Mathematical Models and Methods in Applied Sciences
Vol. 34, No. 10 (2024) 1861–1904

DOI: 10.1142/S0218202524500386

Reduced energies for thin ferromagnetic films with perpendicular anisotropy

Giovanni Di Fratta

Dipartimento di Matematica e Applicazioni,
"R. Caccioppoli", Università degli Studi
di Napoli "Federico II", Via Cintia,
80126 Napoli, Italy
giovanni.difratta@unina.it

Cyrill B. Muratov*

Dipartimento di Matematica,
Università di Pisa, Largo B. Pontecorvo, 5,
56127 Pisa, Italy
cyrill.muratov@unipi.it

Valeriy V. Slastikov

School of Mathematics, University of Bristol,
Bristol BS8 1TW, UK
valeriy.slastikov@bristol.ac.uk

Received 5 September 2023
Revised 12 May 2024
Accepted 3 June 2024
Published 30 July 2024
Communicated by I. Fonseca

We derive four reduced two-dimensional models that describe, at different spatial scales, the micromagnetics of ultrathin ferromagnetic materials of finite spatial extent featuring perpendicular magnetic anisotropy and interfacial Dzyaloshinskii–Moriya interaction. Starting with a microscopic model that regularizes the stray field near the material's lateral edges, we carry out an asymptotic analysis of the energy by means of Γ-convergence. Depending on the scaling assumptions on the size of the material domain versus the strength of dipolar interaction, we obtain a hierarchy of the limit energies that exhibit progressively stronger stray field effects of the material edges. These limit energies feature, respectively, a renormalization of the out-of-plane anisotropy, an additional local boundary penalty term forcing out-of-plane alignment of the magnetization at the edge, a pinned magnetization at the edge, and, finally, a pinned magnetization and an additional field-like term that blows up at the edge, as the sample's lateral size is increased.

*Corresponding author.

The pinning of the magnetization at the edge restores the topological protection and enables the existence of magnetic skyrmions in bounded samples.

Keywords: Micromagnetics; magnetic thin films; Dzyaloshinskii–Moriya interaction; Γ-convergence.

AMS Subject Classification 2020: 82D40, 49S05, 35C20

1. Introduction and Motivation

With an ever-increasing control and sophistication of nanofabrication techniques, there is a growing need for a better understanding of the physical phenomena at the nanoscale that are determined by the material geometry. In today's nanomagnetic systems, one typically encounters materials consisting of one or several quasi-two-dimensional magnetic layers interspersed with non-magnetic layers. The presence of magnetic/non-magnetic interfaces gives rise to new physical effects that have the potential to enable the next generation of nanoelectronic devices that harness both the electric charge and the spin degrees of freedom of electron for information technologies.[65] In the context of such *spintronics* applications,[2, 6, 35] one is particularly interested in creating and manipulating spin configurations that are endowed with nontrivial topological characteristics, which make them robust against external influences and noise.[69]

The basic information unit in a topological spintronic device is the *magnetic skyrmion* — a particle-like continuous spin texture with topological degree +1 (under a natural sign convention).[13, 30, 31, 58] For this reason, there has been considerable interest in the behavior of skyrmions in confining geometries, both theoretically, computationally and experimentally (see, e.g. Refs. 1, 12, 18, 24, 52, 59–62 and 19; this list is certainly not meant to be exhaustive).

However, under confinement, the topological protection of non-collinear spin textures is a priori lost since the topological degree of the spin configuration on a bounded domain is generally not well-defined. In this case, a skyrmion-like spin texture may be continuously deformed into a uniform magnetization state by pushing the skyrmion out of the domain through the boundary. A natural solution is to settle the problem in the framework of curvilinear magnetism. Indeed, magnetic thin films with the shape of closed surfaces provide a concrete alternative for degree-preserving confinements and, thus, toward the realization of chiral magnetic textures. The literature on this topic has grown very large. We refer the reader to Refs. 22, 23, 26, 27, 32, 37, 64 and 66, see also the recent monograph,[49] for further reading on the analysis of magnetic skyrmions in curved geometries that are close in spirit to our interests here. But as soon as one is interested in planar thin films, further stabilization mechanisms for the magnetic skyrmions in spintronic nanodevices would be required that provide a repulsive interaction between the skyrmion and the device edge.

In this paper, we explore the additional energetic effects appearing at the edges of two-dimensional ferromagnetic materials of finite spatial extent. Due to the significant role played by the stray field in ferromagnetic materials, these effects are often difficult to predict and cause the emergence of new physical phenomena driven by the material edges. For example, in soft ferromagnets in the form of thin films, the additional contribution of the stray field may penalize the normal component of the magnetization at the film edge,[16, 43] causing the appearance of boundary vortices,[39, 44, 46, 54] edge-curling domain walls,[47, 48] interior walls,[38, 40, 53] etc. (for a review, see Ref. 21). In the current materials for spintronics applications, additional physical mechanisms contribute at the film edge,[56, 57, 61] further complicating the situation. Therefore, to better understand the energetics of the material edge, we carry out an asymptotic analysis of the micromagnetic model of a two-dimensional ferromagnet exhibiting perpendicular magnetic anisotropy and interfacial Dzyaloshinskii–Moriya interaction (DMI),[61] using the techniques of Γ-convergence. The singularity of the stray field near the film edge in two-dimensional micromagnetics requires a regularization of the magnetostatic problem near the edge and gives rise to a hierarchy of reduced models appearing in the limit of the vanishing stray field interaction strength. This regularization, however, does not affect in any way the obtained limits, demonstrating the universality of the asymptotic behavior of two-dimensional ferromagnets.

We demonstrate that depending on the scaling of the lateral size of a simply connected ferromagnetic domain with the strength of the effective stray field interaction and for suitable renormalizations of the other parameters, there are four distinct asymptotic regimes in which the stray field acts differently at the domain edge and in the ferromagnet's interior. In the first regime reminiscent of the thin film limit studied by Gioia and James[34] (see also Refs. 16, 22 and 25), the edge does not exert any influence on the magnetization, resulting in a free boundary condition and a renormalization of the magnetocrystalline anisotropy constant (Theorem 3.1). In the second regime reminiscent of the one studied by Kohn and Slastikov for soft ferromagnets[43] and characterized by a larger lateral film extent, the edge begins to exert an additional penalization of the deviation of the magnetization from either one of the out-of-plane directions (Theorem 3.2). In the third regime at yet larger film's lateral extent, the magnetization becomes rigidly pinned to a single out-of-plane direction at the film edge, while the stray field still contributes locally in the film's interior (Theorem 3.3). Finally, in the fourth regime at yet larger film's extent, the magnetization is also rigidly pinned at the film's edge, but a non-local interaction term appears in the interior, as well as an additional geometry-induced external field-like term (Theorem 3.4). We note that the third regime corresponds precisely to the one studied in Ref. 52, where the topological protection of single Néel skyrmions was shown to be restored in a minimal micromagnetic setting.

The proofs of the main results are rather technical and require a careful asymptotic analysis of the stray field close to the film edge. For this purpose, we find it convenient to reformulate the leading order expansion for the dipolar interaction energy[42] in terms of the gradients of the magnetization. This is then used to compare the contributions of the mollified material edge with that of the trace of the magnetization at a fixed material boundary of the limit domain. The proofs proceed by a divide-and-conquer strategy, in which the different parts of the dipolar interaction are progressively isolated and ultimately estimated by the limiting energy up to error terms that are bounded by a small fraction of the exchange energy. The Γ-limits are, in turn, proved by suitably combining the different terms in the energy and passing to the limit directly. We note that considerably finer estimates are required here to establish our results compared to those needed in Refs. 16, 22, 25, 34 and 43.

1.1. *Outline*

This paper is organized as follows. In Sec. 2, we introduce the micromagnetic model of a two-dimensional ferromagnetic film exhibiting perpendicular magnetocrystalline anisotropy and an interfacial DMI. Here we also introduce the edge regularization, which is further derived from first principles in Sec. 2.1, and state our results informally, so that they can be easily related to the original physical model. Then, in Sec. 3, we state the precise assumptions and definitions, and proceed to present the statements of the main results of this paper. The rest of this paper is devoted to the proofs of the theorems. In Sec. 4, we prove some preliminary technical results. In Sec. 5, we carry out the necessary asymptotic expansions of the magnetostatic energy. Finally, in Sec. 6, we complete the proofs of Γ-convergence.

2. The Micromagnetic Model

We start by considering a reduced two-dimensional micromagnetic energy for an extended ferromagnetic thin film of effective thickness $\delta > 0$, with the lengths measured in the units of the exchange length $\ell = \sqrt{A/K_{\mathrm{d}}}$, where $K_{\mathrm{d}} = \frac{1}{2}\mu_0 M_{\mathrm{s}}^2$, μ_0 is the vacuum permeability, M_{s} is the saturation magnetization and A is the exchange stiffness. The magnetization is characterized by a non-dimensional vector $\boldsymbol{m}(x) \in \mathbb{R}^3$ at each point $x \in \mathbb{R}^2$ of the film. We assume that the film exhibits perpendicular magnetic anisotropy, interfacial DMI and in the presence of an applied field perpendicular to the film plane, so that the energy functional $E(\boldsymbol{m})$ has the form

$$E(\boldsymbol{m}) = E_{\mathrm{ex}}(\boldsymbol{m}) + E_{\mathrm{a}}(\boldsymbol{m}) + E_{\mathrm{Z}}(\boldsymbol{m}) + E_{\mathrm{DMI}}(\boldsymbol{m}) + E_{\mathrm{s}}(\boldsymbol{m}). \tag{2.1}$$

Here, in order of appearance, the terms are the exchange, anisotropy, Zeeman, the DMI and the stray field energies measured in the units of $A\ell\delta$. As was discussed in Refs. 4, 5, 42 and 56, in an extended film, where

$$\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{S}^2 \tag{2.2}$$

is sufficiently smooth and goes to, say, $\boldsymbol{m}_0 = (0, 0, -1)$ sufficiently fast at infinity, with the notations

$$\boldsymbol{m} = (\boldsymbol{m}_\perp, m_\parallel), \quad \boldsymbol{m}_\perp : \mathbb{R}^2 \to \mathbb{R}^2, \quad m_\parallel : \mathbb{R}^2 \to \mathbb{R}, \tag{2.3}$$

where $\boldsymbol{m}_\perp$ is the in-plane component and $m_\parallel$ is the out-of-plane component of $\boldsymbol{m}$, respectively, these terms take the following form[3, 4]:

$$E_{\mathrm{ex}}(\boldsymbol{m}) := \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 \mathrm{d}x, \tag{2.4}$$

$$E_{\mathrm{a}}(\boldsymbol{m}) := Q \int_{\mathbb{R}^2} |\boldsymbol{m}_\perp|^2 \mathrm{d}x, \tag{2.5}$$

$$E_{\mathrm{Z}}(\boldsymbol{m}) := -2h \int_{\mathbb{R}^2} (1 + m_\parallel) \mathrm{d}x, \tag{2.6}$$

$$E_{\mathrm{DMI}}(\boldsymbol{m}) := \kappa \int_{\mathbb{R}^2} (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel) \mathrm{d}x, \tag{2.7}$$

$$E_{\mathrm{s}}(\boldsymbol{m}) := -\int_{\mathbb{R}^2} |\boldsymbol{m}_\perp|^2 \mathrm{d}x + \frac{\delta}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\operatorname{div} \boldsymbol{m}_\perp(x) \cdot \operatorname{div} \boldsymbol{m}_\perp(y)}{|x-y|} \mathrm{d}x \mathrm{d}y, \tag{2.8}$$

$$- \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(m_\parallel(x) - m_\parallel(y))^2}{|x-y|^3} \mathrm{d}x \mathrm{d}y, \tag{2.9}$$

where

$$Q = \frac{K_{\mathrm{u}}}{K_{\mathrm{d}}}, \quad \kappa = \frac{D}{\sqrt{AK_{\mathrm{d}}}}, \quad h = \frac{H}{M_{\mathrm{s}}}, \tag{2.10}$$

with Q, κ and h being the dimensionless quality factor of the out-of-plane anisotropy, the dimensionless DMI strength and the dimensionless applied field strength, corresponding to the dimensional magnetocrystalline anisotropy constant K_{u}, DMI strength D normalized per unit volume and the out-of-plane field H, respectively. Note that κ and h may change sign, while for a perpendicular magnetic anisotropy material we have $Q > 1$. Under suitable conditions, the above energy exhibits local minimizers in the form of the topologically nontrivial magnetization configurations — *magnetic skyrmions*.[3–5, 7–11, 19, 41, 50, 52]

