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The Γ-Limit of the Two-Dimensional Ohta–Kawasaki Energy. Droplet Arrangement via the Renormalized Energy

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Abstract

This is the second in a series of papers in which we derive a Γ-expansion for the two-dimensional non-local Ginzburg–Landau energy with Coulomb repulsion known as the Ohta–Kawasaki model in connection with diblock copolymer systems. In this model, two phases appear, which interact via a nonlocal Coulomb type energy. Here we focus on the sharp interface version of this energy in the regime where one of the phases has very small volume fraction, thus creating small "droplets" of the minority phase in a "sea" of the majority phase. In our previous paper, we computed the Γ-limit of the leading order energy, which yields the averaged behavior for almost minimizers, namely that the density of droplets should be uniform. Here we go to the next order and derive a next order Γ-limit energy, which is exactly the Coulombian renormalized energy obtained by Sandier and Serfaty as a limiting interaction energy for vortices in the magnetic Ginzburg–Landau model. The derivation is based on the abstract scheme of Sandier-Serfaty that serves to obtain lower bounds for 2-scale energies and express them through some probabilities on patterns via the multiparameter ergodic theorem. Thus, without appealing to the Euler–Lagrange equation, we establish for all configurations which have "almost minimal energy" the asymptotic roundness and radius of the droplets, and the fact that they asymptotically shrink to points whose arrangement minimizes the renormalized energy in some averaged sense. Via a kind of Γ-equivalence, the obtained results also yield an expansion of the minimal energy and a characterization of the zero super-level sets of the minimizers for the original Ohta–Kawasaki energy. This leads to the expectation of seeing triangular lattices of droplets as energy minimizers.

1. Introduction

This is our second paper devoted to the Γ-convergence study of the two-dimensional Ohta–Kawasaki energy functional [28] in two space dimensions in the regime near the onset of non-trivial minimizers. The energy functional has the following form:

$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right) \mathrm{d}x + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x, y)(u(y) - \bar{u}) \, \mathrm{d}x \, \mathrm{d}y, \tag{1.1}$$

where Ω is the domain occupied by the material, $u : \Omega \to \mathbb{R}$ is the scalar order parameter, $V(u)$ is a symmetric double-well potential with minima at $u = \pm 1$, such as the usual Ginzburg–Landau potential $V(u) = \frac{9}{32}(1 - u^2)^2$ (for simplicity, the overall coefficient in V is chosen to make the associated surface tension constant to be equal to ε, that is, we have $\int_{-1}^{1} \sqrt{2V(u)} \, du = 1$; see also the discussion at the beginning of [18, Sec. 3]), $\varepsilon > 0$ is a parameter characterizing interfacial thickness, $\bar{u} \in (-1, 1)$ is the background charge density, and G_0 is the Neumann Green's function of the Laplacian, that is, G_0 solves

$$-\Delta G_0(x, y) = \delta(x - y) - \frac{1}{|\Omega|}, \qquad \int_\Omega G_0(x, y) \, \mathrm{d}x = 0, \tag{1.2}$$

where Δ is the Laplacian in x and $\delta(x)$ is the Dirac delta-function, with Neumann boundary conditions. Note that u is also assumed to satisfy the "charge neutrality" condition

$$\frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x = \bar{u}. \tag{1.3}$$

For a discussion of the motivation and the main quantitative features of this model, see our first paper [18], as well as [25,26]. For specific applications to physical systems, we refer the reader to [15,17,22,24,25,27,28,41].

In our first paper [18], we established the leading order term in the Γ-expansion of the energy in (1.1) in the scaling regime corresponding to the threshold between trivial and non-trivial minimizers. More precisely, we studied the behavior of the energy as $\varepsilon \to 0$ when the background charge density scales like

$$\bar{u}^\varepsilon := -1 + \varepsilon^{2/3} |\ln \varepsilon|^{1/3} \bar{\delta}, \tag{1.4}$$

for some fixed $\bar{\delta} > 0$ and when Ω is a flat two-dimensional torus of side length ℓ, that is, when $\Omega = \mathbb{T}_\ell^2 = [0, \ell)^2$, with periodic boundary conditions. This is the relevant regime to observe a transition between the case in which minimizers are trivial ($u^\varepsilon \simeq -1$) and the case in which droplets of $+1$ phase appear. As follows from [18, Corollary 2.3 and Theorem 4], this transition happens for the critical value of $\bar{\delta}$ given by

$$\bar{\delta}_c := \frac{1}{2} 3^{2/3} \kappa^2, \tag{1.5}$$

where $\kappa = 1/\sqrt{V''(1)} = \frac{2}{3}$. For $\bar{\delta} > \bar{\delta}_c$, minimizers of $\mathcal{E}$ consist of many small "droplets" (regions where $u > 0$) and their number blows up as $\varepsilon \to 0$. We showed that, after a suitable rescaling the energy functional in (1.1) Γ-converges in the sense of convergence of the (suitably normalized) droplet densities, to the limit functional $E^0[\mu]$ defined for all measures $\mu \in \mathcal{M}^+(\mathbb{T}^2_\ell) \cap H^{-1}(\mathbb{T}^2_\ell)$ by:

$$E^0[\mu] = \frac{\bar{\delta}^2 \ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2}\right) \int_{\mathbb{T}^2_\ell} \mathrm{d}\mu + 2 \iint_{\mathbb{T}^2_\ell \times \mathbb{T}^2_\ell} G(x-y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y), \tag{1.6}$$

where $G(x)$ is the *screened* Green's function of the Laplacian, that is, it solves the periodic problem for the equation

$$-\Delta G + \kappa^2 G = \delta(x) \quad \text{in} \quad \mathbb{T}^2_\ell. \tag{1.7}$$

Here we note that the double integral in (1.6) is well defined (for technical details associated with this point, see [18, Lemma 3.2]).

In particular, for $\bar{\delta} > \bar{\delta}_c$ the limit energy $E^0[\mu]$ is minimized by $\mathrm{d}\mu(x) = \bar{\mu}\, \mathrm{d}x$, where

$$\bar{\mu} = \tfrac{1}{2}(\bar{\delta} - \bar{\delta}_c) \qquad \text{and} \qquad E^0[\bar{\mu}] = \tfrac{\bar{\delta}_c}{2\kappa^2}(2\bar{\delta} - \bar{\delta}_c). \tag{1.8}$$

When $\bar{\delta} \leq \bar{\delta}_c$, the limit energy is minimized by $\mu = 0$, with $E^0[0] = \bar{\delta}^2/(2\kappa^2)$. The value of $\bar{\delta} = \bar{\delta}_c$ thus serves as the threshold separating the trivial and the non-trivial minimizers of the energy in (1.1) together with (1.4) for sufficiently small ε. Above that threshold, the droplet density of energy-minimizers converges to the uniform density $\bar{\mu}$.

The key point that enables the analysis above is a kind of Γ-equivalence between the energy functional in (1.1) and its screened sharp interface analog (for general notions of Γ-equivalence or variational equivalence, see [3,8]):

$$E^\varepsilon[u] = \frac{\varepsilon}{2} \int_{\mathbb{T}^2_\ell} |\nabla u|\, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^2_\ell} \int_{\mathbb{T}^2_\ell} (u(x) - \bar{u}^\varepsilon) G(x-y)(u(y) - \bar{u}^\varepsilon)\, \mathrm{d}x\, \mathrm{d}y. \tag{1.9}$$

Here, G is the screened potential as in (1.7), and $u \in \mathcal{A}$, where

$$\mathcal{A} := BV(\mathbb{T}^2_\ell; \{-1, 1\}), \tag{1.10}$$

and we note that on the level of E^ε the neutrality condition in (1.3) has been removed. As we showed in [18] (see Remark 6.1 there), following the approach of [26], for $\mathcal{E}^\varepsilon$ given by (1.1) in which $\bar{u} = \bar{u}^\varepsilon$ and $\bar{u}^\varepsilon$ is defined in (1.4), we have

$$\min \mathcal{E}^\varepsilon = \min E^\varepsilon + O(\varepsilon^\alpha \min E^\varepsilon), \tag{1.11}$$

for some $\alpha > 0$. In particular, as was noted in the proof of [18, Theorem 4], the estimate in (1.11) allows one to identify, upon a modification on a small set, the zero super-level set of the minimizer of $\mathcal{E}^\varepsilon$ with an almost minimizer of E^ε to all orders in the expansion in $|\ln \varepsilon|^{-1}$. Therefore, in order to understand the leading order asymptotic expansion of the minimal energy $\min \mathcal{E}^\varepsilon$ in terms of $|\ln \varepsilon|^{-1}$ and the asymptotic behavior of the zero super-level sets of the minimizers, it is sufficient to obtain such an expansion for min E^ε and a characterization of almost minimizers of E^ε. This is precisely what we will do in the present paper.

In view of the discussion above, in this paper we concentrate our efforts on the analysis of the sharp interface energy E^ε in (1.9). Here we wish to extract the next order non-trivial term in the Γ-expansion of the sharp interface energy E^ε after (1.6). In contrast to [26], we will not use the Euler–Lagrange equation associated to (1.9), so our results about minimizers will also be valid for "almost minimizers" (cf. Theorem 2).

We recall that for $\varepsilon \ll 1$ the energy minimizers for E^ε and $\bar{\delta} > \bar{\delta}_c$ consist of $O(|\ln \varepsilon|)$ nearly circular droplets of radius $r \simeq 3^{1/3}\varepsilon^{1/3}|\ln \varepsilon|^{-1/3}$ uniformly distributed throughout $\mathbb{T}^2_\ell$ [26, Theorem 2.2]. This is in contrast with the study of [11,12] for a closely related energy, where the number of droplets remains bounded as $\varepsilon \to 0$, and the authors extract a limiting interaction energy for a finite number of points. By considering a regime with $\bar{\delta}$ very close to $\bar{\delta}_c$ (at a suitable rate depending of ε) we also find ourselves in a similar situation where the number of droplets remains bounded as $\varepsilon \to 0$, which can be treated in a similar way as [11,12], (see [26, Section 3.4.] for the case of minimizers). Here, we consider instead the regime where $\bar{\delta} - \bar{\delta}_c$ is of the order of a constant, which leads to an unbounded number of droplets.

By Γ-convergence, we obtained in [18, Theorem 1] the convergence of the droplet density of almost minimizers (u^ε) of E^ε:

$$\mu^\varepsilon(x) := \frac{1}{2}\varepsilon^{-2/3}|\ln \varepsilon|^{-1/3}(1 + u^\varepsilon(x)), \tag{1.12}$$

to the uniform density $\bar{\mu}$ defined in (1.8). However, this result does not say anything about the microscopic placement of droplets in the limit $\varepsilon \to 0$. In order to understand the asymptotic arrangement of droplets in an energy minimizer, our plan is to blow-up the coordinates by a factor of $\sqrt{|\ln \varepsilon|}$, which is the inverse of the scale of the typical inter-droplet distance, and to extract the next order term in the Γ-expansion of the energy in terms of the limits as $\varepsilon \to 0$ of the blown-up configurations (which will consist of an infinite number of point charges in the plane with identical charge). Because the effect of the blow-up is to make the size of the domain tend to infinity as $\varepsilon \to 0$, we will use the framework of Sandier-Serfaty introduced in [35], itself suggested by Varadhan, which allows us to obtain lower bounds for "two-scale energies" via the use of the ergodic theorem and a formalism of probabilities on patterns: it allows the derivation of limiting energies defined as averages over large boxes (as a "cell problem" in a homogenization or a thermodynamic limit") and formulated as a function of the probability with respect to the blow-up center of obtaining a given limit profile or pattern.

We will show that the arrangement of the limit point configurations is governed by the Coulombic renormalized energy W, which was introduced in [35]. That energy W was already derived as a next order Γ-limit for the magnetic Ginzburg–Landau model of superconductivity [35,36], and also for two-dimensional Coulomb gases [38]. Our results here follow the same method of [36], and yield almost identical conclusions.

The "Coulombic renormalized energy" is a way of computing a total Coulomb interaction between an infinite number of point charges in the plane, neutralized by a uniform background charge (for more details see Section 2). It is shown in [36] that its minimum is achieved. It is also shown there that the minimum among simple lattice patterns (of fixed volume) is uniquely achieved by the triangular lattice (for a closely related result, see [9]), and it is conjectured that the triangular lattice is also a global minimizer. This triangular lattice is called "Abrikosov lattice" in the context of superconductivity and is observed in experiments in superconductors [42].

The next order limit of E^ε that we shall derive below is in fact the average of the energy W over all limits of blown-up configurations (that is average with respect to the blow up center as mentioned above). Our result says that limits of blow-ups of (almost) minimizers should minimize this average of W. This permits one to distinguish between different patterns at the microscopic scale and it leads, in view of the conjecture above, to expecting to see triangular lattices of droplets (in the limit $\varepsilon \to 0$), around almost every blow-up center (possibly with defects). Note that the selection of triangular lattices was also considered in the context of the Ohta–Kawasaki energy by Chen and Oshita [9], but there they were only obtained as minimizers among simple lattice configurations consisting of non-overlapping ideally circular droplets.

It is somewhat expected that minimizers of the Ohta–Kawasaki energy in the macroscopic setting are periodic patterns in all space dimensions (in fact in the original paper [28] only periodic patterns are considered as candidates for minimizers). This fact has never been proved rigorously, except in one dimension by Müller [23] (see also [31,43]), and at the moment seems very difficult. For higher-dimensional problems, some recent results in this direction were obtained in [2,26,39] establishing equidistribution of energy in various versions of the Ohta–Kawasaki model on macroscopically large domains. Several other results [1,11,12,14,40] were also obtained to characterize the geometry of minimizers on smaller domains. The results we obtain here, in the regime of small volume fraction and in dimension two, provide more quantitative and qualitative information (since we are able to distinguish between the cost of various patterns, and have an idea of what the minimizers should be like) and a first setting where periodicity can be expected to be proved.

The Ohta–Kawasaki setting differs from that of the magnetic Ginzburg–Landau model in the fact that the droplet "charges" (that is, their volume) are all positive, in contrast with the vortex degrees in Ginzburg–Landau, which play an analogous role and can be both positive and negative integers. This makes the setting of the Ohta–Kawasaki energy somewhat simpler. However, this advantage is counterbalanced by the fact that the droplets carry some geometry and their volumes are not quantized, contrarily to the degrees in the Ginzburg–Landau model. This creates difficulties and the major difference in the proofs. In particular, we have to account for the possibility of many very small droplets or of very elongated droplets, and we have to show that the isoperimetric terms in the energy suffice to force (almost) all the droplets to be round and of fixed volume. This has to be done at the same time as the lower bound for the other terms in the energy, for example an adapted "ball construction" for non-quantized quantities has to be re-implemented, and the interplay between these two effects turns out to be delicate. Another technical point is that since we are looking at next-to-leading order terms in the energy, we need to use refined estimates for the expected radius of the droplets, containing precise logarithmic corrections.

Our paper is organized as follows. In Section 2 we formulate the problem and state our main results concerning the Γ-limit of the next order term in the energy (1.9) after the zeroth order energy derived in [18] is subtracted off. Here we also give a detailed sketch of the proof that is intended to guide the reader through the arguments, while omitting the great many technicalities. In Section 3, we derive a lower bound on this next order energy via an energy expansion as done in [18] however isolating lower order terms obtained via the process. We then proceed via a ball construction as in [19,33,35] to obtain lower bounds on this energy in Section 4 and consequently obtain an energy density bounded from below with almost the same energy via energy displacement as in [36] in Section 5. In Section 6 we obtain explicit lower bounds on this density on bounded sets in the plane in terms of the renormalized energy for a finite number of points. We are then in the appropriate setting to apply the multiparameter ergodic theorem as in [36] to extend the lower bounds obtained to global bounds, which we present at the end of Section 6. Finally the corresponding upper bound (cf. Part (ii) of Theorem 1) is presented in Section 7.

Some notations We use the notation $(u^\varepsilon) \in \mathcal{A}$ to denote sequences of functions $u^\varepsilon \in \mathcal{A}$ as $\varepsilon = \varepsilon_n \to 0$, where $\mathcal{A}$ is an admissible class. We also use the notation $\mu \in \mathcal{M}^+(\Omega)$ to denote a non-negative finite Radon measure on the domain Ω. With a slight abuse of notation, we will often speak of μ as the "density" on Ω and set $\mathrm{d}\mu(x) = \mu(x)\,\mathrm{d}x$ whenever $\mu \in L^1(\Omega)$. With some more abuse of notation, for a measurable set E we use $|E|$ to denote its Lebesgue measure, $|\partial E|$ to denote its perimeter (in the sense of De Giorgi), and $\mu(E)$ to denote $\int_E \mathrm{d}\mu$. The symbols $H^1(\Omega)$, $BV(\Omega)$, $C^k(\Omega)$ and $H^{-1}(\Omega)$ denote the usual Sobolev space, the space of functions of bounded variation, the space of k-times continuously differentiable functions, and the dual of $H^1(\Omega)$, respectively. The symbol $o_\varepsilon(1)$ stands for the quantities that tend to zero as $\varepsilon \to 0$ with the rate of convergence depending only on ℓ, $\bar{\delta}$ and κ.

2. Problem Formulation and Main Results

2.1. *Formulation and Preliminary Definitions*

In the following, we fix the parameters $\kappa > 0$, $\bar{\delta} > 0$ and $\ell > 0$, and work with the energy E^ε in (1.9), which can be equivalently rewritten in terms of the connected components Ω_i^ε of the family of sets of finite perimeter $\Omega^\varepsilon := \{u^\varepsilon = +1\}$, where $(u^\varepsilon) \in \mathcal{A}$ are almost minimizers of E^ε, for sufficiently small ε (see also the discussion in [18, Sec. 3]). The sets Ω^ε can be decomposed into countable unions of connected disjoint sets, that is, $\Omega^\varepsilon = \bigcup_i \Omega_i^\varepsilon$, whose boundaries $\partial\Omega_i^\varepsilon$ are rectifiable and can be decomposed (up to negligible sets) into countable unions of disjoint simple closed curves. Then the density μ^ε in (1.12) can be rewritten as

$$\mu^\varepsilon(x) := \varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}\sum_i \chi_{\Omega_i^\varepsilon}(x), \tag{2.1}$$

where $\chi_{\Omega_i^\varepsilon}$ are the characteristic functions of Ω_i^ε. Motivated by the scaling analysis in the discussion preceding equation (1.12), we define the rescaled areas and perimeters of the droplets:

$$A_i^\varepsilon := \varepsilon^{-2/3}|\ln\varepsilon|^{2/3}|\Omega_i^\varepsilon|, \qquad P_i^\varepsilon := \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}|\partial\Omega_i^\varepsilon|. \tag{2.2}$$

Using these definitions, we obtain (see [18,26]) the following equivalent definition of the energy of the family (u^ε):

$$E^\varepsilon[u^\varepsilon] = \varepsilon^{4/3}|\ln\varepsilon|^{2/3}\left(\frac{\bar\delta^2\ell^2}{2\kappa^2} + \bar E^\varepsilon[u^\varepsilon]\right), \tag{2.3}$$

where

$$\bar E^\varepsilon[u^\varepsilon] := \frac{1}{|\ln\varepsilon|}\sum_i\left(P_i^\varepsilon - \frac{2\bar\delta}{\kappa^2}A_i^\varepsilon\right) + 2\iint_{\mathbb{T}_\ell^2\times\mathbb{T}_\ell^2} G(x-y)\mathrm{d}\mu^\varepsilon(x)\mathrm{d}\mu^\varepsilon(y). \tag{2.4}$$

Also note the relation

$$\mu^\varepsilon(\mathbb{T}_\ell^2) = \frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon. \tag{2.5}$$

As was shown in [18,26], in the limit $\varepsilon \to 0$ the minimizers of E^ε are non-trivial if and only if $\bar\delta > \bar\delta_c$, and we have asymptotically

$$\min E^\varepsilon \simeq \frac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c)\varepsilon^{4/3}|\ln\varepsilon|^{2/3}\ell^2 \qquad \text{as } \varepsilon \to 0. \tag{2.6}$$

Furthermore, if μ^ε is as in (2.1) and we let v^ε be the unique solution of

$$-\Delta v^\varepsilon + \kappa^2 v^\varepsilon = \mu^\varepsilon \qquad \text{in} \quad W^{2,p}(\mathbb{T}_\ell^2), \tag{2.7}$$

for any $p < \infty$, then we have

$$v^\varepsilon \rightharpoonup \bar v := \frac{1}{2\kappa^2}(\bar\delta - \bar\delta_c) \qquad \text{in} \quad H^1(\mathbb{T}_\ell^2). \tag{2.8}$$

To extract the next order terms in the Γ-expansion of E^ε we, therefore, subtract this contribution from E^ε to define a new rescaled energy F^ε (per unit area):

$$\begin{aligned} F^\varepsilon[u] := {} & \varepsilon^{-4/3}|\ln\varepsilon|^{1/3}\ell^{-2}E^\varepsilon[u] - |\ln\varepsilon|\frac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c) \\ & + \frac{1}{4\cdot 3^{1/3}}(\bar\delta - \bar\delta_c)(\ln|\ln\varepsilon| + \ln 9). \end{aligned} \tag{2.9}$$

Note that we also added the third term into the bracket in the right-hand side of (2.9) to subtract the next-to-leading order contribution of the droplet self-energy, and we have scaled F^ε in a way that allows us to extract a non-trivial $O(1)$ contribution to the minimal energy (see details in Section 3). The main result of this paper is in fact to establish Γ-convergence of F^ε to the renormalized energy W which we now define.

In [36], the renormalized energy W was introduced and defined in terms of the superconducting current j, which is particularly suited to studying the magnetic Ginzburg–Landau model of superconductivity. Here, instead, we give an equivalent definition, which is expressed in terms of the limiting electrostatic potential of the charged droplets, after blow-up, which is the limit of some proper rescaling of v^ε (see below). However, this limiting electrostatic potential will only be known up to additive constants, due to the fact that we will take limits over larger and larger tori. This issue can be dealt with in a natural way by considering *equivalence classes* of potentials, whereby two potentials differing by a constant are not distinguished:

$$[\varphi] := \{\varphi + c \mid c \in \mathbb{R}\}. \tag{2.10}$$

This definition turns the homogeneous spaces $\dot{W}^{1,p}(\mathbb{R}^d)$ into Banach spaces of equivalence classes of functions in $W^{1,p}_{\text{loc}}(\mathbb{R}^d)$ defined in (2.10) (see, for example, [29]). Here we similarly define the local analog of the homogeneous Sobolev spaces as

$$\dot{W}^{1,p}_{\text{loc}}(\mathbb{R}^2) := \left\{[\varphi] \mid \varphi \in W^{1,p}_{\text{loc}}(\mathbb{R}^2)\right\}, \tag{2.11}$$

with the notion of convergence to be that of the L^p_{loc} convergence of gradients. In the following, we will omit the brackets in $[\cdot]$ to simplify the notation and will write $\varphi \in \dot{W}^{1,p}_{\text{loc}}(\mathbb{R}^2)$ to imply that φ is any member of the equivalence class in (2.10).

We define the admissible class of the renormalized energy as follows:

Definition 2.1. For given $m > 0$ and $p \in (1, 2)$, we say that φ belongs to the admissible class $\mathcal{A}_m$, if $\varphi \in \dot{W}^{1,p}_{\text{loc}}(\mathbb{R}^2)$ and φ solves distributionally

$$-\Delta\varphi = 2\pi \sum_{a\in\Lambda} \delta_a - m, \tag{2.12}$$

where $\Lambda \subset \mathbb{R}^2$ is a discrete set and

$$\lim_{R\to\infty} \frac{2}{R^2} \int_{B_R(0)} \sum_{a\in\Lambda} \delta_a(x)\, \mathrm{d}x = m. \tag{2.13}$$

Remark 2.2. Observe that if $\varphi \in \mathcal{A}_m$, then for every $x \in B_R(0)$ we have

$$\varphi(x) = \sum_{a\in\Lambda_R} \ln|x-a|^{-1} + \varphi_R(x), \tag{2.14}$$

where $\Lambda_R := \Lambda \cap \bar{B}_R(0)$ is a finite set of distinct points and $\varphi_R \in C^\infty(\mathbb{R}^2)$ is analytic in $B_R(0)$. In particular, the definition of $\mathcal{A}_m$ is independent of p.

