|\nabla(u-h)|^2\,dx \geq C_b|B_\rho|,$$

for some $C_b > 0$, depending only on R,η.

At this point, using also (38) and (37), we have

$$\begin{aligned} C_b|B_\rho| &\leq \frac{1}{\eta}|B_\rho\cap\{u=0\}| + C_aq|B_\rho| + (M+2\gamma_1)\rho|B_\rho| \\ &\leq \frac{1}{\eta}|B_\rho\cap\{u=0\}| + C_aq|B_\rho| + C_g\rho|B_\rho|, \end{aligned}$$

for a universal constant $C_g > 0$, recalling that M is fixed (see Remark 3.3) and that γ_1 is uniformly bounded by a constant depending only on $\|u\|_{L^\infty(B_R)}$, which in turn is uniform in q (see Lemma 2.4). It is then enough to take

$$q_1 \leq \min\left\{q_0, \frac{C_b}{4C_a}\right\}, \qquad \rho_0 \leq \frac{C_b}{4C_g},$$

and we obtain that

$$\frac{C_b}{4}|B_\rho| \leq \frac{1}{\eta}|B_\rho\cap\{u=0\}|,$$

which entails the density estimate from above, so the claim is proved. ☐

Remark 3.13. Notice that the constants determining the Lipschitz regularity and the density estimates of the previous result do not depend on q for q small enough.

From now on, q_1 always refers to the constant defined in Lemma 3.12.

3.4. Equivalence between the minimizations of E_q and $E_{q,M,\eta}$

In this section, we show that unconstrained minima of $E_{q,M,\eta}$ and volume constrained minima of E_q (or equivalently $E_{q,M}$) are actually the same. We begin by showing that for q small, the minimizers of $E_{q,M,\eta}$ in B_R are close to a ball in L^∞. To do that, we first start with an estimate that assures the L^1−proximity of an optimal set for problem (16) to a ball with radius not too large.

Lemma 3.14. *Let $R > 2$, $q \in (0, q_1]$ and $\eta \in (0, 1)$. Let $\Omega = \Omega_{q,M,\eta}$ be an optimal set for (16) such that $\Omega = \{u_\Omega > 0\}$, where $u_\Omega \in H^1_0(\Omega)$ is the (positive) function attaining $E_q(\Omega) = E_{q,M}(\Omega)$, and $B = B_{q,M,\eta}$ a ball of measure $|\Omega|$ attaining the Fraenkel asymmetry for Ω, namely such that*

$$\mathcal{A}(\Omega) = \frac{|\Omega \Delta B|}{|\Omega|}.$$

Then, setting $u_B \in H^1_0(B)$ the function attaining $E_q(B)$, normalized so that $\|u_B\|_{L^2(B)} = \|u_\Omega\|_{L^2(\Omega)} = 1$, we have

$$|\{u_B > 0\}\Delta\{u_\Omega > 0\}|^2 \le Cq|\Omega|^{\frac{7}{3}}.$$

for some universal constant $C > 0$.

Proof. Let w_B be the normalized first eigenfunction of the Dirichlet Laplacian in B, and note that it is an admissible competitor for $E_q(B)$. We note that $\{w_B > 0\} = \{u_B > 0\} = B$, see also Lemma 2.7. Thanks to the quantitative Faber–Krahn inequality (Theorem 2.11), we have

$$\begin{aligned} |\Omega|^{2/3}\int_\Omega |\nabla u_\Omega|^2\,dx &\ge |\Omega|^{2/3}\lambda_0(\Omega) \\ &\ge |\Omega|^{2/3}\lambda_0(B) + \widehat{\sigma}\frac{|\{u_B > 0\}\Delta\{u_\Omega > 0\}|^2}{|\Omega|^2} \\ &= |\Omega|^{2/3}\int_B |\nabla w_B|^2\,dx + \widehat{\sigma}\frac{|\{u_B > 0\}\Delta\{u_\Omega > 0\}|^2}{|\Omega|^2}. \end{aligned} \tag{39}$$

From the optimality of Ω, we deduce,

$$\begin{aligned} \int_\Omega |\nabla u_\Omega|^2\,dx + \frac{q}{2}\int_\Omega\int_\Omega \frac{u_\Omega^2(x)u_\Omega^2(y)}{|x-y|}\,dx\,dy + f_\eta(|\Omega|) &\le E_q(B) + f_\eta(|B|) \\ &\le \int_B |\nabla w_B|^2\,dx + \frac{q}{2}\int_B\int_B \frac{w_B^2(x)w_B^2(y)}{|x-y|}\,dx\,dy + f_\eta(|B|), \end{aligned}$$

and using also (39) we obtain

$$\begin{aligned} \widehat{\sigma}\frac{|\{u_B > 0\}\Delta\{u_\Omega > 0\}|^2}{|\Omega|^2} &\le |\Omega|^{2/3}\int_\Omega |\nabla u_\Omega|^2\,dx - |\Omega|^{2/3}\int_B |\nabla w_B|^2\,dx \\ &\le |\Omega|^{2/3}\frac{q}{2}\Big(D(w_B^2, w_B^2) - D(u_\Omega^2, u_\Omega^2)\Big) \le |\Omega|^{2/3}\frac{q}{2}D(w_B^2, w_B^2). \end{aligned}$$

Now, using the L^∞ bound on the first eigenfunction w_B (see [34, Example 2.1.8]), and also the scaling of the Riesz functional,

$$D(w_B^2, w_B^2) \le C|\Omega|^{-2} \int_B \int_B \frac{dx\,dy}{|x-y|} \le C|\Omega|^{-\frac{1}{3}},$$

for some universal constant $C > 0$.

The previous two estimates lead to

$$|\{u_B > 0\}\Delta\{u_\Omega > 0\}|^2 \le \frac{C}{\tilde{\sigma}}|\Omega|^{\frac{7}{3}}\frac{q}{2},$$

so the claim is proved. □

A simple but important consequence of the previous result is the following lemma, stating that the measure of the ball $B = B_{q,M,\eta}$ (to which any optimal set $\Omega_{q,M,\eta}$ is L^1-close) is not too large, as we show in the next result.

Lemma 3.15. *Let $R > 2$. There exists $\eta_1 \in (0,1)$ such that for all $q \in (0, q_1]$ and $\eta \le \eta_1$, we have that any optimal set $\Omega_{q,M,\eta}$ for problem (16), such that $\Omega_{q,M,\eta} = \{u > 0\}$, where $u \in H_0^1(\Omega)$ is any (positive) function attaining $E_q(\Omega) = E_{q,M}(\Omega)$, satisfies*

$$|\Omega_{q,M,\eta}| \le |B_2|, \qquad |\Omega_{q,M,\eta}\Delta B_{q,M,\eta}| \le c_1 q, \tag{40}$$

for some universal constant $c_1 > 0$.

Proof. Let us suppose for the sake of contradiction that $|\Omega_{q,M,\eta}| > |B_2|$. We are then going to reach a contradiction as long as

$$1/\eta \ge E_{q_0}(B_1).$$

Since the functional

$$q \mapsto E_{q,M,\eta}(\Omega_{q,M,\eta}),$$

is nondecreasing, we obtain

$$\sup_{q\in(0,q_0)} E_{q,M,\eta}(\Omega_{q,M,\eta}) = E_{q_0,M,\eta}(\Omega_{q_0,M,\eta}) \le E_{q_0,M,\eta}(B_1) = E_{q_0}(B_1),$$

recalling that the optimal function for $E_{q_0}(B_1)$ is with unit L^2 norm. On the other hand, using the positivity of E, since $|\Omega_{q,M,\eta}| > |B_2|$ we have

$$E_{q_0}(B_1) \ge E_{q_0,M,\eta}(\Omega_{q_0,M,\eta}) \ge \frac{1}{\eta}(|\Omega_{q_0,M,\eta}| - |B_1|) \ge \frac{1}{\eta}(|B_2| - |B_1|).$$

By choosing η_1 such that $\eta_1 < 1$ and

$$\frac{(|B_2| - |B_1|)}{\eta_1} > E_{q_0}(B_1),$$

we reach the desired contradiction. The second part of the claim then follows from Lemma 3.14. □

We note that in the above lemma, η_1 does not depend on R. Next we show that, for q small, the boundary of any optimizer $\Omega_{q,M,\eta}$ (such that $\Omega_{q,M,\eta} = \{u > 0\}$, where $u \in H_0^1(\Omega)$ is the (positive) function attaining $E_q(\Omega) = E_{q,M}(\Omega)$) is close to the one of the corresponding optimal ball of the same measure $B_{q,M,\eta}$ in the definition of Fraenkel asymmetry, with respect to the Hausdorff distance d_H (see [35, Definition 4.4.9] for more details about the Hausdorff distance).

Lemma 3.16. *Under the assumptions of Lemma 3.15, for all $\delta > 0$ there exists $q_\delta = q_\delta(R, \eta) \in (0, q_1]$ such that for all $q \le q_\delta$, we have*

$$\mathrm{dist}_H(\partial\Omega_{q,M,\eta}, \partial B_{q,M,\eta}) \le \delta.$$

Proof. This follows exactly as in [19, Lemma 5.4], we report here the proof for the sake of completeness.