Observe that the stray field energy in (2.9) admits the following representation with the help of the Fourier transform:

$$\widehat{\boldsymbol{m}}(k) = \int_{\mathbb{R}^2} e^{-ik\cdot x} (\boldsymbol{m}(x) - \boldsymbol{m}_0) \mathrm{d}x \tag{2.11}$$

of $\boldsymbol{m} - \boldsymbol{m}_0 \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^3)$:

$$E_s(\boldsymbol{m}) = -\int_{\mathbb{R}^2} |\widehat{\boldsymbol{m}}_\perp(k)|^2 \frac{\mathrm{d}k}{(2\pi)^2} + \frac{\delta}{2} \int_{\mathbb{R}^2} \frac{|k \cdot \widehat{\boldsymbol{m}}_\perp(k)|^2}{|k|} \frac{\mathrm{d}k}{(2\pi)^2}$$
$$- \frac{\delta}{2} \int_{\mathbb{R}^2} |k| |\widehat{m}_\parallel(k)|^2 \frac{\mathrm{d}k}{(2\pi)^2}. \tag{2.12}$$

In particular, the first term on the right-hand side of (2.12), also referred to as the shape anisotropy term, may be combined with $E_{\mathrm{a}}(\boldsymbol{m})$ to define an effective out-of-plane anisotropy with strength $Q - 1$ (going back to Ref. 68); the second term on the right-hand side of (2.12) represents the effect of the bulk charges and can be seen to be non-negative; and the third term represents the effect of the surface charges and is non-positive. The Fourier representation in (2.12) also arises as a relaxation of the energy in (2.9) in the natural class of configurations in which $\boldsymbol{m} - \boldsymbol{m}_0 \in H^1(\mathbb{R}^2; \mathbb{R}^3)$. Notice that by (2.12) and simple interpolation inequalities the energy $E(\boldsymbol{m})$ is always well defined in this class (for further details, see Ref. 5).

The expression for the stray field energy in (2.9) or (2.12) may be rigorously obtained as the leading order terms in the asymptotic expansion of the full micromagnetic energy of a three-dimensional ferromagnetic film of thickness $\delta \ll 1$, with the errors controlled by the exchange energy at the next order.[42] It represents a suitably renormalized dipolar interaction between different spins in an infinitesimally thin ferromagnetic layer. Care, however, is needed when extending this definition to samples of *finite* spatial extent. Indeed, as the stray field energy involves non-local terms, one cannot simply restrict the integration in (2.9) to a *bounded* spatial domain $\Omega^\delta \subset \mathbb{R}^2$ (whose size may depend on δ), as this would disregard the dipolar interactions that contribute to the first term in (2.9). A more systematic approach would instead consist of extending the magnetization $\boldsymbol{m} : \Omega^\delta \to \mathbb{S}^2$ to the whole plane by zero outside Ω^δ. However, such an approach also presents difficulties, as a jump discontinuity in $\boldsymbol{m}$ across $\partial\Omega^\delta$ would then generally make the non-local terms in (2.9) infinite. Thus, a regularization at the scale of the film thickness is necessary close to the film edge to make sense of the energy in (2.9). Such a regularization was first introduced in Ref. 17 (see also Ref. 55, for further discussion see Refs. 47 and 48) in the context of reduced thin film energies for soft ferromagnetic materials, in which the magnetization tends to lie in the film plane.

We note that several regularizations are, in fact, possible that can lead to slightly different reduced thin film energies. The precise model would inevitably depend on the specific physics at the film edge, which may be governed by a number of physical effects such as a different material composition in an as-grown film near the edge, changes in the crystalline structure near the edge, edge roughness, etc. We point out, however, that the magnetization, which in the physical space rotates on the scale of the exchange length that exceeds by an order of magnitude the atomic scale,[36] should experience the effect of the edge via some sort of effective boundary terms. This is indeed confirmed by rigorous studies of the thin film limit of soft-three-dimensional ferromagnetic layers.[43] This paper aims to derive these boundary terms via Γ-convergence for ferromagnetic films with perpendicular magnetocrystalline anisotropy that are relevant to the studies of magnetic skyrmions.

Our starting point will be the regularization in which for, say, a given $\boldsymbol{m} \in C^\infty(\mathbb{R}^2; \mathbb{S}^2)$ we define the physically observable magnetization $\boldsymbol{m}_\delta$ in the film:

$$\boldsymbol{m}_\delta(x) := \eta_\delta(x)\boldsymbol{m}(x) \quad \forall\, x \in \mathbb{R}^2, \tag{2.13}$$

where for a bounded open, simply connected set Ω^δ with boundary of class C^2 representing the film of finite extent we defined a cutoff function

$$\eta_\delta(x) := \eta\left(\frac{d_{\partial\Omega^\delta}(x)}{\delta}\right), \tag{2.14}$$

in which $d_{\partial\Omega^\delta}$ is the signed distance from the boundary (cf. (3.4)) and η is a non-increasing, sufficiently regular function that goes from $\eta(-\infty) = 1$ to $\eta(+\infty) = 0$ (see Sec. 3 for details). For a microscopic derivation of this condition, see Sec. 2.1. Notice that $\boldsymbol{m}_\delta$ thus defined automatically lies in $H^1(\mathbb{R}^2;\mathbb{R}^3)$ if $\boldsymbol{m} \in H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{S}^2)$. We then replace all the instances of $\boldsymbol{m}$ in (2.1) with $\boldsymbol{m}_\delta$ to define the reduced thin film energy $\mathsf{E}_\delta(\boldsymbol{m}) := E(\boldsymbol{m}_\delta)$.

We next specify the asymptotic regimes in which the obtained energy E_δ can be significantly simplified. In the first two regimes the limit energy becomes local, with the edge effects either disappearing or appearing as a boundary term that generalizes the regime for soft ferromagnetic films identified by Kohn and Slastikov.[43] We will also identify the scalings of the parameters for which the resulting limit energy still exhibits the terms that are needed to produce skyrmion-type solutions. To this end, we introduce a small parameter $\varepsilon > 0$ and make all the model parameters, as well as the domain, depend on ε as follows:

$$Q_\varepsilon = 1 + \frac{\varepsilon|\ln\varepsilon|}{2\pi\gamma_\varepsilon}\alpha, \quad h_\varepsilon = \frac{\varepsilon|\ln\varepsilon|}{2\pi\gamma_\varepsilon}\beta, \quad \kappa_\varepsilon = \left(\frac{\varepsilon|\ln\varepsilon|}{2\pi\gamma_\varepsilon}\right)^{1/2}\lambda, \quad \delta_\varepsilon = \left(\frac{2\pi\varepsilon\gamma_\varepsilon}{|\ln\varepsilon|}\right)^{1/2}, \tag{2.15}$$

together with

$$\Omega_\varepsilon^\delta := \varepsilon^{-1}\delta_\varepsilon\Omega, \tag{2.16}$$

for some fixed $\lambda > 0$, $\alpha, \beta \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^2$. Notice that at this point (2.15) and (2.16) simply represent a reparametrization that imposes a certain dependence on two parameters, ε and γ_ε (with the dependence of the latter on ε to be specified), and forces suitable balances between different terms in the energy as $\varepsilon \to 0$ depending on the choices of γ_ε.

With the above choices and after some algebra, we have $\mathsf{E}_{\delta_\varepsilon}(\boldsymbol{m}(\varepsilon^{-1}\delta_\varepsilon\cdot)) = E_\varepsilon(\boldsymbol{m}) + C_\varepsilon$, where

$$\begin{aligned} E_\varepsilon(\boldsymbol{m}) := &\int_{\mathbb{R}^2} (\eta_\varepsilon^2|\nabla\boldsymbol{m}|^2 + \alpha\eta_\varepsilon^2|\boldsymbol{m}_\perp|^2 - 2\beta\eta_\varepsilon m_\parallel)\mathrm{d}x \\ &+ \lambda\int_{\mathbb{R}^2} \eta_\varepsilon^2(m_\parallel \operatorname{div}\boldsymbol{m}_\perp - \boldsymbol{m}_\perp\cdot\nabla m_\parallel)\mathrm{d}x \\ &+ \frac{\gamma_\varepsilon}{2|\ln\varepsilon|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\operatorname{div}(\eta_\varepsilon\boldsymbol{m}_\perp)(x)\operatorname{div}(\eta_\varepsilon\boldsymbol{m}_\perp)(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \\ &- \frac{\gamma_\varepsilon}{4|\ln\varepsilon|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(\eta_\varepsilon(x)m_\parallel(x) - \eta_\varepsilon(y)m_\parallel(y))^2}{|x-y|^3}\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{2.17}$$

and the additive constant C_ε is independent of $\boldsymbol{m}$ and, therefore, is inconsequential for the variational problem associated with $\mathsf{E}_{\delta_\varepsilon}$. We note that for $\lambda = 0$, only a slightly different version of this type of energy with $\varepsilon \sim 1$ can be shown to arise from the full micromagnetic energy of a three-dimensional thin ferromagnetic film with variable thickness equal to $\eta_\varepsilon(x)$, which tapers off at the film edge.[63]

Now, with the choice $\gamma_\varepsilon \to \gamma$ as $\varepsilon \to 0$ for $\gamma > 0$, the limit functional will be shown to be

$$F(\boldsymbol{m}) := \int_\Omega (|\nabla \boldsymbol{m}|^2 + \alpha |\boldsymbol{m}_\perp|^2 - 2\beta m_\parallel)\mathrm{d}x + \lambda \int_\Omega (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel)\mathrm{d}x$$
$$+ \gamma \int_{\partial\Omega} ((\boldsymbol{m}_\perp \cdot \boldsymbol{n})^2 - m_\parallel^2)\mathrm{d}\mathcal{H}^1(\sigma), \tag{2.18}$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$, and $F(\boldsymbol{m})$ is defined for $\boldsymbol{m} \in H^1(\Omega; \mathbb{S}^2)$. The limit functional sees only the limit domain Ω and in addition to the expected local terms inside Ω it features a boundary term that penalizes the deviations of the in-plane component of the magnetization from tangential to $\partial\Omega$, and another boundary term that favors the out-of-plane component of the magnetization to be ± 1. These terms arise, respectively, as the limits of the next-to-last and the last term in the definition of E_ε in (2.17) due to the logarithmic divergence of the respective integrals as $\varepsilon \to 0$. We remark that the limit energy with $\gamma = 0$ similarly arises when $\gamma_\varepsilon \to 0$ and $\varepsilon \to 0$, a result analogous to a well-known result of Gioia and James.[34]

We will also consider two other scaling regimes, which lead to different limit behaviors. First, we define

$$E_\varepsilon^0(\boldsymbol{m}) := E_\varepsilon(\boldsymbol{m}) + \gamma_\varepsilon \mathcal{H}^1(\partial\Omega), \tag{2.19}$$

where γ is replaced with γ_ε in (2.17), and we will be interested in the limit in which $\gamma_\varepsilon \to +\infty$ as $\varepsilon \to 0$ with α, β and λ, as well as the domain Ω, fixed (note that the limit $\gamma_\varepsilon \to 0$ is much simpler and is obtained by just setting $\gamma = 0$ in (2.18)). When $\gamma_\varepsilon \to +\infty$, we show that for $\gamma_\varepsilon = o(|\ln \varepsilon|)$ the limit energy for E_ε^0 is given by

$$F_0(\boldsymbol{m}) := \int_\Omega (|\nabla \boldsymbol{m}|^2 + \alpha |\boldsymbol{m}_\perp|^2 - 2\beta m_\parallel)\mathrm{d}x + \lambda \int_\Omega (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel)\mathrm{d}x, \tag{2.20}$$

specified for all $\boldsymbol{m} \in H^1(\Omega; \mathbb{S}^2)$ such that $\boldsymbol{m} = \boldsymbol{e}_3$ or $\boldsymbol{m} = -\boldsymbol{e}_3$ on $\partial\Omega$ in the sense of trace. This could be thought of in some sense as the limit case of the energy in (2.18) with $\gamma = \infty$, after a suitable renormalization.