We next define the renormalized energy.

Definition 2.3. For a given $\varphi \in \bigcup_{m>0} \mathcal{A}_m$, the renormalized energy W of φ is defined as

$$W(\varphi) := \limsup_{R\to\infty} \lim_{\eta\to 0} \frac{1}{|K_R|}\left(\int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda} B_\eta(a)} \frac{1}{2}|\nabla\varphi|^2\chi_R \,\mathrm{d}x + \pi\ln\eta \sum_{a\in\Lambda}\chi_R(a)\right), \tag{2.15}$$

where $K_R = [-R, R]^2$, χ_R is a smooth cutoff function with the properties that $0 < \chi_R < 1$, in $K_R\setminus(\partial K_R \cup K_{R-1})$, $\chi_R(x) = 1$ for all $x \in K_{R-1}$, $\chi_R(x) = 0$ for all $x \in \mathbb{R}^2\setminus K_R$, and $|\nabla\chi_R| \le C$ for some $C > 0$ independent of R.

Various properties of W are established in [36], we refer the reader to that paper. The most relevant to us here are

1. $\min_{\mathcal{A}_m} W$ is achieved for each $m > 0$.
2. If $\varphi \in \mathcal{A}_m$ and $\varphi'(x) := \varphi(\frac{x}{\sqrt{m}})$, then $\varphi' \in \mathcal{A}_1$ and

$$W(\varphi) = m\left(W(\varphi') - \frac{1}{4}\ln m\right), \tag{2.16}$$

hence

$$\min_{\mathcal{A}_m} W = m\left(\min_{\mathcal{A}_1} W - \frac{1}{4}\ln m\right).$$

3. W is minimized over potentials in $\mathcal{A}_1$ generated by charge configurations Λ consisting of simple lattices by the potential of a triangular lattice, that is [36, Theorem 2 and Remark 1.5],

$$\min_{\substack{\varphi\in\mathcal{A}_1\\ \Lambda \text{ simple lattice}}} W(\varphi) = W(\varphi^\triangle) = -\frac{1}{2}\ln(\sqrt{2\pi b}\,|\eta(\tau)|^2) \simeq -0.2011,$$

where $\tau = a + ib$, $\eta(\tau) = q^{1/24}\prod_{n\ge1}(1 - q^n)$ is the Dedekind eta function, $q = \mathrm{e}^{2\pi i\tau}$, $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$ are real numbers such that $\Lambda^*_\triangle = \frac{1}{\sqrt{2\pi b}}((1, 0)\mathbb{Z} \oplus (a, b)\mathbb{Z})$ is the dual lattice to a triangular lattice $\Lambda^\triangle$ whose unit cell has area 2π, and $\varphi^\triangle$ solves (2.12) with $\Lambda = \Lambda^\triangle$.

In particular, from property 2 above it is easy to see that the role of m in the definition of W is inconsequential.

2.2. Main Results

Let $\ell^\varepsilon := |\ln\varepsilon|^{1/2}\ell$. For a given $u^\varepsilon \in \mathcal{A}$, we then introduce the potential (recall that φ^ε is a representative in the equivalence class defined in (2.10))

$$\varphi^\varepsilon(x) := 2\cdot 3^{-2/3}|\ln\varepsilon|\,\tilde{v}^\varepsilon(x|\ln\varepsilon|^{-1/2}), \tag{2.17}$$

where $\tilde{v}^\varepsilon$ is a periodic extension of v^ε from $\mathbb{T}^2_{\ell^\varepsilon}$ to the whole of $\mathbb{R}^2$. The notion of convergence for which we will establish Γ-convergence will involve the following embedding i of $\mathcal{A}$ into the set of probability measures on $\dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$: for each u^ε we define $i(u^\varepsilon)$ to be the probability P^ε which is the pushforward of the normalized uniform measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by the map $x \mapsto \varphi^\varepsilon(x+\cdot)$, where φ^ε is as in (2.17), or equivalently

$$i(u^\varepsilon) := P^\varepsilon = \fint_{\mathbb{T}^2_{\ell^\varepsilon}} \delta_{\varphi^\varepsilon(x+\cdot)}\, \mathrm{d}x. \tag{2.18}$$

It will also involve the analogous (modulo composition with the Laplacian) embedding j from $\mathcal{A}$ to the set of probability measures on $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2)$

$$j(u^\varepsilon) := Q^\varepsilon = \fint_{\mathbb{T}^2_{\ell^\varepsilon}} \delta_{-\Delta\varphi^\varepsilon(x+\cdot)}\, \mathrm{d}x. \tag{2.19}$$

Note that the notion of Γ-convergence can easily be generalized to be defined with respect to the convergence of nonlinear functions of the argument (see for example [20,34]). Here we will consider the convergence of the $i(u^\varepsilon)$ (for the weak topology on probability measures) for the lower bound, and the convergence of $j(u^\varepsilon)$ for the upper bound.

We also define $\mathcal{P}$ to be the family of translation-invariant probability measures on $\dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ concentrated on $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$.

Theorem 1. (Γ-convergence of F^ε) *Fix $\kappa > 0$, $\bar{\delta} > \bar{\delta}_c$, $p \in (1,2)$ and $\ell > 0$, and let F^ε be defined by (2.9). Then, as $\varepsilon \to 0$ we have*

$$F^\varepsilon \xrightarrow{\Gamma} F^0[P] := 3^{4/3} \int W(\varphi)\, \mathrm{d}P(\varphi) + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}, \tag{2.20}$$

where $P \in \mathcal{P}$. More precisely:

(i) (*Lower Bound*) *Let $(u^\varepsilon) \in \mathcal{A}$ be such that*

$$\limsup_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] < +\infty, \tag{2.21}$$

and let P^ε be the probability measure on $\dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ which is the pushforward of the normalized uniform measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by the map $x \mapsto \varphi^\varepsilon(x+\cdot)$, where φ^ε is as in (2.17). Then, upon extraction of a subsequence, (P^ε) converges weakly to some $P \in \mathcal{P}$, in the sense of measures on $\dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ and

$$\liminf_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] \geq F^0[P]. \tag{2.22}$$

(ii) (*Upper Bound*) *Conversely, for any probability measure $P \in \mathcal{P}$, letting Q be its push-forward under $-\Delta$, there exists $(u^\varepsilon) \in \mathcal{A}$ such that letting Q^ε be the push-forward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x \mapsto -\Delta\varphi^\varepsilon(x+\cdot)$,* *where* φ^ε *is as in* (2.17)*, we have* $Q^\varepsilon \rightharpoonup Q$*, in the sense of measures on* $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2)$*, and*

$$\limsup_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \le F^0[P], \tag{2.23}$$

as $\varepsilon \to 0$*.*

We will prove that the minimum of F^0 is achieved. Moreover, it is achieved for any $P \in \mathcal{P}$ which is concentrated on minimizers of $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar\delta - \bar\delta_c)$.

Remark 2.4. The phrasing of the theorem does not exactly fit the framework of Γ-convergence, since the lower bound result and the upper bound result are not expressed with the same notion of convergence. However, since weak convergence of P_ε to P implies weak convergence of Q_ε to Q, the theorem implies a result of Γ-convergence in the sense of the weak convergence of $j(u^\varepsilon)$ as probabilities.

The next theorem expresses the consequence of Theorem 1 for almost minimizers:

Theorem 2. *Let* $m = 3^{-2/3}(\bar\delta - \bar\delta_c)$ *and let* $(u^\varepsilon) \in \mathcal{A}$ *be a family of almost minimizers of* F^0*, that is, let*

$$\lim_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] = \min_{P\in\mathcal{P}} F^0[P].$$

Then, if P *is the limit measure from Theorem* 1*,* P*-almost every* φ *minimizes* W *over* $\mathcal{A}_m$*. In addition*

$$\min_{P\in\mathcal{P}} F^0[P] = 3^{4/3} \min_{\varphi\in\mathcal{A}_m} W(\varphi) + \frac{3^{2/3}(\bar\delta - \bar\delta_c)}{8}. \tag{2.24}$$

Note that the formula in (2.24) is not totally obvious, since the probability measure concentrated on a single minimizer $\varphi \in \mathcal{A}_m$ of W does not belong to $\mathcal{P}$.

Let us point out that by the arguments in the proof of [18, Theorem 4] the statement of Theorem 2 should hold for some superlevel sets of the non-trivial minimizers (u^ε) of $\mathcal{E}^\varepsilon$. In addition, the result in Theorem 2 allows us to establish the expansion of the minimal value of the original energy $\mathcal{E}^\varepsilon$ by combining it with (2.9) and (1.11). Thus, we have the following result.

Theorem 3. (Asymptotic behavior of minimizers of $\mathcal{E}^\varepsilon$) *Let* $V = \frac{9}{32}(1-u^2)^2$, $\kappa = \frac{2}{3}$ *and* $m = 3^{-2/3}(\bar\delta - \bar\delta_c)$*. Fix* $\bar\delta > \bar\delta_c$ *and* $\ell > 0$*, let* $\mathcal{E}^\varepsilon$ *be defined by* (1.1) *with* $\bar u = \bar u^\varepsilon$ *from* (1.4)*, let* u^ε *be a minimizer of* $\mathcal{E}^\varepsilon$ *over* $\mathcal{A}^\varepsilon$*, and let*

$$u_0^\varepsilon(x) := \begin{cases} +1, & u^\varepsilon(x) > 0, \\ -1, & u^\varepsilon(x) \le 0, \end{cases} \qquad \forall x \in \mathbb{T}^2_\ell. \tag{2.25}$$

Then, as $\varepsilon \to 0$ *we have:*

(i) (*Asymptotic expansion of the minimal energy*)

$$\begin{aligned}\ell^{-2}\mathcal{E}^\varepsilon[u^\varepsilon] = {} & \frac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c)\varepsilon^{4/3}|\ln\varepsilon|^{2/3} \\ & -\frac{1}{4\cdot 3^{1/3}}(\bar\delta - \bar\delta_c)\varepsilon^{4/3}|\ln\varepsilon|^{-1/3}(\ln|\ln\varepsilon| + \ln 9) \\ & +\varepsilon^{4/3}|\ln\varepsilon|^{-1/3}\left(3^{4/3}\min_{\varphi\in\mathcal{A}_m} W(\varphi) + \frac{3^{2/3}(\bar\delta - \bar\delta_c)}{8}\right) \\ & +o(\varepsilon^{4/3}|\ln\varepsilon|^{-1/3}). \end{aligned} \tag{2.26}$$

(ii) (*Behavior of zero superlevel sets of the minimizes*) *There exists* $\tilde u^\varepsilon \in \mathcal{A}$ *such that*

$$\|\tilde u^\varepsilon - u_0^\varepsilon\|_{L^1(\mathbb{T}_\ell^2)} = O(\varepsilon^{2/3+\alpha}), \tag{2.27}$$

for some $\alpha > 0$*, and* $(\tilde u^\varepsilon) \in \mathcal{A}$ *satisfies the assumptions of Theorem 2.*

The proof of Theorem 3 is obtained by a straightforward adaptation of the arguments in the proof of [18, Theorem 4].

Remark 2.5. The estimate in (2.27) implies that $\tilde u^\varepsilon$ and u_0^ε differ on a set whose area is much smaller than the area of one optimal droplet. Recall that the latter is of order $\varepsilon^{2/3}|\ln\varepsilon|^{-2/3}$.

As mentioned above, the Γ-limit in Theorem 1 cannot be expressed in terms of a single limiting function φ, but rather it effectively averages W over all the blown-up limits of φ^ε, with respect to all the possible blow-up centers. Consequently, for almost minimizers of the energy, we cannot guarantee that each blown-up potential φ^ε converges to a minimizer of W, but only that this is true after blow-up except around points that belong to a set with asymptotically vanishing volume fraction. Indeed, one could easily imagine a configuration with some small regions where the configuration does not resemble any minimizer of W, and this would not contradict the fact of being an almost minimizer since these regions would contribute only a negligible fraction to the energy. Near all the good blow-up centers, we will know some more about the droplets: it will be shown in Theorem 4 that they are asymptotically round and of optimal radii.

2.3. Sketch of the Proof

Most of the proof consists in proving the lower bound, that is Part (i) of Theorem 1. The first step, accomplished in Section 3 is, following the ideas of [26], to extract from F^ε some positive terms involving the sizes and shapes of the droplets and which are minimized by round droplets of fixed appropriate radius, and to express the remainder in blown-up coordinates $x' = x\sqrt{|\ln\varepsilon|}$. We arrive at a lower bound which is roughly of the following form:

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq M_\varepsilon + \frac{2}{|\ln \varepsilon|}\int_{\mathbb{T}^2_{\ell\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|}|h'_\varepsilon|^2\right) \mathrm{d}x' - \frac{1}{\pi \bar{r}_\varepsilon^3} \sum_{\text{nonsmall droplets}} |A_i^\varepsilon|^2 + o_\varepsilon(1). \tag{2.28}$$

Here $\bar{r}_\varepsilon$ is the corrected expected radius of a droplet (seen at suitable scale), it is a number approaching $3^{1/3}$ as $\varepsilon \to 0$. The A_i^ε are the rescaled areas of the droplets as in (2.2). The function h'_ε is $\frac{3^{2/3}}{2}\varphi^\varepsilon$, where φ^ε is the potential defined in (2.17). Finally, the term M_ε is a sum of positive expressions, which, as announced, control the discrepancy between the droplets and the ideal round droplets of optimal sizes:

$$M_\varepsilon := \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\right) + c_1 \sum_{\text{large droplets}} A_i^\varepsilon + c_2 \sum_{\text{medium droplets}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 + c_3 \sum_{\text{small droplets}} A_i^\varepsilon. \tag{2.29}$$

Indeed, for all these terms to be small, we need the droplets to be roughly round (as controlled by the isoperimetric deficit terms in the first sum) and to all have area close to $\pi \bar{r}_\varepsilon^2$, that is, radius close to $\bar{r}_\varepsilon$.

We then consider $\ell^2 F^\varepsilon - M_\varepsilon$, and observe that it is an energy functional with no sign, which ressembles very much the one studied in [36]. Since h'_ε satisfies

$$-\Delta h'_\varepsilon + \frac{\kappa^2}{|\ln \varepsilon|} = \mu'_\varepsilon - \bar{\mu}^\varepsilon,$$

the positive term $\frac{2}{|\ln \varepsilon|}\int_{\mathbb{T}^2_{\ell\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|}|h'_\varepsilon|^2\right) \mathrm{d}x'$ corresponds to the total interaction of the droplets and the fixed "background charge" of density $\bar{\mu}^\varepsilon$, via the screened Coulomb kernel. The negative term $-\frac{1}{\pi \bar{r}_\varepsilon^3}\sum_{\text{nonsmall droplets}} |A_i^\varepsilon|^2$ in fact corresponds to subtracting off the self-energy of each droplet, provided one knows that they have the optimal size, which will be known only later. As $\varepsilon \to 0$ and the droplets become smaller and smaller, the sum of the two is thus expected to converge to the (average of the) renormalized energy W, precisely designed to be the total Coulomb energy of a set of point charges neutralized by a constant background, after "renormalizing" by subtracting off the self-interaction energy of each charge. To find this limit, we apply the strategy of [36].

Defining the energy density (up to the factor of 2):

$$f_\varepsilon := |\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|}|h'_\varepsilon|^2 - \frac{|\ln \varepsilon|}{2\pi \bar{r}_\varepsilon^3} \sum_{\text{nonsmall droplets}} |A_i^\varepsilon|^2 \frac{1}{|\Omega'_{i,\varepsilon}|}\mathbf{1}_{\Omega'_{i,\varepsilon}},$$

the energy to study is $2 ⨍_{\mathbb{T}^2_{\ell\varepsilon}} f_\varepsilon(x)\,\mathrm{d}x$. The starting point of the abstract method of [36] for obtaining lower bounds on "two-scale energies" is a sort of "continuous partition of unity" based on a simple application of Fubini's theorem: given a smooth cut-off function χ with integral 1 supported in $B_1(0)$, we may write

$$\fint_{\mathbb{T}^2_{\ell^\varepsilon}} f_\varepsilon(x)\,\mathrm{d}x = \fint_{\mathbb{T}^2_{\ell^\varepsilon}} \left(\int_{\mathbb{R}^2} \chi(x-y)\,\mathrm{d}y\right) f_\varepsilon(x)\,\mathrm{d}x$$
$$= \fint_{\mathbb{T}^2_{\ell^\varepsilon}} \int_{\mathbb{R}^2} \chi(y) f_\varepsilon(x+y)\,\mathrm{d}y\,\mathrm{d}x. \qquad (2.30)$$

This rewriting can be interpreted as the average over blow-up centers $x \in \mathbb{T}^2_{\ell^\varepsilon}$ of the local energies $\int_{\mathbb{R}^2} \chi(y) f_\varepsilon(x+y)\,\mathrm{d}y$. The method of [36] consists in obtaining a limit as $\varepsilon \to 0$ to these local energies (which are now restricted to an $O(1)$ scale, meaning integrals over an $O(1)$ size domain) of the form $\int_{\mathbb{R}^2} \chi(y) f(x+y)$ and then to retrieve a lower bound for (2.30) as the average of the limits, with respect to a measure which is nothing but P, the limit of P^ε in (2.18). Using the fact that P is translation invariant by construction, Wiener's multiparameter theorem allows us to replace this by the average with respect to P of $\lim_{R\to\infty} \frac{1}{|K_R|} \int_{\mathbb{R}^2} \chi * \mathbf{1}_{K_R} f(y)\,\mathrm{d}y$, which we need to show coincides with $W(\varphi)$. To show this, we may restrict ourselves to "good blow-up centers x" defined as those for which

$$\forall R, \quad \limsup_{\varepsilon\to 0} \fint_{K_R} f_\varepsilon(x+y) \le C \qquad (2.31)$$

for, otherwise, the desired lower bound is automatically true. We may also without loss reintroduce at this stage the energy density associated with the M_ε terms into the density f_ε. We need to show that if the integral of the energy density is locally bounded (which serves to provide a priori bounds), then it can be bounded from below by the integral of the renormalized energy density W. Using the terms of M_ε that they contain, the a priori bounds ensure that near these good blow-up centers all the droplets behave as expected: they are almost round and of optimal sizes. Once this is known, the desired lower bound then follows as in [36] from "ball construction estimates" and convergence arguments. This application of the abstract method and the proof of the local lower bounds are described in Section 6.

One of the main obstacles to the above reasoning is that the abstract scheme of [36] requires the energy density f_ε to be bounded from below by some constant independent of ε. However, f_ε is not bounded below independently of ε. This difficulty is already encountered in [36], and we use here the same fix: we show that f_ε can be transformed into an energy-density g_ε which is bounded below, at the expense of a small error. More precisely, g_ε is obtained from f_ε by "mass displacement" (or mass transport) at $O(1)$ distances, that is by absorbing the negative part of f_ε into its positive part. Th resulting g_ε is constructed such that $\|f_\varepsilon - g_\varepsilon\|_{\mathrm{Lip}^*}$ is bounded, by terms that will be relatively small, once we know that M_ε is small (in other words bounded by terms that depend on M_ε in an explicit manner).

In order to prove that this is possible, we first need to establish sharp lower bounds for the energy carried by the droplets, with an error only $o(1)$ per droplet. This is done in Section 4 via a ball construction as in [19,33,35]. Here we face two additional difficulties compared to the Ginzburg–Landau case of [36]. First we need to distinguish between the small droplets, which will be ignored during the construction and the lower bound, and the other droplets. Second, the ball construction is well suited for bounding below the energy starting from initial balls and growing them, obtaining estimates in $\ln \frac{r}{r(\mathcal{B}_0)}$, where r is a final $O(1)$ radius and $r(\mathcal{B}_0)$ is the sum of the radii of the initial balls. However, here we would like to start rather from the droplets as initial set, and we do not know yet that the droplets are almost balls, so instead we cover them by disjoint balls and lose in the estimate a factor depending on the difference between the droplets and balls, which we show can be controlled by M_ε again. This is the content of Section 4.

In order to obtain optimal lower bounds (with an error $o(1)$ per droplet in the end), as in [36] the ball construction has to be performed locally, more precisely, U_α by U_α, where $(U_\alpha)_\alpha$ is a covering of $\mathbb{R}^2$ by balls of the same size centered on a lattice, and then supplemented by lower bounds on "annuli" surrounding the U_α's, which allow one to compensate for the possible loss of energy during each ball construction. It is after this procedure only, which again introduces errors expressed in terms of M_ε, that the displacement from f_ε to g_ε can be performed. This is the content of Section 5.

As we have seen, all along the steps of Sections 4 and 5, we have to deal with error terms that are due to the possible discrepancy between the droplets and ideal round droplets of optimal size. The error terms that they create, which have to be carried throughout, can fortunately all be controlled in terms of M_ε. At the very end of the procedure, we obtain that the errors are bounded below by $-C\ln^2 M_\varepsilon$, which is compensated by the M_ε term in the energy (2.28), so that the total is bounded below by a universal constant. As a result, once an optimal upper bound for the minimal energy is obtained (this is done in Section 7, via an explicit construction of a test-configuration, following again the method of [36]), we deduce that for almost-minimizers of the energy, M_ε must be bounded, which allows us to conclude that the droplets are, for the most part, close to round and of optimal sizes (an application of Bonnesen's inequality allows us to control their difference to balls in terms of the isoperimetric terms in M_ε). Knowing that M_ε is bounded allows in the end for all the steps mentioned above to go through.

3. Derivation of the Leading Order Energy

In preparation for the proof of Theorem 1, we define

$$\rho_\varepsilon := 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{1/6} \quad \text{and} \quad \bar{r}_\varepsilon := \left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3}. \tag{3.1}$$

Recall that to leading order the droplets are expected to be circular with radius $3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}$. Thus ρ_ε is the expected radius, once we have blown up coordinates by the factor of $\sqrt{|\ln\varepsilon|}$, which will be done below. Also, we know that the expected normalized area A_i is $3^{2/3}\pi$, but this is only true up to lower order terms which were negligible in [18]; as we show below, a more precise estimate is $A_i \simeq \pi\bar{r}_\varepsilon^2$, so $\bar{r}_\varepsilon$ above can be viewed as a "corrected" normalized droplet radius. Since our estimates must be accurate up to $o_\varepsilon(1)$ per droplet and the self-energy of a droplet is of order $A_i^2 \ln\rho_\varepsilon$, we can no longer ignore these corrections.

The goal of the next subsection is to obtain an explicit lower bound for F_ε defined by (2.9) in terms of the droplet areas and perimeters, which will then be studied in Section 4 and onward. We follow the analysis of [18], but isolate higher order terms.