We fix $\delta > 0$ and call $B_\delta(B_{q,M,\eta}) := B_{q,M,\eta} + B_\delta$ the δ-neighborhood of $B_{q,M,\eta}$. First of all, we aim to prove that $\Omega_{q,M,\eta} \subset B_\delta(B_{q,M,\eta})$. If $\Omega_{q,M,\eta} \setminus B_\delta(B_{q,M,\eta})$ is empty, then there is nothing to prove. Otherwise there exists $x \in \Omega_{q,M,\eta} \setminus B_\delta(B_{q,M,\eta})$ so that by Lemma 3.12 there exists $\rho_0(R, \eta)$ such that for $\rho \le \rho_1 := \min\{\rho_0(R, \eta), \delta\}$ it holds

$$|B_1|\theta\rho^3 \le |B_\rho(x) \cap \Omega_{q,M,\eta}| \le |\Omega_{q,M,\eta} \setminus B_{q,M,\eta}| \le c_1 q,$$

where the last estimate follows from Lemma 3.15 and precisely (40). Notice that the choice of $\rho_1 \le \delta$ assures that $|B_\rho(x) \cap \Omega_{q,M,\eta}| \le |\Omega_{q,M,\eta} \setminus B_{q,M,\eta}|$. In conclusion, choosing $\rho = \rho_1$, we have

$$|B_1|\theta\rho_1^3 \le c_1 q,$$

which is not possible as soon as

$$q \le q_\delta := \frac{|B_1|\theta}{c_1}\rho_1^3.$$

With the same argument, thanks to the outer density estimate from Lemma 3.12 and again to the L^1 proximity from Lemma 3.15, we can show also that $B_R \setminus \Omega_{q,M,\eta} \subset B_\delta(B_R \setminus B_{q,M,\eta})$, with the same notation as above. This concludes the proof. □

Remark 3.17. It is worth noting that the constant q_δ in the lemma above depends also on R. This is one of the main obstacles while trying to get rid of the equiboundedness assumption of Theorem 3.1.

Remark 3.18. In view of the previous result, first of all we not that the energy is invariant under translations and thus we can assume that $\Omega \subset B_3$, then we fix $q_2(R) := \min\{q_1, q_\delta\}$, where q_δ is the constant from Lemma 3.16 with the choice of $\delta := 1/2$.

Therefore, if $q \in (0, q_2]$, in the proof of the next Theorem 3.19, we are allowed to inflate a set without touching the boundary of the geometric constraint $\Omega \subset B_R$.

We can now show the equivalence between the constrained and the unconstrained problems. We recall that the constant M has been already fixed, see Remark 3.3.

Theorem 3.19. *There exists a universal constant $R_0 \ge 10$ such that, for all $R \ge R_0$, there exists $q_3 = q_3(R) \le q_2$ and $\eta_2 = \eta_2(R) \le \eta_1$ such that, for all $\eta \le \eta_2$ and $q \in (0, q_3]$, we have that*

$$\min\left\{E_{q,M,\eta}(\Omega) : \Omega \subset B_R\right\} = \inf\left\{E_q(\Omega) : \Omega \subset B_R,\ |\Omega| = |B_1|\right\}.$$

As a consequence, problems (10) and (16) are equivalent for these values of q and η.

Proof. It is easy to check that

$$\min\left\{E_{q,M,\eta}(\Omega) : \Omega \subset B_R\right\} \leq \inf\left\{E_q(\Omega) : \Omega \subset B_R,\ |\Omega| = |B_1|\right\} =: \mu(q,R),$$

as the two functionals coincide on sets of measure $|B_1|$, thanks to the definition of f_η. Then, if the reversed inequality holds, it follows that on the set of minimizers (of the first or of the second problem) the two functionals do coincide, that is, problems (10) and (16) are equivalent.

We prove the claim of the theorem by contradiction. Let $R > R_0$ to be chosen later and

$$\Omega_{q,M,\eta} \subset B_R, \quad \sigma_{q,M,\eta} \in \mathbb{R}, \quad |\Omega_{q,M,\eta}| = |B_1| + \sigma_{q,M,\eta}, \quad E_{q,M,\eta}(\Omega_{q,M,\eta}) < \mu,$$

and we also note that, $\mu \leq E_q(B_1)$, by definition of infimum. We moreover assume, without loss of generality, that $\Omega_{q,M,\eta}$ are minimizers for problem (16). We treat separately the case $\sigma_{q,M,\eta} > 0$ and $\sigma_{q,M,\eta} < 0$.

Case $\sigma_{q,M,\eta} > 0$**.** We first observe that $\sigma_{q,M,\eta} \to 0$ as $\eta \to 0$. Indeed (recalling also that $E_q = E_{q,M}$ thanks to Proposition 3.2)

$$E_{q,M,\eta}(\Omega_{q,M,\eta}) = E_q(\Omega_{q,M,\eta}) + \frac{1}{\eta}\sigma_{q,M,\eta}$$

and so

$$0 \leq \frac{\sigma_{q,M,\eta}}{\eta} = E_{q,M,\eta}(\Omega_{q,M,\eta}) - E_q(\Omega_{q,M,\eta}) \leq E_q(B_1),$$

using the assumption $E_{q,M,\eta}(\Omega_{q,M,\eta}) \leq \mu \leq E_q(B_1)$ and the positivity of the energy. This implies that $\sigma_{q,M,\eta} \to 0$ as $\eta \to 0$.

Let now $\rho_{q,M,\eta} < 1$ be such that $|\rho_{q,M,\eta}\Omega_{q,M,\eta}| = |B_1|$, therefore

$$\rho_{q,M,\eta} = 1 - \frac{\sigma_{q,M,\eta}}{3|B_1|} + C\sigma_{q,M,\eta}^2,$$

for some $C = C(\sigma_{q,M,\eta}) \in \mathbb{R}$ such that $|C| \leq C_0$ for all $|\sigma_{q,M,\eta}| < \frac{1}{2}|B_1|$ some $C_0 > 0$ universal.

We call $u = u_{q,M,\eta}$ an optimal normalized function attaining $E_q(\Omega_{q,M,\eta})$, thus the function

$$\widetilde{u}(y) = {\rho_{q,M,\eta}}^{-\frac{3}{2}} u\left(\frac{y}{\rho_{q,M,\eta}}\right), \qquad y \in \rho_{q,M,\eta}\Omega_{q,M,\eta},$$

is an admissible competitor with unitary L^2-norm for $E_q(\rho_{q,M,\eta}\Omega_{q,M,\eta})$. We have the following scalings

$$\int_{\rho_{q,M,\eta}\Omega_{q,M,\eta}} |\nabla\widetilde{u}(y)|^2\,dy = {\rho_{q,M,\eta}}^{-2}\int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\,dx,$$

$$\begin{aligned} D(\widetilde{u}^2, \widetilde{u}^2) &= \int_{\rho_{q,M,\eta}\Omega_{q,M,\eta}}\int_{\rho_{q,M,\eta}\Omega_{q,M,\eta}} \frac{\widetilde{u}^2(x)\widetilde{u}^2(y)}{|x-y|}\,dxdy \\ &= {\rho_{q,M,\eta}}^{-1}\int_{\Omega_{q,M,\eta}}\int_{\Omega_{q,M,\eta}} \frac{u^2(w)u^2(z)}{|w-z|}\,dwdz \\ &= {\rho_{q,M,\eta}}^{-1} D(u^2, u^2). \end{aligned}$$

Since the new set $\rho_{q,M,\eta}\Omega_{q,M,\eta}$ is now admissible in the constrained minimization problem (10), using the above scaling we obtain

$$\begin{aligned}
E_{q,M,\eta}(\Omega_{q,M,\eta}) &= E_q(\Omega_{q,M,\eta}) + \frac{\sigma_{q,M,\eta}}{\eta} \\
&< \mu \\
&\le E_q(\rho_{q,M,\eta}\Omega_{q,M,\eta}) \\
&\le \int_{\rho_{q,M,\eta}\Omega_{q,M,\eta}} |\nabla \widetilde{u}(y)|^2\, dy + \frac{q}{2} D(\widetilde{u}^2, \widetilde{u}^2) \\
&= \rho_{q,M,\eta}{}^{-2} \int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\, dx + \rho_{q,M,\eta}{}^{-1} \frac{q}{2} D(u^2, u^2) \\
&= \int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\, dx \left(1 + \frac{2\sigma_{q,M,\eta}}{3|B_1|} + C\sigma_{q,M,\eta}^2\right) \\
&\quad + \frac{q}{2} D(u^2, u^2) \left(1 + \frac{\sigma_{q,M,\eta}}{3|B_1|} + C\sigma_{q,M,\eta}^2\right),
\end{aligned}$$

we deduce that (up to increasing C, recalling also that $E_q(\Omega_{q,M,\eta}) \le E_q(B_1)$)

$$\begin{aligned}
\frac{\sigma_{q,M,\eta}}{\eta} &< \int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\, dx \left(\frac{2\sigma_{q,M,\eta}}{3|B_1|}\right) + \frac{q}{2} D(u^2, u^2) \left(\frac{\sigma_{q,M,\eta}}{3|B_1|}\right) + 2E_q(B_1) C\sigma_{q,M,\eta}^2 \\
&\le \frac{\sigma_{q,M,\eta}}{3|B_1|} 2E_q(\Omega_{q,M,\eta}) + C\sigma_{q,M,\eta}^2.
\end{aligned}$$

Thus, for some universal $C > 0$,

$$\frac{1}{\eta} \le CE_q(\Omega_{q,M,\eta}) + C\sigma \le CE_q(B_1),$$

which leads to a contradiction as soon as $\eta_2 < \frac{1}{C(E_q(B_1))}$.