Finally, we consider the regime in which for $\nu > 0$ and α, β, λ real we have

$$Q_\varepsilon = 1 + \frac{\varepsilon}{2\pi\nu}\alpha, \quad h_\varepsilon = \frac{\varepsilon}{2\pi\nu}\beta, \quad \kappa_\varepsilon = \left(\frac{\varepsilon}{2\pi\nu}\right)^{1/2}\lambda, \quad \delta_\varepsilon = (2\pi\varepsilon\nu)^{1/2}, \tag{2.21}$$

once again together with (2.16), which corresponds to the choice of $\gamma_\varepsilon = \nu|\ln\varepsilon|$ in (2.19). Here we find the following limit energy for E_ε^0, up to an additive constant:

$$F_\nu(\boldsymbol{m}) := \int_\Omega (|\nabla \boldsymbol{m}|^2 + \alpha |\boldsymbol{m}_\perp|^2 - 2\beta m_\parallel)\mathrm{d}x$$
$$+ \lambda \int_\Omega (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel)\mathrm{d}x + \nu \int_\Omega \boldsymbol{b} \cdot \nabla m_\parallel \mathrm{d}x$$

$$+\frac{\nu}{2}\int_\Omega\int_\Omega \frac{\operatorname{div}\boldsymbol{m}_\perp(x)\operatorname{div}\boldsymbol{m}_\perp(y)}{|x-y|}\mathrm{d}x\mathrm{d}y$$
$$-\frac{\nu}{2}\int_\Omega\int_\Omega \frac{\nabla m_\parallel(x)\cdot\nabla m_\parallel(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{2.22}$$

where we defined the vector field

$$\boldsymbol{b}(x) := \int_{\partial\Omega}\frac{m_\parallel(y)\boldsymbol{n}(y)}{|x-y|}\mathrm{d}y, \quad x\in\Omega, \tag{2.23}$$

in which $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$. This vector field encodes the stray field effect of the film edge. The energy F_ν is defined for all $\boldsymbol{m}\in H^1(\Omega;\mathbb{S}^2)$ such that $\boldsymbol{m}=\boldsymbol{e}_3$ or $\boldsymbol{m}=-\boldsymbol{e}_3$ on $\partial\Omega$ in the sense of trace. Note that for $\boldsymbol{m}=\pm\boldsymbol{e}_3$ on $\partial\Omega$ we have $\boldsymbol{b}\in C^\infty(\Omega)$ and $\boldsymbol{b}(x)$ diverges logarithmically with distance as $x\in\Omega$ approaches $\partial\Omega$. In particular, the term in the energy involving $\boldsymbol{b}$ is under control by the gradient squared term.

All of the aforementioned statements are made precise within the framework of Γ-convergence in Sec. 3.

2.1. *A microscopic derivation of the reduced two-dimensional model*

As was already mentioned, the precise behavior of the magnetization near the film edge depends on the detailed physics at the edge of the film. Here we use a particular model that illustrates how an energy of the form given in (2.17) may be obtained from a more microscopic description.

To avoid dealing with truly discrete models of ferromagnetism at the atomic scale, we pick a model that still allows to describe the film as a continuum, but retains the thermodynamic essence of the ferromagnetic phase and allows to evaluate the additional effects of the film edge. Namely, we consider a mean-field model of a Heisenberg ferromagnet with a long-range Kac attractive interaction. Such models have been rigorously derived in the context of theories of phase transitions, going back to Lebowitz and Penrose for the liquid–gas phase transition[45] and Thompson and Silver for the classical Heisenberg magnet.[67] Moreover, in the considered limit the metastable spatially varying states may be understood via minimization of a free energy functional,[33] which in the case of the Heisenberg model with the interaction kernel $J_\delta(|x|)$ takes the form

$$\mathcal{F}(\rho) = -\frac{1}{2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_\Omega\int_\Omega J_\delta(|x-y|)(\boldsymbol{m}\cdot\boldsymbol{m}')\rho(x,\boldsymbol{m})\rho(y,\boldsymbol{m}')\mathrm{d}x\,\mathrm{d}y\mathrm{d}\mathcal{H}^2(\boldsymbol{m})\mathrm{d}\mathcal{H}^2$$
$$\times(\boldsymbol{m}') + \beta^{-1}\int_{\mathbb{S}^2}\int_\Omega \rho(x,\boldsymbol{m})\ln\rho(x,\boldsymbol{m})\mathrm{d}x\,\mathrm{d}\mathcal{H}^2(\boldsymbol{m}). \tag{2.24}$$

Here $\rho\in L^1(\mathbb{R}^2\times\mathbb{S}^2;[0,\infty])$ is the probability density to observe a spin at point $x\in\mathbb{R}^2$ in the direction $\boldsymbol{m}\in\mathbb{S}^2$, $J_\delta\in C_c^\infty(\mathbb{R})$ is a positive, even interaction potential such that

$$\operatorname{supp}(J_\delta) \subset B_\delta(0) \quad \text{and} \quad \int_0^\infty 2\pi r J_\delta(r) \mathrm{d}r = J_0 > 0 \tag{2.25}$$

for a $J_0 > 0$ fixed independently of δ, and $\beta > 0$ is the inverse temperature. The function ρ satisfies the following normalization conditions:

$$\int_{\mathbb{S}^2} \rho(x, \boldsymbol{m}) \mathrm{d}\mathcal{H}^2(\boldsymbol{m}) = 1 \quad \text{if } x \in \Omega, \quad \rho(x, \boldsymbol{m}) = 0 \quad \text{if } x \in \mathbb{R}^2 \backslash \Omega, \tag{2.26}$$

expressing the fact that the ferromagnet occupies the spatial domain $\Omega \subset \mathbb{R}^2$. For our purposes, all other terms in the energy, which are all small perturbations to the Heisenberg exchange, have been neglected. Notice that the parameter δ measures the finite range of the ferromagnetic coupling and physically corresponds to the extent of the exchange interaction of several lattice spacings.

The free energy in (2.24) admits a moments closure, allowing to reduce the minimization problem to that of a functional of the average magnetization (see also Ref. 28)

$$\overline{\boldsymbol{m}}(x) := \int_{\mathbb{S}^2} \boldsymbol{m} \rho(x, \boldsymbol{m}) \mathrm{d}\mathcal{H}^2(\boldsymbol{m}). \tag{2.27}$$

For a fixed value of $\overline{\boldsymbol{m}}$ the entropic term in the free energy is easily seen to be minimized pointwise by

$$\bar{\rho}(x, \boldsymbol{m}) = \exp(\beta(\mu(x) + \boldsymbol{\lambda}(x) \cdot \boldsymbol{m})), \tag{2.28}$$

where the functions μ and $\boldsymbol{\lambda}$ are obtained by enforcing (2.26) and (2.27) with $\rho = \bar{\rho}$ in Ω:

$$1 = 4\pi e^{\beta\mu} \frac{\sinh(\beta|\boldsymbol{\lambda}|)}{\beta|\boldsymbol{\lambda}|}, \tag{2.29}$$

$$\overline{\boldsymbol{m}} = 4\pi e^{\beta\mu} \frac{\beta|\boldsymbol{\lambda}| \cosh(\beta|\boldsymbol{\lambda}|) - \sinh(\beta|\boldsymbol{\lambda}|)}{\beta^2|\boldsymbol{\lambda}|^3} \boldsymbol{\lambda}. \tag{2.30}$$

This yields $\boldsymbol{\lambda} = \beta^{-1} \overline{\boldsymbol{m}} f(|\overline{\boldsymbol{m}}|)/|\overline{\boldsymbol{m}}|$, where the function $f(s) \geq 0$ is the unique positive solution of the equation

$$s = \coth f(s) - \frac{1}{f(s)}, \quad 0 < s < 1, \tag{2.31}$$

vanishing at $s = 0$ and diverging as $s \to 1^-$. The plot of $f(s)$ is presented in Fig. 1(a). Note that $f \in C^\infty([0, 1))$ and is strictly monotone increasing. Substituting this back to the entropy term results in

$$\int_{\mathbb{S}^2} \bar{\rho}(x, \boldsymbol{m}) \ln \bar{\rho}(x, \boldsymbol{m}) \mathrm{d}\mathcal{H}^2(\boldsymbol{m}) = \ln\left(\frac{f(|\overline{\boldsymbol{m}}(x)|)}{4\pi \sinh f(|\overline{\boldsymbol{m}}(x)|)}\right) + \overline{\boldsymbol{m}}(x) f(|\overline{\boldsymbol{m}}(x)|). \tag{2.32}$$

Thus, for $\overline{\boldsymbol{m}}(x)$ fixed the free energy satisfies $\mathcal{F}(\rho) \geq \bar{\mathcal{F}}(\overline{\boldsymbol{m}})$, where

$$\bar{\mathcal{F}}(\overline{\boldsymbol{m}}) := -\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J_\delta(|x-y|)\overline{\boldsymbol{m}}(x)\cdot\overline{\boldsymbol{m}}(y)\mathrm{d}x\mathrm{d}y + \frac{J_0}{2}\int_{\mathbb{R}^2}|\overline{\boldsymbol{m}}|^2\mathrm{d}x + \int_{\mathbb{R}^2} U_\beta(|\overline{\boldsymbol{m}}|)\mathrm{d}x, \tag{2.33}$$

with equality holding if and only if $\rho(x,\boldsymbol{m}) = \bar{\rho}(x,\boldsymbol{m})$. Here the effective potential U_β is given by

$$U_\beta(s) := \beta^{-1}\ln\left(\frac{f(s)}{4\pi\sinh f(s)}\right) + \beta^{-1}sf(s) - \frac{1}{2}J_0\, s^2, \tag{2.34}$$

with the convention that $U_\beta(0) := -\beta^{-1}\ln 4\pi$. The plots of $U_\beta(s)$ for several values of β are illustrated in Fig. 1(b).