3.1. Energy Extraction

We begin with the original energy $\bar{E}^\varepsilon$ (cf. (2.4)) while adding and subtracting the *truncated* self interaction: first we define, for $\gamma \in (0,1)$, truncated droplet volumes by

$$\tilde{A}_i^\varepsilon := \begin{cases} A_i^\varepsilon & \text{if } A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1}, \\ (3^{2/3}\pi\gamma^{-1}|A_i^\varepsilon|)^{1/2} & \text{if } A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1}, \end{cases} \tag{3.2}$$

as in [18]. The motivation for this truncation will become clear in the proof of Proposition 5.1, when we obtain lower bounds on the energy on annuli. In [18] the self-interaction energy of each droplet extracted from $\bar{E}^\varepsilon$ was $\frac{|\tilde{A}_i^\varepsilon|^2}{3\pi|\ln\varepsilon|}$, yielding in the end the leading order energy $E^0[\mu]$ in (1.6). A more precise calculation of the self-interaction energy corrects the coefficient of $|\tilde{A}_i^\varepsilon|^2$ by an $O(\ln|\ln\varepsilon|/|\ln\varepsilon|)$ term, yielding the following corrected leading order energy for E^ε:

$$E_\varepsilon^0[\mu] := \frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} \mathrm{d}\mu + 2\iint_{\mathbb{T}_\ell^2\times\mathbb{T}_\ell^2} G(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y). \tag{3.3}$$

The energy in (3.3) is explicitly minimized by $\mathrm{d}\mu(x) = \bar{\mu}_\varepsilon\,\mathrm{d}x$ (again a correction to the previously known $\bar{\mu}$ from (1.8)) where

$$\bar{\mu}_\varepsilon := \frac{1}{2}\left(\bar{\delta} - \frac{3\kappa^2}{2\bar{r}_\varepsilon}\right) \qquad \text{for} \qquad \bar{\delta} > \frac{3\kappa^2}{2\bar{r}_\varepsilon}, \tag{3.4}$$

and

$$\min E_\varepsilon^0 = \frac{\bar{\delta}_c\ell^2}{2\kappa^2}\left\{2\bar{\delta}\left(\frac{3}{\bar{r}_\varepsilon^3}\right)^{1/3} - \bar{\delta}_c\left(\frac{3}{\bar{r}_\varepsilon^3}\right)^{2/3}\right\}. \tag{3.5}$$

Observing that $\bar{r}_\varepsilon \to 3^{1/3}$ we immediately check that

$$\bar{\mu}_\varepsilon \to \bar{\mu} \quad \text{as } \varepsilon \to 0, \tag{3.6}$$

and in addition that (3.5) converges to the second expression in (1.8). To obtain the next order term, we Taylor-expand the obtained formulas upon substituting the definition of $\bar{r}_\varepsilon$. After some algebra, we obtain

$$\ell^{-2}\min E_\varepsilon^0 = \frac{\bar{\delta}_c}{2\kappa^2}\left(2\bar{\delta}-\bar{\delta}_c\right) - \frac{1}{4\cdot 3^{1/3}}(\bar{\delta}-\bar{\delta}_c)\frac{\ln|\ln\varepsilon| + \ln 9}{|\ln\varepsilon|} + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|^2}\right). \tag{3.7}$$

Recalling once again the definition of F^ε from (2.9), we then find

$$F^\varepsilon[u^\varepsilon] = |\ln\varepsilon|(\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}E^\varepsilon[u^\varepsilon] - \ell^{-2}\min E^0_\varepsilon) + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|}\right),$$

and in view of the definition of $\bar{E}^\varepsilon$ from (2.3), we thus may write

$$F^\varepsilon[u^\varepsilon] = |\ln\varepsilon|\ell^{-2}\left(\bar{E}^\varepsilon[u^\varepsilon] + \frac{\bar{\delta}^2\ell^2}{2\kappa^2} - \min E^0_\varepsilon\right) + O\left(\frac{(\ln|\ln\varepsilon|)^2}{|\ln\varepsilon|}\right). \tag{3.8}$$

Thus obtaining a lower bound for the first term in the right-hand side of (3.8) implies, up to $o_\varepsilon(1)$, a lower bound for F^ε. This is how we proceed to prove Lemma 3.1 below.

With this in mind, we begin by setting

$$v^\varepsilon = \bar{v}^\varepsilon + \frac{h_\varepsilon}{|\ln\varepsilon|}, \qquad \bar{v}^\varepsilon = \frac{1}{2\kappa^2}\left(\bar{\delta} - \frac{3\kappa^2}{2\bar{r}_\varepsilon}\right), \tag{3.9}$$

where $\bar{v}^\varepsilon$ is the solution to (2.7) with right side equal to $\bar{\mu}_\varepsilon$ in (3.4).

3.2. Blowup of Coordinates

We now rescale the domain $\mathbb{T}^2_\ell$ by making the change of variables

$$\begin{aligned} x' &= x\sqrt{|\ln\varepsilon|}, \\ h'_\varepsilon(x') &= h_\varepsilon(x), \\ \Omega'_{i,\varepsilon} &= \Omega^\varepsilon_i\sqrt{|\ln\varepsilon|}, \\ \ell^\varepsilon &= \ell\sqrt{|\ln\varepsilon|}. \end{aligned} \tag{3.10}$$

Observe that

$$\varphi^\varepsilon(x') = 2\cdot 3^{-2/3}h'_\varepsilon(x') \qquad \forall x' \in \mathbb{T}^2_{\ell^\varepsilon}, \tag{3.11}$$

where φ^ε is defined by (2.17). It turns out to be more convenient to work with h'_ε and rescale only at the end back to φ^ε.

3.3. Main Result

We are now ready to state the main result of this section, which provides an explicit lower bound on F^ε. The strategy, in particular for dealing with droplets that are too small or too large is the same as [18], except that we need to go to higher order terms.

Proposition 3.1. *There exist universal constants* $\gamma \in (0, \frac{1}{6})$, $c_1 > 0, c_2 > 0$, $c_3 > 0$ *and* $\varepsilon_0 > 0$ *such that if* $\bar{\delta} > \bar{\delta}_c$ *and* $(u^\varepsilon) \in \mathcal{A}$ *with* $\Omega^\varepsilon := \{u^\varepsilon > 0\}$, *then for all* $\varepsilon < \varepsilon_0$

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq M_\varepsilon + \frac{2}{|\ln \varepsilon|} \int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' \\ - \frac{1}{\pi \bar{r}_\varepsilon^3} \sum_{A_i^\varepsilon \geq 3^{2/3}\pi\gamma} |\tilde{A}_i^\varepsilon|^2 + o_\varepsilon(1), \tag{3.12}$$

where $M_\varepsilon \geq 0$ *is defined by*

$$M_\varepsilon := \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right) + c_1 \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} A_i^\varepsilon \\ + c_2 \sum_{3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 + c_3 \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon. \tag{3.13}$$

Remark 3.2. Defining $\beta := 3^{2/3}\pi\gamma$, by isoperimetric inequality applied to each connected component of Ω^ε separately every term in the first sum in the definition of M_ε in (3.13) is non-negative. In particular, M_ε measures the discrepancy between the droplets Ω_i^ε with $A_i^\varepsilon \geq \beta$ and disks of radius $\bar{r}_\varepsilon$.

The proposition will be proved below, but before let us examine some of its further consequences. The result of the proposition implies that our a priori assumption $\limsup_{\varepsilon \to 0} F^\varepsilon[u^\varepsilon] < +\infty$ translates into

$$M_\varepsilon + \frac{2}{|\ln \varepsilon|} \int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' - \frac{1}{\pi \bar{r}_\varepsilon^3} \sum_{A_i^\varepsilon \geq \beta} |\tilde{A}_i^\varepsilon|^2 \leq C,$$

for some $C > 0$ independent of $\varepsilon \ll 1$, which, in view of (3.1) is also

$$M_\varepsilon + \frac{2}{|\ln \varepsilon|} \left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' - \frac{1}{2\pi} |\ln \rho_\varepsilon| \sum_{A_i^\varepsilon \geq \beta} |\tilde{A}_i^\varepsilon|^2 \right) \leq C. \tag{3.14}$$

A major goal of the next sections is to obtain the following estimate

$$\frac{1}{|\ln \varepsilon|} \left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' - \frac{1}{2\pi} |\ln \rho_\varepsilon| \sum_{A_i^\varepsilon \geq \beta} |\tilde{A}_i^\varepsilon|^2 \right) \\ \geq -C \ln^2(M_\varepsilon + 2), \tag{3.15}$$

for some $C > 0$ independent of $\varepsilon \ll 1$, so that the a priori bound (3.14) in fact implies that M_ε is uniformly bounded independently of ε for small ε. This will be used crucially in Section 6.2.

We note that $h'_\varepsilon(x')$ satisfies the equation

$$-\Delta h'_\varepsilon + \frac{\kappa^2}{|\ln\varepsilon|} h'_\varepsilon = \mu'_\varepsilon - \bar{\mu}^\varepsilon \quad \text{in } W^{2,p}(\mathbb{T}^2_{\ell^\varepsilon}) \tag{3.16}$$

where we define in $\mathbb{T}^2_{\ell^\varepsilon}$

$$\mu'_\varepsilon(x') := \sum_i A_i^\varepsilon \tilde{\delta}_i^\varepsilon(x'), \tag{3.17}$$

and

$$\tilde{\delta}_i^\varepsilon(x') := \frac{\chi_{\Omega'_{i,\varepsilon}}(x')}{|\Omega'_{i,\varepsilon}|}, \tag{3.18}$$

which will be used in what follows. Notice that each $\tilde{\delta}_i^\varepsilon(x')$ approximates the Dirac delta concentrated on some point in the support of $\Omega'_{i,\varepsilon}$ and, hence, $\mu'_\varepsilon(x')\,\mathrm{d}x'$ approximates the measure associated with the collection of point charges with magnitude A_i^ε. In particular, the measure $\mathrm{d}\mu'_\varepsilon$ evaluated over the whole torus equals the total charge: $\mu'_\varepsilon(\mathbb{T}^2_{\ell^\varepsilon}) = \sum_i A_i^\varepsilon$.

3.4. Proof of Proposition 3.1

Step 1 We are first going to show that for universally small $\varepsilon > 0$ and all $\gamma \in (0, \frac{1}{6})$ we have

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq T_1 + T_2 + T_3 + T_4 + T_5 + o_\varepsilon(1), \tag{3.19}$$

where

$$T_1 = \sum_i \left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon} \right), \tag{3.20}$$

$$T_2 = \frac{\gamma^{7/2}}{4\pi} \sum_{3^{2/3}\pi\gamma \leq A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2, \tag{3.21}$$

$$T_3 = \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon (A_i^\varepsilon - \pi\bar{r}_\varepsilon^2)^2, \tag{3.22}$$

$$T_4 = \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} \left(6^{-1}\gamma^{-1} - 1\right) A_i^\varepsilon, \tag{3.23}$$

$$T_5 = \frac{2}{|\ln\varepsilon|} \int_{\mathbb{T}^2_\ell} \left(|\nabla h_\varepsilon|^2 + \kappa^2 |h_\varepsilon|^2 \right) \mathrm{d}x - \frac{1}{\pi\bar{r}_\varepsilon^3} \sum_i |\tilde{A}_i^\varepsilon|^2. \tag{3.24}$$

To bound $F^\varepsilon[u^\varepsilon]$ from below, we start from (3.8). In particular, in view of (2.7) we may rewrite (2.4) as

$$\begin{aligned}\bar{E}^\varepsilon[u^\varepsilon] &= \frac{1}{|\ln\varepsilon|}\sum_i\left(P_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon\right) + 2\int_{\mathbb{T}_\ell^2}(|\nabla v^\varepsilon|^2 + \kappa^2|v^\varepsilon|^2)\,\mathrm{d}x \\ &= \frac{1}{|\ln\varepsilon|}\sum_i\left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\right) \\ &\quad + \frac{1}{|\ln\varepsilon|}\sum_i\left(\sqrt{4\pi A_i^\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon + \frac{1}{\pi\bar{r}_\varepsilon^3}|\tilde{A}_i^\varepsilon|^2\right) \qquad (3.25) \\ &\quad + 2\int_{\mathbb{T}_\ell^2}(|\nabla v^\varepsilon|^2 + \kappa^2|v^\varepsilon|^2)\,\mathrm{d}x - \frac{1}{\pi\bar{r}_\varepsilon^3|\ln\varepsilon|}\sum_i|\tilde{A}_i^\varepsilon|^2. \qquad (3.26)\end{aligned}$$

We start by focusing on (3.25). First, in the case $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$ we have $|\tilde{A}_i^\varepsilon|^2 = 3^{2/3}\pi\gamma^{-1}A_i^\varepsilon$ and hence, recalling that $\bar{r}_\varepsilon = 3^{1/3} + o_\varepsilon(1)$, where $o_\varepsilon(1)$ depends only on ε, we have for ε universally small and $\gamma < \frac{1}{6}$:

$$\begin{aligned}\frac{|\tilde{A}_i^\varepsilon|^2}{\pi\bar{r}_\varepsilon^3} &= \frac{A_i^\varepsilon}{\pi\bar{r}_\varepsilon^3}(3^{2/3}\pi\gamma^{-1} - 3\pi\bar{r}_\varepsilon^2 + 3\pi\bar{r}_\varepsilon^2) = A_i^\varepsilon\left(\frac{3}{\bar{r}_\varepsilon} + \frac{3^{2/3}}{\bar{r}_\varepsilon^3}\left(\gamma^{-1} - 3\left(\frac{\bar{r}_\varepsilon}{3^{1/3}}\right)^2\right)\right) \\ &\geq A_i^\varepsilon\left(\frac{3}{\bar{r}_\varepsilon} + \frac{1}{6}\left(\gamma^{-1} - 6\right)\right). \qquad (3.27)\end{aligned}$$

We conclude that for $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$, we have

$$\left(\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi\bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon\right) \geq \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} + \frac{1}{6}(\gamma^{-1} - 6)\right)A_i^\varepsilon. \qquad (3.28)$$

On the other hand, when $A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}$ we have $\tilde{A}_i^\varepsilon = A_i^\varepsilon$ and we proceed as follows. Let us begin by defining, similarly to [18], the function

$$f(x) = \frac{2\sqrt{\pi}}{\sqrt{x}} + \frac{x}{\pi\bar{r}_\varepsilon^3}$$

for $x \in (0, +\infty)$ and observe that f is convex and attains its minimum of $\frac{3}{\bar{r}_\varepsilon}$ at $x = \pi\bar{r}_\varepsilon^2$, with

$$f''(x) = \frac{3\sqrt{\pi}}{2x^{5/2}} > 0.$$

By a second order Taylor expansion of f around $\pi\bar{r}_\varepsilon^2$, using the fact that f'' is decreasing on $(0, +\infty)$, we then have for all $x \leq x_0$

$$\sqrt{4\pi x} + \frac{x^2}{\pi\bar{r}_\varepsilon^3} = xf(x) \geq x\left(\frac{3}{\bar{r}_\varepsilon} + \frac{3\sqrt{\pi}}{4x_0^{5/2}}(x - \pi\bar{r}_\varepsilon^2)^2\right). \qquad (3.29)$$

We, hence, conclude that when $3^{2/3}\pi\gamma \le A_i^\varepsilon \le 3^{2/3}\pi\gamma^{-1}$, we have

$$\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi \bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon \ge \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right) A_i^\varepsilon + \frac{\gamma^{5/2}}{4\pi^2 \cdot 3^{2/3}} A_i^\varepsilon (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2, \tag{3.30}$$

and when $A_i^\varepsilon < 3^{2/3}\pi\gamma$, we have

$$\sqrt{4\pi A_i^\varepsilon} + \frac{|\tilde{A}_i^\varepsilon|^2}{\pi \bar{r}_\varepsilon^3} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon \ge \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right) A_i^\varepsilon + \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} A_i^\varepsilon (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2, \tag{3.31}$$

Combining (3.28), (3.30) and (3.31), summing over all i, and distinguishing the different cases, we can now bound (3.25) from below as follows:

$$\begin{aligned}\sum_i \left(\sqrt{4\pi A_i^\varepsilon} - \frac{2\bar{\delta}}{\kappa^2} A_i^\varepsilon + \frac{1}{\pi \bar{r}_\varepsilon^3} |\tilde{A}_i^\varepsilon|^2\right) &\ge \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right) \sum_i A_i^\varepsilon \\ &+ \frac{\gamma^{7/2}}{4\pi} \sum_{3^{2/3}\pi\gamma \le A_i^\varepsilon \le 3^{2/3}\pi\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 + \frac{\gamma^{-5/2}}{4\pi^2 \cdot 3^{2/3}} \\ &\times \sum_{A_i^\varepsilon < 3^{2/3}\pi\gamma} A_i^\varepsilon (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 + \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} (6^{-1}\gamma^{-1} - 1) A_i^\varepsilon. \end{aligned} \tag{3.32}$$

We now focus on the term in (3.26). Using (3.9), we can write the integral in (3.26) as:

$$\begin{aligned} 2\int_{\mathbb{T}_\ell^2} (\nabla v^\varepsilon|^2 + \kappa^2 |v^\varepsilon|^2)\,\mathrm{d}x = \frac{2}{|\ln\varepsilon|^2} \int_{\mathbb{T}_\ell^2} (|\nabla h_\varepsilon|^2 + \kappa^2 h_\varepsilon^2)\,\mathrm{d}x \\ + \frac{4\kappa^2 \bar{v}^\varepsilon}{|\ln\varepsilon|} \int_{\mathbb{T}_\ell^2} h_\varepsilon\,\mathrm{d}x + 2\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2. \end{aligned} \tag{3.33}$$

Integrating (2.7) over $\mathbb{T}_\ell^2$ and recalling the definition of h_ε in (3.9), as well as (2.5), leads to

$$\frac{4\kappa^2 \bar{v}^\varepsilon}{|\ln\varepsilon|} \int_{\mathbb{T}_\ell^2} h_\varepsilon\,\mathrm{d}x = \frac{4\bar{v}^\varepsilon}{|\ln\varepsilon|} \sum_i A_i^\varepsilon - 4\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2. \tag{3.34}$$

Combining (3.33) and (3.34), we then find

$$\begin{aligned} 2\int_{\mathbb{T}_\ell^2} (|\nabla v^\varepsilon|^2 + \kappa^2 |v^\varepsilon|^2)\,\mathrm{d}x = \frac{2}{|\ln\varepsilon|^2} \int_{\mathbb{T}_\ell^2} (|\nabla h_\varepsilon|^2 + \kappa^2 h_\varepsilon^2)\,\mathrm{d}x \\ - \frac{1}{|\ln\varepsilon|} \left(\frac{3}{\bar{r}_\varepsilon} - \frac{2\bar{\delta}}{\kappa^2}\right) \sum_i A_i^\varepsilon - 2\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2. \end{aligned} \tag{3.35}$$

Also, by direct computation using (3.5) and (3.9) we have

$$2\kappa^2 |\bar{v}^\varepsilon|^2 \ell^2 = \frac{\bar{\delta}^2 \ell^2}{2\kappa^2} - \min E_\varepsilon^0. \tag{3.36}$$

Therefore, combining this with (3.8), (3.32) and (3.35), after passing to the rescaled coordinates and performing the cancellations we find that

$$\ell^2 F^\varepsilon[u^\varepsilon] \geq T_1+T_2+T_3+T_4+\frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}}\left(|\nabla h'_\varepsilon(x')|^2+\frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon(x')|^2\right)\mathrm{d}x' \\ -\frac{1}{\pi\bar r_\varepsilon^3}\sum_i |\tilde A_i^\varepsilon|^2 + o_\varepsilon(1), \tag{3.37}$$

which is nothing but (3.19).

Step 2 We proceed to absorbing the contributions of the small droplets in (3.24) by (3.21). To that effect, we observe that, for the function

$$\Phi_\varepsilon(x) := \frac{\gamma^{-5/2}}{4\pi^2\cdot 3^{2/3}}x(x-\pi\bar r_\varepsilon^2)^2 - \frac{1}{\bar r_\varepsilon^3}x^2 \\ \geq \frac{\gamma^{-5/2}x}{4\pi^2\cdot 3^{2/3}}\left\{\pi^2\bar r_\varepsilon^4 - \left(2\pi\bar r_\varepsilon^2+\frac{\gamma^{5/2}}{\bar r_\varepsilon^3}\right)x\right\}, \tag{3.38}$$

there exists a universal $\gamma \in (0, \frac{1}{6})$ such that $\Phi_\varepsilon(x) \geq x$ whenever $0 \leq x < 3^{2/3}\pi\gamma$ and ε is universally small. Using this observation, we may absorb all the terms with $A_i^\varepsilon < 3^{2/3}\pi\gamma$ appearing in the second term in (3.24) into (3.22) by suitably reducing the coefficient in front of the latter. This proves the result. □

4. Ball Construction

The goal of this section is to show (3.15) using the abstract framework of Theorem 3 in [36]. The difficulty in doing this, as in the case of the Ginzburg–Landau model treated in [36], is that the energy density $e'_\varepsilon - \frac{1}{\pi}|\ln\rho_\varepsilon|\sum_{A_i^\varepsilon\geq\beta}|\tilde A_i^\varepsilon|^2\tilde\delta_i^\varepsilon$ is not positive (or bounded below independently of (u^ε)). The next two subsections are meant to get around this difficulty by showing that this energy density can be modified, by displacing a part of the energy from the regions where the energy density is positive into regions where the energy density is negative in order to bound the modified energy density from below while making only a small enough error. This is achieved by obtaining sharp lower bounds on the energy of the droplets. Since their volumes and shapes are a priori unknown, the terms in M_ε are used to control in a quantitative way the deviations from the droplets being balls of fixed volume.

In this section we perform a ball construction which follows the procedure of [36]. The goal is to cover the droplets $\{\Omega'_{i,\varepsilon}\}$ whose volumes are bounded from below by a given $\beta > 0$ with a finite collection of disjoint closed balls whose radii are smaller than 1, on which we have a good lower bound for the energy in the left-hand side of (3.15). This is possible for sufficiently small ε in view of the fact that $\ell^\varepsilon \to \infty$ and that the leading order asymptotic behavior of the energy from (2.6) yields control on the perimeter and, therefore, the essential diameter of each of $\Omega'_{i,\varepsilon}$. The precise statements are given below. We will also need the following basic result, which holds for sufficiently small ε ensuring that the droplets are smaller than the sidelength of the torus (see [18, Lemma 3.1]).

Lemma 4.1. *There exists* $\varepsilon_0 > 0$ *depending only on* ℓ, κ, $\bar{\delta}$ *and* $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ *such that for all* $\varepsilon \leq \varepsilon_0$ *we have*

$$\operatorname{ess\,diam}(\Omega'_{i,\varepsilon}) \leq c|\partial\Omega'_{i,\varepsilon}|, \tag{4.1}$$

for some universal $c > 0$.

From now on and for the rest of the paper we fix γ to be the constant given in Proposition 3.1 and, as in the previous section, we define $\beta = 3^{2/3}\pi\gamma$. We also introduce the following notation which will be used repeatedly below. To index the droplets, we will use the following definitions:

$$I_\beta := \{i \in \mathbb{N} : A_i^\varepsilon \geq \beta\}, \quad I_E := \{i \in \mathbb{N} : |\Omega'_{i,\varepsilon} \cap (\mathbb{T}^2_{\ell^\varepsilon} \backslash E)| = 0\},$$
$$I_{\beta,E} := I_\beta \cap I_E, \tag{4.2}$$

where $E \subset \mathbb{T}^2_{\ell^\varepsilon}$. I_E essentially corresponds to the droplets that are in E, and $I_{\beta,E}$ to the nonsmall droplets that are in E. For a collection of balls $\mathcal{B}$, the number $r(\mathcal{B})$ (also called the total radius of the collection) denotes the sum of the radii of the balls in $\mathcal{B}$. For simplicity, we will say that a ball B *covers* $\Omega'_{i,\varepsilon}$, if $i \in I_B$.

The principle of the ball construction introduced by Jerrard [19] and Sandier [33] and adapted to the present situation is to start from an initial set, here $\bigcup_{i\in I_{\beta,U}} \Omega'_{i,\varepsilon}$ for a given $U \subseteq \mathbb{T}^2_{\ell^\varepsilon}$ and cover it by a union of finitely many closed balls with sufficiently small radii. This collection can then be transformed into a collection of *disjoint* closed balls by the procedure, whereby every pair of intersecting balls is replaced by a larger ball whose radius equals the sum of the radii of the smaller balls and which contains the smaller balls. This process is repeated until all the balls are disjoint. The obtained collection will be denoted $\mathcal{B}_0$, its total radius is $r(\mathcal{B}_0)$. Then each ball is dilated by the same factor with respect to its corresponding center. As the dilation factor increases, some balls may touch. If that happens, the above procedure of ball merging is applied again to obtain a new collection of disjoint balls of the same total radius. The construction can be stopped when any desired total radius r is reached, provided that r is universally small compared to ℓ^ε. This yields a collection $\mathcal{B}_r$ covering the initial set and containing a logarithmic energy [19,33].