Case $\sigma_{q,M,\eta} < 0$**.** For this case let us call

$$\rho_{q,M,\eta} := \left(1 + \frac{\sigma_{q,M,\eta}}{|B_1|}\right)^{-1/3},$$

so that $|\rho_{q,M,\eta}\Omega_{q,M,\eta}| = |B_1|$.

We recall from the previous sections that a minimizer $\Omega_{q,M,\eta}$ for $E_{q,M,\eta}$ exists, and by Lemma 3.16, up to taking $q_3 \le q_2$ as in Remark 3.18, and $\eta_2 < \eta_1$ as in Lemma 3.15, the rescaled set $\rho_{q,M,\eta}\Omega_{q,M,\eta}$ is still contained in B_R, as soon as R_0 is big enough.

In fact, we show that $\sigma_{q,M,\eta} \ge -\frac{3}{4}|B_1|$, thus $\rho_{q,M,\eta} \le 4^3$ (hence, recalling Remark 3.18 we can take any $R_0 > 2 \cdot 4^3$).

If, for the sake of contradiction, $\sigma_{q,M,\eta} < -\frac{3}{4}|B_1|$, then $|\Omega_{q,M,\eta}| \le \frac{1}{4}|B_1|$ and $|2^{1/3}\Omega_{q,M,\eta}| \le \frac{1}{2}|B_1|$. The optimality of $\Omega_{q,M,\eta}$ entails,

$$\begin{aligned}
E_{q,M,\eta}(\Omega_{q,M,\eta}) = E_q(\Omega_{q,M,\eta}) + \eta\sigma_{q,M,\eta} &\le E_q(2^{1/3}\Omega_{q,M,\eta}) + \eta(2|\Omega_{q,M,\eta}| - |B_1|) \\
&= E_q(2^{1/3}\Omega_{q,M,\eta}) + \eta(|B_1| + 2\sigma_{q,M,\eta}),
\end{aligned}$$

which is equivalent to say

$$E_q(\Omega_{q,M,\eta}) - E_q(2^{1/3}\Omega_{q,M,\eta}) \le \eta(|B_1| + \sigma_{q,M,\eta}).$$

On the other hand (calling u_Ω the L^2 normalized function attaining $E_q(\Omega_{q,M,\eta})$), we have the upper bound on $D(\cdot,\cdot)$ thanks to Lemma 2.1 (for a universal constant $C > 0$)

$$D(u_\Omega^2, u_\Omega^2) \le 2\|u_\Omega\|_{L^2}\|\nabla u_\Omega\|_{L^2}\int_{\Omega_{q,M,\eta}} u_\Omega^2\, dx \le C\|\nabla u_\Omega\|_{L^2}.$$

Then, thanks to the above estimate and using also the positivity of $D(\cdot,\cdot)$, the Faber-Krahn inequality and up to a decrease of q_3

$$\begin{aligned}
E_q(\Omega_{q,M,\eta}) - E_q(2^{1/3}\Omega_{q,M,\eta}) &= (1-2^{-2/3})\int_{\Omega_{q,M,\eta}} |\nabla u_\Omega|^2\, dx + (1-2^2)\frac{q}{2}\int_{\Omega_{q,M,\eta}}\int_{\Omega_{q,M,\eta}} \frac{u_\Omega^2(x)u_\Omega^2(y)}{|x-y|}\, dxdy \\
&\ge (1-2^{-2/3})\int_{\Omega_{q,M,\eta}} |\nabla u_\Omega|^2\, dx - 2Cq\left(\int_{\Omega_{q,M,\eta}} |\nabla u_\Omega|^2\, dx\right)^{1/2} \\
&\ge (1-2^{-2/3})\int_{\Omega_{q,M,\eta}} |\nabla u_\Omega|^2\, dx \\
&\quad - 2Cq \max\left\{1, \left(\int_{\Omega_{q,M,\eta}} |\nabla u_\Omega|^2\, dx\right)^{1/2}\right\} \\
&\ge \frac{(1-2^{-2/3})}{2}\lambda_0(\Omega_{q,M,\eta}) \ge \frac{(1-2^{-2/3})}{2}\lambda_0(B_1).
\end{aligned}$$

Finally, $(|B_1| + \sigma_{q,M,\eta}) \le |B_1|$, so we reach a contradiction as soon as $\eta \le \eta_2$ and

$$\eta_2 < \frac{(1-2^{-\frac{2}{3}})}{2}\frac{\lambda_0(B_1)}{|B_1|}.$$

Let us define the function, using the same notations for $u, \widetilde{u}$ as in the previous case,

$$g\colon [1,\rho_{q,M,\eta}] \to \mathbb{R}, \quad g(r) = \int_{r\Omega_{q,M,\eta}} |\nabla \widetilde{u}|^2\, dx + \frac{q}{2}\int_{r\Omega_{q,M,\eta}}\int_{r\Omega_{q,M,\eta}} \frac{\widetilde{u}^2(x)\widetilde{u}^2(y)}{|x-y|}\, dxdy + \eta(r^3|\Omega_{q,M,\eta}| - |B_1|).$$

We show that the minimum of the function g is attained at $r = \rho := \rho_{q,M,\eta}$. Then the proof is concluded because this implies that $E_q(\rho_{q,M,\eta}\Omega_{q,M,\eta}) = E_{q,M,\eta}(\rho_{q,M,\eta}\Omega_{q,M,\eta}) \le E_{q,M,\eta}(\Omega_{q,M,\eta}) \le E_q(\Omega')$ for all $\Omega' \subset B_R$ with $|\Omega'| = |B_1|$. This is equivalent to show that for some η the inequality

$$g(r) \ge \int_{\rho\Omega_{q,M,\eta}} |\nabla \widetilde{u}|^2\, dx + \frac{q}{2}\int_{\rho\Omega_{q,M,\eta}}\int_{\rho\Omega_{q,M,\eta}} \frac{\widetilde{u}^2(x)\widetilde{u}^2(y)}{|x-y|}\, dxdy, \qquad \text{for all } r \in [1,\rho],$$

holds true. Up to rearranging the terms, and by the rescaling of the involved integrals, such an inequality reads as

$$\eta\left(1 - \left(\frac{r}{\rho}\right)^3\right) \le \int_{\rho\Omega_{q,M,\eta}} |\nabla \widetilde{u}|^2\left(\left(\frac{r}{\rho}\right)^{-2} - 1\right) + \frac{q}{2}D(\widetilde{u}^2, \widetilde{u}^2)\left(\left(\frac{r}{\rho}\right)^{-1} - 1\right).$$

Setting $t := \frac{r}{\rho} < 1$, and observing that $r^3|\Omega_{q,M,\eta}| = t^3$, the last inequality is equivalent to

$$\eta \leq \frac{\int_{\rho\Omega_{q,M,\eta}} |\nabla\widetilde{u}|^2\, dx(t^{-2}-1) + \frac{q}{2}D(\widetilde{u}^2,\widetilde{u}^2)(t^{-1}-1)}{1-t^3}.$$

It is easy to check that the right hand side is bounded from below by the function

$$t \mapsto \lambda_0(B)\frac{t^{-2}-1}{1-t^3}, \qquad t \in (0,1),$$

which is a function strictly decreasing in its domain and with infimum given by

$$\lim_{t\to 1^-} \lambda_0(B)\frac{t^{-2}-1}{1-t^3} = \frac{2}{3} > 0.$$

Thus it is enough to take $\eta \leq \eta_2 \leq 2/3$ and we immediately deduce that g has minimum for $r = \rho$. This concludes the proof. □

We highlight that if $\eta > 0$ is such that Theorem 3.19 holds true, we can freely assume the equivalence of the volume-constrained minimization of E_q and the unconstrained one for $E_{q,M,\eta}$ (since M has been already fixed, see Remark 3.3). On the other hand, we stress that this choice of η depends, in all our estimates, on R.

4. Optimality conditions and improvement of flatness

We have now the following picture. We know that minimizers for the auxiliary functional $E_{q,M,\eta}$ exist and (with the choices we made for M, η) are the same of those of the volume constrained functional E_q. These minimizers satisfy density estimates which are uniform with respect to $q \leq q_3$ (see Lemma 3.12 for the density estimate and Theorem 3.19 for the constant q_3). Moreover in Lemma 3.16 we have shown that such minimizers are close in Hausdorff distance to a given ball (any ball achieving the minimum in the definition of Fraenkel asymmetry of Ω). We now improve such a regularity with the final scope of showing that optimal sets are uniform $C^{2,\alpha}$ parametrizations on the boundary of the ball. The results and the proofs in this section are borrowed (with nontrivial adjustments) from results in [32, Theorem 4.5 and Theorem 4.8], [22, Theorem 2]. We begin with the following theorem, in which we use the notation ∂^*F for the reduced boundary of a set of finite perimeter F.