Notice that the reduced energy in (2.33) may be rewritten in a more convenient form as

$$\bar{\mathcal{F}}(\overline{\boldsymbol{m}}) = \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J_\delta(|x-y|)|\overline{\boldsymbol{m}}(x)-\overline{\boldsymbol{m}}(y)|^2\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^2} U_\beta(|\overline{\boldsymbol{m}}|)\mathrm{d}x, \tag{2.35}$$

which is a vectorial, non-local analog of the classical Cahn–Hilliard functional, since U_β has a form of a Mexican hat potential for $\beta > \beta_c := 3J_0$. It is also easy to see that in a periodic setting the energy functional $\bar{\mathcal{F}}$ admits a unique minimizer $\overline{\boldsymbol{m}} = \mathbf{0}$ whenever $\beta \leq \beta_c$, and a family of minimizers $|\overline{\boldsymbol{m}}| = s_0(\beta)$ with $0 < s_0(\beta) < 1$ for $\beta > \beta_c$ (see also Ref. 28). To simplify matters further, we employ the usual gradient approximation to the non-local term in (2.35) to obtain $\bar{\mathcal{F}}(\overline{\boldsymbol{m}}) \simeq \bar{\mathcal{F}}_0(\overline{\boldsymbol{m}})$, where (see also Ref. 29 for a closely related problem)

$$\bar{\mathcal{F}}_0(\overline{\boldsymbol{m}}) := \int_\Omega (g_\delta|\nabla\overline{\boldsymbol{m}}|^2 + U_\beta(|\overline{\boldsymbol{m}}|))\mathrm{d}x, \tag{2.36}$$

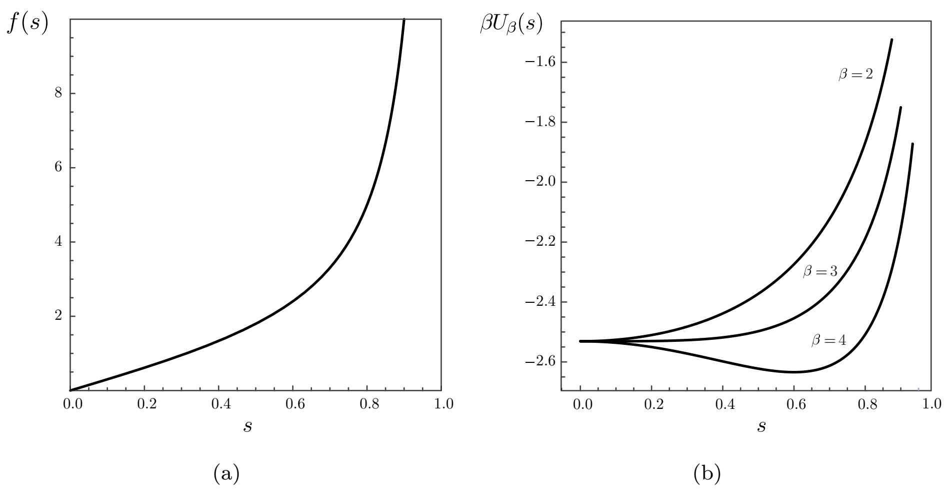

Fig. 1. (a) Plot of $f(s)$. (b) Plots of $\beta U_\beta(s)$ for the indicated values of β and $J_0 = 1$.

and $g_\delta := \frac{\pi}{4}\int_0^\infty r^3 J_\delta(r)\mathrm{d}r = O(\delta^2)$. The expression in (2.36) is specified for $\overline{\boldsymbol{m}} \in H_0^1(\Omega;\mathbb{R}^3)$, inheriting the zero boundary condition from the assumption that $\overline{\boldsymbol{m}} = \mathbf{0}$ in $\mathbb{R}^2\backslash\Omega$.

Near the edge of the sample the curvature of the edge is negligible to the leading order in δ. Hence, the problem of minimizing the energy in (2.36) reduces to a one-dimensional problem on half-line, i.e. to minimizing the energy

$$\bar{\mathcal{F}}_0^{1\mathrm{d}}(\overline{\boldsymbol{m}}) := \int_0^\infty (g_\delta|\overline{\boldsymbol{m}}'|^2 + U_\beta(|\overline{\boldsymbol{m}}|) - U_\beta(s_0(\beta)))\mathrm{d}x \tag{2.37}$$

over $\overline{\boldsymbol{m}} \in H^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}^3) \cap C(\overline{\mathbb{R}}^+;\mathbb{R}^3)$ such that $\overline{\boldsymbol{m}}(0) = \mathbf{0}$. An explicit energy-minimizing profile may be obtained from (2.37), using the polar representation $\overline{\boldsymbol{m}}(x) = \phi(x)\boldsymbol{u}(x)$, where $|\boldsymbol{u}| = 1$, for which we get

$$\bar{\mathcal{F}}_0^{1\mathrm{d}}(\overline{\boldsymbol{m}}) = \int_0^\infty (g_\delta|\phi'|^2 + U_\beta(\phi) - U_\beta(s_0(\beta)))\mathrm{d}x + \int_0^\infty g_\delta\phi^2|\boldsymbol{u}'|^2\mathrm{d}x. \tag{2.38}$$

Thus, the energy $\bar{\mathcal{F}}_0^{1\mathrm{d}}$ is minimized by $\boldsymbol{u} = \mathrm{const}$ and $\phi = \phi_\delta$, where by the Modica–Mortola trick[51] we have

$$\min \bar{\mathcal{F}}_0^{1\mathrm{d}} = \bar{\mathcal{F}}_0^{1\mathrm{d}}(\phi_\delta\boldsymbol{u}) = 2\int_0^{s_0(\beta)} \sqrt{g_\delta(U_\beta(\phi) - U_\beta(s_0(\beta)))}\mathrm{d}\phi, \tag{2.39}$$

$$\int_0^{\phi_\delta(x)} \frac{\mathrm{d}\phi}{\sqrt{g_\delta(U_\beta(\phi) - U_\beta(s_0(\beta)))}} = x, \quad \boldsymbol{u} \in \mathbb{S}^2 \text{ arbitrary.} \tag{2.40}$$

In particular, we have $|\overline{\boldsymbol{m}}(x)| = \phi_\delta(x)$, where $\phi_\delta(x)$ is a monotone increasing function such that $\phi_\delta(0) = 0$ and $\phi'_\delta(0) > 0$, and $\phi_\delta(x)$ approaches exponentially the "saturation magnetization" $s_0(\beta)$ for $x \gg \delta$. Therefore, as a matter of modeling convenience one could replace the function ϕ_δ with that of $s_0\eta_\delta$, where η_δ is defined in the following section.

3. Mathematical Setup and Statement of the Main Results

We now give the precise mathematical formulation of the considered problem and state our main theorems.

3.1. *Film geometry*

Throughout this paper we assume that $\Omega \subset \mathbb{R}^2$ is a bounded, simply connected open set with boundary of class C^2. We parametrize $\partial\Omega$ by a C^2 regular curve, and we denote by

$$\varphi : s \in I_{\partial\Omega} \mapsto \varphi(s) \in \partial\Omega, \quad I_{\partial\Omega} := [0, \mathcal{H}^1(\partial\Omega)], \tag{3.1}$$

its positive parameterization by the arc length (extended periodically, if necessary). We denote by $(\boldsymbol{\tau}(x), \boldsymbol{n}(x)) \in \mathbb{S}^1 \times \mathbb{S}^1$ the respective Frenet frame at $x \in \partial\Omega$. The vector fields $\boldsymbol{\tau} : \partial\Omega \to \mathbb{S}^1$ and $\boldsymbol{n} : \partial\Omega \to \mathbb{S}^1$ represent, respectively, the unit tangent vector field to $\partial\Omega$ given by $\boldsymbol{\tau}(\varphi(s)) := \varphi'(s)$, and the outer unit normal vector field.

To avoid cumbersome notations, we write $\boldsymbol{\tau}(s)$ and $\boldsymbol{n}(s)$ to mean the compositions $\boldsymbol{\tau}(\varphi(s))$ and $\boldsymbol{n}(\varphi(s))$ from now on. Clearly, for every $s \in I_{\partial\Omega}$ we have $\boldsymbol{\tau}(s)\cdot\boldsymbol{n}(s) = 0$, and the Frenet–Serret formulas hold:

$$\boldsymbol{\tau}'(s) = -\kappa(s)\boldsymbol{n}(s), \tag{3.2}$$

$$\boldsymbol{n}'(s) = \kappa(s)\boldsymbol{\tau}(s), \tag{3.3}$$

where $\kappa(s) := -\varphi''(s)\cdot\boldsymbol{n}(s)$ stands for the signed curvature of $\partial\Omega$ at the point $\varphi(s)$. Note that since we assume $\partial\Omega$ of class C^2, the vector fields $\boldsymbol{\tau}(s)$ and $\boldsymbol{n}(s)$ are of class $C^1(I_{\partial\Omega};\mathbb{S}^1)$ and $\kappa(s)$ is a continuous function.

In order to define a cutoff function near the boundary of the ferromagnet, we first introduce the signed distance function from $\partial\Omega$ which assigns positive values to points in the exterior of Ω and negative values in the interior of Ω:

$$d_{\partial\Omega}(x) := \begin{cases} +\inf\limits_{y\in\partial\Omega} |x-y| & \text{if } x \in \mathbb{R}^2\backslash\Omega, \\ -\inf\limits_{y\in\partial\Omega} |x-y| & \text{if } x \in \Omega. \end{cases} \tag{3.4}$$

Since Ω is a C^2-domain, there exists $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$ the set

$$\mathcal{O}_\varepsilon := \{x \in \mathbb{R}^2 : |d_{\partial\Omega}(x)| < \varepsilon\} \tag{3.5}$$

is in the tubular neighborhood of $\partial\Omega$ of radius $\bar{\varepsilon}$, namely in $\mathcal{O}_{\bar{\varepsilon}} := \{x \in \mathbb{R}^2 : |d_{\partial\Omega}(x)| < \bar{\varepsilon}\}$. For any $0 < \varepsilon < \bar{\varepsilon}$ we also set $\mathcal{O}_\varepsilon^+ := \{x \in \mathbb{R}^2 : 0 \le d_{\partial\Omega}(x) < \varepsilon\}$. Since Ω is a simply connected C^2 domain, there exists a C^1 projection map $\boldsymbol{\pi} : \mathcal{O}_{\bar{\varepsilon}} \to \partial\Omega$ such that $x = \boldsymbol{\pi}(x) + d_{\partial\Omega}(x)\boldsymbol{n}(\boldsymbol{\pi}(x))$ for every $x \in \mathcal{O}_{\bar{\varepsilon}}$ and

$$\nabla d_{\partial\Omega}(x) = \boldsymbol{n}(\boldsymbol{\pi}(x)). \tag{3.6}$$

In particular, $|\nabla d_{\partial\Omega}(x)| = 1$ for every $x \in \mathcal{O}_{\bar{\varepsilon}}$, and the values of $\boldsymbol{\pi}$ may be parametrized by the arclength of $\partial\Omega$. In what follows, we always assume that $0 < \varepsilon < \bar{\varepsilon}$ so that the tubular neighborhood theorem holds.

For any $0 < \varepsilon < \bar{\varepsilon}$ we set $\Omega_\varepsilon := \Omega \cup \mathcal{O}_\varepsilon^+$, which represents the domain in the plane occupied by the ferromagnetic film, and consider the family of cutoff functions

$$\eta_\varepsilon(x) := \eta\left(\frac{d_{\partial\Omega}(x)}{\varepsilon}\right) \tag{3.7}$$

defined by a *non-increasing* function $\eta \in C^{0,1}(\mathbb{R})$ such that

$$\eta(t) \equiv 1 \quad \text{for } t \in (-\infty, 0), \quad \eta(t) \equiv 0 \quad \text{for } t \in (1, +\infty). \tag{3.8}$$

We further assume that on $(-\infty, 1]$ the function η is continuously differentiable, but allow $\eta'(t)$ to jump at $t = 1$ in accordance with the behavior of $|\overline{\boldsymbol{m}}|$ at the film edge observed in Sec. 2.1.

Note that for every $\varepsilon > 0$ we have $\eta_\varepsilon(x) \equiv 1$ whenever $d_{\partial\Omega}(x) \le 0$ and $\eta_\varepsilon(x) \equiv 0$ when $d_{\partial\Omega}(x) \ge \varepsilon$. In other words, η_ε is a cutoff function whose support is included in the closure of Ω_ε (i.e. $\mathrm{supp}(\eta_\varepsilon) \subseteq \overline{\Omega}_\varepsilon$) such that $\eta_\varepsilon \equiv 1$ on Ω. We observe the following identities:

$$\begin{aligned} \nabla \eta_\varepsilon(x) &= \frac{1}{\varepsilon}\eta' \left(\frac{d_{\partial\Omega}(x)}{\varepsilon} \right) \boldsymbol{n}(\boldsymbol{\pi}(x)) \\ &= -|\nabla \eta_\varepsilon(x)| \boldsymbol{n}(\boldsymbol{\pi}(x)). \end{aligned} \tag{3.9}$$

In particular, $\nabla\eta_\varepsilon \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and $\mathrm{supp}(\nabla\eta_\varepsilon) \subseteq \overline{\Omega}_\varepsilon \backslash \Omega = \overline{\mathcal{O}_\varepsilon^+}$.