We now give the statement of our result concerning the ball construction and the associated lower bounds. Throughout the rest of the paper we use the notation $f^+ := \max(f, 0)$ and $f^- := -\min(f, 0)$.

Proposition 4.2. *Let* $U \subseteq \mathbb{T}^2_{\ell^\varepsilon}$ *be an open set such that* $I_{\beta,U} \neq \varnothing$, *and assume that* (2.21) *holds.*

- *There exists* $\varepsilon_0 > 0$, $r_0 \in (0, 1)$ *and* $C > 0$ *depending only on* ℓ, κ, $\bar{\delta}$ *and* $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ *such that for all* $\varepsilon < \varepsilon_0$ *there exists a collection of finitely many disjoint closed balls* $\mathcal{B}_0$ *whose union covers* $\bigcup_{i\in I_{\beta,U}} \Omega'_{i,\varepsilon}$ *and such that*

$$r(\mathcal{B}_0) \leq c\varepsilon^{1/3}|\ln\varepsilon|^{1/6} \sum_{i\in I_{\beta,U}} P_i^\varepsilon < r_0, \tag{4.3}$$

for some universal $c > 0$. *Furthermore, for every* $r \in [r(\mathcal{B}_0), r_0]$ *there is a family of disjoint closed balls* $\mathcal{B}_r$ *of total radius* r *covering* $\mathcal{B}_0$.

- *For every $B \in \mathcal{B}_r$ such that $B \subset U$ we have*

$$\int_B \left(|\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' \geq \frac{1}{2\pi} \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right)^+ \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2,$$

for some $c > 0$ depending only on κ and $\bar{\delta}$.
- *If $B \in \mathcal{B}_r$, for any non-negative Lipschitz function χ with support in U, we have*

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) dx' - \frac{1}{2\pi} \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right)^+ \times \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}_i^\varepsilon|^2 \geq -C \|\nabla \chi\|_\infty \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2,$$

where $\chi_i := \int_U \chi \tilde{\delta}_i^\varepsilon \, dx'$, with $\tilde{\delta}_i^\varepsilon(x')$ defined in (3.18), for some $c > 0$ depending only on κ and $\bar{\delta}$, and a universal $C > 0$.

Remark 4.3. The explanation for the factor of $\frac{1}{4}$ in front of $\frac{\kappa^2}{|\ln \varepsilon|} |h'_\varepsilon|^2$ is that we must "save" a fraction of this term for the mass displacement argument in Section 5 and in the convergence result in Section 6.

Proof of the first item Choose an arbitrary $r_0 \in (0, 1)$. As in [18], from the basic lower bound on $\bar{E}^\varepsilon$ (see [18, Lemma 3.4]):

$$\bar{E}^\varepsilon[u^\varepsilon] \geq \frac{1}{|\ln \varepsilon|} \sum_i P_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2 |\ln \varepsilon|} \sum_i A_i^\varepsilon + \frac{c}{|\ln \varepsilon|^2} \left(\sum_i A_i^\varepsilon \right)^2, \tag{4.4}$$

where A_i^ε and P_i^ε are defined in (2.2) and $c > 0$ depends only on κ and ℓ, we obtain with the help of (2.21) that

$$\limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_i A_i^\varepsilon \leq C, \qquad \limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_i P_i^\varepsilon \leq C, \tag{4.5}$$

for some $C > 0$ depending only on ℓ, κ, $\bar{\delta}$ and $\sup_{\varepsilon > 0} F^\varepsilon[u^\varepsilon]$.

In view of Lemma 4.1, the definition of P_i^ε in (2.2) and the second of (4.5), for sufficiently small ε it is possible to cover each $\Omega'_{i,\varepsilon}$ with $i \in I_{\beta,U}$ by a closed ball B_i, so that the collection $\tilde{\mathcal{B}}_0$ consisting of all B_i's (possibly intersecting) has total radius

$$r_0(\tilde{\mathcal{B}}_0) \leq C \varepsilon^{1/3} |\ln \varepsilon|^{1/6} \sum_{i \in I_{\beta,U}} P_i^\varepsilon, \tag{4.6}$$

for some universal $C > 0$. Furthermore, by the first inequality in (4.5) and the fact that $A_i^\varepsilon \geq \beta$ for all $i \in I_{\beta,U}$ the collection $\tilde{\mathcal{B}}_0$ consists of only finitely many balls. Therefore, we can apply the construction à la Jerrard and Sandier outlined at the beginning of this section to obtain the desired family of balls $\mathcal{B}_0$ and $\mathcal{B}_r$, with $r(\mathcal{B}_0) = r(\tilde{\mathcal{B}}_0)$. The estimate on the radii follows by combining the second of (4.5) and (4.6) and the fact that $\ell^\varepsilon \to \infty$ with the rate depending only on ℓ, for sufficiently small ε depending on ℓ, κ, $\bar{\delta}$, $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ and r_0. □

Proof of the second item Let $B \subset U$ be a ball in the collection $\mathcal{B}_r$. Denote the radius of B by r_B and set

$$X_\varepsilon := \frac{\kappa^2}{|\ln\varepsilon|}\int_B h'_\varepsilon \, \mathrm{d}x'.$$

Integrating (3.16) over B and applying the divergence theorem, we have

$$\int_{\partial B} \frac{\partial h'_\varepsilon}{\partial \nu} \, \mathrm{d}\mathcal{H}^1(x') = m_{B,\varepsilon} - X_\varepsilon, \tag{4.7}$$

where

$$m_{B,\varepsilon} := \int_B (\mu'_\varepsilon(x') - \bar{\mu}^\varepsilon)\, \mathrm{d}x' = \sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B} \theta_i A_i^\varepsilon - \bar{\mu}^\varepsilon |B|,$$

for some $\theta_i \in [0, 1)$ representing the volume fraction in B of those droplets that are not covered completely by B, and ν is the inward normal to ∂B. Using the Cauchy–Schwarz inequality, we then deduce from (4.7) that

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 \, \mathrm{d}\mathcal{H}^1(x') \geq \frac{1}{2\pi r_B}(m_{B,\varepsilon} - X_\varepsilon)^2 \geq \frac{m_{B,\varepsilon}^2 - 2m_{B,\varepsilon}X_\varepsilon}{2\pi r_B}. \tag{4.8}$$

By another application of the Cauchy–Schwarz inequality, we may write

$$\frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, \mathrm{d}x' \geq \frac{X_\varepsilon^2}{4\pi r_B^2}\frac{|\ln\varepsilon|}{\kappa^2}. \tag{4.9}$$

We now add (4.8) and (4.9) and optimize the right-hand side over X_ε. We obtain

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 \, \mathrm{d}\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, \mathrm{d}x' \geq \frac{m_{B,\varepsilon}^2}{2\pi r_B}\left(1 - \frac{Cr_B}{|\ln\varepsilon|}\right), \tag{4.10}$$

for $C = \kappa^4$. Recalling that $r_B \leq r \leq r_0 < 1$, we can choose ε sufficiently small depending only on κ so that the term in parentheses above is positive.

Inserting the definition of $m_{B,\varepsilon}$ into (4.10) and discarding some positive terms yields

$$\begin{aligned}
&\int_{\partial B} |\nabla h'_\varepsilon|^2 \mathrm{d}\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2 \, \mathrm{d}x' \\
&\geq \frac{1}{2\pi r_B}\left(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B}\theta_i A_i^\varepsilon - \bar{\mu}^\varepsilon|B|\right)^2\left(1 - \frac{Cr_B}{|\ln\varepsilon|}\right) \\
&\geq \frac{1}{2\pi r_B}\left(\sum_{i\in I_B} A_i^\varepsilon + \sum_{i\notin I_B}\theta_i A_i^\varepsilon\right)^2\left(1 - 2\bar{\mu}^\varepsilon|B|\left(\sum_{i\in I_B} A_i^\varepsilon\right)^{-1} - \frac{Cr_B}{|\ln\varepsilon|}\right).
\end{aligned} \tag{4.11}$$

We now use the fact that by construction B covers at least one $\Omega'_{i,\varepsilon}$ with $A_i^\varepsilon \geq \beta$. This leads us to

$$\begin{aligned}
&\int_{\partial B} |\nabla h'_\varepsilon|^2 \mathrm{d}\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln \varepsilon|} \int_B |h'_\varepsilon|^2 \,\mathrm{d}x' \\
&\quad \geq \frac{1}{2\pi r_B} \left(\sum_{i \in I_B} A_i^\varepsilon + \sum_{i \notin I_B} \theta_i A_i^\varepsilon \right)^2 \left(1 - \frac{2\pi \bar{\mu}^\varepsilon r_B^2}{\beta} - \frac{C r_B}{|\ln \varepsilon|} \right) \\
&\quad \geq \frac{1}{2\pi r_B} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \,(1 - c r_B)\,, \qquad (4.12)
\end{aligned}$$

for some $c > 0$ depending only on κ and $\bar{\delta}$, where in the last line we used that $A_i^\varepsilon \geq \tilde{A}_i^\varepsilon$. Hence there exists $r_0 \in (0,1)$ depending only on κ, and $\bar{\delta}$ such that the right-hand side of (4.12) is positive.

Finally, let us define $\mathcal{F}(x,r) := \int_{B(x,r)} |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \frac{r\kappa^2}{4|\ln \varepsilon|} \int_{B(x,r)} |h'_\varepsilon|^2 \,dx'$, where $B(x,r)$ is the ball centered at x of radius r. The relation (4.12) then reads for $B(x,r) = B \in \mathcal{B}_r$ and almost everywhere $r \in (r(\mathcal{B}_0), r_0]$:

$$\frac{\partial \mathcal{F}}{\partial r} \geq \frac{1}{2\pi r} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 (1 - cr), \qquad (4.13)$$

with c as before. Then using [35, Proposition 4.1], for every $B \in \mathcal{B}(s) := \mathcal{B}_r$ with $r = \mathrm{e}^s r(\mathcal{B}_0)$ (using the notation of [35, Theorem 4.2]) we have

$$\begin{aligned}
&\int_{B \setminus \mathcal{B}_0} |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \frac{r_B \kappa^2}{4|\ln \varepsilon|} \int_B |h'_\varepsilon|^2 \,\mathrm{d}x' \\
&\quad \geq \int_0^s \sum_{\substack{B' \in \mathcal{B}(t) \\ B' \subset B}} \frac{1}{2\pi} \sum_{i \in I_{\beta,B'}} |\tilde{A}_i^\varepsilon|^2 \,(1 - c r(\mathcal{B}(t)))\,\mathrm{d}t \\
&\quad = \int_0^s \sum_{\substack{B' \in \mathcal{B}(t) \\ B' \subset B}} \frac{1}{2\pi} \sum_{i \in I_{\beta,B'}} |\tilde{A}_i^\varepsilon|^2 (1 - c\mathrm{e}^t r(\mathcal{B}_0))\,\mathrm{d}t \\
&\quad \geq \frac{1}{2\pi} \sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{r}{r(\mathcal{B}_0)} - cr \right), \qquad (4.14)
\end{aligned}$$

where we observed that the double summation appearing in the first and second lines is simply the summation over $I_{\beta,B}$. Once again, in view of the fact that $r_B \leq 1$ and that both terms in the integrand of the left-hand side of (4.14) are non-negative, this completes the proof of the second item. □

Proof of the third item This follows [36]. Let χ be a non-negative Lipschitz function with support in U. By the "layer-cake" theorem [21], for any $B \in \mathcal{B}_r$ we have

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' = \int_0^{+\infty} \int_{E_t \cap B} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' \,\mathrm{d}t, \qquad (4.15)$$

where $E_t := \{\chi > t\}$. If $i \in I_{\beta,B}$, then by construction for any $s \in [r(\mathcal{B}_0), r]$ there exists a unique closed ball $B_{i,s} \in \mathcal{B}_s$ containing $\Omega'_{i,\varepsilon}$. Therefore, for $t > 0$ we can define

$$s(i,t) := \sup\left\{s \in [r(\mathcal{B}_0), r] \,:\, B_{i,s} \subset E_t\right\},$$

with the convention that $s(i,t) = r(\mathcal{B}_0)$ if the set is empty. We also let $B_i^t := B_{i,s(i,t)}$ whenever $s(i,t) > r(\mathcal{B}_0)$. Note that for each $i \in I_{\beta,B}$ we have that $t \mapsto s(i,t)$ is a non-increasing function. In particular, we can define $t_i \geq 0$ to be the supremum of the set of t's at which $s(i,t) = r$ (or zero, if this set is empty).

If $t > t_i$ and $s(i,t) > r(\mathcal{B}_0)$, then for any $x \in \Omega'_{i,\varepsilon}$ and any $y \in B_i^t \backslash E_t$ (which is not empty) we have

$$\chi(x) - t \leq \chi(x) - \chi(y) \leq 2s(i,t)\|\nabla\chi\|_\infty. \tag{4.16}$$

Averaging over all $x \in \Omega'_{i,\varepsilon}$, we hence deduce

$$\chi_i - t \leq 2s(i,t)\|\nabla\chi\|_\infty. \tag{4.17}$$

Now, for any $t \geq 0$ the collection $\{B_i^t\}_{i \in I_{\beta,B,t}}$, where $I_{\beta,B,t} := \{i \in I_{\beta,B} : s(i,t) > r(\mathcal{B}_0)\}$ is disjoint. Indeed if $i, j \in I_{\beta,B,t}$ and $s(i,t) \geq s(j,t)$ then, since $\mathcal{B}_{s(i,t)}$ is disjoint, the balls $B_{i,s(i,t)}$ and $B_{j,s(i,t)}$ are either equal or disjoint. If they are disjoint we note that $s(i,t) \geq s(j,t)$ implies that $B_{j,s(j,t)} \subseteq B_{j,s(i,t)}$, and, therefore, $B_j^t = B_{j,s(j,t)}$ and $B_i^t = B_{i,s(i,t)}$ are disjoint. If they are equal and $s(i,t) > s(j,t)$, then $B_{j,s(j,t)} \subset E_t$, contradicting the definition of $s(j,t)$. So $s(j,t) = s(i,t)$ and then $B_j^t = B_i^t$.

Now assume that $B' \in \{B_i^t\}_{i \in I_{\beta,B,t}}$ and let s be the common value of $s(i,t)$ for i's in $I_{\beta,B'}$. Then, the previous item of the proposition yields

$$\int_{B'} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) \mathrm{d}x' \geq \frac{1}{2\pi}\left(\ln\frac{s}{r(\mathcal{B}_0)} - cs\right)^+ \sum_{i \in I_{\beta,B',t}} |\tilde{A}_i^\varepsilon|^2.$$

Summing over $B' \in \{B_i^t\}_{i \in I_{\beta,B,t}}$, we deduce

$$\begin{aligned}\int_{B \cap E_t} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) \mathrm{d}x' &\geq \frac{1}{2\pi}\sum_{i \in I_{\beta,B,t}} |\tilde{A}_i^\varepsilon|^2 \left(\ln\frac{s(i,t)}{r(\mathcal{B}_0)} - cs(i,t)\right)^+ \\ &= \frac{1}{2\pi}\sum_{i \in I_{\beta,B}} |\tilde{A}_i^\varepsilon|^2 \left(\ln\frac{s(i,t)}{r(\mathcal{B}_0)} - cs(i,t)\right)^+,\end{aligned} \tag{4.18}$$

where in the last inequality we took into consideration that all the terms corresponding to $i \in I_{\beta,B} \backslash I_{\beta,B,t}$ give no contribution to the sum in the right-hand side.

Integrating the above expression over t and using the fact that $r_0(\mathcal{B}_0) \le s(i,t) \le r$ yields

$$\int_0^{+\infty}\int_{E_t\cap B}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathrm{d}x'\,\mathrm{d}t \ge \frac{1}{2\pi}\sum_{i\in I_{\beta,B}}|\tilde{A}_i^\varepsilon|^2$$
$$\times\int_0^{\chi_i}\left(\ln\frac{s(i,t)}{r(\mathcal{B}_0)} - cr\right)^+\mathrm{d}t \ge \frac{1}{2\pi}\sum_{i\in I_{\beta,B}}\chi_i|\tilde{A}_i^\varepsilon|^2\left(\ln\frac{r}{r(\mathcal{B}_0)} - cr\right)^+$$
$$+\frac{1}{2\pi}\sum_{i\in I_{\beta,B}}|\tilde{A}_i^\varepsilon|^2\int_0^{\chi_i}\ln\frac{s(i,t)}{r}\,\mathrm{d}t. \tag{4.19}$$

We now concentrate on the last term in (4.19). Using the estimate in (4.17) and the definition of t_i, we can bound the integral in this term as follows

$$\int_0^{\chi_i}\ln\frac{s(i,t)}{r}\,\mathrm{d}t \ge \int_{t_i}^{\chi_i}\ln\left(\frac{\chi_i - t}{2r\|\nabla\chi\|_\infty}\right)\mathrm{d}t \ge -C\|\nabla\chi\|_\infty, \tag{4.20}$$

for some universal $C > 0$, which is obtained by an explicit computation and the fact that $r \le r_0 < 1$. Finally, combining (4.20) with (4.19), the statement follows from (4.15). □

Remark 4.4. Inspecting the proof, we note that the statements of the proposition are still true with the left-hand sides replaced by $\int_{B\setminus\mathcal{B}_0}\chi|\nabla h'_\varepsilon|^2\,\mathrm{d}x' + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B\chi|h'_\varepsilon|^2\,\mathrm{d}x'$ (with $\chi \equiv 1$ or χ Lipschitz, respectively).

5. Energy Displacement

In this section, we follow the idea of [36] of localizing the ball construction and combine it with an "energy displacement" which allows us to reduce to the situation where the energy density in (3.15) is bounded below. For the proposition below we define for all $x' \in \mathbb{T}^2_{\ell^\varepsilon}$:

$$\nu^\varepsilon(x') := \sum_{i\in I_\beta}|\tilde{A}_i^\varepsilon|^2\tilde{\delta}_i^\varepsilon(x'), \tag{5.1}$$

where $\tilde{\delta}_i^\varepsilon(x')$ is given by (3.18). We also recall that ρ_ε defined in (3.1) is the expected radius of droplets in a minimizing configuration in the blown up coordinates.

We cover $\mathbb{T}^2_{\ell^\varepsilon}$ by the balls of radius $\frac{1}{4}r_0$ whose centers are in $\frac{r_0}{8}\mathbb{Z}^2$. We call this cover $\{U_\alpha\}_\alpha$ and $\{x_\alpha\}_\alpha$ the centers. We also introduce $D_\alpha := B(x_\alpha, \frac{3r_0}{4})$.

Proposition 5.1. *Let h'_ε satisfy (3.16), assume (2.21) holds, and set*

$$f_\varepsilon := |\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\,\nu^\varepsilon. \tag{5.2}$$

Then there exist $\varepsilon_0 > 0$ as in Proposition 4.2 and constants $c, C > 0$ depending only on $\bar{\delta}$ and κ such that for all $\varepsilon < \varepsilon_0$, there exists a family of integers $\{n_\alpha\}_\alpha$ and a density g_ε on $\mathbb{T}^2_{\ell^\varepsilon}$ with the following properties.

- *g_ε is bounded below:*

$$g_\varepsilon \geq -c \ln^2(M_\varepsilon + 2) \quad on \quad \mathbb{T}^2_{\ell^\varepsilon}.$$

- *For any α,*

$$n_\alpha^2 \leq C(g_\varepsilon(D_\alpha) + c \ln^2(M_\varepsilon + 2)).$$

- *For any Lipschitz function χ on $\mathbb{T}^2_{\ell^\varepsilon}$ we have*

$$\left| \int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(f_\varepsilon - g_\varepsilon)\, \mathrm{d}x' \right| \leq C \sum_\alpha (\nu^\varepsilon(U_\alpha) + (n_\alpha + M_\varepsilon) \ln(n_\alpha + M_\varepsilon + 2)) \|\nabla \chi\|_{L^\infty(D_\alpha)}. \tag{5.3}$$

Proof. The proof follows the method of [32], involving a localization of the ball construction followed by energy displacement. Here we follow [36, Proposition 4.9]. One key difference is the restriction to I_β which means we cover only those $\Omega'_{i,\varepsilon}$ satisfying $A_i^\varepsilon \geq \beta$ as in Proposition 4.2.

Step 1 Localization of the ball construction.
We use U_α defined above as the cover on $\mathbb{T}^2_{\ell^\varepsilon}$. For each U_α covering at least one droplet whose volume is greater or equal than β and for any $r \in (r(\mathcal{B}_0), \frac{1}{4} r_0)$ we construct disjoint balls $\mathcal{B}_r^\alpha$ covering all $\Omega'_{i,\varepsilon}$ with $i \in I_{\beta, U_\alpha}$, using Proposition 4.2. Then choosing a small enough $\rho \in (r(\mathcal{B}_0), \frac{1}{4} r_0)$ independent of ε (to be specified below), we may extract from $\cup_\alpha \mathcal{B}_\rho^\alpha$ a disjoint family which covers $\cup_{i \in I_\beta} \Omega'_{i,\varepsilon}$ as follows. Denoting by $\mathcal{C}$ a connected component of $\cup_\alpha \mathcal{B}_\rho^\alpha$, we claim that there exists α_0 such that $\mathcal{C} \subset U_{\alpha_0}$. Indeed if $x \in \mathcal{C}$ and letting λ be a Lebesgue number of the covering of $\mathbb{T}^2_{\ell^\varepsilon}$ by $\{U_\alpha\}_\alpha$ (it is easy to see that in our case $\frac{1}{4} r_0 < \lambda < \frac{1}{2} r_0$), there exists α_0 such that $B(x, \lambda) \subset U_{\alpha_0}$.[1] If $\mathcal{C}$ intersected the complement of U_{α_0}, there would exist a chain of balls connecting x to $(U_{\alpha_0})^c$, each of which would intersect U_{α_0}. Each of the balls in the chain would belong to some $\mathcal{B}_\rho^{\alpha'}$ with α' such that dist $(U_{\alpha'}, U_{\alpha_0}) \leq 2\rho < \frac{1}{2} r_0$. Thus, calling k the universal maximum number of α''s such that dist $(U_{\alpha'}, U_{\alpha_0}) < \frac{1}{2} r_0$, the length of the chain is at most $2k\rho$ and thus $\lambda \leq 2k\rho$. If we choose $\rho < \lambda/(2k)$, this is impossible and the claim is proved. Let us then choose $\rho = \lambda/(4k)$. By the above, each $\mathcal{C}$ is included in some U_α.

We next obtain a disjoint cover of $\cup_{i \in I_\beta} \Omega'_{i,\varepsilon}$ from $\cup_\alpha \mathcal{B}_\rho^\alpha$. Let $\mathcal{C}$ be a connected component of $\cup_\alpha \mathcal{B}_\rho^\alpha$. By the discussion of the preceding paragraph, there exists an index α_0 such that $\mathcal{C} \subset U_{\alpha_0}$. We then remove from $\mathcal{C}$ all the balls which do not belong to $\mathcal{B}_\rho^{\alpha_0}$ and still denote by $\mathcal{B}_\rho^{\alpha_0}$ the obtained collection. We repeat this process for all the connected components and obtain a disjoint cover $\mathcal{B}_\rho = \cup_\alpha \mathcal{B}_\rho^\alpha$ of $\cup_{i \in I_\beta} \Omega'_{i,\varepsilon}$. Note that this procedure uniquely associates an α to a given $B \in \mathcal{B}_\rho$, as well as to each $\Omega'_{i,\varepsilon}$ for a given $i \in I_\beta$ by assigning to it the ball in $\mathcal{B}_\rho$ that covers it, and then the α of this ball. We will use this repeatedly below. We also slightly abuse the notation by sometimes using $\mathcal{B}_\rho^\alpha$ to denote the union of the balls in the family $\mathcal{B}_\rho^\alpha$.