Theorem 4.1. *Let $q \in (0, q_3]$, let Ω be a minimizer of (2), and let u be an optimal function attaining $E_q(\Omega)$, thus also solution of (5). Then we have that:*

(i) There is a Borel function $\mu_u \colon \partial\Omega \to \mathbb{R}$ such that, in the sense of the distributions, one has

$$-\Delta u = (\lambda_q - q v_u)u - \mu_u \mathcal{H}^2 \llcorner \partial\Omega, \qquad \text{in } B_R. \tag{41}$$

(ii) There exist constants $0 < c < C < +\infty$, depending on R, such that $c \leq \mu_u \leq C$.

(iii) For all points $\overline{x} \in \partial^*\Omega = \partial^*\{u > 0\}$, the measure theoretic inner unit normal $\nu_u(\overline{x})$ is well defined and, as $\rho \to 0$,

$$\frac{\Omega - \overline{x}}{\rho} \to \{x : x \cdot \nu_u(\overline{x}) \geq 0\}, \qquad \text{in } L^1(B_R).$$

(iv) For $\mathcal{H}^2$ almost all $\overline{x} \in \partial^*\{u>0\}$ we have

$$\frac{u(\overline{x}+\rho x)}{\rho} \longrightarrow \mu_u(\overline{x})(x\cdot\nu_u(\overline{x}))_+, \qquad \text{in } W^{1,p}(B_R) \text{ for every } p\in[1,+\infty).$$

(v) $\mathcal{H}^2(\partial\Omega\setminus\partial^*\Omega)=0.$

Proof. The proof is essentially identical to that in [32, Section 4]. We only have to check that our hypotheses match with those in [32]. First by Lemma 2.4 u satisfies

$$-\Delta u - Q(x) = 0 \qquad \text{in } \mathcal{D}'(\Omega),$$

where $Q=(\lambda_q - q v_u)u \in L^\infty(\Omega)$ and $u\in H^1_0(\Omega)$. Hence, by repeating the proof of [32, Theorem 4.5] or by directly applying [36, Proposition 2.3] one obtains that there exists a positive Radon measure concentrated on $\partial\Omega$ that we denote $\mu_u\mathcal{H}^2\llcorner\partial\Omega$. Moreover, thanks to the non-degeneracy, see Remark 3.10 and the Lipschtiz continuity of u we have that there exist constant $C>c>0$ depending on q and R such that

$$c \le \frac{1}{r}\fint_{\partial B_r} u\, d\mathcal{H}^2 \le C.$$

Hence we can work under the hypotheses of [32, Theorem 4.5] so that μ_u is a density of a Radon measure on $\partial\Omega$ and, denoting still with μ_u the function defininig it, μ_u satisfies $(i)-(v)$. □

4.1. The structure of μ_u: blow up limits

We show now the following result.

Proposition 4.2. *Let Ω be a minimizer of (10). Then function $\mu_u:\partial\Omega\to\mathbb{R}$ found in Theorem 4.1 is constant on $\partial^*\Omega$.*

Proof. The proof follows the path of [19, Theorem 6.5], in turn inspired by [22]. Due to the nonlocal term, we will have to perform some new and non-straightforward computations.

We reason by contradiction and we assume that there exists $x_0,x_1\in\partial^*\Omega$ such that

$$\mu_u(x_0) < \mu_u(x_1).$$

Then we construct a family of volume preserving diffeomorphisms as follows: let $\kappa<1$ and $\rho<1$ and let $\varphi\in C^1_0(B_1(0))$ be a non-null, radially symmetric function supported in $B_1(0)$. We define

$$\tau_{\rho,\kappa}(x)=\tau(x)=x+\sum_{i\in\{0,1\}}(-1)^i\kappa\rho\varphi\left(\frac{|x-x_i|}{\rho}\right)\nu_{x_i}\chi_{B_\rho(x_i)},$$

where ν_{x_i} are the measure theoretic inner normals to $\partial^*\Omega$ at x_i, $i=1,2$.

It is easy to notice that τ is indeed a diffeomorphism for ρ and κ small enough and that $\tau(x)-x$ vanishes outside $B_\rho(x_0)\cup B_\rho(x_1)$. Moreover we have:

$$\nabla\tau(x)=Id+\sum_{i\in\{0,1\}}(-1)^i\kappa\varphi'\left(\frac{|x-x_i|}{\rho}\right)\frac{x-x_i}{|x-x_i|}\otimes\nu_{x_i}\chi_{B_\rho(x_i)}, \tag{42}$$

so that[3]

$$\det(\nabla\tau(x)) = 1 + \sum_{i\in\{0,1\}} (-1)^i \kappa\varphi'\left(\frac{|x-x_i|}{\rho}\right)\frac{x-x_i}{|x-x_i|}\cdot \nu_{x_i}\chi_{B_\rho(x_i)} + o(\kappa).$$

We call $\Omega_\rho = \tau(\Omega)$. We aim to show that for κ,ρ small enough it holds $E_{q,M,\eta}(\Omega_\rho) < E_{q,M,\eta}(\Omega)$, hence contradicting the minimality of Ω. To do that, we deal with the first variation of each term of the sum defining $E_{q,M,\eta}$. We stress that the computations regarding the volume and the Dirichlet energy contributions are identical to those performed originally in [22] (see also [20] and [37], where the same idea is applied). Moreover, exactly as in the proof of [19, Theorem 6.5] one obtains that

$$f_\eta(\Omega_\rho) - f_\eta(\Omega) = o(\rho^3), \qquad \text{as } \rho \to 0, \tag{43}$$

and that

$$\frac{1}{\rho^3}\left(\int_{\Omega_\rho} |\nabla u_\rho|^2\,dx - \int_\Omega |\nabla u|^2\,dx\right) \le \kappa(\mu_u^2(x_0) - \mu_u^2(x_1))C(\varphi) + o_\rho(1) + o(\kappa), \tag{44}$$

where u_ρ and u are the functions attaining $E_q(\Omega_\rho)$ and $E_q(\Omega)$ respectively, and

$$C(\varphi) = \int_{B_1(0)\cap\{y\cdot\nu=0\}} \varphi(|y|)\,d\mathcal{H}^2(y) = -\int_{B_1(0)\cap\{y\cdot\nu>0\}} \varphi'(|y|)\frac{y\cdot\nu}{|y|}\,dy,$$

with the last equality that follows from the Divergence Theorem, recalling that ν is a inner normal and

$$\operatorname{div}(\varphi(|y|)\nu) = \varphi'(|y|)\frac{y\cdot\nu}{|y|}.$$

Notice also that by the radial symmetry of φ the value of $C(\varphi)$ is not affected by the choice of ν.

We are left to compute the variation of the nonlocal term $D(\cdot,\cdot)$. This is the major technical difference with respect to the proof of [19, Theorem 6.5]. We claim that (recalling that u_ρ and u are the functions attaining $E_q(\Omega_\rho)$ and $E_q(\Omega)$ respectively)

$$\frac{1}{\rho^3}\left(D(u_\rho^2,u_\rho^2) - D(u^2,u^2)\right) = o(\kappa) + o(\rho). \tag{45}$$

Once that (45) is proved, the conclusion then readily follows: by minimality of Ω and thanks to (43), (44), and (45) we have that

$$\begin{aligned} 0 &\le E_{q,M,\eta}(\Omega_\rho) - E_{q,M,\eta}(\Omega) \\ &\le \kappa\rho^3 C(\varphi)\Big((\mu_u(x_0)^2 - \mu_u(x_1)^2)\Big) + o(\rho^3) + \rho^3 o(\kappa). \end{aligned}$$

Since from the assumptions there holds $\mu_u(x_0)^2 - \mu_u(x_1)^2 < 0$ we get the desired contradiction by choosing ρ and κ small enough.

It remains to show the validity of (45). To do so, we set

$$\widetilde{u}(x) = u(\tau^{-1}(x)) \quad \text{and} \quad \widetilde{w}(x) = v_{\widetilde{u}}(x)\widetilde{u}(x)^2,$$

[3]We are using the formula $\det(Id + \xi A) = 1 + trace(A)\xi + o(\xi)$ for a matrix $A \in \mathbb{R}^{N\times N}$.

where $v_u(x) = \int_\Omega \frac{u^2(y)}{|x-y|} dy$. With such a notation in force we compute, using also formula (42),

$$
\begin{aligned}
\frac{1}{\rho^3}\left(D(u_\rho^2, u_\rho^2) - D(u^2,u^2)\right) &= \frac{1}{\rho^3}\left(\int_{\Omega_\rho} \widetilde{w}\, dx - \int_\Omega w\, dx\right) \\
&= \frac{1}{\rho^3}\int_\Omega \left(\widetilde{w}(\tau(x))\det(\nabla\tau(x)) - w(x)\right)\, dx \\
&= \frac{1}{\rho^3}\int_\Omega (\widetilde{w}(\tau(x)) - w(x))\, dx \\
&\quad + \frac{1}{\rho^3}\int_\Omega \widetilde{w}(\tau(x)) \sum_{i\in\{0,1\}} (-1)^i \kappa\varphi' \\
&\quad \left(\frac{|x-x_i|}{\rho}\right)\frac{x-x_i}{|x-x_i|}\cdot \nu_{x_i}\chi_{B_\rho(x_i)}\, dx + o(\kappa).
\end{aligned} \tag{46}
$$