3.2. *The micromagnetic energy*

Given a configuration $\boldsymbol{m}^\varepsilon \in H^1_{\mathrm{loc}}(\Omega_\varepsilon;\mathbb{S}^2)$, we extend it by zero outside Ω_ε to define the magnetization in the whole plane, so that the physically observable magnetization is $\boldsymbol{m}_\varepsilon(x) = \eta_\varepsilon(x)\boldsymbol{m}^\varepsilon(x)$ for all $x \in \mathbb{R}^2$, after a rescaling of length from (2.16). The rescaled cutoff length is now $\varepsilon \ll 1$ and the domain Ω_ε has $O(1)$ size, converging to a fixed domain Ω from outside as $\varepsilon \to 0$. As we will see later, the precise shape of the cutoff function η will prove not to play any role in the limiting behavior of the energy analyzed in this paper. Further care is needed, however, to specify a representation of the micromagnetic energy that is well-suited for the analysis of those limits.

We first need to define a space where the different terms in our micromagnetic energy are all bounded. To make sure this is the case, we clearly need that $\int_{\Omega_\varepsilon} \eta_\varepsilon^2 |\nabla \boldsymbol{m}^\varepsilon|^2 \mathrm{d}x < \infty$. Therefore, we assume that $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$, where $H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ stands for the weighted Sobolev (metric) space defined by

$$\begin{aligned} H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2) := \{&\boldsymbol{m}^\varepsilon \in L^2(\mathbb{R}^2;\mathbb{R}^3) : \boldsymbol{m}^\varepsilon_{|\Omega_\varepsilon} \in H^1_{\mathrm{loc}}(\Omega_\varepsilon, \mathbb{S}^2), \\ &\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_{|\Omega_\varepsilon} \in L^2(\Omega_\varepsilon), \boldsymbol{m}^\varepsilon \equiv 0 \text{ in } \mathbb{R}^2 \backslash \Omega_\varepsilon\}. \end{aligned} \tag{3.10}$$

Note that elements of $H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ are identically zero outside of Ω_ε. We view $H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ as a metric subspace of $H_\varepsilon(\mathbb{R}^2)$, where

$$\begin{aligned} H_\varepsilon(\mathbb{R}^2) := \big\{&\boldsymbol{u}^\varepsilon \in L^2(\mathbb{R}^2;\mathbb{R}^3) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^3) : \boldsymbol{u}^\varepsilon_{|\Omega_\varepsilon} \in H^1_{\mathrm{loc}}(\Omega_\varepsilon;\mathbb{R}^3), \\ &\eta_\varepsilon \nabla \boldsymbol{u}^\varepsilon_{|\Omega_\varepsilon} \in L^2(\Omega_\varepsilon;\mathbb{R}^6), \boldsymbol{u}^\varepsilon \equiv 0 \text{ in } \mathbb{R}^2 \backslash \Omega_\varepsilon \big\}. \end{aligned} \tag{3.11}$$

We can similarly introduce the "limit" space

$$H_0(\mathbb{R}^2;\mathbb{S}^2) := \{\boldsymbol{m} \in L^2(\mathbb{R}^2;\mathbb{R}^3) : \boldsymbol{m}_{|\Omega} \in H^1(\Omega, \mathbb{S}^2), \boldsymbol{m} \equiv 0 \text{ in } \mathbb{R}^2 \backslash \Omega\}. \tag{3.12}$$

We next introduce the notation $\boldsymbol{m}^\varepsilon = (\boldsymbol{m}^\varepsilon_\perp, m^\varepsilon_\parallel)$, where $\boldsymbol{m}^\varepsilon_\perp \in \mathbb{R}^2$ and $m^\varepsilon_\parallel \in \mathbb{R}$ give the components of $\boldsymbol{m}^\varepsilon$ that are perpendicular and parallel to the material easy axis $\pm\boldsymbol{e}_3$, respectively. The non-local contribution from the stray field energy can then be seen to be proportional to

$$\mathcal{W}_\varepsilon(\boldsymbol{m}^\varepsilon) = \frac{1}{2|\ln\varepsilon|}\mathcal{V}(\eta_\varepsilon \boldsymbol{m}^\varepsilon_\perp) - \frac{1}{2|\ln\varepsilon|}\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel), \tag{3.13}$$

defined for every $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$, with

$$\mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\mathrm{div}\,(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon)(x)\mathrm{div}\,(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon)(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{3.14}$$

$$\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla(\eta_\varepsilon m_\parallel^\varepsilon)(x)\cdot\nabla(\eta_\varepsilon m_\parallel^\varepsilon)(y)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{3.15}$$

The equivalence of the above expression with the one appearing in (2.17) for smooth functions can be seen via the Fourier representation. Note that both $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are non-negative, a result that can be easily shown in the Fourier domain by Parseval–Plancherel identity.

The DMI contribution to the energy is proportional to

$$\mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon) := \int_{\mathbb{R}^2} \eta_\varepsilon^2 (m_\parallel^\varepsilon \mathrm{div}\,\boldsymbol{m}_\perp^\varepsilon - \boldsymbol{m}_\perp^\varepsilon\cdot\nabla m_\parallel^\varepsilon)\mathrm{d}x. \tag{3.16}$$

The remaining terms may be defined analogously. Note that the space $H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ depends on ε and, therefore, is not well-suited for Γ-convergence arguments. Therefore, since $\boldsymbol{m}_\varepsilon \in L^2(\mathbb{R}^2;\mathbb{R}^3)$, we can use a penalization to formulate the Γ-convergence results in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ with the agreement that the energy is infinite outside $H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$. Furthermore, the space $L^2(\mathbb{R}^2;\mathbb{R}^3)$ also provides a natural topology for the Γ-convergence.

The simplified micromagnetic energy defined for $\boldsymbol{m}^\varepsilon \in L^2(\mathbb{R}^2;\mathbb{R}^3)$ that disregards the anisotropy and the Zeeman terms takes the form

$$\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) := \begin{cases} \displaystyle\int_{\mathbb{R}^2} \eta_\varepsilon^2|\nabla \boldsymbol{m}^\varepsilon|^2\mathrm{d}x + \lambda\mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon) + \gamma_\varepsilon\mathcal{W}_\varepsilon(\boldsymbol{m}^\varepsilon) & \text{if } \boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.17}$$

where the precise dependence of γ_ε on ε will be specified in the sequel for various Γ-limits. As is common in the studies of Γ-convergence, we will simply write $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ as $\varepsilon \to 0$, always tacitly implying that for any sequence of $\varepsilon_n \to 0$ we have $\boldsymbol{m}^{\varepsilon_n} \to \boldsymbol{m}$ as $n \to \infty$.

Remark 3.1. We note that the functional $\mathcal{G}_\varepsilon$ in (3.17) provides a mathematically suitable formulation for the main terms in the energy E_ε defined in (2.17) in the considered topology of Γ-convergence. In particular, we have $\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) = E_\varepsilon(\boldsymbol{m}^\varepsilon)$ whenever $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ with $\alpha = 0$ and $\beta = 0$. Hence throughout the rest of this paper we formulate our results in terms of the functional $\mathcal{G}_\varepsilon$ and note that the anisotropy and the Zeeman terms are continuous perturbations to $\mathcal{G}_\varepsilon$ under L^2 convergence and hence can be trivially included in the statements of the theorems.

3.3. *Main results*

In this section, we formulate the main results of this paper. We split our results into four theorems corresponding to different magnetic regimes previously studied for ferromagnets with strong in-plane anisotropy.

The first regime we consider is the analogue of the Gioia and James regime.[34] We have the following theorem.

Theorem 3.1. (Free boundary) *Let $\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon)$ be defined on $L^2(\mathbb{R}^2;\mathbb{R}^3)$ by* (3.17) *and let $\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$. We define $\mathcal{G}_0(\boldsymbol{m})$ on $L^2(\mathbb{R}^2;\mathbb{R}^3)$ by*

$$\mathcal{G}_0(\boldsymbol{m}) := \begin{cases} \displaystyle\int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x + \lambda \int_\Omega (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel)\mathrm{d}x & \text{if } \boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.18}$$

Then the following statements hold:

(1) (Compactness) *If $\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) < +\infty$ then $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ and $\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m}$ weakly in $H^1(\Omega;\mathbb{S}^2)$ for some $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ as $\varepsilon \to 0$ (possibly up to a subsequence).*

(2) (Γ-liminf inequality) *Let $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ be such that $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ for some $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ as $\varepsilon \to 0$. Then*

$$\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) \geq \mathcal{G}_0(\boldsymbol{m}). \tag{3.19}$$

(3) (Γ-limsup inequality) *Let $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$. Then there exists $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ such that $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ as $\varepsilon \to 0$ and*

$$\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) \leq \mathcal{G}_0(\boldsymbol{m}). \tag{3.20}$$

The second regime we study corresponds to the result of Kohn and Slastikov,[43] where in the limit of small thickness, the magnetization prefers to stay in-plane, and a local boundary contribution corresponding to shape anisotropy replaces the non-local magnetostatic energy. The limiting behavior we obtain for materials with perpendicular anisotropy is contained in the following theorem. Here and everywhere below, the values of $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ on $\partial\Omega$ are understood in the sense of trace of the Sobolev function $\boldsymbol{m}_{|\Omega} \in H^1(\Omega;\mathbb{S}^2)$.

Theorem 3.2. (Boundary penalty) *Let $\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon)$ be defined on $L^2(\mathbb{R}^2;\mathbb{R}^3)$ by* (3.17) *and $\gamma_\varepsilon \to \gamma$ for some $\gamma > 0$ as $\varepsilon \to 0$. We define $\mathcal{G}_0^\gamma(\boldsymbol{m})$ on $L^2(\mathbb{R}^2;\mathbb{R}^3)$ by*

$$\mathcal{G}_0^\gamma(\boldsymbol{m}) := \begin{cases} \displaystyle\int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x + \lambda \int_\Omega (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel)\mathrm{d}x & \\ \displaystyle\qquad + \gamma \int_{\partial\Omega} ((\boldsymbol{m}_\perp \cdot \boldsymbol{n})^2 - m_\parallel^2)\mathrm{d}\mathcal{H}^1(x) & \text{if } \boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.21}$$

Then the following statements hold:

(1) (Compactness) *If* $\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) < +\infty$, *then* $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *and* $\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m}$ *weakly in* $H^1(\Omega;\mathbb{S}^2)$ *for some* $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ *as* $\varepsilon \to 0$ (*possibly up to a subsequence*).

(2) (Γ-liminf inequality) *Let* $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ *be such that* $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *for some* $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ *as* $\varepsilon \to 0$. *Then*

$$\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) \geq \mathcal{G}_0^\gamma(\boldsymbol{m}). \tag{3.22}$$

(3) (Γ-limsup inequality) *Let* $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$. *Then there exists* $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ *such that* $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *as* $\varepsilon \to 0$ *and*

$$\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) \leq \mathcal{G}_0^\gamma(\boldsymbol{m}). \tag{3.23}$$

The following two results are fundamentally different from what exists in the micromagnetic literature for in-plane materials. This is due to the fact that they correspond to magnetic regimes, where the shape anisotropy of the micromagnetic energy is penalized in the limit and, as a result, the magnetization acquires Dirichlet conditions at the boundary. For in-plane materials this regime is impossible as it leads to a singular behavior of the micromagnetic energy due to a topological obstruction. For materials with perpendicular anisotropy, however, these regimes provide the micromagnetic energy describing the behavior of magnetic skyrmions (see Ref. 52) and therefore are of utter importance.

We formulate our results in the following theorems corresponding to local and non-local versions of the micromagnetic energies. Note that everywhere below the statement $m_\parallel = \pm 1$ on $\partial\Omega$ means that either $m_\parallel = 1$ on $\partial\Omega$ or $m_\parallel = -1$ on $\partial\Omega$, in the sense of trace. In other words, the trace of $\boldsymbol{m}$ is constant on $\partial\Omega$ and is equal to either $\boldsymbol{e}_3$ or $-\boldsymbol{e}_3$.

Our first theorem yields a local limiting micromagnetic energy with the Dirichlet boundary condition.