We now proceed to the energy displacement.

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[1] A Lebesgue number of a covering of a compact set is a number $\lambda > 0$ such that every subset of diameter less than λ is contained in some element of the covering.

Step 2: Energy displacement in the balls.

Note that by construction every ball in $\mathcal{B}^\alpha_\rho$ is included in U_α. From the last item of Proposition 4.2 applied to a ball $B \in \mathcal{B}^\alpha_\rho$, if ε is small enough then, for any Lipschitz non-negative χ we have for some $c > 0$ depending only on κ and $\bar{\delta}$ and a universal $C > 0$

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|}|h'_\varepsilon|^2 \right) \mathrm{d}x' - \frac{1}{2\pi}\left(\ln \frac{\rho}{r(\mathcal{B}^\alpha_0)} - c\rho \right)^+ \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}^\varepsilon_i|^2$$
$$\geq -C\nu^\varepsilon(B)\|\nabla\chi\|_{L^\infty(B)},$$

where ν^ε is defined by (5.1). Rewriting the above, recalling the definition (3.1) and defining $n_\alpha \geq 1$ to be the number of droplets included in $U_\alpha \supset B$ and satisfying $A^\varepsilon_i \geq \beta$, we have

$$\int_B \chi \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|}|h'_\varepsilon|^2 \right) \mathrm{d}x' - \frac{1}{2\pi}\left(\ln \frac{\rho}{n_\alpha \rho_\varepsilon} - c \right) \sum_{i \in I_{\beta,B}} \chi_i |\tilde{A}^\varepsilon_i|^2$$
$$+ \int_B \chi \omega_\varepsilon \, \mathrm{d}x' \geq -C\nu^\varepsilon(B)\|\nabla\chi\|_{L^\infty(B)},$$

where we set $\bar{r}_\alpha := \frac{r(\mathcal{B}^\alpha_0)}{\rho_\varepsilon}$ and define (recall that α implicitly depends on $i \in I_\beta$)

$$\omega_\varepsilon(x') := \frac{1}{2\pi}\sum_{i \in I_\beta} |\tilde{A}^\varepsilon_i|^2 \ln\left(\frac{r(\mathcal{B}^\alpha_0)}{n_\alpha \rho_\varepsilon} \right) \tilde{\delta}^\varepsilon_i(x') = \frac{1}{2\pi}\sum_{i \in I_\beta} |\tilde{A}^\varepsilon_i|^2 \ln\left(\frac{\bar{r}_\alpha}{n_\alpha} \right) \tilde{\delta}^\varepsilon_i(x'). \tag{5.4}$$

The quantity ω_ε in some sense measures the discrepancy between the droplets $\Omega'_{i,\varepsilon}$ and balls of radius ρ_ε. We will thus naturally use M_ε in (3.13) to control it. Note also that it is only supported in the droplets, hence in the balls of $\mathcal{B}_\rho$.

Applying Lemma 3.1 of [32] to

$$f_{B,\varepsilon} = \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}\left(\ln \frac{\rho}{\rho_\varepsilon n_\alpha} - c \right) \sum_{i \in I_{\beta,B}} |\tilde{A}^\varepsilon_i|^2 \tilde{\delta}^\varepsilon_i + \omega_\varepsilon \right) \mathbf{1}_B$$

we deduce the existence of a positive measure $g_{B,\varepsilon}$ such that

$$\|f_{B,\varepsilon} - g_{B,\varepsilon}\|_{\mathrm{Lip}^*} \leq C\nu^\varepsilon(B), \tag{5.5}$$

where Lip* denotes the dual norm to the space of Lipschitz functions and $C > 0$ is universal.

Step 3: Energy displacement on annuli and definition of g_ε.

We define a set C_α as follows: recall that ρ was assumed equal to $\lambda/(4k)$, where $\lambda \leq \frac{1}{4}r_0$ and k bounds the number of α''s such that dist $(U_{\alpha'}, U_\alpha) < \frac{1}{2}r_0$ for any given α. Therefore the total radius of the balls in $\mathcal{B}_\rho$ which are at distance less than r_0 from U_α is at most $k\rho = \frac{1}{16}r_0$. In particular, letting T_α denote the set of $t \in (\frac{r_0}{2}, \frac{3r_0}{4})$ such that the circle of center x_α (where we recall x_α is the center of U_α) and radius

t does not intersect $\mathcal{B}_\rho^\alpha$, we have $|T_\alpha| \geq \frac{3}{16} r_0$. We let $C_\alpha = \{x \mid |x - x_\alpha| \in T_\alpha\}$ and recall that $D_\alpha = B(x_\alpha, \frac{3r_0}{4})$.

Let $t \in T_\alpha$. Arguing exactly as in the proof of (4.10), we find that

$$\int_{\partial B(x_\alpha,t)} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(x_\alpha,t)} |h'_\varepsilon|^2 \, dx' \geq \frac{m_{\varepsilon,t}^2}{2\pi t} \left(1 - \frac{\kappa^2 t}{4|\ln\varepsilon|}\right)$$

with $m_{\varepsilon,t} := \int_{B(x_\alpha,t)} (\mu'_\varepsilon(x') - \bar{\mu}^\varepsilon) \, dx'$. Arguing as in (4.12) and using the fact that $B(x_\alpha, \frac{1}{2} r_0)$ contains all the droplets with $i \in I_{\beta,U_\alpha}$, we find that we can take ε sufficiently small depending on κ, and r_0 sufficiently small depending on κ and $\bar{\delta}$ such that for all $t \in T_\alpha$,

$$\int_{\partial B(x_\alpha,t)} |\nabla h'_\varepsilon|^2 \, d\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{B(x_\alpha,t)} |h'_\varepsilon|^2 \, dx' \geq \frac{1}{4\pi t} \left(\sum_{i \in I_{\beta,U_\alpha}} A_i^\varepsilon \right)^2 .$$

Integrating this over $t \in T_\alpha$, using that $|T_\alpha| \geq \frac{3}{16} r_0$, we obtain that

$$\int_{C_\alpha} |\nabla h'_\varepsilon|^2 \, dx' + \frac{\kappa^2}{4|\ln\varepsilon|} \int_{D_\alpha} |h'_\varepsilon|^2 \, dx' \geq c \left(\sum_{i \in I_{\beta,U_\alpha}} A_i^\varepsilon \right)^2, \tag{5.6}$$

with $c > 0$ depending only on r_0, hence on κ and $\bar{\delta}$.

We now trivially extend the estimate in (5.6) to all α's, including those U_α that contain no droplets of size greater or equal than β. The overlap number of the sets $\{C_\alpha\}_\alpha$, defined as the maximum number of sets to which a given $x' \in \mathbb{T}^2_{\ell^\varepsilon}$ belongs is bounded above by the overlap number of the sets $\{D_\alpha\}_\alpha$, call it k'. Since the latter collection of balls covers the entire $\mathbb{T}^2_{\ell^\varepsilon}$, we have $k' \geq 1$. Then, letting

$$f'_\varepsilon := f_\varepsilon - \sum_{B \in \mathcal{B}_\rho} f_{B,\varepsilon} = \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon} \setminus \mathcal{B}_\rho} + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho}$$
$$+ \frac{1}{2\pi} \sum_{i \in I_\beta} \left(\ln \frac{\rho}{n_\alpha} - c \right) |\tilde{A}_i^\varepsilon|^2 \tilde{\delta}_i^\varepsilon - \omega_\varepsilon, \tag{5.7}$$

and

$$f_{\alpha,\varepsilon} := \frac{1}{2k'} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{C_\alpha} + \frac{1}{2\pi} \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 \left(\ln \frac{\rho}{n_\alpha} - c \right) \tilde{\delta}_i^\varepsilon - \omega_\varepsilon \mathbf{1}_{\mathcal{B}_\rho^\alpha}, \tag{5.8}$$

we have

$$f'_\varepsilon - \sum_\alpha f_{\alpha,\varepsilon} \geq \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon} \setminus \mathcal{B}_\rho}$$
$$- \frac{1}{2k'} \sum_\alpha \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{C_\alpha} + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho}$$
$$\geq \frac{1}{2} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln\varepsilon|} |h'_\varepsilon|^2 \right) \mathbf{1}_{\mathbb{T}^2_{\ell^\varepsilon} \setminus \mathcal{B}_\rho} + \frac{\kappa^2}{4|\ln\varepsilon|} |h'_\varepsilon|^2 \mathbf{1}_{\mathcal{B}_\rho} \geq 0 \tag{5.9}$$

and from (5.6)

$$f_{\alpha,\varepsilon}(D_\alpha) = \frac{1}{2k'} \int_{C_\alpha} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' + \frac{1}{2\pi} \left(\ln \frac{\rho}{n_\alpha} - c \right) \times \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 - \omega_\varepsilon(D_\alpha) \geq c \Bigg(\sum_{i \in I_{\beta,U_\alpha}} A_i^\varepsilon \Bigg)^2 - \frac{1}{2\pi} \ln n_\alpha \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 - \omega_\varepsilon(D_\alpha) - C \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2, \tag{5.10}$$

for some $C, c > 0$ depending only on κ and $\bar{\delta}$. Now we combine the middle two terms, using the definition of $\omega_{\varepsilon,\alpha}$ in (5.4), to obtain

$$f_{\alpha,\varepsilon}(D_\alpha) \geq c \Bigg(\sum_{i \in I_{\beta,U_\alpha}} A_i^\varepsilon \Bigg)^2 - \frac{1}{2\pi} \ln \bar{r}_\alpha \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2 - C \sum_{i \in I_{\beta,\mathcal{B}_\rho^\alpha}} |\tilde{A}_i^\varepsilon|^2. \tag{5.11}$$

□

The next step is to bound $\bar{r}_\alpha$. We separate those $\Omega'_{i,\varepsilon}$ with $A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1}$ and those with $A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1}$. We denote (with s for "small" and b for "big")

$$I_{\beta,\alpha}^s = \{i \in I_{\beta,U_\alpha} : A_i^\varepsilon \leq 3^{2/3}\pi\gamma^{-1}\},$$
$$I_{\beta,\alpha}^b = I_{\beta,U_\alpha} \backslash I_{\beta,\alpha}^s,$$
$$n_{\alpha_s} = \# I_{\beta,\alpha}^s.$$

For the small droplets, we use the obvious bound

$$\sum_{i \in I_{\beta,\alpha}^s} |A_i^\varepsilon|^{1/2} \leq c n_{\alpha_s}, \tag{5.12}$$

with a universal $c > 0$, while for the large droplets we use that in view of the definition of M_ε in (3.13) we have

$$\sum_{i \in I_{\beta,\alpha}^b} |A_i^\varepsilon|^{1/2} \leq C \sum_{i \in I_{\beta,\alpha}^b} A_i^\varepsilon \leq C' M_\varepsilon, \tag{5.13}$$

for some universal $C, C' > 0$. We can now proceed to controlling $\bar{r}_\alpha$. By (3.1) and (4.3), for universally small ε we have

$$\bar{r}_\alpha \leq C \sum_{i \in I_{\beta,U_\alpha}} P_i^\varepsilon, \tag{5.14}$$

for some universal $C > 0$. In view of (3.13), (5.13) and (5.12), we deduce from Remark 3.2 that for universally small ε we have

$$\bar{r}_\alpha \leq C \left(M_\varepsilon + \sqrt{4\pi} \sum_{i \in I_{\beta,U_\alpha}} |A_i^\varepsilon|^{1/2} \right) \leq C(M_\varepsilon + c n_{\alpha_s} + C' M_\varepsilon) \leq C''(n_{\alpha_s} + M_\varepsilon) \leq C''(1 + n_{\alpha_s} + M_\varepsilon), \tag{5.15}$$

where $c, C, C', C'' > 0$ are universal. Therefore, (5.11) becomes

$$f_{\alpha,\varepsilon}(D_\alpha) \geq c\left(\sum_{i\in I^s_{\beta,\alpha}} A^\varepsilon_i\right)^2 + c\left(\sum_{i\in I^b_{\beta,\alpha}} A^\varepsilon_i\right)^2 - C\ln\bar{r}_\alpha \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2$$
$$-C''' \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2, \geq c\beta^2 n^2_{\alpha_s} + c\left(\sum_{i\in I^b_{\beta,\alpha}} A^\varepsilon_i\right)^2$$
$$-C'\ln(C''(1+n_{\alpha_s}+M_\varepsilon))\left(n_{\alpha_s} + \sum_{i\in I^b_{\beta,\alpha}} A^\varepsilon_i\right), \tag{5.16}$$

where $C, C' > 0$ are universal, $c, C'', C''' > 0$ depend only on κ and $\bar{\delta}$, and C' was chosen so that $C|\tilde{A}^\varepsilon_i|^2 \leq C'(A^\varepsilon_i + 1)$.

We now claim that this implies that

$$f_{\alpha,\varepsilon}(D_\alpha) \geq \frac{c}{2}\beta^2 n^2_{\alpha_s} + \frac{c}{2}\left(\sum_{i\in I^b_{\beta,\alpha}} A^\varepsilon_i\right)^2 - C'''\ln^2(M_\varepsilon + 2), \tag{5.17}$$

where $C''' > 0$ depends only on κ and $\bar{\delta}$. This is seen by minimization of the right-hand side, as we now detail. For the rest of the proof, all constants will depend only on κ and $\bar{\delta}$. For shortness, we will set $X := \sum_{i\in I^b_{\beta,\alpha}} A^\varepsilon_i$.

First assume $n_{\alpha_s} = 0$. Then (5.16) can be rewritten

$$f_{\alpha,\varepsilon}(D_\alpha) \geq cX^2 - C'\ln(C''(1+M_\varepsilon)))X,$$

By minimization of the quadratic polynomial in the right-hand side, we easily see that an inequality of the form (5.17) holds. Second, let us consider the case $n_{\alpha_s} \geq 1$. We may use the obvious inequality $\ln(1+x+y) \leq \ln(1+x) + \ln(1+y)$ that holds for all $x \geq 0$ and $y \geq 0$ to bound from below

$$\begin{aligned}\frac{c}{2}\beta^2 n^2_{\alpha_s} + \frac{c}{2}X^2 - C'\ln(C''(1+n_{\alpha_s}+M_\varepsilon))(n_{\alpha_s}+X) \geq \frac{c}{2}\beta^2 n^2_{\alpha_s} + \frac{c}{2}X^2\\ -C(n_{\alpha_s}+X) - Cn_{\alpha_s}\ln(n_{\alpha_s}+1) - CX\ln(n_{\alpha_s}+1)\\ -C\ln(M_\varepsilon+1)(n_{\alpha_s}+X).\end{aligned} \tag{5.18}$$

It is clear that the first three negative terms on the right-hand side can be absorbed into the first two positive terms, at the expense of a possible additive constant, which yields

$$\begin{aligned}&\frac{c}{2}\beta^2 n^2_{\alpha_s} + \frac{c}{2}X^2 - C'\ln(C''(n_{\alpha_s}+M_\varepsilon))(n_{\alpha_s}+X)\\ &\quad\geq \frac{c}{4}\beta^2 n^2_{\alpha_s} + \frac{c}{4}X^2 - C\ln(M_\varepsilon+1)(n_{\alpha_s}+X) - C.\end{aligned} \tag{5.19}$$

Then by quadratic optimization the right hand side of (5.19) is bounded below by $-C\ln^2(M_\varepsilon+2)$ (after possibly changing the constant). Inserting this into (5.16), we obtain (5.17).

We then apply [32, Lemma 3.2] over D_α to $f_{\alpha,\varepsilon}+C'''|D_\alpha|^{-1}\ln^2(M_\varepsilon+2)$, where C''' is the constant in the right-hand side of (5.17). We then deduce the existence of a measure $g_{\alpha,\varepsilon}$ on $\mathbb{T}^2_{\ell^\varepsilon}$ supported in D_α such that $g_{\alpha,\varepsilon}\geq -C'''|D_\alpha|^{-1}\ln^2(M_\varepsilon+2)$ and such that for every Lipschitz function χ

$$\begin{aligned}\left|\int_{D_\alpha}\chi(f_{\alpha,\varepsilon}-g_{\alpha,\varepsilon})\,\mathrm{d}x'\right| &\leq 2\operatorname{diam}(D_\alpha)\|\nabla\chi\|_{L^\infty(D_\alpha)}f^-_{\alpha,\varepsilon}(D_\alpha)\\ &\leq C\ln(n_{\alpha_s}+M_\varepsilon+2)\|\nabla\chi\|_{L^\infty(D_\alpha)}\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}}|\tilde{A}^\varepsilon_i|^2,\end{aligned} \tag{5.20}$$

and we have used the observation that

$$f_{\alpha,\varepsilon}=\frac{1}{2k'}\left(|\nabla h'_\varepsilon|^2+\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathbf{1}_{C_\alpha}+\frac{1}{2\pi}\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}}\left(\ln\frac{\rho}{\bar{r}_\alpha}-C\right)|\tilde{A}^\varepsilon_i|^2\delta^\varepsilon_i, \tag{5.21}$$

and (5.15) to bound the negative part of $f_{\alpha,\varepsilon}$. In particular, taking $\chi=1$, we deduce, in view of (5.17), that

$$g_{\alpha,\varepsilon}(D_\alpha)=f_{\alpha,\varepsilon}(D_\alpha)\geq\frac{c}{2}\beta^2n^2_{\alpha_s}+\frac{c}{2}\Bigg(\sum_{i\in I^b_{\beta,\alpha}}A^\varepsilon_i\Bigg)^2-C'''\ln^2(M_\varepsilon+2), \tag{5.22}$$

from which it follows that

$$g_{\alpha,\varepsilon}(D_\alpha)\geq c'(n^2_{\alpha_s}+(\#I_{\beta,\alpha})^2)-C'''\ln^2(M_\varepsilon+2)\geq\frac{1}{2}c'n^2_\alpha-C'''\ln^2(M_\varepsilon+2). \tag{5.23}$$

Recalling the positivity of $g_{B,\varepsilon}$ introduced in Step 2, we now let

$$g_\varepsilon:=\sum_{B\in\mathcal{B}_\rho}g_{B,\varepsilon}+\sum_\alpha g_{\alpha,\varepsilon}+\Bigg(f'_\varepsilon-\sum_\alpha f_{\alpha,\varepsilon}\Bigg), \tag{5.24}$$

and observe that since $f'_\varepsilon-\sum_\alpha f_{\alpha,\varepsilon}$ is also non-negative by (5.9), and since $\sum_\alpha g_{\alpha,\varepsilon}$ is bounded below by $-k'C'''|D_\alpha|^{-1}\ln^2(M_\varepsilon+2)$, where, as before, k' is the overlap number of $\{D_\alpha\}_\alpha$, we have $g_\varepsilon\geq -c\ln^2(M_\varepsilon+2)$ for some $c>0$ depending only on κ and $\bar{\delta}$, which proves the first item. The second item follows from (5.23), (5.24) and the positiveness of $g_{B,\varepsilon}$ and $\left(f'_\varepsilon-\sum_\alpha f_{\alpha,\varepsilon}\right)$.

Step 4: Proof of the last item.

Using the definition of g_ε in (5.24), for any Lipschitz χ we have

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}}\chi g_\varepsilon\,\mathrm{d}x'=\sum_{B\in\mathcal{B}_\rho}\int_{\mathbb{T}^2_{\ell^\varepsilon}}\chi g_{B,\varepsilon}\,\mathrm{d}x'+\sum_\alpha\int_{\mathbb{T}^2_{\ell^\varepsilon}}\chi(g_{\alpha,\varepsilon}-f_{\alpha,\varepsilon})\,\mathrm{d}x'+\int_{\mathbb{T}^2_{\ell^\varepsilon}}\chi f'_\varepsilon\,\mathrm{d}x'.$$

Hence, in view of (5.5), (5.7) and (5.20) we obtain for some $C > 0$

$$\left|\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(f_\varepsilon - g_\varepsilon)\, \mathrm{d}x'\right| \le \sum_{B\in\mathcal{B}_\rho} \left|\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(g_{B,\varepsilon} - f_{B,\varepsilon})\, \mathrm{d}x'\right)\right|$$
$$+\sum_\alpha \left|\int_{\mathbb{T}^2_{\ell^\varepsilon}} \chi(g_{\alpha,\varepsilon} - f_{\alpha,\varepsilon})\, \mathrm{d}x'\right| \le C \sum_{B\in\mathcal{B}_\rho} \nu^\varepsilon(B)\|\nabla\chi\|_{L^\infty(B)}$$
$$+C\sum_\alpha \ln(n_{\alpha_s} + M_\varepsilon + 2)\|\nabla\chi\|_{L^\infty(D_\alpha)} \sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2. \tag{5.25}$$

Using that $|\tilde{A}^\varepsilon_i|^2 \le C(A^\varepsilon_i + 1)$ for a universal $C > 0$ and (5.13), we have

$$\sum_{i\in I_{\beta,\mathcal{B}^\alpha_\rho}} |\tilde{A}^\varepsilon_i|^2 \le C(n_{\alpha_s} + M_\varepsilon).$$

Since $n_{\alpha_s} \le n_\alpha$, the third item follows from (5.25). □

We now apply Proposition 5.1 to establish uniform bounds on M_ε, which characterizes the deviation of the droplets from the optimal shape.

Proposition 5.2. *If* (2.21) *holds, then M_ε is bounded by a constant depending only on* $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$, κ, $\bar{\delta}$ *and* ℓ.

Proof. From the last item of Proposition 5.1 applied with $\chi \equiv 1$ together with the first item, we have

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} f_\varepsilon\, \mathrm{d}x' = \int_{\mathbb{T}^2_{\ell^\varepsilon}} g_\varepsilon\, \mathrm{d}x' \ge -C|\ln\varepsilon|\ln^2(M_\varepsilon + 2),$$

with some $C > 0$ depending only on κ, $\bar{\delta}$ and ℓ, while from (2.21), (3.12) and (5.2), we have

$$C' \ge \ell^2 F^\varepsilon[u^\varepsilon] \ge M_\varepsilon + \frac{2}{|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}} f_\varepsilon\, \mathrm{d}x' + o(1) \ge M_\varepsilon - C\ln^2(M_\varepsilon + 2) + o_\varepsilon(1),$$

for some $C' > 0$ depending only on $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$, κ, $\bar{\delta}$ and ℓ. The claimed result easily follows.

With the help of Proposition 5.2, an immediate consequence of Proposition 5.1 is the following conclusion.