We observe that since $\widetilde{w}(x) = v_{\widetilde{u}}(x)\widetilde{u}(x)^2$, with $v_{\widetilde{u}}$ uniformly bounded and $\widetilde{u}$ Lipschitz continuous in Ω, then $|\widetilde{w}(\tau(x))| \leq C\rho^2$ in $\Omega \cap B_\rho(x_i)$, since $\widetilde{u}(x_i) = 0$. With this in mind, we can compute

$$
\begin{aligned}
&\left|\frac{1}{\rho^3}\int_\Omega \widetilde{w}(\tau(x)) \sum_{i\in\{0,1\}} (-1)^i\kappa\varphi'\left(\frac{|x-x_i|}{\rho}\right)\frac{x-x_i}{|x-x_i|}\cdot\nu_{x_i}\chi_{B_\rho(x_i)}\, dx\right| \\
&\leq \frac{1}{\rho^3}\sum_{i\in\{0,1\}}\kappa\int_{\Omega\cap B_\rho(x_i)} \left|\widetilde{w}(\tau(x))\right|\cdot\left|\varphi'\left(\frac{|x-x_i|}{\rho}\right)\frac{x-x_i}{|x-x_i|}\cdot\nu_{x_i}\right| dx \\
&\leq \frac{C}{\rho}|B_\rho| = o(\rho).
\end{aligned} \tag{47}
$$

Moreover,

$$
\begin{aligned}
\frac{1}{\rho^3}\int_\Omega(\widetilde{w}(\tau(x)) - w(x))\, dx &= \frac{1}{\rho^3}\int_\Omega u^2(x)\left(\int_{\Omega_\rho}\frac{\widetilde{u}^2(y)}{|\tau(x)-y|}\, dy - \int_\Omega \frac{u^2(y)}{|x-y|}\, dy\right) dx \\
&= \frac{1}{\rho^3}\int_\Omega u^2(x)\left(\int_{\Omega_\rho}\frac{u^2(\tau^{-1}y)}{|\tau(x)-y|}\, dy - \int_\Omega\frac{u^2(y)}{|x-y|}\, dy\right) dx \\
&= \frac{1}{\rho^3}\int_\Omega u^2(x)\int_\Omega\left(\frac{u^2(y)}{|\tau(x)-\tau(y)|}\det(\nabla\tau(y)) - \frac{u^2(y)}{|x-y|}\right) dy\, dx
\end{aligned}
$$

and, with a computation similar to the one done in (47) we obtain that

$$
\begin{aligned}
\frac{1}{\rho^3}\int_\Omega(\widetilde{w}(\tau(x)) - w(x))\, dx &= \frac{1}{\rho^3}\int_\Omega u^2(x)\int_{\Omega\cap(B_\rho(x_0)\cup B_\rho(x_1))} \\
&\qquad u^2(y)\left(\frac{1}{|\tau(x)-\tau(y)|} - \frac{1}{|x-y|}\right) dy\, dx + o(\rho).
\end{aligned}
$$

Now, by Lemma 3.11, we know that u is Lipschitz (with constant L) so that $u^2 \lesssim \rho^2$ in $\Omega \cap B_\rho(x_i)$, for $i = 1, 2$, since $u(x_i) = 0$. Hence, we obtain by the previous formula and an elementary computation that

$$\begin{aligned}&\left|\frac{1}{\rho^3}\int_\Omega(\widetilde{w}(\tau(x))-w(x))\,dx\right|\\&\quad\leq\frac{L}{\rho}\int_\Omega u^2(x)\int_{\Omega\cap(B_\rho(x_0)\cup B_\rho(x_1))}\left|\frac{1}{|x-y|}-\frac{1}{|\tau(x)-\tau(y)|}\right|\,dy\,dx\\&\quad\leq c\rho^2=o(\rho),\end{aligned}\tag{48}$$

for some universal $c > 0$. By (46), (47), and (48) we deduce (45), and the proof is concluded. □

We are now in position to show $C^{2,\alpha}-$regularity of the boundary of a minimizer Ω. This can be done in two steps: first one shows that such a boundary is locally the graph of a $C^{2,\alpha}$ function defined on the boundary of a ball. To do that one exploits the improvement of flatness technique from [32, Section 7 and 8], readapted with minimal changes to our setting as in [38, Appendix]. Then, as we already know by the previous section that the boundary of Ω is close in Hausdorff distance to that of a ball, we obtain that the local parametrization is a global parametrization of class $C^{2,\alpha}$ on the boundary of the ball. We first need a definition (see [32, Definition 7.1]).

Definition 4.3. Let $\gamma_\pm \in (0,1]$ and $k > 0$. A weak solution u of (41) is of class $F(\gamma_-,\gamma_+,k)$ in $B_\rho(x_0)$ with respect to direction $\nu \in \mathbb{S}^{N-1}$ if

(a) $x_0 \in \partial\{u > 0\}$ and

$$\begin{aligned}&u=0,\quad \text{for } (x-x_0)\cdot\nu\leq-\gamma_-\rho,\quad x\in B_\rho(x_0),\\&u(x)\geq\mu_u(x_0)[(x-x_0)\cdot\nu-\gamma_+\rho],\quad \text{for } (x-x_0)\cdot\nu\geq\gamma_+\rho,\quad x\in B_\rho(x_0).\end{aligned}$$

(b) $|\nabla u(x_0)| \leq \mu_u(x_0)(1+k)$ in $B_\rho(x_0)$ and $\operatorname{osc}_{B_\rho(x_0)}\mu_u \leq k\mu_u(x_0)$.

We note that when $k = +\infty$, then condition (b) is automatically satisfied. We can show the following result.

Theorem 4.4. *Let $q \in (0,q_3]$, Ω be an optimal set for (2), and u a function attaining $E_q(\Omega)$ and a weak solution to (41) in B_R. Then there are constants $\overline{\gamma}$ and $\overline{k}$, depending only on R, μ_u, such that if u is of class $F(\gamma,1,+\infty)$ in $B_{4\rho}(x_0)$ with respect to some direction $\nu \in \mathbb{S}^{N-1}$ with $\gamma \leq \overline{\gamma}$ and $\rho \leq \overline{k}\gamma^2$, then there exists a $C^{2,\alpha}$ function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $\|f\|_{C^{2,\alpha}} \leq C(R,\mu_u)$ such that, calling*

$$\operatorname{graph}_\nu f := \{x\in\mathbb{R}^3 : x\cdot\nu = f(x-(x\cdot\nu)\nu)\},$$

then

$$\partial\{u>0\}\cap B_\rho(x_0) = (x_0+\operatorname{graph}_\nu(f))\cap B_\rho(x_0).$$

Moreover, for all $\varepsilon_0 > 0$ there exists $q_\varepsilon \in (0,q_3]$ such that if $q < q_\varepsilon$ then

$$\partial\{u>0\} = \left\{\left(r+\varphi\left(\tfrac{x}{|x|}\right)\right)\frac{x}{|x|}\ :\ x\in\partial B_r\right\}$$

where $\varphi\colon \partial B_1 \to \mathbb{R}$ is a function with the same regularity of f and $\|\varphi\|_{C^{2,\alpha}} \leq \varepsilon_0$.

We omit the proof which is identical to that in [19, Theorems 1.2 and 6.8] (which is in turn inspired by [32, Theorem 8.1] and [38, Theorem 2.17 and Appendix]). We note that in our setting μ_u is constant, thus the requirement to be $C^{1,\alpha}$ regular is trivially satisfied.

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. The existence of a minimizer follows from Lemma 3.5 and Theorem 3.19. On the other hand, the fact that any optimal set is $C^{2,\alpha}$ nearly spherical follows from Theorem 4.4. □

5. The surgery result and the proof of Theorem 1.2

In this section, we remove the equiboundedness assumption that was present in Theorem 3.1. The surgery strategy that we employ is very similar to the one proposed in [21] (see also [39]) and used for the spectral Gamow problem in [19]. We recall here, for the reader's sake, the main notations and the changes that are needed in our setting and we give a proof of the following main result.