Theorem 3.3. (Clamped boundary, local energy) *Let* $\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon)$ *be defined on* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *by* (3.17) *with* $\gamma_\varepsilon \to \infty$ *and* $\gamma_\varepsilon |\ln\varepsilon|^{-1} \to 0$ *as* $\varepsilon \to 0$. *We define* $\tilde{\mathcal{G}}_0(\boldsymbol{m})$ *on* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *by*

$$\tilde{\mathcal{G}}_0(\boldsymbol{m}) := \begin{cases} \displaystyle\int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x \\ \displaystyle\qquad + \lambda \int_\Omega (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel) \mathrm{d}x & \textit{if } \boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2) \\ & \textit{and } m_\parallel = \pm 1 \text{ on } \partial\Omega, \\ +\infty & \textit{otherwise}. \end{cases} \tag{3.24}$$

Then the following statements hold:

(1) (Compactness) *If* $\limsup_{\varepsilon\to 0}(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \gamma_\varepsilon \mathcal{H}^1(\partial\Omega)) < +\infty$ *then* $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *and* $\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m}$ *weakly in* $H^1(\Omega;\mathbb{S}^2)$ *for some* $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ *with* $m_\parallel = \pm 1$ *on* $\partial\Omega$ *as* $\varepsilon \to 0$ (*possibly up to a subsequence*).

(2) (Γ-liminf inequality) *Let* $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ *be such that* $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *for some* $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ *with* $m_\parallel = \pm 1$ *on* $\partial\Omega$ *as* $\varepsilon \to 0$. *Then*

$$\liminf_{\varepsilon\to 0}(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \gamma_\varepsilon \mathcal{H}^1(\partial\Omega)) \geq \tilde{\mathcal{G}}_0(\boldsymbol{m}). \tag{3.25}$$

(3) (Γ-limsup inequality) *Let* $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ *with* $m_\parallel = \pm 1$ *on* $\partial\Omega$. *Then there exists* $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ *such that* $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *as* $\varepsilon \to 0$ *and*

$$\limsup_{\varepsilon\to 0}(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \gamma_\varepsilon \mathcal{H}^1(\partial\Omega)) \leq \tilde{\mathcal{G}}_0(\boldsymbol{m}). \tag{3.26}$$

The next theorem provides a new type of a reduced non-local micromagnetic energy for yet stronger dipolar interaction. To state the theorem, we need to introduce some additional notation. This is due to the fact that in the considered regime the stray field of the film edge continues to contribute to the limit energy and, therefore, needs to be properly accounted for. We define the quantity

$$D_\varepsilon := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\eta_\varepsilon(x)\cdot\nabla\eta_\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{3.27}$$

which will be shown to give, up to the factor of $\frac{1}{2}\nu$, the leading order behavior of the energy $\mathcal{G}_\varepsilon$. In fact, this constant can be seen to be the energy of the ferromagnetic state $\boldsymbol{m} = \pm\boldsymbol{e}_3\chi_\Omega$ and to the leading order satisfies $D_\varepsilon = 2|\ln\varepsilon|\,\mathcal{H}^1(\partial\Omega) + O(1)$ when $\varepsilon \to 0$, as can be seen from Lemma 4.4.

Theorem 3.4. (Clamped boundary, non-local energy) *Let* $\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon)$ *be defined on* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *by* (3.17) *with* $\gamma_\varepsilon = \nu|\ln\varepsilon|$ *for some* $\nu > 0$. *We define* $\tilde{\mathcal{G}}_0^\nu(\boldsymbol{m})$ *on* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *by*

$$\tilde{\mathcal{G}}_0^\nu(\boldsymbol{m}) := \begin{cases} \displaystyle\int_\Omega |\nabla\boldsymbol{m}|^2\mathrm{d}x + \nu\int_\Omega \boldsymbol{b}\cdot\nabla m_\parallel\,\mathrm{d}x & \\ \displaystyle\quad + \lambda\int_\Omega (m_\parallel \operatorname{div}\boldsymbol{m}_\perp - \boldsymbol{m}_\perp\cdot\nabla m_\parallel)\mathrm{d}x & \textit{if } \boldsymbol{m}\in H_0(\mathbb{R}^2;\mathbb{S}^2) \\ \displaystyle\quad + \frac{\nu}{2}\int_\Omega\int_\Omega \frac{\operatorname{div}\boldsymbol{m}_\perp(x)\operatorname{div}\boldsymbol{m}_\perp(y)}{|x-y|}\mathrm{d}x\mathrm{d}y & \textit{and } m_\parallel = \pm 1 \text{ on } \partial\Omega, \\ \displaystyle\quad - \frac{\nu}{2}\int_\Omega\int_\Omega \frac{\nabla m_\parallel(x)\cdot\nabla m_\parallel(y)}{|x-y|}\mathrm{d}x\mathrm{d}y & \\ +\infty & \textit{otherwise}, \end{cases} \tag{3.28}$$

where $\boldsymbol{b}$ *is defined in* (2.23). *Then the following statements hold*:

(1) (Compactness) *If* $\limsup_{\varepsilon\to 0}(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon)+\frac{\nu}{2}D_\varepsilon)<+\infty$ *then* $\boldsymbol{m}^\varepsilon\to\boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *and* $\boldsymbol{m}^\varepsilon\rightharpoonup\boldsymbol{m}$ *weakly in* $H^1(\Omega;\mathbb{S}^2)$ *for some* $\boldsymbol{m}\in H_0(\mathbb{R}^2;\mathbb{S}^2)$ *with* $m_\parallel=\pm 1$ *on* $\partial\Omega$ *as* $\varepsilon\to 0$ (*possibly up to a subsequence*), *where* D_ε *is defined in* (3.27).

(2) (Γ-liminf inequality) *Let* $\boldsymbol{m}^\varepsilon\in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ *satisfy* $\boldsymbol{m}^\varepsilon\to\boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *for some* $m\in H_0(\mathbb{R}^2;\mathbb{S}^2)$ *with* $m_\parallel=\pm 1$ *on* $\partial\Omega$ *as* $\varepsilon\to 0$, *then*

$$\liminf_{\varepsilon\to 0}\left(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon)+\frac{\nu}{2}D_\varepsilon\right)\geq\tilde{\mathcal{G}}_0^\nu(\boldsymbol{m}). \tag{3.29}$$

(3) (Γ-limsup inequality) *Let* $\boldsymbol{m}\in H_0(\Omega;\mathbb{S}^2)$ *with* $m_\parallel=\pm 1$ *on* $\partial\Omega$. *Then there exists* $\boldsymbol{m}^\varepsilon\in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ *that satisfies* $\boldsymbol{m}^\varepsilon\to\boldsymbol{m}$ *strongly in* $L^2(\mathbb{R}^2;\mathbb{R}^3)$ *as* $\varepsilon\to 0$ *and*

$$\limsup_{\varepsilon\to 0}\left(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon)+\frac{\nu}{2}D_\varepsilon\right)\leq\tilde{\mathcal{G}}_0^\nu(\boldsymbol{m}). \tag{3.30}$$

The above results provide us with four comprehensive two-dimensional micromagnetic models to study the magnetization behavior in ultrathin films with perpendicular magnetic anisotropy and DMI. It is also clear that the above results can be supplemented with anisotropy and Zeeman energies as those are just continuous perturbations. In particular, this provides a rigorous justification to the formal limit statements in Sec. 2.

4. Auxiliary Lemmas

Throughout the rest of this paper, unless stated otherwise all the constants in the statements and the proofs depend only on Ω and η. We begin by providing several important technical lemmas. The first two lemmas concern pointwise estimates for the singular integral

$$f_\varepsilon(x):=\int_{\mathcal{O}_\varepsilon^+}\frac{|\nabla\eta_\varepsilon(y)|}{|y-x|}\mathrm{d}y. \tag{4.1}$$

We start with an estimate which for every $0<\varepsilon<\bar{\varepsilon}$ gives a precise control on $f_\varepsilon(x)$ when $x\in\mathcal{O}_\varepsilon$, where $\mathcal{O}_\varepsilon$ is defined in (3.5).

Lemma 4.1. *There exists* $\bar{\varepsilon},C>0$ *such that for any* $0<\varepsilon<\bar{\varepsilon}$ *and* $x\in\mathcal{O}_\varepsilon$ *there holds*

$$2|\ln\varepsilon|-C\leq f_\varepsilon(x)\leq 2|\ln\varepsilon|+C. \tag{4.2}$$

Proof. Recalling the notations of Sec. 3.1, for any $|t|<\bar{\varepsilon}$, the curve

$$\varphi_t: s\in I_{\partial\Omega}\mapsto\varphi(s)+t\boldsymbol{n}(s) \tag{4.3}$$

is a parameterization of the set $\partial\Omega_t:=\{\sigma\in\mathcal{O}_{\bar{\varepsilon}}: d_{\partial\Omega}(\sigma)=t\}$. Since Ω is of class C^2, there exists $\bar{\delta}>0$ sufficiently small such that for any $0<\delta<\bar{\delta}$ there exists a curvature-dependent constant $0 < \alpha(\delta) < \delta$ for which there holds

$$\mathcal{H}^1(\boldsymbol{\pi}(\mathcal{O}_{\bar\varepsilon} \cap B_{\alpha(\delta)}(x))) < \delta \quad \forall x \in \mathcal{O}_{\bar\varepsilon}. \tag{4.4}$$

In particular, there holds that $\mathcal{H}^1(\boldsymbol{\pi}(\partial\Omega_t \cap B_{\alpha(\delta)}(x))) < \delta$ for any $|t| < \bar\varepsilon$, and we can always assume that $\bar\varepsilon$ and $\bar\delta$ are tuned sufficiently small, so that for any $|t| < \bar\varepsilon$ and any $0 < \delta < \bar\delta$, the set $\boldsymbol{\pi}(\partial\Omega_t \cap B_{\alpha(\delta)}(x))$ is connected. In this way, the arc $\boldsymbol{\pi}(\partial\Omega_t \cap B_{\alpha(\delta)}(x))$ can be parameterized through the restriction of φ to a suitable subinterval of $I_{\partial\Omega}$. In other words, we assume that δ is sufficiently small so that for any $|t| < \bar\varepsilon$ and any $x = \varphi_t(s_0) \in \mathcal{O}_{\bar\varepsilon}$ one has

$$\partial\Omega_t \cap B_{\alpha(\delta)}(x) \subseteq \varphi_t(I_\delta(x)), \quad I_\delta(x) := [s_0 - \delta/2, s_0 + \delta/2]. \tag{4.5}$$

Now, let $\varepsilon < \bar\varepsilon$. For what follows, it is convenient to set

$$\kappa_{\partial\Omega} := \sup\{|\kappa(\sigma)| : \sigma \in \partial\Omega\}, \tag{4.6}$$

where $\kappa(\sigma)$ stands for the curvature of $\partial\Omega$ at the point $\sigma \in \partial\Omega$ (cf. (3.2)). Clearly, for any $x \in \mathcal{O}_\varepsilon$ and any $y \in \mathcal{O}_\varepsilon^+ \backslash B_{\alpha(\delta)}(x)$ one has $|y - x| \geq \alpha(\delta)$. We denote by $S_{\alpha(\varepsilon)}(x)$ the small sector around $x \in \mathcal{O}_\varepsilon$ defined by

$$S_{\alpha(\delta)}(x) := \{\varphi_t(I_\delta(x))\}_{|t|<\varepsilon}. \tag{4.7}$$

Clearly, $S_{\alpha(\delta)}(x) \supseteq B_{\alpha(\delta)}(x) \cap \mathcal{O}_\varepsilon^+$, therefore we can decompose $f_\varepsilon(x)$ as

$$f_\varepsilon(x) = \psi_\varepsilon(x) + g^\varepsilon_{\partial\Omega}(x), \tag{4.8}$$

where

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon} \int_{\mathcal{O}_\varepsilon^+ \cap S_{\alpha(\delta)}(x)} \frac{|\eta'(d_{\partial\Omega}(y)/\varepsilon)|}{|y - x|} \mathrm{d}y \tag{4.9}$$

and by the coarea formula the remainder term $g^\varepsilon_{\partial\Omega}(x)$ satisfies the uniform bound

$$\begin{aligned} |g^\varepsilon_{\partial\Omega}(x)| &\leq \frac{1}{\alpha(\delta)} \int_0^1 \int_{\partial\Omega} |\eta'(t)| \cdot |1 + \varepsilon t \kappa(\sigma)| \mathrm{d}\sigma \mathrm{d}t \leq c_\delta, \\ c_\delta &:= \frac{(1 + \bar\varepsilon \kappa_{\partial\Omega}) \mathcal{H}^1(\partial\Omega)}{\alpha(\delta)}. \end{aligned} \tag{4.10}$$