Corollary 5.3. *There exists $C > 0$ depending only on κ, $\bar{\delta}$, ℓ and* $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$ *such that if g_ε is as in Proposition* 5.1 *and* (2.21) *holds, then $g_\varepsilon \ge -C$.*

In the following, we also define the modified energy density $\bar{g}_\varepsilon$, in which we include back the positive terms of M_ε and a half of $\frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2$ that had been “kept aside" instead of being included in f_ε:

$$\bar{g}_\varepsilon := g_\varepsilon + \frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2 + |\ln\varepsilon| \Bigg\{ \sum_i (P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\,\tilde{\delta}_i) + c_1 \sum_{A_i^\varepsilon > \pi 3^{2/3}\gamma^{-1}} A_i^\varepsilon \tilde{\delta}_i \\ + c_2 \sum_{\beta \le A_i^\varepsilon \le \pi 3^{2/3}\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 \tilde{\delta}_i + c_3 \sum_{A_i^\varepsilon < \beta} A_i^\varepsilon \tilde{\delta}_i \Bigg\} \tag{5.26}$$

where we recall $\bar{r}_\varepsilon = \left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3}$ and $\tilde{\delta}_i^\varepsilon$ is defined by (3.18). These extra terms will be used to control the shapes and sizes of the droplets as well as to control h'_ε. We also point out that in view of (5.2), (5.3) and (3.12), we have

$$\ell^2 F^\varepsilon[u^\varepsilon] \ge \frac{2}{|\ln\varepsilon|} \int_{\mathbb{T}^2_{\ell^\varepsilon}} \bar{g}_\varepsilon \,\mathrm{d}x' + o_\varepsilon(1). \tag{5.27}$$

6. Convergence

In this section we study the consequences of the hypothesis

$$\forall R > 0,\ \mathcal{C}_R := \limsup_{\varepsilon\to 0} \int_{K_R} \bar{g}_\varepsilon(x + x_\varepsilon^0)\,\mathrm{d}x < +\infty, \tag{6.1}$$

where $K_R = [-R, R]^2$ and (x_ε^0) is such that $x_\varepsilon^0 + K_R \subset \mathbb{T}^2_{\ell^\varepsilon}$. This corresponds to "good" blow up centers x_ε^0, and will be satisfied for most of them.

In order to obtain $o_\varepsilon(1)$ estimates on the energetic cost of each droplet under this assumption, we need good quantitative estimates for the deviations of the shape of the droplets from balls of the same volume. A convenient quantity that can be used to characterize these deviations is the *isoperimetric deficit*, defined as (in two space dimensions)

$$D(\Omega'_{i,\varepsilon}) := \frac{|\partial\Omega'_{i,\varepsilon}|}{\sqrt{4\pi|\Omega'_{i,\varepsilon}|}} - 1. \tag{6.2}$$

The isoperimetric deficit may be used to bound several types of geometric characteristics of $\Omega'_{i,\varepsilon}$ that measure their deviations from balls. The quantitative isoperimetric inequality, which holds for any set of finite perimeter, may be used to estimate the measure of the symmetric difference between $\Omega'_{i,\varepsilon}$ and a ball. More precisely, we have [16]

$$\alpha(\Omega'_{i,\varepsilon}) \le C\sqrt{D(\Omega'_{i,\varepsilon})}, \tag{6.3}$$

where $C > 0$ is a universal constant and $\alpha(\Omega'_{i,\varepsilon})$ is the Fraenkel asymmetry defined as

$$\alpha(\Omega'_{i,\varepsilon}) := \min_B \frac{|\Omega'_{i,\varepsilon} \triangle B|}{|\Omega'_{i,\varepsilon}|}, \tag{6.4}$$

where $\triangle$ denotes the symmetric difference between the two sets, and the infimum is taken over balls B with $|B| = |\Omega'_{i,\varepsilon}|$. In the following, we will use the notation r_i^ε and a_i^ε for the radii and the centers of the balls that minimize $\alpha(\Omega'_{i,\varepsilon})$, respectively.

On the other hand, in two space dimensions the following inequality due originally to Bonnesen [6] (for a review, see [30]) is applicable to $\Omega'_{i,\varepsilon}$:

$$R_i^\varepsilon \leq r_i^\varepsilon \left(1 + c\sqrt{D(\Omega'_{i,\varepsilon})}\right). \tag{6.5}$$

Here R_i^ε is the radius of the circumscribed circle of the measure theoretic interior of $\Omega'_{i,\varepsilon}$ and $c > 0$ is universal, provided $D(\Omega'_{i,\varepsilon})$ is small. Indeed, simply apply Bonnesen inequality (in the form of [30, Eq. (20)]) to the saturation of $\Omega'_{i,\varepsilon}$ (that is, the set with holes filled in). Since the set $\Omega'_{i,\varepsilon}$ is connected and, therefore, its saturation has, up to negligible sets, a Jordan boundary [4], Bonnesen inequality applies.

6.1. Main Result

We will obtain local lower bounds in terms of the renormalized energy for a finite number of Dirac masses in the manner of [5]:

Definition 6.1. For any function χ and $\varphi \in \mathcal{A}_m$ (cf. Definition 2.1), we denote

$$W(\varphi, \chi) = \lim_{\eta \to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi |\nabla\varphi|^2 \, \mathrm{d}x + \pi \ln \eta \sum_{p\in\Lambda} \chi(p) \right). \tag{6.6}$$

We now state the main result of this section and postpone its proof to Section 6.2. Throughout the section, we use the notation of Section 5. To further simplify the notation, we periodically extend all the measures defined on $\mathbb{T}^2_{\ell^\varepsilon}$ to the whole of $\mathbb{R}^2$, without relabeling them. We also periodically extend the ball constructions to the whole of $\mathbb{R}^2$. This allows us to set, without loss of generality, all $x_\varepsilon^0 = 0$.

Theorem 4. *Under assumption (2.21), the following holds.*

1. *Assume that for any $R > 0$ we have*

$$\limsup_{\varepsilon\to 0} \bar{g}_\varepsilon(K_R) < +\infty, \tag{6.7}$$

where $K_R = [-R, R]^2$. Then, up to a subsequence, the measures μ'_ε, defined in (3.17), converge in $(C_0(\mathbb{R}^2))^*$ to a measure of the form $\nu = 3^{2/3}\pi \sum_{a\in\Lambda} \delta_a$ where Λ is a discrete subset of $\mathbb{R}^2$, and $\{\varphi^\varepsilon\}_\varepsilon$ defined in (2.17) converge weakly in $\dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p \in (1, 2)$ to φ which satisfies

$$-\Delta\varphi = 2\pi \sum_{a\in\Lambda} \delta_a - m \text{ in } \mathbb{R}^2,$$

in the distributional sense, with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$. Moreover, for any sequence $\{\Omega_{i_\varepsilon,\varepsilon}\}_\varepsilon$ which remains in K_R, up to a subsequence, the following two alternatives hold:

i. *Either* $A^{\varepsilon}_{i_\varepsilon} \leq \frac{C_R}{|\ln\varepsilon|}$ *and* $P^{\varepsilon}_{i_\varepsilon} \leq \frac{C_R}{\sqrt{|\ln\varepsilon|}}$ *as* $\varepsilon \to 0$,

ii. *Or* $A^{\varepsilon}_{i_\varepsilon}$ *is bounded below by a positive constant as* $\varepsilon \to 0$, *and*

$$A^{\varepsilon}_{i_\varepsilon} \to 3^{2/3}\pi \text{ and } P^{\varepsilon}_{i_\varepsilon} \to 2\cdot 3^{1/3}\pi \quad \text{as } \varepsilon \to 0,$$

with

$$\alpha(\Omega'_{i_\varepsilon,\varepsilon}) \leq \frac{C_R}{|\ln\varepsilon|^{1/2}} \quad \text{as } \varepsilon \to 0, \tag{6.8}$$

for some $C_R > 0$ *independent of* ε.

2. *If we replace* (6.7) *by the stronger assumption*

$$\limsup_{\varepsilon\to 0} \bar{g}_\varepsilon(K_R) < CR^2, \tag{6.9}$$

where $C > 0$ *is independent of* R, *then we have for any* $p \in (1,2)$,

$$\limsup_{R\to+\infty} \left(\frac{1}{|K_R|} \int_{K_R} |\nabla\varphi|^p \,\mathrm{d}x \right) < +\infty. \tag{6.10}$$

Moreover, for every family $\{\chi_R\}_{R>0}$ *defined in Definition* 2.3 *we have*

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \chi_R \bar{g}_\varepsilon \,\mathrm{d}x \geq \frac{3^{4/3}}{2} W(\varphi,\chi_R) + \frac{3^{4/3}\pi}{8} \sum_{a\in\Lambda} \chi_R(a) + o(|K_R|). \tag{6.11}$$

Remark 6.2. We point out that it is included in Part 1 of Theorem 4 that at most one droplet $\Omega'_{i_\varepsilon,\varepsilon}$ with $A_{i_\varepsilon,\varepsilon}$ bounded from below converges to $a \in \Lambda$. Indeed otherwise in the first item we would have $\mu'_\varepsilon \to 3^{2/3}\pi n_a \sum_{a\in\Lambda}\delta_a$ where $n_a > 1$ is the number of non-vanishing droplets converging to the point a.

Theorem 4 relies crucially on the following proposition which establishes bounds needed for compactness. Each of the bounds relies on (6.7). Throughout the rest of this section, all constants are assumed to implicitly depend on κ, $\bar{\delta}$, ℓ and $\sup_{\varepsilon>0} F^\varepsilon[u^\varepsilon]$.

Lemma 6.3. *Let* $\bar{g}_\varepsilon$ *be as above, assume* (6.7) *holds and denote* $\mathcal{C}_R = \limsup_{\varepsilon\to 0} \bar{g}_\varepsilon(K_R)$. *Then for any* R *and* ε *small enough depending on* R *we have*

$$\sum_{\alpha | U_\alpha \subset K_R} n_\alpha^2 \leq C(\mathcal{C}_{R+C} + R^2), \tag{6.12}$$

$$\sum_{i\in I_{\beta,K_R}} A_i^\varepsilon \leq C(\mathcal{C}_{R+C} + R^2), \tag{6.13}$$

$$\left| \int_{K_R} \chi_R (f_\varepsilon - g_\varepsilon)\,\mathrm{d}x \right| \leq C \sum_{\alpha | U_\alpha \subset K_{R+C}\setminus K_{R-C}} (n_\alpha + 1)\ln(n_\alpha + 2)$$
$$\leq C(\mathcal{C}_{R+C} + R^2), \tag{6.14}$$

where $\{\chi_R\}$ *is as in Definition* 2.3 *and* $n_\alpha = \#I_{\beta,U_\alpha}$, *with* U_α *as in the proof of Proposition* 5.1, *for some* $C > 0$ *independent of* ε *or* R. *Furthermore, for any* $p \in (1, 2)$ *there exists a* $C_p > 0$ *depending on* p *such that for any* $R > 0$ *and* ε *small enough*

$$\int_{K_R} |\nabla h'_\varepsilon|^p \,\mathrm{d}x \le C_p(\mathcal{C}_{R+C} + R^2). \tag{6.15}$$

Proof. First observe that the rescaled droplet volumes and perimeters A_i^ε and P_i^ε are bounded independently of ε, as follows from Proposition 5.2 and the definition of M_ε. Then, (6.12) and (6.13) are a consequence of (6.7), the second item in Proposition 5.1 together with the upper bound on M_ε. The first inequality appearing in (6.14) follows from item 3 of Proposition 5.1 with the bound on M_ε, where we took into consideration that only those D_α that are in the $O(1)$ neighborhood of the support of $|\nabla \chi_R|$ contribute to the sum, along with the observation that the mass of ν^ε (of (5.1)) is now controlled by n_α (a consequence of the above fact that all droplet volumes are uniformly bounded). The second inequality in (6.14) follows from (6.12). The bound (6.15) is a consequence of Proposition 4.2 and follows as in [32] and [36]. We refer the reader to [32], Lemma 4.6 or [36] Lemma 4.6 for the proof in a slightly simpler setting.

6.2. *Lower Bound by the Renormalized Energy* (*Proof of Theorem 4*)

We start by proving the first assertions of the theorem.

Step 1 All limit droplets have optimal sizes. From (5.26), (6.7) and Corollary 5.3, for all ε sufficiently small depending on R we have

$$\int_{K_R} \Bigg(\sum_i \left(P_i - \sqrt{4\pi |A_i^\varepsilon|}\right) \tilde{\delta}_i^\varepsilon + c_1 \sum_{A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}} A_i^\varepsilon \tilde{\delta}_i^\varepsilon \\ + c_2 \sum_{\beta \le A_i^\varepsilon \le \pi 3^{2/3}\gamma^{-1}} (A_i^\varepsilon - \pi \bar{r}_\varepsilon^2)^2 \tilde{\delta}_i^\varepsilon + c_3 \sum_{A_i^\varepsilon < \beta} A_i^\varepsilon \tilde{\delta}_i^\varepsilon \Bigg) \mathrm{d}x \le \frac{C_R}{|\ln \varepsilon|}, \tag{6.16}$$

where we recall that all the terms in the sums are nonnegative. It then easily follows that for all $i \in I_{K_R}$ the droplets with $A_i^\varepsilon > 3^{2/3}\pi\gamma^{-1}$ do not exist when ε is small enough depending on R, and those with $A_i^\varepsilon < \beta$ satisfy $A_i^\varepsilon = C_R |\ln \varepsilon|^{-1}$ and $P_i^\varepsilon \le C_R |\ln \varepsilon|^{-1/2}$, for some $C_R > 0$ independent of ε. This establishes item (i) of Part 1 of the theorem.

It remains to treat the case of $A_i^\varepsilon \in [\beta, 3^{2/3}\pi\gamma^{-1}]$ when ε is small enough. It follows from (6.16) that

$$D(\Omega'_{i,\varepsilon}) \le \frac{C_R}{|\ln \varepsilon|}, \tag{6.17}$$

for some $C_R > 0$ independent of ε, and since $\bar{r}_\varepsilon = 3^{1/3} + o_\varepsilon(1)$, for all these droplets (or equivalently for all droplets with $A_i^\varepsilon \ge \beta$) we must have

$$A_i^\varepsilon \to 3^{2/3}\pi \quad \text{and} \quad P_i^\varepsilon \to 2\cdot 3^{1/3}\pi \quad \text{as } \varepsilon \to 0. \tag{6.18}$$

Using (6.3), (6.8) easily follows from (6.18) and (6.16).

Step 2: Convergence results.

From boundedness of A_i^ε, (6.13) and (6.16) we know that $\# I_{\beta,K_R}$ and $\mu'_\varepsilon(K_R)$ are both bounded independently of ε as $\varepsilon \to 0$. We easily deduce from this, the previous step and the definition of μ'_ε that up to extraction, μ'_ε converges in each K_R to at most finitely many point masses which are integer multiples of $3^{2/3}\pi$ and, hence, to a measure of the form $\nu = 3^{2/3}\pi \sum_{a\in\Lambda} d_a\delta_a$, where $d_a \in \mathbb{N}$ and Λ is a discrete set in the whole of $\mathbb{R}^2$. In view of (6.15), we also have $h'_\varepsilon \rightharpoonup h \in \dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$, up to extraction (recall that we work with equivalence classes from (2.10)). Finally, from the definition of $\bar{g}_\varepsilon$ in (5.26) and the bound (6.7) we deduce that

$$\frac{\kappa^2}{|\ln\varepsilon|}\int_{K_R} |h'_\varepsilon|^2 \le C_R \tag{6.19}$$

from which it follows that $|\ln\varepsilon|^{-1}h'_\varepsilon$ tends to 0 in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Passing to the limit in the sense of distributions in (3.16), we then deduce from the above convergences that we must have

$$-\Delta h = 3^{2/3}\pi \sum_{a\in\Lambda} d_a\delta_a - \bar{\mu} \quad \text{on } \mathbb{R}^2. \tag{6.20}$$

We will show below that $d_a = 1$ for every $a \in \Lambda$, and when this is done, this will complete the proof of the first item after recalling $\varphi^\varepsilon = 2\cdot 3^{-2/3}h'_\varepsilon$ and $m = 2\cdot 3^{-2/3}\bar{\mu}$.

Step 3: There is only one droplet converging to any limit point a.
In order to prove this statement, we examine lower bounds for the energy. Fix $R > 1$ such that $\partial K_R \cap \Lambda = \varnothing$ and consider $a \in \Lambda \cap K_R$. From Step 1, (2.2) and Lemma 4.1, for any $\eta \in (0, \frac{1}{2})$ such that $\eta < \frac{1}{2}\min_{b\in\Lambda\cap K_R\setminus\{a\}} |a-b|$ and for all $r < \eta$, all the droplets converging to a are covered by $B(a,r)$, and $B(a,\eta)$ contains no other droplets with $A_i^\varepsilon \ge \beta$, for ε small enough. There are $d_a \ge 1$ droplets in $B(a,r)$ such that $A_i^\varepsilon \to 3^{2/3}\pi$ as $\varepsilon \to 0$, let us relabel them as $\Omega'_{1,\varepsilon}, \ldots, \Omega'_{d_a,\varepsilon}$.

Let $U = B(a,\eta)$. Arguing as in the proof of the first item of Proposition 4.2, by (6.18), we may construct a collection $\mathcal{B}_0$ of disjoint closed balls covering $\bigcup_{i\in I_{\beta,U}} \Omega'_{i,\varepsilon}$ and satisfying

$$r(\mathcal{B}_0) \le C d_a \rho_\varepsilon < \eta, \tag{6.21}$$

for some universal $C > 0$, provided ε is small enough, and a collection of disjoint balls $\mathcal{B}_r$ covering $\mathcal{B}_0$ of total radius $r \in [r(\mathcal{B}_0), \eta]$. Choosing $r = \eta^3$, which is always possible for small enough ε, it is clear that $\mathcal{B}_{\eta^3}$ consists of only a single ball contained in $B(a, \frac{3}{2}\eta^3)$ for ε small enough. Applying the second item of Proposition 4.2 to that ball, we then obtain

$$\int_{\mathcal{B}_{\eta^3}} \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right) \mathrm{d}x' \ge \frac{1}{2\pi}\left(\ln\frac{\eta^3}{r(\mathcal{B}_0)} - c\eta^3\right)\sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2. \tag{6.22}$$

Therefore, we have

$$\int_{\mathcal{B}_{\eta^3}} \chi_R \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln \varepsilon|} |h'_\varepsilon|^2 \right) \mathrm{d}x' \\ \geq \frac{1}{2\pi} \left(\ln \frac{\eta^3}{r(\mathcal{B}_0)} - c\eta^3 \right) \big(\min_{B(a,\eta)} \chi_R \big) \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2. \tag{6.23}$$

On the other hand, we can estimate the contribution of the remaining part of $B(a, \eta)$ as

$$\int_{B(a,\eta)\setminus\mathcal{B}_{\eta^3}} \chi_R |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{B(a,\eta)} \chi_R |h'_\varepsilon|^2 \,\mathrm{d}x' \\ \geq \left(\min_{B(a,\eta)} \chi_R \right) \left(\int_{B(a,\eta)\setminus B(a,2\eta^3)} |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{B(a,\eta)} |h'_\varepsilon|^2 \,\mathrm{d}x' \right) \\ \geq \left(\min_{B(a,\eta)} \chi_R \right) \int_{2\eta^3}^{\eta} \left(\int_{\partial B(a,r_B)} |\nabla h'_\varepsilon|^2 \,\mathrm{d}\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{B(a,r_B)} |h'_\varepsilon|^2 \,\mathrm{d}x' \right) \mathrm{d}r_B. \tag{6.24}$$

Arguing as in (4.12) and using the fact that $\eta < \frac{1}{2}$, we obtain

$$\int_{B(a,\eta)\setminus\mathcal{B}_{\eta^3}} \chi_R |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \frac{\kappa^2}{4|\ln \varepsilon|} \int_{B(a,\eta)} \chi_R |h'_\varepsilon|^2 \,\mathrm{d}x' \\ \geq \frac{1}{2\pi} \left(\min_{B(a,\eta)} \chi_R \right) \ln \frac{1}{2\eta^2} \left(\sum_{i=1}^{d_a} A_i^\varepsilon \right)^2 (1 - C\eta), \tag{6.25}$$

where $C > 0$ is independent of η and ε, for small enough ε.

We will now use crucially the fact shown in Step 1 that all $A_i^\varepsilon \geq \beta$ approach the same limit as $\varepsilon \to 0$. We begin by adding (6.22) and (6.25) and subtracting $\frac{1}{2\pi} |\ln \rho_\varepsilon| \sum_{i=1}^{d_a} |\tilde{A}_i^\varepsilon|^2 \chi_R^i$ from both sides. With the help of (6.21) we can cancel out the leading order $O(|\ln \rho_\varepsilon|)$ term in the right-hand side of the obtained inequality. Replacing $\tilde{A}_i^\varepsilon$ and A_i^ε with $3^{2/3}\pi + o_\varepsilon(1)$ in the remaining terms and using the fact that $\min_{B(a,\eta)} \chi_R \geq \chi_R(a) - 2\eta \|\nabla \chi_R\|_\infty$ on $B(a, \eta)$, we then find

$$\int_{B(a,\eta)} \chi_R \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{2|\ln \varepsilon|} |h'_\varepsilon|^2 - \frac{1}{2\pi} |\ln \rho_\varepsilon| \nu^\varepsilon \right) \mathrm{d}x' \\ \geq \frac{3^{4/3}\pi}{2} \chi_R(a) \left(d_a^2 \ln \frac{1}{2\eta^2} + d_a \ln \frac{\eta^3}{2} \right) - C, \tag{6.26}$$

where $C > 0$ is independent of ε or η.

Now, adding up the contributions of all $a \in \Lambda \cap K_R$ and recalling the definition of f_ε in (5.2), we conclude that on the considered sequence

$$\limsup_{\varepsilon\to 0} \int_{K_R} \chi_R f_\varepsilon \,\mathrm{d}x' \geq \limsup_{\varepsilon\to 0} \sum_{a\in\Lambda\cap K_R} \int_{B(a,\eta)} \chi_R f_\varepsilon \,\mathrm{d}x' \\ \geq \frac{3^{4/3}\pi}{2} |\ln \eta| \sum_{a\in\Lambda\cap K_R} (2d_a^2 - 3d_a) \chi_R(a) - C, \tag{6.27}$$

for some $C > 0$ is independent of ε or η. In particular, since $\chi_R(a) > 0$ for all $a \in \Lambda \cap K_R$, the right-hand side of (6.27) goes to plus infinity as $\eta \to 0$, unless all $d_a = 1$. But by the estimate (6.14) of Proposition 6.3, Corollary 5.3 and our assumption in (6.7) together with (5.26), the left-hand side of (6.27) is bounded independently of η, which yields the conclusion.