Lemma 5.1. *There exist universal constants D, $\overline{\delta} < 1$ and $\overline{q} \in (0, q_3]$ such that if $q \leq \overline{q}$ then for any open and connected set $\Omega \subset \mathbb{R}^3$ of measure $|B_1|$ satisfying $E_q(\Omega) - \lambda_0(B) \leq \overline{\delta}$ there exists an open, connected set $\widehat{\Omega}$ of measure $|B_1|$ with diameter bounded by D and such that*

$$E_q(\widehat{\Omega}) \leq E_q(\Omega).$$

Let us introduce some notation. Let Ω be a connected set of measure $|B_1|$ such that $\lambda_0(\Omega) - \lambda_0(B_1) \leq E_q(\Omega) - \lambda_0(B_1) \leq \overline{\delta}$, and we fix B_1 the ball attaining the minimum in the Fraenkel asymmetry for Ω (see (9)). We can clearly assume (up to a traslation of Ω) that B_1 is centered at the origin. Then, by the quantitative Faber–Krahn inequality (see Theorem 2.11 or [20]), we have

$$|\Omega \Delta B_1| = \mathcal{A}(\Omega) \leq |B_1|^{1/3} \left(\frac{\overline{\delta}}{\widehat{\sigma}}\right)^{1/2},$$

where $\widehat{\sigma}$ is the constant from Theorem 2.11. By defining

$$K := \lambda_0(B_1) + 1 \geq \lambda_0(B_1) + \overline{\delta}$$

we obtain immediately

$$E_q(\Omega) \leq K, \qquad \text{and in particular,} \qquad \int_\Omega |\nabla u|^2 \, dx \leq K,$$

where $u = u_{q,\Omega}$ from now on is the function attaining $E_q(\Omega)$. We then note that (since B_1 has unit radius)

$$|\Omega \setminus [-t, t]^3| \leq |\Omega \Delta B| = \mathcal{A}(\Omega), \qquad \text{for all } t \geq 1.$$

Let $\widehat{m} \in (0, 1/4)$ be such that

$$\frac{(4\widehat{m})^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}} K \leq \frac{1}{2}. \tag{49}$$

Moreover, we choose $\overline{\delta}$ small enough so that

$$|\Omega \setminus [-1,1]^3| \le \mathcal{A}(\Omega) \le |B_1| \left(\frac{\overline{\delta}}{\widehat{\sigma}}\right) \le \frac{\widehat{m}}{2^6}.$$

We first focus on the direction e_1 and detail the construction in this case. We shall denote $z = (x, y) \in \mathbb{R} \times \mathbb{R}^2$ and by z_i the i-th component of $z \in \mathbb{R}^3$. For any $t \in \mathbb{R}$, we define

$$\Omega_t := \left\{ y \in \mathbb{R}^2 : (t, y) \in \Omega \right\},$$

and given any set $\Omega \subseteq \mathbb{R}^3$, we define its 1-dimensional projections for $p \in \{1, 2, 3\}$ as

$$\pi_p(\Omega) := \left\{ t \in \mathbb{R} : \exists\, (z_1,\, z_2, z_3) \in \Omega,\ z_p = t \right\}.$$

For every $t \le -1$ we call

$$\Omega^+(t) := \left\{ (x, y) \in \Omega : x > t \right\}, \Omega^-(t) := \left\{ (x, y) \in \Omega : x < t \right\}, \qquad \varepsilon(t) := \mathcal{H}^2(\Omega_t)\,.$$

Observe that

$$m(t) := \left|\Omega^-(t)\right| = \int_{-\infty}^{t} \varepsilon(s)\, ds \le 2\widehat{m}\,.$$

We call u the optimizer for $E_q(\Omega)$ (we note that it is unique since $\Omega = \{u > 0\}$ is connected). We define then also, for every $t \le -1$,

$$\delta(t) := \int_{\Omega_t} |\nabla u(t, y)|^2\, d\mathcal{H}^2(y)\,, \mu(t) := \int_{\Omega_t} u(t, y)^2\, d\mathcal{H}^2(y)\,,$$

which makes sense since u is smooth inside Ω. Applying the Faber–Krahn inequality in $\mathbb{R}^2$ to the set Ω_t, and using the rescaling property of eigenvalues on $\mathbb{R}^2$, we know that

$$\varepsilon(t)\lambda_0(\Omega_t) = \mathcal{H}^2(\Omega_t)\lambda_0(\Omega_t) \ge \lambda_0(B_{\mathbb{R}^2})\,,$$

calling $B_{\mathbb{R}^2}$ the ball of unit measure in $\mathbb{R}^2$. As a trivial consequence, we can estimate μ in terms of ε and δ: in fact, noting that $u(t, \cdot) \in H^1_0(\Omega_t)$ and writing $\nabla u = (\nabla_1 u, \nabla_y u)$, we have

$$\mu(t) = \int_{\Omega_t} u(t, \cdot)^2\, d\mathcal{H}^2 \le \frac{1}{\lambda_0(\Omega_t)} \int_{\Omega_t} |\nabla_y u(t, \cdot)|^2\, d\mathcal{H}^2 \le C\varepsilon(t)\delta(t). \tag{50}$$

We can now present two estimates which assure that u and ∇u cannot be too big in $\Omega^-(t)$.

Lemma 5.2. *Let $\Omega \subseteq \mathbb{R}^3$ and u be as in Lemma 5.1. For every $t \le -1$ the following inequalities hold:*

$$\int_{\Omega^-(t)} u^2\, dx \le C_1 \varepsilon(t)^{\frac{1}{2}} \delta(t)\,, \int_{\Omega^-(t)} |\nabla u|^2\, dx \le C_1 \varepsilon(t)^{\frac{1}{2}} \delta(t)\,, \tag{51}$$

for some universal constant $C_1 > 0$.

The proof of the above Lemma follows, up to a few minor changes, as in [21, Lemma 2.3], by working on u (and recalling it solves the PDE (7)) instead of the first eigenfunction of the Dirichlet Laplacian in Ω. We reproduce it here for the sake of completeness.

Proof. Let us fix $t \le -1$. Consider the set Ω_S^- obtained by the union of $\Omega^-(t)$ and its reflection with respect to the plane $\{x = t\}$, and call $u_S \in H_0^1(\Omega_S)$ the function obtained by reflecting u. Using the Faber-Krahn inequality, we find then

$$\frac{\lambda_0(B_1)|B_1|^{\frac{2}{3}}}{\big(2m(t)\big)^{\frac{2}{3}}} = \frac{\lambda_0(B_1)|B_1|^{\frac{2}{3}}}{|\Omega_S^-|^{\frac{2}{3}}} \le \lambda_0(\Omega_S^-)$$
$$\le \frac{\displaystyle\int_{\Omega_S^-} |\nabla u_S|^2\,dx}{\displaystyle\int_{\Omega_S^-} u_S^2\,dx} = \frac{\displaystyle\int_{\Omega^-(t)} |\nabla u|^2\,dx}{\displaystyle\int_{\Omega^-(t)} u^2\,dx} = \frac{\displaystyle\int_{\Omega^-(t)} |\nabla u|^2\,dx}{\displaystyle\int_{\Omega^-(t)} u^2\,dx},$$

by the symmetry of Ω_S^-, and using the scaling. This estimate gives

$$\int_{\Omega^-(t)} u^2\,dx \le \frac{\big(2m(t)\big)^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}} \int_{\Omega^-(t)} |\nabla u|^2\,dx \tag{52}$$

which in particular, being $m(t) \le 2\widehat{m}$ and recalling (49), implies

$$\int_{\Omega^-(t)} u^2\,dx \le \frac{1}{2}\,.$$

On the other hand, recalling that $-\Delta u \le \lambda_q u$ in Ω, by Schwarz inequality and using (50) we have

$$\begin{aligned}\int_{\Omega^-(t)} |\nabla u|^2\,dx &\le \int_{\Omega^-(t)} \lambda_q u^2\,dx + \int_{\Omega_t} u\,\frac{\partial u}{\partial \nu}\,d\mathcal{H}^2\\ &\le K\int_{\Omega^-(t)} u^2\,dx + \sqrt{\int_{\Omega_t} u^2\,d\mathcal{H}^2 \int_{\Omega_t} |\nabla u|^2\,d\mathcal{H}^2}\\ &\le K\int_{\Omega^-(t)} u^2\,dx + C\varepsilon(t)^{\frac{1}{2}}\delta(t)\,.\end{aligned} \tag{53}$$

It is now easy to obtain (51) combining (52) and (53). In fact, by inserting the latter into the first, we find

$$\int_{\Omega^-(t)} u^2\,dx \le \frac{\big(2m(t)\big)^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}}\left(K\int_{\Omega^-(t)} u^2\,dx + C\varepsilon(t)^{\frac{1}{2}}\delta(t)\right),$$

which by (49) again yields

$$\frac{1}{2}\int_{\Omega^-(t)} u^2\,dx \le \frac{\big(2m(t)\big)^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}}\,C\varepsilon(t)^{\frac{1}{2}}\delta(t) \le C\varepsilon(t)^{\frac{1}{2}}\delta(t)\,. \tag{54}$$

The left estimate in (51) is then obtained. To get the right one, one has then just to insert (54) into (53). □

Let us go further into the construction, giving some additional definitions. For any $t \le -1$ and $\sigma(t) > 0$, we define the cylinder $Q(t)$ as

$$Q(t) := \Big\{(x,y) \in \mathbb{R}^3 : t - \sigma(t) < x < t,\ (t,y) \in \Omega\Big\} = \big(t - \sigma(t), t\big) \times \Omega_t\,,$$

where for any $t \le -1$ we set

$$\sigma(t) = \varepsilon(t)^{\frac{1}{2}}.$$

We let also $\widetilde{\Omega}(t) = \Omega^+(t) \cup Q(t)$, and we introduce $\widetilde{u} \in H^1_0\big(\widetilde{\Omega}(t)\big)$ as

$$\widetilde{u}(x,y) := \begin{cases} u(x,y) & \text{if (x,y)} \in \Omega^+(t)\,, \\ \dfrac{x-t+\sigma(t)}{\sigma(t)}\,u(t,y) & \text{if (x,y)} \in Q(t)\,. \end{cases}$$

The fact that $\widetilde{u}$ vanishes on $\partial\widetilde{\Omega}(t)$ is obvious; moreover, $\nabla u = \nabla\widetilde{u}$ on $\Omega^+(t)$, while on $Q(t)$ one has

$$\nabla\widetilde{u}(x,y) = \left(\frac{u(t,y)}{\sigma(t)}, \frac{x-t+\sigma(t)}{\sigma(t)}\,\nabla_y u(t,y)\right).$$

A simple calculation allows us to estimate the integrals of $\widetilde{u}$ and $\nabla\widetilde{u}$ on $Q(t)$.