It remains to estimate $\psi_\varepsilon(x)$. For that we observe that with $x = \sigma + \varepsilon t \boldsymbol{n}(\sigma)$, $|t| < 1$, and by the coarea formula we have

$$\psi_\varepsilon(x) = \int_0^1 \int_{\varphi(I_\delta(x))} |\eta'(r)| \frac{1 + \varepsilon r \kappa(\omega)}{|\sigma + \varepsilon t \boldsymbol{n}(\sigma) - \omega - \varepsilon r \boldsymbol{n}(\omega)|} \mathrm{d}\mathcal{H}^1(\omega) \mathrm{d}r. \tag{4.11}$$

We estimate the denominator of the integrand as

$$\begin{aligned} \omega - \sigma + \varepsilon(r\boldsymbol{n}(\omega) - t\boldsymbol{n}(\sigma)) &= \varphi(s) - \varphi(s_0) + \varepsilon(r\boldsymbol{n}(s) - t\boldsymbol{n}(s_0)) \\ &= (s - s_0)\boldsymbol{\tau}(s_0) + \varepsilon(r - t)\boldsymbol{n}(s_0) \\ &\quad + \varepsilon r(\boldsymbol{n}(s) - \boldsymbol{n}(s_0)) + O(|s - s_0|^2), \end{aligned} \tag{4.12}$$

where $|O(|s-s_0|^2)| \le \kappa_{\partial\Omega}(s-s_0)^2$. On the other hand, since $0<r<1$ and $|t|<1$, and $\boldsymbol{n}(s)$ is Lipschitz continuous with constant $\kappa_{\partial\Omega}$, we also have

$$\begin{aligned}&|\varepsilon r(\boldsymbol{n}(s)-\boldsymbol{n}(s_0))+O(|s-s_0|^2)|\\ &\qquad\le \kappa_{\partial\Omega}(\varepsilon+|s-s_0|)|s-s_0|\\ &\qquad\le \kappa_{\partial\Omega}(\varepsilon+|s-s_0|)|(s-s_0)\boldsymbol{\tau}(s_0)+\varepsilon(r-t)\boldsymbol{n}(s_0)|. \qquad (4.13)\end{aligned}$$

Hence, combining (4.12) and (4.13) we obtain that for $\bar{\varepsilon},\delta<\frac{1}{2}\kappa_{\partial\Omega}^{-1}$ and $|s-s_0|<\delta$ we have

$$\begin{aligned}&|\omega-\sigma+\varepsilon(r\boldsymbol{n}(\omega)-t\boldsymbol{n}(\sigma))|\\ &\qquad\le (1+\kappa_{\partial\Omega}\varepsilon+\kappa_{\partial\Omega}|s-s_0|)|(s-s_0)\boldsymbol{\tau}(s_0)+\varepsilon(r-t)\boldsymbol{n}(s_0)|, \qquad (4.14)\\ &|\omega-\sigma+\varepsilon(r\boldsymbol{n}(\omega)-t\boldsymbol{n}(\sigma))|\\ &\qquad\ge (1-\kappa_{\partial\Omega}\varepsilon-\kappa_{\partial\Omega}|s-s_0|)|(s-s_0)\boldsymbol{\tau}(s_0)+\varepsilon(r-t)\boldsymbol{n}(s_0)|. \qquad (4.15)\end{aligned}$$

Overall, we get that for any $x=\sigma+\varepsilon t\boldsymbol{n}(\sigma)$, $|t|<1$, there holds

$$\begin{aligned}&\frac{1-\varepsilon\kappa_{\partial\Omega}}{1+\varepsilon\kappa_{\partial\Omega}}\int_0^1|\eta'(r)|\varrho_\varepsilon^+\left(\frac{\varepsilon^2}{\delta^2}(r-t)^2\right)\mathrm{d}r\\ &\qquad\le\psi_\varepsilon(x)\le\frac{1+\varepsilon\kappa_{\partial\Omega}}{1-\varepsilon\kappa_{\partial\Omega}}\int_0^1|\eta'(r)|\varrho_\varepsilon^-\left(\frac{\varepsilon^2}{\delta^2}(r-t)^2\right)\mathrm{d}r, \qquad (4.16)\end{aligned}$$

with

$$\varrho_\varepsilon^\pm\left(\frac{\varepsilon^2}{\delta^2}(r-t)^2\right):=\int_{I_\delta(x)}\frac{1}{(1\pm a_\varepsilon^\pm|s-s_0|)|(s-s_0)\boldsymbol{\tau}(s_0)+\varepsilon(r-t)\boldsymbol{n}(s_0)|}\mathrm{d}s \qquad (4.17)$$

$$=\int_{s_0-\delta/2}^{s_0+\delta/2}\frac{1}{(1\pm a_\varepsilon^\pm|s-s_0|)\sqrt{(s-s_0)^2+\varepsilon^2(r-t)^2}}\mathrm{d}s \qquad (4.18)$$

$$=2\int_0^{\frac{1}{2}}\frac{1}{(1\pm a_\varepsilon^\pm\delta s)\sqrt{s^2+\frac{\varepsilon^2}{\delta^2}(r-t)^2}}\mathrm{d}s, \qquad (4.19)$$

where $a_\varepsilon^\pm=\frac{\kappa_{\partial\Omega}}{1\pm\varepsilon\kappa_{\partial\Omega}}>0$.

A direct integration yields

$$\varrho_\varepsilon^\pm(\beta)=\rho(\beta)\mp\int_0^{\frac{1}{2}}\frac{2a_\varepsilon^\pm\delta s}{(1\pm a_\varepsilon^\pm\delta s)\sqrt{s^2+\beta}}\mathrm{d}s, \qquad (4.20)$$

where

$$\varrho(\beta):=\ln\left(\frac{1+2\beta+\sqrt{1+4\beta}}{2\beta}\right). \qquad (4.21)$$

By inspection, for all $\beta<1$ we have

$$-\ln\beta-C\le\varrho_\varepsilon^\pm(\beta)\le-\ln\beta+C, \qquad (4.22)$$

for some $C > 0$ universal. Also note that

$$\int_0^1 |\eta'(r)| \cdot |\ln (r-t)^2| \mathrm{d}r \leq C\|\eta'\|_\infty, \tag{4.23}$$

for some $C > 0$ universal. Therefore, given (4.22) and (4.23), from the relation (4.16) we infer the estimate

$$2|\ln \varepsilon| - C \leq \psi_\varepsilon(x) \leq 2|\ln \varepsilon| + C, \tag{4.24}$$

for some $C > 0$, assuming that $\varepsilon < \delta$ and that $\delta < \bar{\delta}$ is chosen sufficiently small. The conclusion of the lemma then follows from the previous estimate, (4.8), and (4.10). □

Remark 4.1. We point out that Lemma 4.1 and all the subsequent results, which rely on Lemma 4.1, remain valid if $\eta(t)$ is only Hölder continuous at $t = 1$ (with an arbitrary Hölder exponent).

We next prove a pointwise bound on the function $f_\varepsilon(x)$ outside $\mathcal{O}_\varepsilon$.

Lemma 4.2. *There exist $\bar{\varepsilon}, C > 0$ depending only on Ω, such that for any $0 < \varepsilon < \bar{\varepsilon}$ and $x \in \Omega \backslash \mathcal{O}_\varepsilon$ there holds*

$$0 \leq f_\varepsilon(x) \leq C\big(1 + |\ln(\operatorname{dist}(x, \partial\Omega))|\big). \tag{4.25}$$

Proof. As in the proof of Lemma 4.1, using the coarea formula we infer that for any $x \in \Omega \backslash \mathcal{O}_\varepsilon$ and sufficiently small ε there holds

$$f_\varepsilon(x) = \int_{\partial\Omega} \int_0^1 |\eta'(t)| \frac{1 + \varepsilon t \kappa(\sigma)}{|x - \sigma - \varepsilon t \boldsymbol{n}(\sigma)|} \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^1(\sigma), \tag{4.26}$$

where $\kappa(\sigma)$ is a curvature at point $\sigma \in \partial\Omega$. Furthermore, we claim that $|x - \sigma - \varepsilon t \boldsymbol{n}(\sigma)| \geq \frac{1}{2}|x - \sigma|$ for every $x \in \Omega \backslash \mathcal{O}_\varepsilon$, $\sigma \in \partial\Omega$ and $t \in [0, 1]$, provided that ε is sufficiently small. To see this, notice that the estimate trivially holds when $|x - \sigma| \geq 2\varepsilon$ or when $(x - \sigma) \cdot \boldsymbol{n}(\sigma) < 0$. At the same time, in the opposite case we can estimate $|x - \sigma| \leq 2|(x - \sigma) \cdot \boldsymbol{\tau}(\sigma)| \leq 2|x - \sigma - \varepsilon t \boldsymbol{n}(\sigma)|$ for all ε sufficiently small in view of the regularity of $\partial\Omega$. Thus, we have

$$0 \leq f_\varepsilon(x) \leq 4 \int_{\partial\Omega} \frac{1}{|x - \sigma|} \mathrm{d}\mathcal{H}^1(\sigma) \leq C\big(1 + |\ln(\operatorname{dist}(x, \partial\Omega))|\big), \tag{4.27}$$

for some $C > 0$ and all ε small enough. □

As an immediate consequence of Lemmas 4.1 and 4.2 we have the following result.

Lemma 4.3. *There exist $\bar{\varepsilon}, C > 0$ such that for every $0 < \varepsilon < \bar{\varepsilon}$ and every $x \in \Omega_\varepsilon$ there holds*

$$|f_\varepsilon(x)| \leq C\big(1 + |\ln(\varepsilon + \operatorname{dist}(x, \partial\Omega))|\big). \tag{4.28}$$

In particular, for every $p \geq 1$ there is $C_p > 0$ such that $\|f_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq C_p$.

We will also need a sharp estimate for the quantity D_ε introduced in (3.27).

Lemma 4.4. *There exist $\bar{\varepsilon}, C > 0$ such that for every $0 < \varepsilon < \bar{\varepsilon}$ there holds*

$$2|\ln \varepsilon|\mathcal{H}^1(\partial\Omega) - C \le D_\varepsilon \le 2|\ln \varepsilon|\mathcal{H}^1(\partial\Omega) + C. \tag{4.29}$$

Proof. Observe that for all $\bar{\varepsilon}$ sufficiently small we have

$$D_\varepsilon = \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{|\nabla\eta_\varepsilon(x)||\nabla\eta_\varepsilon(y)|}{|x-y|}\,\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)|\,|\nabla\eta_\varepsilon(y)| \\ \times \frac{|\boldsymbol{n}(x) - \boldsymbol{n}(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{4.30}$$

with an abuse of notation $\boldsymbol{n}(x) := \boldsymbol{n}(\boldsymbol{\pi}(x))$ for $x \in \mathcal{O}_\varepsilon^+$. Notice that by the C^2 regularity of $\partial\Omega$ the second integral in (4.30) is uniformly bounded when $\varepsilon \to 0$, as $\boldsymbol{n}(x)$ is Lipschitz continuous. At the same time, by the coarea formula and Lemma 4.1 the first integral in (4.30) is

$$\begin{aligned}
&\int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{|\nabla\eta_\varepsilon(x)|\,|\nabla\eta_\varepsilon(y)|}{|x-y|}\,\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)| f_\varepsilon(x)\mathrm{d}x \\
&\quad = \int_0^1 \int_{\partial\Omega} |\eta'(t)| f_\varepsilon(\sigma + \varepsilon t\boldsymbol{n}(\sigma))(1 + \varepsilon t\kappa(\sigma))\mathrm{d}\mathcal{H}^1(\sigma)\mathrm{d}t \\
&\quad = 2|\ln \varepsilon|\,\mathcal{H}^1(\partial\Omega) + O(1),
\end{aligned} \tag{4.31}$$

as $\varepsilon \to 0$, which implies the statement of the lemma. □

The next key technical lemma provides a comparison of an integral involving $\nabla\eta_\varepsilon$ tested against a bounded Sobolev function with that of the same integral evaluated on the trace of that Sobolev function on $\partial\Omega$.