Step 4: Energy of each droplet. Now that we know that for each $a_i \in \Lambda \cap K_R$ there exists exactly one droplet $\Omega'_{i,\varepsilon}$ such that $a_i^\varepsilon \to a_i$ and $A_i^\varepsilon \to 3^{2/3}\pi$, we can extract more precisely the part of energy that concentrates in a small ball around each such droplet. Let B_i be a ball that minimizes Fraenkel asymmetry defined in (6.4), that is, let $B_i = B(a_i^\varepsilon, r_i^\varepsilon)$, and let B be a ball of radius r_B centered at a_i^ε. Arguing as in (4.12) in the proof of the second item of Proposition 4.2, we can write

$$\int_{\partial B} |\nabla h'_\varepsilon|^2 \mathrm{d}\mathcal{H}^1(x') + \frac{\kappa^2}{4|\ln\varepsilon|}\int_B |h'_\varepsilon|^2\,\mathrm{d}x' \geq \frac{\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}|\Omega'_{i,\varepsilon}\cap B|^2}{2\pi r_B}(1 - cr_i^\varepsilon). \tag{6.28}$$

Observe that by the definition of Fraenkel asymmetry

we have $|\Omega'_{i,\varepsilon}\cap B| \geq |B| - \frac{1}{2}\alpha(\Omega'_{i,\varepsilon})|B_i|$ for all $r_B < r_i^\varepsilon$. Hence, denoting by $\tilde{r}_i^\varepsilon$ the smallest value of r_B for which the right-hand side of this inequality is non-negative and integrating from $\tilde{r}_i^\varepsilon$ to r_i^ε, we find

$$\int_{B_i}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathrm{d}x' \geq \frac{\pi}{2}(1 + o_\varepsilon(1))\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3} \times \int_{\tilde{r}_i^\varepsilon}^{r_i^\varepsilon} r_B^{-1}(r_B^2 - |\tilde{r}_i^\varepsilon|^2)^2 \mathrm{d}r_B. \tag{6.29}$$

Since by (6.8) and (6.18) we have $\tilde{r}_i^\varepsilon/r_i^\varepsilon \to 0$ and $\varepsilon^{-1/3}|\ln\varepsilon|^{-1/6}r_i^\varepsilon \to 3^{1/3}$ as $\varepsilon \to 0$, after an elementary computation we find

$$\int_{\Omega'_{i,\varepsilon}}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathrm{d}x' \geq \frac{3^{4/3}\pi}{8} + o_\varepsilon(1). \tag{6.30}$$

On the other hand, by (6.5) and (6.17) it is possible to choose a collection $\mathcal{B}_0 \subset B(a_i, \eta)$, actually consisting of only a single ball $B(\tilde{a}_i^\varepsilon, R_i^\varepsilon)$ circumscribing $\Omega'_{i,\varepsilon}$, so that

$$r(\mathcal{B}_0) = R_i^\varepsilon \leq r_i^\varepsilon(1 + C_R|\ln\varepsilon|^{-1/2}) = \rho_\varepsilon + o_\varepsilon(\rho_\varepsilon). \tag{6.31}$$

The corresponding ball construction $\mathcal{B}_r$ of the first item of Proposition 4.2, with $U = B(a_i, \eta)$ and η as in Step 3 of the proof (again, just a single ball $B(\tilde{a}_i^\varepsilon, r)$), exists and is contained in U for all $r \in [r(\mathcal{B}_0), \eta']$, for any $\eta' \in (r(\mathcal{B}_0), \eta)$, provided ε is sufficiently small depending on η'. In view of the fact that for small enough η'

and small enough ε depending on η' we have $\chi_R(x) \geq \chi_R(\tilde{a}_i^\varepsilon) - c|x - \tilde{a}_i^\varepsilon| > 0$, with $c > 0$ independent of ε, η' or R, we obtain that

$$\begin{aligned}
&\int_{B(\tilde{a}_i^\varepsilon,\eta')\setminus\mathcal{B}_0} \chi_R |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \frac{\kappa^2}{4|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} \chi_R |h'_\varepsilon|^2 \,\mathrm{d}x' \\
&\geq \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)\left(\int_{\partial\mathcal{B}_r} |\nabla h'_\varepsilon|^2 \mathrm{d}\mathcal{H}^1(x)\right)\mathrm{d}r + \frac{\kappa^2 \chi_R(\tilde{a}_i^\varepsilon)}{8|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} |h'_\varepsilon|^2 \,\mathrm{d}x' \\
&\geq \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)\left(\int_{\partial\mathcal{B}_r} |\nabla h'_\varepsilon|^2 \mathrm{d}\mathcal{H}^1(x) + \frac{\kappa^2}{8\eta'|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} |h'_\varepsilon|^2 \,\mathrm{d}x'\right)\mathrm{d}r \\
&\geq \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)\left(\int_{\partial\mathcal{B}_r} |\nabla h'_\varepsilon|^2 \mathrm{d}\mathcal{H}^1(x) + \frac{\kappa^2}{4|\ln\varepsilon|}\int_{\mathcal{B}_r} |h'_\varepsilon|^2 \,\mathrm{d}x'\right)\mathrm{d}r \\
&\geq \frac{1}{2\pi}|\tilde{A}_i^\varepsilon|^2 \int_{r(\mathcal{B}_0)}^{\eta'} (\chi_R(\tilde{a}_i^\varepsilon) - cr)(1 - Cr)\,\frac{\mathrm{d}r}{r},
\end{aligned} \tag{6.32}$$

for η' and ε sufficiently small, arguing as in (4.12) in the proof of Proposition 4.2 and taking into account Remark 4.4 in deducing the last line. Performing integration in (6.32) and using (6.31), we then conclude

$$\begin{aligned}
&\int_{B(\tilde{a}_i^\varepsilon,\eta')\setminus\mathcal{B}_0} \chi_R |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \frac{\kappa^2}{4|\ln\varepsilon|}\int_{B(\tilde{a}_i^\varepsilon,\eta')} \chi_R |h'_\varepsilon|^2 \,\mathrm{d}x' \\
&\quad\geq \frac{1}{2\pi}|\tilde{A}_i^\varepsilon|^2 \chi_R(\tilde{a}_i^\varepsilon) \ln\left(\frac{\eta'}{\rho_\varepsilon}\right) - C\eta',
\end{aligned} \tag{6.33}$$

for ε sufficiently small.

Step 5: Convergence. Using the fact, seen in Step 2, that $h'_\varepsilon \rightharpoonup h$ in $\dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$, we have, by lower semi-continuity,

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda} B(a,\eta)} \chi_R |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' \geq \int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda} B(a,\eta)} \chi_R |\nabla h|^2 \,\mathrm{d}x'. \tag{6.34}$$

On the other hand, in view of $\chi_R(\tilde{a}_i^\varepsilon) = \chi_R^i + O(\rho_\varepsilon)$ by (6.31), from (6.33) we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{B(a_i^\varepsilon,\eta)\setminus\mathcal{B}_0} \chi_R |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \int_{B(a_i^\varepsilon,\eta)} \chi_R \left(\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right)\mathrm{d}x' \\
&\geq \liminf_{\varepsilon\to 0} \int_{B(\tilde{a}_i^\varepsilon,\eta')\setminus\mathcal{B}_0} \chi_R |\nabla h'_\varepsilon|^2 \,\mathrm{d}x' + \int_{B(\tilde{a}_i^\varepsilon,\eta')} \chi_R \\
&\quad\times \left(\frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2 - \frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right)\mathrm{d}x' \geq \frac{3^{4/3}\pi}{2}\chi_R(a_i)\ln\eta' - C\eta',
\end{aligned} \tag{6.35}$$

where we also used that $\chi_R^i \to \chi_R(a)$ as $\varepsilon \to 0$.

We now convert the estimate in (6.30) to one over $\mathcal{B}_0$ and involving χ_R as well. Observing that $\Omega'_{i,\varepsilon} \subseteq \mathcal{B}_0$ and that $\chi_R(x') \geq \chi_i^R - 4\rho_\varepsilon\|\nabla\chi_R\|_\infty$ for all $x' \in \Omega'_{i,\varepsilon}$ and ε small enough by (6.31), from (6.30) and (3.1) we obtain

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{B}_0} \chi_R \left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{4|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathrm{d}x' \geq \frac{3^{4/3}\pi}{8}\chi_R(a_i), \tag{6.36}$$

where we used the fact that by (3.2), (3.12) and (4.5) the integral in the left-hand side of (6.30) may be bounded by $C|\ln\varepsilon|$, for some $C > 0$ independent of ε and R. Adding up (6.34) with (6.35) and (6.36) summed over all $a_i \in K_R$, in view of the arbitrariness of $\eta' < \eta$ we then obtain

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{R}^2}\chi_R\left(|\nabla h'_\varepsilon|^2+\frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2-\frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right)\mathrm{d}x' \geq \int_{\mathbb{R}^2\setminus\cup_{a\in\Lambda}B(a,\eta)}\chi_R|\nabla h|^2\,\mathrm{d}x'+\frac{3^{4/3}\pi}{2}\sum_{a\in\Lambda}\chi_R(a)\left(\ln\eta+\frac{1}{4}\right)-C\eta. \tag{6.37}$$

Letting now $\eta\to 0$ in (6.37), and recalling that $\varphi = 2\cdot 3^{-2/3}h$ and that the definition of $W(\varphi,\chi)$ is given by Definition 6.1, we obtain

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{R}^2}\chi_R\left(|\nabla h'_\varepsilon|^2+\frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2-\frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon\right)\mathrm{d}x' \geq \frac{3^{4/3}}{2}W(\varphi,\chi_R)+\frac{3^{4/3}\pi}{8}\sum_{a\in\Lambda}\chi_R(a). \tag{6.38}$$

From (6.14) we may replace $f_\varepsilon = |\nabla h'_\varepsilon|^2+\frac{\kappa^2}{2|\ln\varepsilon|}|h'_\varepsilon|^2-\frac{1}{2\pi}|\ln\rho_\varepsilon|\nu^\varepsilon$ by g_ε in (6.38) with an additional error term:

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{R}^2}\chi_R g_\varepsilon\,\mathrm{d}x' \geq \frac{3^{4/3}}{2}W(\varphi,\chi_R)+\frac{3^{4/3}\pi}{8}\sum_{a\in\Lambda}\chi_R(a)-c\Delta(R), \tag{6.39}$$

where

$$\Delta(R)=\limsup_{\varepsilon\to 0}\sum_{\alpha|K_{R-C}\subset U_\alpha\subset K_{R+C}}(n_\alpha+1)\ln(n_\alpha+2),$$

for some $c, C > 0$ independent of R. Under hypothesis (6.9), from (6.12) we have

$$\limsup_{\varepsilon\to 0}\sum_{\alpha|U_\alpha\subset K_R}n_\alpha^2\leq CR^2,$$

and thus, using Hölder inequality and bounding the number of α's involved in the sum by CR we find

$$\Delta(R)\leq C\limsup_{\varepsilon\to 0}\sum_{\alpha|U_\alpha\subset K_{R+C}\setminus K_{R-C}}(n_\alpha^{3/2}+1) \leq C'R^{1/4}\limsup_{\varepsilon\to 0}\left(\sum_{\alpha|U_\alpha\subset K_{R+C}}n_\alpha^2\right)^{3/4}+CR\leq C''R^{7/4},$$

for some $C, C', C'' > 0$ independent of R. Hence

$$\limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\frac{\Delta(R)}{R^2}=0,$$

which together with (6.39) and the fact that $\bar{g}_\varepsilon \geq g_\varepsilon$ establishes (6.11). □

6.3. Local to Global Bounds via the Ergodic Theorem: Proof of Theorem 1, item i.

The proof follows the procedure outlined in [36]. We refer the reader to Sections 4 and 6 of [36] for the proof adapted to the case of the magnetic Ginzburg–Landau energy, which is essentially identical to the present one, with some simplifications due to the fact that we work on the torus. As in [36], we say that $\mu \in \mathcal{M}_0(\mathbb{R}^2)$, if the measure $\mathrm{d}\mu + C\,\mathrm{d}x$ is a positive locally bounded measure on $\mathbb{R}^2$, where C is the constant appearing in Corollary 5.3. The measures $\mathrm{d}\bar{g}_\varepsilon$ and the functions φ_ε will be alternatively seen as functions on $\mathbb{T}^2_{\ell^\varepsilon}$ or as periodically extended to the whole of $\mathbb{R}^2$, which will be clear from the context. We let χ be a smooth non-negative function on $\mathbb{R}^2$ with support in $B(0,1)$ and with $\int_{\mathbb{R}^2}\chi(x)\,\mathrm{d}x = 1$. We set $X = \dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2) \times \mathcal{M}_0(\mathbb{R}^2)$, and define for every $\mathbf{x} = (\varphi, g) \in X$ the following functional

$$\mathbf{f}(\mathbf{x}) := 2\int_{\mathbb{R}^2}\chi(y)\,\mathrm{d}g(y). \tag{6.40}$$

We note that from (5.27) we have for ε sufficiently small

$$F^\varepsilon[u^\varepsilon] + o_\varepsilon(1) \geq \frac{2}{\ell^2|\ln\varepsilon|}\int_{\mathbb{T}^2_{\ell^\varepsilon}}\mathrm{d}\bar{g}_\varepsilon = \fint_{\mathbb{T}^2_{\ell^\varepsilon}}\mathbf{f}(\theta_\lambda\mathbf{x}_\varepsilon)\,\mathrm{d}\lambda, \tag{6.41}$$

where $\mathbf{x}_\varepsilon := (\varphi^\varepsilon, \bar{g}_\varepsilon)$, θ_λ denotes the translation operator by $\lambda \in \mathbb{R}^2$, that is, $\theta_\lambda f(x) := f(x+\lambda)$, and $\fint$ stands for the average. Here the last equality follows by an application of Fubini's theorem and the fact that $\int_{\mathbb{R}^2}\chi(x)\,\mathrm{d}x = 1$.

It can be easily shown as in [36] that $\mathbf{f}_\varepsilon = \mathbf{f}$ satisfies the coercivity and Γ-liminf properties required for the application of Theorem 3 in [36] on sequences consisting of $\mathbf{x}_\varepsilon = (\varphi^\varepsilon, \bar{g}_\varepsilon)$ obtained from (u^ε) obeying (2.21). This is done by starting with a sequence $\{\mathbf{x}_\varepsilon\}_\varepsilon$ in X such that

$$\limsup_{\varepsilon\to 0}\int_{K_R}\mathbf{f}(\theta_\lambda\mathbf{x}_\varepsilon)d\lambda < +\infty, \tag{6.42}$$

for every $R > 0$, which implies that the integral is finite whenever ε is small enough. Consequently $\mathbf{f}_\varepsilon(\theta_\lambda\mathbf{x}_\varepsilon) < +\infty$ for almost every $\lambda \in K_R$. Applying Fubini's theorem again, (6.42) becomes

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^2}\chi_R(y)\,\mathrm{d}\bar{g}_\varepsilon(y) < +\infty,$$

where $\chi_R = \chi * \mathbf{1}_{K_R}$, and "$*$" denotes convolution. Then since $\chi_R = 1$ in K_{R-1} and $\bar{g}_\varepsilon$ is bounded below by a constant, the assumption (6.7) in Part 1 of Theorem 4 is satisfied, and we deduce from that theorem that φ^ε and $\bar{g}_\varepsilon$ converge, upon extraction of a subsequence, weakly in $\dot{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ and weakly in the sense of measures, respectively. Furthermore, if $\mathbf{x}_\varepsilon \to \mathbf{x} = (\varphi, g)$ on this subsequence, we have $2\int_{\mathbb{R}^2}\chi(y)\,\mathrm{d}\bar{g}_\varepsilon(y) = \mathbf{f}(\mathbf{x}_\varepsilon) \to \mathbf{f}(\mathbf{x}) = 2\int_{\mathbb{R}^2}\chi(y)\,\mathrm{d}\bar{g}(y)$.

We may then apply Theorem 3 of [36] to $\mathbf{f}$ on $\mathbb{T}^2_{\ell^\varepsilon}$ and conclude that the measure $\{\widetilde{P}^\varepsilon\}_\varepsilon$ defined as the push-forward of the normalized uniform measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by

$$\lambda \mapsto (\theta_\lambda\varphi^\varepsilon, \theta_\lambda\bar{g}_\varepsilon),$$

converges to a translation-invariant probability measure $\widetilde{P}$ on X with

$$\liminf_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \geq \int \mathbf{f}(\mathbf{x})\,\mathrm{d}\widetilde{P}(\mathbf{x}) = \int \mathbf{f}^*(\mathbf{x})\,\mathrm{d}\widetilde{P}(\mathbf{x}), \tag{6.43}$$

where

$$\mathbf{f}^*(\varphi, g) = \lim_{R\to\infty} \fint_{K_R} \mathbf{f}(\theta_\lambda \mathbf{x})\,\mathrm{d}\lambda = \lim_{R\to+\infty}\left(\frac{2}{|K_R|}\int_{\mathbb{R}^2} \chi_R(y)\,\mathrm{d}g(y)\right), \tag{6.44}$$

provided that $\mathbf{x}$ is in the support of $\widetilde{P}$.

The next step is to show that for $\widetilde{P}$-almost everywhere $\mathbf{x}$ we have $\varphi \in \mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$, and $\mathbf{f}^*$ can be computed. By [36, Remark 1.6], we have that for $\widetilde{P}$-a.e $\mathbf{x}$, there exists a sequence $\{\lambda_\varepsilon\}_\varepsilon$ such that $\mathbf{x}_\varepsilon = (\theta_{\lambda_\varepsilon}\varphi^\varepsilon, \theta_{\lambda_\varepsilon}\bar{g}_\varepsilon)$ converges to $\mathbf{x}$ in X. In addition, from (6.43)–(6.44), for $\widetilde{P}$-almost everywhere $\mathbf{x}$, we have

$$\lim_{R\to+\infty} \fint_{K_R} \mathbf{f}(\theta_\lambda \mathbf{x})\,\mathrm{d}\lambda < +\infty,$$

for $\widetilde{P}$-almost every $\mathbf{x}$. Using Fubini's theorem again, together with the definition of $\mathbf{f}$, we then find

$$\lim_{R\to+\infty}\left(\frac{1}{|K_R|}\int_{\mathbb{R}^2} \chi_R(y)\,\mathrm{d}g(y)\right) < +\infty.$$

Therefore, since

$$\int_{\mathbb{R}^2} \chi_R(y)\,\mathrm{d}\bar{g}_\varepsilon(y) \to \int_{\mathbb{R}^2} \chi_R(y)\,\mathrm{d}g(y) \qquad \text{as } \varepsilon \to 0, \tag{6.45}$$

a bound of the type (6.9) holds, and the results of Part 2 of Theorem 4 hold for $\mathbf{x}_\varepsilon$. In particular, we find that

$$-\Delta\varphi = 2\pi \sum_{a\in\Lambda} \delta_a - m, \tag{6.46}$$

with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$, and that

$$\mathbf{f}^*(\varphi, g) = \lim_{R\to\infty}\left(\frac{2}{|K_R|}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \chi_R \bar{g}_\varepsilon\,\mathrm{d}x\right) \geq 3^{4/3} W(\varphi) + \frac{3^{4/3}}{8} m. \tag{6.47}$$

The result in (6.47) follows from the definition of $\mathbf{f}^*$, (6.45), (6.11), the definition of W, provided we can show that

$$\lim_{R\to+\infty} \frac{1}{|K_R|}\sum_{a\in\Lambda} \chi_R(a) = \lim_{R\to+\infty} \frac{\nu(K_R)}{2\pi|K_R|} = \frac{m}{2\pi}. \tag{6.48}$$

The latter can be obtained from (6.15), exactly as in Lemma 4.11 of [36], so we omit the proof. Note that with (6.46), it proves that $\varphi \in \mathcal{A}_m$, and we thus have the claimed result. Combining (6.43) and (6.47), we obtain

$$\liminf_{\varepsilon\to 0} F^\varepsilon[u^\varepsilon] \geq \int \left(3^{4/3} W(\varphi) + \frac{3^{2/3}}{8}(\bar{\delta} - \bar{\delta}_c)\right) \mathrm{d}\widetilde{P}(\varphi, g).$$

Letting now P^ε and P be the first marginals of $\widetilde{P}^\varepsilon$ and $\widetilde{P}$ respectively, this proves (2.22) and the fact that P-almost every φ is in $\mathcal{A}_m$ with $m = 3^{-2/3}(\bar{\delta} - \bar{\delta}_c)$. □

7. Upper Bound Construction: Proof of Part (ii) of Theorem 1

We follow closely the construction performed for the magnetic Ginzburg–Landau energy in [36], but our situation is somewhat simpler, since we work on a torus (instead of a domain bounded by a free boundary). The construction given in [36] relies on a result stated as Corollary 4.5 in [36], which we repeat below with slight modifications to adapt it to our setting. These results imply, in particular, that the minimum of W may be approximated by sequences of periodic configurations of larger and larger period. Below for any discrete set of points Λ, $|\Lambda|$ will denote its cardinal.

Proposition 7.1. (Corollary 4.5 in [36]) *Let $p \in (1,2)$ and let P be a probability measure on $\dot{W}^{1,p}_{\text{loc}}(\mathbb{R}^2)$ which is invariant under the action of translations and concentrated on $\mathcal{A}_1$. Let Q be the push-forward of P under $-\Delta$. Then there exists a sequence $R \to \infty$ with $R^2 \in 2\pi\mathbb{N}$ and a sequence $\{b_R\}_R$ of $2R$-periodic vector fields such that:*

- *There exists a finite subset Λ_R of the interior of K_R such that*

$$\begin{cases} -\text{div}\, b_R = 2\pi \sum_{a\in\Lambda_R} \delta_a - 1 & \text{in } K_R \\ b_R \cdot \nu = 0 & \text{on } \partial K_R. \end{cases}$$

- *Letting Q_R be the probability measure on $W^{-1,p}_{\text{loc}}(\mathbb{R}^2)$, which is defined as the image of the normalized Lebesgue measure on K_R by $x \mapsto -\text{div}\, b_R(x+\cdot)$, we have $Q_R \to Q$ weakly as $R \to \infty$.*
- $\displaystyle \limsup_{R\to\infty} \frac{1}{|K_R|} \lim_{\eta\to 0} \left(\frac{1}{2} \int_{K_R \setminus \cup_{a\in\Lambda_R} B(a,\eta)} |b_R|^2 \, \mathrm{d}x + \pi |\Lambda_R| \ln \eta \right) \leq \int W(\varphi)\, \mathrm{d}P(\varphi).$

Remark 7.2. We would like to make the following observations concerning the vector field b_R constructed in Proposition 7.1.

1. By construction, the vector fields b_R has no distributional divergence concentrating on ∂K_R and its translated copies since $b_R \cdot \nu$ is continuous across ∂K_R. However, $b_R \cdot \tau$ may not be, and this may create a singular part of the distributional curl b_R. This is the difficulty that prevents us from stating the convergence result for P directly in Theorem 1, Part ii).
2. We also note that an inspection of the construction in [36] shows that b_R is curl-free in a neighborhood of each point $a \in \Lambda_R$ and that curl b_R belongs to $W^{-1,p}_{\text{loc}}(\mathbb{R}^2)$ for $p < \infty$.

7.1. Definition of the Test Configuration

We take R the sequence given by Proposition 7.1. The first thing to do is to change the density 1 into a suitably chosen density $m_{\varepsilon,R}$, in order to ensure the compatibility of the functions with the torus volume. Recalling that $\bar{\mu}^\varepsilon > 0$ for $\bar{\delta} > \bar{\delta}_c$ and ε small enough, we set

$$m_{\varepsilon,R} = \frac{4R^2}{|\ell^\varepsilon|^2} \left\lfloor \frac{\ell^\varepsilon \sqrt{2\bar{\mu}^\varepsilon}}{2R\bar{r}_\varepsilon} \right\rfloor^2 \tag{7.1}$$

where, as usual, $\lfloor x \rfloor$ denotes the integer part of a x. We note for later that

$$\left| m_{\varepsilon,R} - \frac{2\bar{\mu}^{\varepsilon}}{\bar{r}_{\varepsilon}^2} \right| \leq \frac{CR}{\ell^{\varepsilon}} = o_{\varepsilon}(1). \tag{7.2}$$

Recalling also that $\bar{r}_\varepsilon = 3^{1/3} + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)$ and $\bar{\mu}^\varepsilon - \bar{\mu} = O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)$, we deduce that $m_{\varepsilon,R} \to m$, where $m := 2\cdot 3^{-2/3}\bar{\mu}$, as $\varepsilon \to 0$, for each R. In particular, $m_{\varepsilon,R}$ is bounded above and below by constants independent of ε and R. The choice of $m_{\varepsilon,R}$ ensures that we can split the torus into an integer number of translates of the square $K_{R'}$ with $R' := \frac{R}{\sqrt{m_{\varepsilon,R}}}$, each of which containing an identical configuration of $\frac{2R^2}{\pi}$ points.