Lemma 5.3. *For every $t \le -1$, one has*

$$\int_{Q(t)} |\nabla\widetilde{u}|^2\,dx \le C_2\varepsilon(t)^{\frac{1}{2}}\delta(t)\,, \int_{Q(t)} \widetilde{u}^2\,dx \le C_2\varepsilon(t)^{\frac{3}{2}}\delta(t)\,,$$

for a universal constant $C_2 > 0$.

The proof of the above Lemma follows as [21, Lemma 2.4].

Another simple but useful estimate concerns the Rayleigh quotients of the functions $\widetilde{u}$ on the sets $\widetilde{\Omega}(t)$: notice that, while u has unit L^2 norm, the modifed function $\widetilde{u}$ in general is not normalized so we need to take care also of its norm.

Lemma 5.4. *There exists a universal constant $C_3 > 0$ such that for every $t \le -1$, one has*

$$\int_{\widetilde{\Omega}(t)} |\nabla\widetilde{u}|^2\,dx \le \int_\Omega |\nabla u|^2\,dx + C_3\varepsilon(t)^{\frac{1}{2}}\delta(t)\,, \qquad \int_{\widetilde{\Omega}(t)} \widetilde{u}^2\,dx \ge \int_\Omega u^2\,dx - C_3\varepsilon(t)^{\frac{1}{2}}\delta(t)\,.$$

Proof. It is enough to note that, by definition of $\widetilde{\Omega}(t)$ and using Lemma 5.2 and 5.3, we obtain for the gradient term

$$\begin{aligned}\int_{\widetilde{\Omega}(t)} |\nabla\widetilde{u}|^2\,dx &= \int_{\Omega^+(t)} |\nabla u|^2\,dx + \int_{Q(t)} |\nabla\widetilde{u}|^2\,dx \\ &= \int_\Omega |\nabla u|^2\,dx + \int_{Q(t)} |\nabla\widetilde{u}|^2\,dx - \int_{\Omega^-(t)} |\nabla u|^2\,dx \\ &\le \int_\Omega |\nabla u|^2\,dx + C_2\varepsilon(t)^{\frac{1}{2}}\delta(t)\,,\end{aligned}$$

while for the function, we have

$$\begin{aligned}\int_{\widetilde{\Omega}(t)} \widetilde{u}^2\,dx &= \int_{\Omega^+(t)} u^2\,dx + \int_{Q(t)} \widetilde{u}^2\,dx = \int_\Omega u^2\,dx + \int_{Q(t)} \widetilde{u}^2\,dx \\ &\quad - \int_{\Omega^-(t)} u^2\,dx \ge \int_\Omega u^2\,dx - C_1\varepsilon(t)^{\frac{1}{2}}\delta(t)\,.\end{aligned}$$

□

We can now enter in the central part of our construction. Basically, we aim to show that either Ω already has bounded left "tail" in direction e_1, or some rescaling of $\widetilde{\Omega}(t)$ has energy lower than that of Ω.

Lemma 5.5. *Let Ω be as in the assumptions of Lemma 5.2, and let $t \le -1$. There exist universal $\overline{q} \in (0, q_3]$ and $C_4 > 2$ such that, for all $q \le \overline{q}$ exactly one of the three following conditions hold:*

(1) $\max\{\varepsilon(t), \delta(t)\} > 1$;
(2) (1) does not hold and $m(t) \le C_4\big(\varepsilon(t) + \delta(t)\big)\varepsilon(t)^{\frac{1}{2}}$;
(3) (1) and (2) do not hold and one has that

$$\frac{\int_{\widehat{\Omega}(t)} |\nabla \widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)} \widehat{u}^2\,dx} \le \int_\Omega |\nabla u|^2\,dx, \qquad \text{and} \qquad E\big(\widehat{\Omega}(t)\big) < E_q(\Omega),$$

where for $t \le -1$ we set

$$\widehat{\Omega}(t) := |B_1|^{\frac{1}{3}}\big|\widetilde{\Omega}(t)\big|^{-\frac{1}{3}}\widetilde{\Omega}(t), \qquad \text{and} \qquad \widehat{u}(x) = \widetilde{u}\big(|B_1|^{-\frac{1}{3}}|\widetilde{\Omega}(t)|^{\frac{1}{3}}x\big), \quad \text{for } x \in \widehat{\Omega}(t).$$

Proof. Assume (1) is false. Then it is possible to apply Lemma 5.4, to obtain

$$\int_{\widetilde{\Omega}(t)} |\nabla \widetilde{u}|^2\,dx \le \int_\Omega |\nabla u|^2\,dx + C_3\varepsilon(t)^{\frac{1}{2}}\delta(t),$$
$$\int_{\widetilde{\Omega}(t)} \widetilde{u}^2\,dx \ge \int_\Omega u^2\,dx - C_3\varepsilon(t)^{\frac{1}{2}}\delta(t) = 1 - C_3\varepsilon(t)^{\frac{1}{2}}\delta(t)\,. \tag{55}$$

By the scaling properties of the eigenvalue and the fact that $\big|\widehat{\Omega}(t)\big| = |B_1|$, we know that

$$\frac{\int_{\widehat{\Omega}(t)} |\nabla \widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)} \widehat{u}^2\,dx} = \frac{\big|\widetilde{\Omega}(t)\big|^{\frac{2}{3}}}{|B_1|^{\frac{2}{3}}}\,\frac{\int_{\widetilde{\Omega}(t)} |\nabla \widetilde{u}|^2\,dx}{\int_{\widetilde{\Omega}(t)} \widetilde{u}^2\,dx}\,.$$

By construction,

$$\big|\widetilde{\Omega}(t)\big| = \big|\Omega^+(t)\big| + \big|Q(t)\big| = |B_1| - m(t) + \varepsilon(t)^{\frac{3}{2}}\,,$$

hence the above estimates, the scaling of the integrals due to the definition of $\widehat{u}$ and (55) lead to

$$\begin{aligned}
\frac{\int_{\widehat{\Omega}(t)} |\nabla \widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)} \widehat{u}^2\,dx} &= \left(1 - \frac{m(t)}{|B_1|} + \frac{\varepsilon(t)^{\frac{3}{2}}}{|B_1|}\right)^{\frac{2}{3}} \frac{\int_{\widetilde{\Omega}(t)} |\nabla \widetilde{u}|^2\,dx}{\int_{\widetilde{\Omega}(t)} \widetilde{u}^2\,dx},\\
&\le \left(1 - \frac{2}{3|B_1|}\,m(t) + \frac{2}{3|B_1|}\,\varepsilon(t)^{\frac{3}{2}}\right)\left(1 + C_3\varepsilon^{\frac{1}{2}}(t)\delta(t)\right)\\
&\quad\left(\int_\Omega |\nabla u|^2\,dx + C_3\varepsilon(t)^{\frac{1}{2}}\delta(t)\right)\\
&\le \left(\int_\Omega |\nabla u|^2\,dx - \frac{2\lambda_0(B_1)}{3|B_1|}\,m(t) + \frac{2K}{3|B_1|}\,\varepsilon(t)^{\frac{3}{2}}\right.\\
&\quad\left. + \left(2C_3 + KC_3 + \frac{2}{3|B_1|}\right)\varepsilon(t)^{\frac{1}{2}}\delta(t)\right).
\end{aligned}$$

At this point, defining $C_4 := \max\{\frac{2(K+1)}{3|B_1|} + 2C_3 + KC_3, 2\}$, if

$$m(t) \le C_4\big(\varepsilon(t) + \delta(t)\big)\varepsilon(t)^{\frac{1}{2}},$$

then condition (2) holds true. Otherwise, we immediately have that

$$\frac{\int_{\widehat{\Omega}(t)} |\nabla \widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)} \widehat{u}^2\,dx} \le \left(\int_\Omega |\nabla u|^2\,dx - \left(\frac{2\lambda_0(B_1)}{3|B_1|} - 1\right) m(t)\right) \le \int_\Omega |\nabla u|^2\,dx - C_5 m(t), \quad (56)$$

for a universal constant $C_5 > 0$, therefore the first part of the third claim is verified.