Let $P \in \mathcal{P}$ be given as in the assumption of Part 2 of Theorem 1, that is, let P be a probability measure concentrated on $\mathcal{A}_m$. Letting $\bar{P}$ be the push-forward of P by $\varphi \mapsto \varphi(\frac{\cdot}{\sqrt{m}})$, it is clear that $\bar{P}$ is concentrated on $\mathcal{A}_1$, and by the change of scales formula (2.16) we have

$$\int W(\varphi)\,\mathrm{d}\bar{P}(\varphi) = \frac{1}{m}\int W(\varphi)\,\mathrm{d}P(\varphi) + \frac{1}{4}\ln m. \tag{7.3}$$

We may then apply Proposition 7.1 to $\bar{P}$. It yields a vector field $\bar{b}_R$. We may then rescale it by setting

$$b_{\varepsilon,R}(x) = \sqrt{m_{\varepsilon,R}}\,\bar{b}_R(\sqrt{m_{\varepsilon,R}}x).$$

We note that $b_{\varepsilon,R}$ is a well-defined periodic vector-field on $\mathbb{T}^2_{\ell^\varepsilon}$ because $\frac{\ell^\varepsilon\sqrt{m_{\varepsilon,R}}}{2R}$ is an integer. This new vector field satisfies

$$-\operatorname{div} b_{\varepsilon,R} = 2\pi\sum_{a\in\Lambda_{\varepsilon,R}}\delta_a - m_{\varepsilon,R} \quad \text{in } \mathbb{T}^2_{\ell^\varepsilon} \tag{7.4}$$

for some set of points that we denote $\Lambda_{\varepsilon,R}$, and

$$\frac{1}{|K_R|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{\frac{R}{\sqrt{m_{\varepsilon,R}}}}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2\,\mathrm{d}x + \pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln(\eta\sqrt{m_{\varepsilon,R}})\right)$$
$$\leq \int W(\varphi)\,\mathrm{d}\bar{P}(\varphi) + o_R(1) \quad \text{as } R\to\infty.$$

Using (7.3) and $|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}| = \frac{2R^2}{\pi}$, this can be rewritten as

$$\frac{m_{\varepsilon,R}}{|K_R|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{\frac{R}{\sqrt{m_{\varepsilon,R}}}}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)}|b_{\varepsilon,R}|^2\,\mathrm{d}x + \pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln\eta\right)$$
$$+\frac{m_{\varepsilon,R}}{4}\ln\frac{m_{\varepsilon,R}}{m} \leq \frac{m_{\varepsilon,R}}{m}\int W(\varphi)\,\mathrm{d}P(\varphi) + o_R(1).$$

But we saw that $m_{\varepsilon,R} \to m$ as $\varepsilon \to 0$ hence $\ln\left(\frac{m_{\varepsilon,R}}{m}\right) \to 0$. Therefore, recalling the definition of R' we have

$$\frac{1}{|K_{R'}|}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_{R'}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} |b_{\varepsilon,R}|^2\,\mathrm{d}x + \pi|\Lambda_{\varepsilon,R}\cap K_{R/\sqrt{m_{\varepsilon,R}}}|\ln\eta\right)$$
$$\leq \int W(\varphi)\,\mathrm{d}P(\varphi) + o_R(1) + o_\varepsilon(1). \tag{7.5}$$

It thus follows that

$$\frac{1}{(\ell^\varepsilon)^2}\lim_{\eta\to 0}\left(\frac{1}{2}\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} |b_{\varepsilon,R}|^2\,\mathrm{d}x' + \pi|\Lambda_{\varepsilon,R}|\ln\eta\right)$$
$$\leq \int W(\varphi)\,\mathrm{d}P(\varphi) + o_R(1) + o_\varepsilon(1). \tag{7.6}$$

Note that $\Lambda_{\varepsilon,R}$ is a dilation by the factor $1/\sqrt{m_{\varepsilon,R}}$, uniformly bounded above and below, of the set of points Λ_R, hence the minimal distance between the points in $\Lambda_{\varepsilon,R}$ is bounded below by a constant which may depend on R but does not depend on ε. For the same reason, estimates on $b_{R,\varepsilon}$ are uniform with respect to ε.

In addition, we have that $\bar{Q}_{\varepsilon,R}$, the push-forward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x \mapsto -\mathrm{div}\, b_{\varepsilon,R}(x+\cdot)$ converges to Q, the push-forward of P by $-\Delta$, as $\varepsilon \to 0$ and $R \to \infty$. The final step is to replace the Dirac masses appearing above by their non-singular approximations:

$$\tilde{\delta}_a := \frac{\chi_{B(a,r'_\varepsilon)}}{\pi|r'_\varepsilon|^2} \qquad r'_\varepsilon := \varepsilon^{1/3}|\ln\varepsilon|^{1/6}\bar{r}_\varepsilon, \tag{7.7}$$

where $\bar{r}_\varepsilon$ was defined in (3.1). Note also that in view of the discussion of Section 3 it is crucial to use droplets with the corrected radius $\varepsilon^{1/3}|\ln\varepsilon|^{1/6}\bar{r}_\varepsilon$ instead of its leading order value $\rho_\varepsilon = 3^{1/3}\varepsilon^{1/3}|\ln\varepsilon|^{1/6}$.

Once the set $\Lambda_{\varepsilon,R}$ has been defined, the definition of the test function $u^\varepsilon \in \mathcal{A}$ follows: it suffices to take

$$u^\varepsilon(x) = -1 + 2\sum_{a\in\Lambda_{\varepsilon,R}} \chi_{B(a,r'_\varepsilon)}(x|\ln\varepsilon|^{1/2}),$$

which means (after blow up) that all droplets are round of identical radii r'_ε and centered at the points of $\Lambda_{\varepsilon,R}$. We now need to compute $F^\varepsilon[u^\varepsilon]$ and check that all the desired properties are satisfied. This is done by working with the associated function h'_ε defined in (3.16), that is the solution in $\mathbb{T}^2_{\ell^\varepsilon}$ to

$$-\Delta h'_\varepsilon + \frac{\kappa^2}{|\ln\varepsilon|}h'_\varepsilon = \pi\bar{r}^2_\varepsilon \sum_{a\in\Lambda_{\varepsilon,R}} \tilde{\delta}_a - \bar{\mu}^\varepsilon, \tag{7.8}$$

obtained from (3.16) by explicitly setting all $A^\varepsilon_i = \pi\bar{r}^2_\varepsilon$.

7.2. Reduction to Auxiliary Functions

Let us introduce ϕ_ε, which is the solution with mean zero of

$$-\Delta\phi_\varepsilon = 2\pi \sum_{a\in\Lambda_{\varepsilon,R}} \tilde{\delta}_a - m_{\varepsilon,R} \text{ in } \mathbb{T}^2_{\ell^\varepsilon}, \tag{7.9}$$

where $m_{\varepsilon,R}$ is as in (7.1), and f_ε the solution with mean zero of

$$-\Delta f_\varepsilon = 2\pi \sum_{a\in\Lambda_{\varepsilon,R}} \delta_a - m_{\varepsilon,R} \text{ in } \mathbb{T}^2_{\ell^\varepsilon}. \tag{7.10}$$

We note that f_ε is a rescaling by the factor $m_{\varepsilon,R} \to m$ of a function independent of ε, so all estimates on f_ε can be made uniform with respect to ε.

Lemma 7.3. *Let h'_ε and ϕ_ε be as above. We have as $\varepsilon \to 0$*

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} |h'_\varepsilon|^2 \,\mathrm{d}x' \le C_R |\ln\varepsilon| \tag{7.11}$$

and for any $1 \le q < \infty$

$$\left\| \nabla \left(h'_\varepsilon - \frac{\bar{r}^2_\varepsilon}{2}\phi_\varepsilon \right) \right\|_{L^q(\mathbb{T}^2_{\ell^\varepsilon})} \le C_{R,q}, \tag{7.12}$$

for some constant $C_{R,q} > 0$ independent of ε.

Proof. Since $\Lambda_{\varepsilon,R}$ is $2R'$-periodic, h'_ε is too, and thus

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}} |h'_\varepsilon|^2 \,\mathrm{d}x' = \ell^2 |\ln\varepsilon| \fint_{K_{R'}} |h'_\varepsilon|^2 \,\mathrm{d}x' \le C_R |\ln\varepsilon|.$$

For the second assertion, let

$$h_\varepsilon(x) = h'_\varepsilon(x\sqrt{|\ln\varepsilon|}) \qquad \hat{\phi}_\varepsilon(x) = \phi_\varepsilon(x\sqrt{|\ln\varepsilon|})$$

be the rescalings of h'_ε and ϕ_ε onto the torus $\mathbb{T}^2_\ell$. Rescaling (7.11) gives

$$\|h_\varepsilon\|_{L^2(\mathbb{T}^2_\ell)} \le C_R. \tag{7.13}$$

Furthermore, the function $w_\varepsilon := h_\varepsilon - \frac{1}{2}\bar{r}^2_\varepsilon \hat{\phi}_\varepsilon$ is easily seen to solve

$$-\Delta w_\varepsilon = -\kappa^2 \left(h_\varepsilon - \fint_{\mathbb{T}^2_\ell} h_\varepsilon \,\mathrm{d}x \right) \text{ in } \mathbb{T}^2_\ell.$$

But from elliptic regularity, Cauchy–Schwarz inequality and (7.13), we must have

$$\|\nabla w_\varepsilon\|_{L^q(\mathbb{T}^2_\ell)} \le C \left\| h_\varepsilon - \fint_{\mathbb{T}^2_\ell} h_\varepsilon \right\|_{L^2(\mathbb{T}^2_\ell)} \le C_{R,q},$$

which yields (7.12).

The next lemma consists in comparing ϕ_ε and f_ε.

Lemma 7.4. *We have*

$$\|\nabla(f_\varepsilon - \phi_\varepsilon)\|_{L^\infty\left(\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_a B(a,r'_\varepsilon)\right)} \le C_R \varepsilon^{1/4}.$$

Proof. We observe that f_ε and ϕ_ε are both $2R'$-periodic. We may thus write

$$\phi_\varepsilon(x) - f(x) = 2\pi \int_{\mathbb{T}^2_{2R'}} G_{2R'}(x-y) \sum_{a\in\Lambda_{\varepsilon,R}} d(\tilde{\delta}_a - \delta_a)(y),$$

where $G_{2R'}$ is the zero mean Green's function for the Laplace's operator on the square torus of size $2R'$ with periodic boundary conditions, that is the solution to

$$-\Delta G_{2R'} = \delta_0 - \frac{1}{|\mathbb{T}^2_{2R'}|} \quad \text{in } \mathbb{T}^2_{2R'} \tag{7.14}$$

which we may be split as $G_{2R'}(x) = -\frac{1}{2\pi}\ln|x| + S_{2R'}(x)$ with $S_{2R'}$ a smooth function. By Newton's theorem (or equivalently by the mean value theorem for harmonic functions applied to the function $\ln|\cdot|$ away from the origin), the contribution due to the logarithmic part is zero outside of $\cup_{a\in\Lambda_{\varepsilon,R}} B(a, r'_\varepsilon)$. Differentiating the above we may thus write that for all $x \notin \cup_{a\in\Lambda_{\varepsilon,R}} B(a, r'_\varepsilon)$,

$$\nabla(\phi_\varepsilon - f)(x) = 2\pi \int_{\mathbb{T}^2_{2R'}} \nabla S_{2R'}(x-y) \sum_{a\in\Lambda_{\varepsilon,R}} \mathrm{d}(\tilde{\delta}_a - \delta_a)(y). \tag{7.15}$$

Using the C^2 character of $S_{2R'}$ we deduce that

$$\|\nabla(f_\varepsilon - \phi_\varepsilon)\|_{L^\infty\left(\mathbb{T}^2_{\ell^\varepsilon}\setminus\cup_a B(a,r'_\varepsilon)\right)} \le C_{R'}|\Lambda_{\varepsilon,R}\cap K_{R'}|r'_\varepsilon$$

and the result follows in view of (7.7).

The next step involves a comparison of the energy of ϕ_ε and that of $b_{\varepsilon,R}$ and leads to the following conclusion.

Lemma 7.5. *Given $\Lambda_{\varepsilon,R}$ as constructed above, and h'_ε the solution to* (7.8)*, we have*

$$\frac{1}{(\ell^\varepsilon)^2}\lim_{\eta\to 0}\left(\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{2}{\bar{r}_\varepsilon^4}|\nabla h'_\varepsilon|^2\,\mathrm{d}x' + \pi|\Lambda_{\varepsilon,R}|\ln\eta\right)$$
$$\le \int W(\varphi)\,\mathrm{d}P(\varphi) + o_\varepsilon(1) + o_R(1).$$

Proof. In view of Lemmas 7.3 and 7.4, it suffices to show the corresponding result for $\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}} B(a,\eta)} \frac{1}{2}|\nabla f_\varepsilon|^2\,\mathrm{d}x'$ instead of the one for h'_ε. From (7.10) and (7.4), we have div $(b_{\varepsilon,R} - \nabla f_\varepsilon) = 0$ hence by Poincaré's lemma we may write $\nabla f_\varepsilon = b_{\varepsilon,R} + \nabla^\perp\xi_\varepsilon$. We note that $-\Delta\xi_\varepsilon = \mathrm{curl}\, b_{\varepsilon,R}$, which is in $W^{-1,p}_{\mathrm{loc}}$ for any $p < +\infty$

as mentioned in Remark 7.2. By elliptic regularity we find that $\nabla\xi_\varepsilon \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for all $1 \leq p < +\infty$, uniformly with respect to ε. We may thus write

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} \frac{1}{2}|b_{\varepsilon,R}|^2\,\mathrm{d}x' = \int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} \left(\frac{1}{2}|\nabla f_\varepsilon|^2 + \frac{1}{2}|\nabla\xi_\varepsilon|^2 - \nabla f_\varepsilon\cdot\nabla^\perp\xi_\varepsilon\right)\mathrm{d}x', \tag{7.16}$$

where $\nabla f_\varepsilon\cdot\nabla^\perp\xi_\varepsilon$ makes sense in the duality $\nabla\xi_\varepsilon \in L^p,\ p > 2,\ \nabla f_\varepsilon \in L^q,\ q < 2$. In addition, by the same duality, we have for any $a \in \Lambda_{\varepsilon,R}$,

$$\lim_{\eta\to 0}\int_{B(a,\eta)} \nabla f_\varepsilon\cdot\nabla^\perp\xi_\varepsilon = 0$$

uniformly with respect to ε.

Therefore, we may extend the domain of integration in the last integral in (7.16) to the whole of $\mathbb{T}^2_{\ell^\varepsilon}$ at the expense of an error $o_\eta(1)$ multiplied by the number of points, and obtain

$$\int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} \frac{1}{2}|\nabla f_\varepsilon|^2\,\mathrm{d}x' \leq \int_{\mathbb{T}^2_{\ell^\varepsilon}\setminus\bigcup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} \frac{1}{2}|b_{\varepsilon,R}|^2\,\mathrm{d}x' + \int_{\mathbb{T}^2_{\ell^\varepsilon}} \nabla f_\varepsilon\cdot\nabla^\perp\xi\,\mathrm{d}x' + o_\eta(|\ln\varepsilon|). \tag{7.17}$$

Noting that the last integral on the right-hand side vanishes by Stokes' theorem (and by approximating ∇f_ε and $\nabla^\perp\xi_\varepsilon$ by smooth functions), adding $\pi|\Lambda_{\varepsilon,R}|\ln\eta$ to both sides, and combining with (7.6) we obtain the result.

In view of (7.4) and (7.9) we have that $-\mathrm{div}\, b_{\varepsilon,R} + \Delta\phi_\varepsilon = 2\pi\sum_{a\in\Lambda_{\varepsilon,R}}(\delta_a - \tilde\delta_a) \to 0$ in $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2)$, so we deduce, since the push-forward of the normalized Lebesgue measure on $\mathbb{T}^2_{\ell^\varepsilon}$ by $x \mapsto -\mathrm{div}\, b_{\varepsilon,R}(x+\cdot)$ converges to Q, that the push-forward of it by $x \mapsto -\Delta\varphi^\varepsilon(x+\cdot)$ also converges to Q. Thus, part ii) of Theorem 1 is established modulo (2.23), which remains to be proved.

7.3. Calculating the Energy

We begin by calculating the exact amount of energy contained in a ball of radius η.

Lemma 7.6. *Let h'_ε be as above. Then we have for any $a \in \Lambda_{\varepsilon,R}$,*

$$\int_{B(a,r'_\varepsilon)} |\nabla h'_\varepsilon|^2\,\mathrm{d}x' = \frac{3^{4/3}\pi}{8} + o_\varepsilon(1) \tag{7.18}$$

and

$$\int_{B(a,\eta)\setminus B(a,r'_\varepsilon)} |\nabla h'_\varepsilon|^2\,\mathrm{d}x' \leq \frac{\pi}{2}\bar r_\varepsilon^4 \ln\frac{\eta}{\rho_\varepsilon} + o_\varepsilon(1) + o_\eta(1). \tag{7.19}$$

Proof. In view of (7.12) applied with $q > 2$ and using Hölder's inequality, we have that for all $a \in \Lambda_{\varepsilon,R}$,

$$\int_{B(a,\eta)} \left|\nabla(h'_\varepsilon - \frac{\bar{r}_\varepsilon^2}{2}\phi_\varepsilon)\right|^2 \mathrm{d}x' \leq o_\eta(1). \tag{7.20}$$

Thus it suffices to compute the corresponding integrals for ϕ_ε. Using again the $2R'$-periodicity of ϕ_ε, we may write, with the same notation as in the proof of Lemma 7.4

$$\phi_\varepsilon(x) = \int_{\mathbb{T}^2_{2R'}} G_{2R'}(x-y)\left(2\pi \sum_{a\in\bar{\Lambda}_{\varepsilon,R}} \tilde{\delta}_a(y) - m_{\varepsilon,R}\right)\mathrm{d}y.$$

Since the distances between the points in $\bar{\Lambda}_{\varepsilon,R}$ are bounded below independently of ε, and the number of points is bounded as well, we may write ϕ_ε in $B(a,\eta)$ as

$$\phi_\varepsilon(x) = \psi_\varepsilon(x) - \int_{\mathbb{T}^2_{2R'}} \ln|x-y|\,\tilde{\delta}_a(y)\,\mathrm{d}y \tag{7.21}$$

where $\psi_\varepsilon(x)$ is smooth and its derivative is bounded independently of ε (but depending on R).

Thus the contribution of ψ_ε to the integrals $\int_{B(a,\eta)} |\nabla\phi_\varepsilon|^2$ is $o_\eta(1)$, and its contribution to $\int_{B(a,r'_\varepsilon)} |\nabla\phi_\varepsilon|^2$ is $o_\varepsilon(1)$. There remains to compute the contribution of the logarithmic term in (7.21). But this is almost exactly the same computation as in (6.28)–(6.30), and with (7.20) it yields (7.18), while it yields as well that

$$\int_{B(a,\eta)\setminus B(a,r'_\varepsilon)} |\nabla\phi_\varepsilon|^2\,\mathrm{d}x' \leq 2\pi \ln\frac{\eta}{r'_\varepsilon} + o_\eta(1). \tag{7.22}$$

Now

$$\frac{r'_\varepsilon}{\rho_\varepsilon} = \frac{1}{3^{1/3}}\left(\frac{|\ln\varepsilon|}{|\ln\rho_\varepsilon|}\right)^{1/3} = \left(1 + O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)\right)^{1/3}.$$

Consequently $\ln\frac{r'_\varepsilon}{\rho_\varepsilon} = o_\varepsilon(1)$, and so we may replace r'_ε with ρ_ε at an extra cost of $o_\varepsilon(1)$ in (7.22), and the result follows with (7.20).

We can now combine all the previous results to compute the energy of the test-function u^ε. By following the lower bounds of Proposition 3.1, it is easy to see that in our case (all the droplets being balls of radius r'_ε) all the inequalities in that proof become equalities, and thus recalling (3.1):

$$F^\varepsilon[u^\varepsilon] = \frac{1}{|\ell^\varepsilon|^2}\left(2\int_{\mathbb{T}^2_{\ell^\varepsilon}}\left(|\nabla h'_\varepsilon|^2 + \frac{\kappa^2}{|\ln\varepsilon|}|h'_\varepsilon|^2\right)\mathrm{d}x' + \pi\bar{r}_\varepsilon^4|\Lambda_{\varepsilon,R}|\ln\rho_\varepsilon\right) + o_\varepsilon(1),$$

with the help of Lemma 7.6 we have for every R

$$F^{\varepsilon}[u^{\varepsilon}] \le \frac{1}{|\ell^{\varepsilon}|^2}\left(2\int_{\mathbb{T}^2_{\ell^{\varepsilon}}\setminus\cup_{a\in\Lambda_{\varepsilon,R}}B(a,\eta)} |\nabla h'_{\varepsilon}|^2\,\mathrm{d}x' + \pi\bar{r}_{\varepsilon}^4|\Lambda_{\varepsilon,R}|\ln\eta + \frac{3^{4/3}\pi}{4}|\Lambda_{\varepsilon,R}|\right) + o_{\varepsilon}(1) + o_{\eta}(1).$$

In view of Lemma 7.5, letting $\eta \to 0$, we obtain

$$F^{\varepsilon}[u^{\varepsilon}] \le \bar{r}_{\varepsilon}^4\left(\int W(\varphi)\,\mathrm{d}P(\varphi) + o_{\varepsilon}(1) + o_R(1)\right) + \frac{3^{4/3}\pi}{4|\ell^{\varepsilon}|^2}|\Lambda_{\varepsilon,R}| + o_{\varepsilon}(1).$$

Letting $\varepsilon \to 0$, using that $\bar{r}_{\varepsilon} \to 3^{1/3}$ and the fact that $|\Lambda_{\varepsilon,R}| = \frac{1}{2\pi}m_{\varepsilon,R}|\ell^{\varepsilon}|^2$ with $m_{\varepsilon,R} \to m$, and then finally letting $R \to \infty$, we conclude that

$$\limsup_{\varepsilon\to 0} F^{\varepsilon}[u^{\varepsilon}] \le 3^{4/3}\int W(\varphi)\,\mathrm{d}P(\varphi) + \frac{3^{4/3}m}{8}.$$

Since $\frac{1}{8}3^{2/3}m = \frac{1}{8}(\bar{\delta} - \bar{\delta}_c)$, this completes the proof of part ii) of Theorem 1. □

7.4. Proof of Theorem 2

In order to prove Theorem 2, it suffices to show that

$$\min_{P\in\mathcal{P}} F^0[P] = 3^{4/3}\min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}. \tag{7.23}$$

For the proof, we use the following result, adapted from Corollary 4.4 in [36].

Proposition 7.7. (Corollary 4.4 in [36]) *Let $\varphi \in \mathcal{A}_1$ be given, such that $W(\varphi) < \infty$. For any R such that $R^2 \in 2\pi\mathbb{N}$, there exists a $2R$-periodic φ_R such that*

$$\begin{cases} -\Delta\varphi_R = 2\pi\sum_{a\in\Lambda_R}\delta_a - 1 & \text{in } K_R, \\ \dfrac{\partial\varphi_R}{\partial\nu} = 0 & \text{on } \partial K_R, \end{cases}$$

where Λ_R is a finite subset of the interior of K_R, and such that

$$\limsup_{R\to\infty}\frac{W(\varphi_R, \mathbf{1}_{K_R})}{|K_R|} \le W(\varphi).$$

Let us take φ to be a minimizer of W over $\mathcal{A}_m$ (which exists from [36]). We may rescale it to be an element of $\mathcal{A}_1$. Then Proposition 7.7 yields a φ_R, which can be extended periodically. We can then repeat the same construction as in the beginning of this section, starting from $\nabla\varphi_R$ instead of b_R, and in the end it yields a u^{ε} with

$$\limsup_{\varepsilon\to 0} F^{\varepsilon}[u^{\varepsilon}] \le 3^{4/3}\min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta} - \bar{\delta}_c)}{8}.$$

It follows that

$$\limsup_{\varepsilon\to 0} \min_{\mathcal{A}} F^\varepsilon \leq 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta}-\bar{\delta}_c)}{8},$$

but by part i) of Theorem 1 applied to a sequence of minimizers of F^ε, we also have

$$\liminf_{\varepsilon\to 0} \min_{\mathcal{A}} F^\varepsilon \geq \inf_{\mathcal{P}} F^0 \geq 3^{4/3} \min_{\mathcal{A}_m} W + \frac{3^{2/3}(\bar{\delta}-\bar{\delta}_c)}{8}$$

where the last inequality is an immediate consequence of the definition of F^0. Comparing the inequalities yields that there must be equality and (7.23) is proved, which completes the proof of Theorem 2. □

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