On the other hand, we note that, using the L^∞ bound of u, see Lemma 2.4, the fact that $\widetilde{u} \le u$ by construction and also [40, Lemma 2.4],

$$\begin{aligned} D(\widetilde{u}^2, \widetilde{u}^2) &= D(u^2, u^2) + 2\int_{\Omega^+(t)}\int_{Q(t)} \frac{\widetilde{u}^2(x)\widetilde{u}^2(y)}{|x-y|}\,dxdy \\ &\quad + \int_{Q(t)}\int_{Q(t)} \frac{\widetilde{u}^2(x)\widetilde{u}^2(y)}{|x-y|}\,dxdy \le D(u^2,u^2) + C_{fp}\varepsilon^{\frac{3}{2}}(t). \end{aligned}$$

Then we can estimate, using the appropriate scalings,

$$\begin{aligned} \frac{D(\widehat{u}^2, \widehat{u}^2)}{\left(\int_{\widehat{\Omega}(t)} \widehat{u}^2\,dx\right)^2} &\le \frac{D(\widetilde{u}^2, \widetilde{u}^2)}{\left(\int_{\widetilde{\Omega}(t)} \widetilde{u}^2\,dx\right)^2} \left(1 - \frac{m(t)}{|B_1|} + \frac{\varepsilon(t)^{\frac{3}{2}}}{|B_1|}\right)^{-\frac{2}{3}} \\ &\le \left(1 + \frac{2}{3|B_1|} m(t)\right) \frac{D(\widetilde{u}^2, \widetilde{u}^2)}{\left(\int_{\widetilde{\Omega}(t)} \widetilde{u}^2\,dx\right)^2} \\ &\le \left(1 + \frac{2}{3|B_1|} m(t)\right)\left(1 + C_3\varepsilon^{\frac{1}{2}}(t)\delta(t)\right)\left(D(u^2,u^2) + C_{fp}\varepsilon^{\frac{3}{2}}(t)\right) \\ &\le D(u^2,u^2) + C\|u\|_{L^\infty}^2 m(t) + C_{fp}\varepsilon^{\frac{3}{2}}(t) + C_3\|u\|_{L^\infty}^2\varepsilon^{\frac{1}{2}}(t)\delta(t) \\ &\le D(u^2,u^2) + C\|u\|_{L^\infty}^2 m(t) + (C_{fp} + C_3\|u\|_{L^\infty}^2) m(t) \\ &= D(u^2,u^2) + C_6 m(t). \end{aligned} \quad (57)$$

Then, putting together (56) and (57), recalling also Remark 2.3 for the equivalence of the scale invariant energy,

$$\begin{aligned} E_q(\widehat{\Omega}(t)) &\le \frac{\int_{\widehat{\Omega}(t)} |\nabla \widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)} \widehat{u}^2\,dx} + \frac{q}{2}\frac{D(\widehat{u}^2, \widehat{u}^2)}{\left(\int_{\widehat{\Omega}(t)} \widehat{u}^2\,dx\right)^2} \\ &\le \int_\Omega |\nabla u|^2\,dx + \frac{q}{2} D(u^2,u^2) - (C_5 - \frac{q}{2}C_6) m(t) \\ &\le \int_\Omega |\nabla u|^2\,dx + \frac{q}{2} D(u^2,u^2) - \frac{C_5}{2} m(t), \end{aligned}$$

up to taking $q \le \overline{q} < \sqrt{\frac{C_5}{2C_6}}$, so that in this case condition (3) holds and the proof is concluded. □

Once we have Lemma 5.5, the rest of the proof follows as in [21] or [19] as we detail here below.

Proof of Lemma 5.1. It is enough to repeat the analogs of [19, Lemma 8.7, Lemma 8.8, Proposition 8.1 and Section 9.2], noting that it is only a geometric argument and having $\int_\Omega |\nabla u|^2\,dx$ instead of $\lambda_0(\Omega)$ does not change anything. □

Proof of Theorem 1.2. We aim to apply the surgery result Lemma 5.1 and then to employ Theorem 3.1. Precisely, first, as in Section 9.2 of [19] we select a minimizing sequence for problem (2) made of connected sets. Then, by Lemma 5.1 we select another minimizing sequence of equibounded sets. At this point we are in position to apply Theorem 3.1 and we conclude. □

6. The case of *q* large: proof of Theorem 1.5

In this final section, we show that the energy of a minimizer cannot exceed a value of order $q^{3/2}$. This is done by a simple estimate on the energy of a suitably chosen union of balls with mutual distance large enough. As a consequence, we show that for large values of q any minimizer has a bound on the diameter from below and that the ball B_1 can not be optimal. We also formulate the following conjecture, motivated by the proof of Lu-Otto [23] for the Thomas-Fermi-Dirac-Von Weizsäcker energy.

Conjecture 6.1. There exists a threshold $M > 0$ such that for $q > M$ no minimizer occurs for (2).

Proof of Theorem 1.5. We construct a competitor Ω_N made up of a suitably chosen quantity of disjoint balls with mutual distance diverging to infinity.

Let $\Omega_N = \cup_{i=1}^N B_r(x_i)$, where $|x_i - x_j|$ is diverging sufficiently fast to infinity for $i \neq j$ as $q \to \infty$. We select $N \in \mathbb{N}$ and $r > 0$ so that $|\Omega_N| = |B_1|$; this implies in particular that $Nr^3 = 1$. Calling $w_B \in H_0^1(B_1)$ the first Dirichlet eigenfunction of B_1 extended by zero to all of $\mathbb{R}^3$ and normalized with $\int_{B_1} w_B^2(z)\,dz = 1$, then we can define as test function supported on Ω_N the function $\widetilde{w}_N = \sum_{i=1}^N w_B((x - x_i)/r)$, so that

$$\int_{\Omega_N} \widetilde{w}_N^2(z)\,dz = N\int_{B_r} w_B^2(z/r)\,dz = Nr^3\int_{B_1} w_B^2(y)dy = \int_{B_1} w_B^2 = 1.$$

Thus, using the minimality of Ω_N we obtain

$$\begin{aligned} E_q(\Omega_N) \le E_q(\widetilde{w}_N, \Omega_N) &\le N\int_{B_r} |\nabla w_B(z/r)|^2\,dz \\ &\quad + \frac{Nq}{2}\int_{B_r}\int_{B_r} \frac{w_B^2(z/r)w_B^2(w/r)}{|z-w|}\,dzdw + \frac{Cq}{\min_{1\le i<j\le N}|x_i - x_j|} \\ &\le C\left(N^{2/3}\int_{B_1} |\nabla w_B|^2\,dy + \frac{q}{N^{2/3}} D(w_B^2, w_B^2)\right) \\ &= C\left(\lambda_0(B_1)N^{2/3} + \frac{q}{N^{2/3}} D(w_B^2, w_B^2)\right). \end{aligned}$$

Minimizing with respect to N for q sufficiently large universal, we obtain that the optimal number of balls to be $N = Cq^{3/4}$ for some $C = C(q) > 0$ approaching a universal constant as $q \to \infty$, leading to

$$E_q(\Omega_N) \le Cq^{1/2}$$

for all q sufficiently large. As a consequence, if Ω is an optimal set for problem (2), then we have the bound from above

$$E_q(\Omega) \le Cq^{1/2} \tag{58}$$

for all q sufficiently large.

On the other hand, we can estimate from below the energy of the ball of unit radius, for $q \ge 1$:

$$E_q(B_1) = \min_{u \in H^1_0(B_1)} \left\{ \int_\Omega |\nabla u|^2 dx + \frac{q}{2} D(u^2, u^2) : \int_{B_1} u^2 dx = 1 \right\} \ge \frac{q}{4}.$$

As a consequence, for q sufficiently large the unit ball cannot be the optimal set.

Finally, let Ω be an optimal set for problem (2). Then

$$E_q(\Omega) \ge \frac{q}{2} \int_\Omega \int_\Omega \frac{u^2(x)u^2(y)}{|x-y|}\, dxdy \ge \frac{q}{2} \int_\Omega \int_\Omega \frac{u^2(x)u^2(y)}{\mathrm{diam}(\Omega)}\, dxdy \ge \frac{1}{2} \frac{q}{\mathrm{diam}(\Omega)}.$$

Thus, thanks to (58), we deduce that

$$\mathrm{diam}(\Omega) \ge Cq^{1/2},$$

for all q sufficiently large and a universal constant $C > 0$. □

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Data availability

The paper has no associated data.

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A. The physical model and non-dimensionalization

In its dimensional form, the ground state bosonic Hartree energy for the Cooper pairs takes the form (in the SI units)

$$\mathcal{E}(u,\Omega) = \frac{N\hbar^2}{2m^*}\int_\Omega |\nabla u(x)|^2\,dx + \frac{N(N-1)e^2}{2\pi\varepsilon_0\varepsilon}\int_\Omega\int_\Omega \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy,$$

where m^* is the effective mass of a Cooper pair and $-2|e|$ is its charge, where $-|e|$ is the elementary charge, ε_0 is the vacuum permittivity, ε is the dielectric constant of the surrounding matrix (within a simplified local treatment of the dielectric), and N is the number of Cooper pairs in the island and u is a single-orbital wave function subject to the normalization

$$\int_\Omega u^2(x)\,dx = 1. \tag{59}$$

We now perform a rescaling

$$x \to Lx, \qquad u \to L^{-3/2}u,$$

that keeps the normalization condition in (59) unchanged. After some simple algebra, we arrive at

$$\mathcal{E}(L^{-3/2}u(\cdot/L), L\Omega) = \frac{N\hbar^2}{2m^*L^2}E_q(u,\Omega),$$

where

$$q = \frac{2e^2(N-1)m^*L}{\pi\hbar^2\varepsilon_0\varepsilon}.$$

With the choice of $L = \left(\frac{3V}{4\pi}\right)^{1/3} = (\frac{V}{|B_1|})^{1/3}$, we then arrive at the shape optimization problem in (2).