Lemma 4.5. *There exist constants $\bar{\varepsilon}, c > 0$ such that for every $0 < \mu < 1$ there holds*

$$\frac{1}{c}\int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)||u(x) - u(\boldsymbol{\pi}(x))|\mathrm{d}x \le \mu\|u\|_{L^\infty(\mathcal{O}_\varepsilon^+)} + |\ln \mu|\|\eta_\varepsilon\nabla u\|_{L^1(\mathcal{O}_\varepsilon^+)}, \tag{4.32}$$

for any $u \in L^\infty(\mathcal{O}_\varepsilon^+) \cap W^{1,1}_{\mathrm{loc}}(\Omega_\varepsilon)$.

Proof. Again, with the usual abuse of notation $\boldsymbol{n}(x) := \boldsymbol{n}(\boldsymbol{\pi}(x))$, we have $x = \boldsymbol{\pi}(x) + d_{\partial\Omega}(x)\boldsymbol{n}(x)$ for every $x \in \mathcal{O}_\varepsilon^+$. Moreover, as noted in (3.9), we have that

$$|\nabla\eta_\varepsilon(x)| = \frac{1}{\varepsilon}|\eta'(d_{\partial\Omega}(x)/\varepsilon)|. \tag{4.33}$$

Therefore, taking the precise representative of u, by its differentiability on the lines $\boldsymbol{\pi}(x) = y$ for $\mathcal{H}^1$-a.e. $y \in \partial\Omega$ and monotonicity of $\eta(t)$ we have

$$|(u(x) - u(\boldsymbol{\pi}(x)))\nabla\eta_\varepsilon(x)| = |u(\boldsymbol{\pi}(x) + d_{\partial\Omega}(x)\boldsymbol{n}(x)) - u(\boldsymbol{\pi}(x))||\nabla\eta_\varepsilon(x)|$$

$$\leq |\nabla \eta_\varepsilon(x)| \cdot \int_0^{d_{\partial\Omega}(x)} |\partial_t[u(\boldsymbol{\pi}(x) + t\boldsymbol{n}(x))]| \mathrm{d}t$$
$$\leq \frac{|\eta'(d_{\partial\Omega}(x)/\varepsilon)|}{\varepsilon\eta(d_{\partial\Omega}(x)/\varepsilon)} \int_0^{d_{\partial\Omega}(x)} \eta(t/\varepsilon)|\nabla u(\boldsymbol{\pi}(x) + t\boldsymbol{n}(x))| \mathrm{d}t. \tag{4.34}$$

For $0 < \lambda < 1$ we decompose $\mathcal{O}_\varepsilon^+$ as $\mathcal{O}_\varepsilon^+ = \mathcal{O}_{\lambda\varepsilon}^+ \cup (\mathcal{O}_\varepsilon^+ \backslash \mathcal{O}_{\lambda\varepsilon}^+)$ and focus separately on $\mathcal{O}_{\lambda\varepsilon}^+$ and $\mathcal{O}_\varepsilon^+ \backslash \mathcal{O}_{\lambda\varepsilon}^+$. Starting with $\mathcal{O}_{\lambda\varepsilon}^+$, we observe that both $\boldsymbol{\pi}(x)$ and $\boldsymbol{n}(x)$ are constant along the normal direction, so by (4.33), (4.34) and the coarea formula we infer that

$$\int_{\mathcal{O}_{\lambda\varepsilon}^+} |\nabla \eta_\varepsilon(x)||u(x) - u(\boldsymbol{\pi}(x))| \mathrm{d}x$$
$$\leq \int_{\mathcal{O}_{\lambda\varepsilon}^+} \frac{|\eta'(d_{\partial\Omega}(x)/\varepsilon)|}{\varepsilon\eta(d_{\partial\Omega}(x)/\varepsilon)} \left(\int_0^{\lambda\varepsilon} \eta(t/\varepsilon)|\nabla u(\boldsymbol{\pi}(x) + t\boldsymbol{n}(x)\,)\,| \mathrm{d}t \right) \mathrm{d}x$$
$$= \int_0^\lambda \int_{\partial\Omega} \frac{|\eta'(h)|}{\eta(h)} \left(\int_0^{\lambda\varepsilon} \eta(t/\varepsilon)|\nabla u(\sigma + t\boldsymbol{n}(\sigma)\,)\,| \mathrm{d}t \right)$$
$$\times |1 + \varepsilon h\kappa(\sigma)| \mathrm{d}\mathcal{H}^1(\sigma) \mathrm{d}h$$
$$\leq 3|\ln \eta(\lambda)| \int_0^{\lambda\varepsilon} \int_{\partial\Omega} \eta(t/\varepsilon)|\nabla u(\boldsymbol{\pi}(x) + t\boldsymbol{n}(x)\,)\,||1 + t\kappa(\sigma)| \mathrm{d}\mathcal{H}^1(\sigma) \mathrm{d}t, \tag{4.35}$$

provided $\bar{\varepsilon} < \frac{1}{2}\kappa_{\partial\Omega}^{-1}$. Thus

$$\int_{\mathcal{O}_{\lambda\varepsilon}^+} |\nabla \eta_\varepsilon(x)||u(x) - u(\boldsymbol{\pi}(x))| \mathrm{d}x$$
$$\leq 3|\ln \eta(\lambda)| \int_0^{\lambda\varepsilon} \int_{\partial\Omega} \eta_\varepsilon(t)|\nabla u(\boldsymbol{\pi}(x) + t\boldsymbol{n}(x)\,)\,||1 + t\kappa(\sigma)| \mathrm{d}\mathcal{H}^1(\sigma) \mathrm{d}t$$
$$= 3|\ln \eta(\lambda)| \cdot \|\eta_\varepsilon \nabla u\|_{L^1(\mathcal{O}_{\lambda\varepsilon}^+)}. \tag{4.36}$$

On the other hand, for the part on $\mathcal{O}_\varepsilon^+ \backslash \mathcal{O}_{\lambda\varepsilon}^+$, we have, again by (4.33) and coarea formula, that

$$\int_{\mathcal{O}_\varepsilon^+ \backslash \mathcal{O}_{\lambda\varepsilon}^+} |\nabla \eta_\varepsilon(x)||u(x) - u(\boldsymbol{\pi}(x))| \mathrm{d}x$$
$$= \frac{1}{\varepsilon} \int_{\mathcal{O}_\varepsilon^+ \backslash \mathcal{O}_{\lambda\varepsilon}^+} |\eta'\,(d_{\partial\Omega}(x)/\varepsilon)||u(x) - u(\boldsymbol{\pi}(x))| \mathrm{d}x$$
$$\leq 4|\partial\Omega| \left(\int_\lambda^1 |\eta'(t)| \mathrm{d}t \right) \|u\|_{L^\infty(\mathcal{O}_\varepsilon^+)}$$
$$= 4|\partial\Omega|\eta(\lambda)\|u\|_{L^\infty(\mathcal{O}_\varepsilon^+)}. \tag{4.37}$$

Overall, combining the estimates (4.36) and (4.37), we get that

$$\frac{1}{c}\int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)||u(x)-u(\boldsymbol{\pi}(x))|\mathrm{d}x \le \eta(\lambda)\|u\|_{L^\infty(\mathcal{O}_\varepsilon^+)} + |\ln\eta(\lambda)|\,\|\eta_\varepsilon\nabla u\|_{L^1(\mathcal{O}_\varepsilon^+)}, \tag{4.38}$$

for some $c > 0$. The previous estimate holds for every $0 < \lambda < 1$. Since $\eta(\lambda)$ maps $[0,1]$ surjectively onto $[0,1]$, setting $\lambda := \eta^{-1}(\mu)$ we get that for every $\mu \in (0,1)$ there holds:

$$\frac{1}{c}\int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)||u(x)-u(\boldsymbol{\pi}(x))|\mathrm{d}x \le \mu\|u\|_{L^\infty(\mathcal{O}_\varepsilon^+)} + |\ln\mu|\|\eta_\varepsilon\nabla u\|_{L^1(\mathcal{O}_\varepsilon^+)}, \tag{4.39}$$

which proves the L^1-estimate (4.32). □

Our next lemma gives a bound that will be useful to estimate the interior contribution of the bulk charges to the micromagnetic energy. Note that for $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^2)$ a straightforward interpolation estimate between the $\mathring{H}^{-1/2}$ norm of $\operatorname{div}\boldsymbol{u}$ and the L^2 norms of $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$ would have held true if $\boldsymbol{u}$ vanished at the boundary of Ω. However, the presence of a nonzero trace on $\partial\Omega$ requires some additional care due to a logarithmic failure of this interpolation. A counterexample to the latter is provided by $\boldsymbol{u}$, which is equal to the outward unit normal at the projection point to the boundary of Ω multiplied by a cutoff function making $\boldsymbol{u}$ zero at distances greater than δ from $\partial\Omega$ (for a related phenomenon, see Ref. 20).

Lemma 4.6. *There exists a constant $C > 0$ depending only on Ω such that for any $\delta \in (0, \frac{1}{2})$ and any $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^2)$ there holds*

$$\int_\Omega\int_\Omega \frac{\operatorname{div}\boldsymbol{u}(x)\operatorname{div}\boldsymbol{u}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le \delta\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} + C\delta^{-1}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + C|\ln\delta|\,\|\boldsymbol{u}\|^2_{L^2(\partial\Omega)}. \tag{4.40}$$

Proof. For a non-negative cutoff function $\omega \in C^\infty(\mathbb{R})$ satisfying $\omega(t) = 1$ for all $t \le 1$ and $\omega(t) = 0$ for all $t \ge 2$, we write

$$\frac{1}{|x-y|} = G_\delta(x-y) + H_\delta(x-y), \tag{4.41}$$

where $H_\delta(x) := |x|^{-1}\omega(c|x|/\delta)$ with $c > 0$ to be fixed shortly and $G_\delta(x) := |x|^{-1}(1-\omega(c|x|/\delta))$. For the contribution of H_δ, by Young's inequality for convolutions we have

$$\left|\int_\Omega\int_\Omega H_\delta(x-y)\operatorname{div}\boldsymbol{u}(x)\operatorname{div}\boldsymbol{u}(y)\mathrm{d}x\mathrm{d}y\right| \le 2\|H_\delta\|_{L^1(\mathbb{R}^2)}\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} \le \delta\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)}, \tag{4.42}$$

for a suitable choice of $c > 0$ depending only on ω. For the contribution of the even function G_δ, we integrate by parts and apply the Cauchy–Schwarz inequality to obtain

$$
\begin{aligned}
&\int_\Omega \int_\Omega G_\delta(x-y)\operatorname{div}\boldsymbol{u}(x)\operatorname{div}\boldsymbol{u}(y)\mathrm{d}x\mathrm{d}y \\
&\quad = \int_\Omega \int_\Omega \boldsymbol{u}(x)\cdot\nabla^2_{xy}G_\delta(x-y)\boldsymbol{u}(y)\mathrm{d}x\mathrm{d}y \\
&\qquad -2\int_\Omega \int_{\partial\Omega} (\boldsymbol{u}(y)\cdot\boldsymbol{n}(y))\boldsymbol{u}(x)\cdot\nabla_x G_\delta(x-y)\mathrm{d}\mathcal{H}^1(y)\mathrm{d}x \\
&\qquad +\int_{\partial\Omega} \int_{\partial\Omega} (\boldsymbol{u}(x)\cdot\boldsymbol{n}(x))(\boldsymbol{u}(y)\cdot\boldsymbol{n}(y))G_\delta(x-y)\mathrm{d}\mathcal{H}^1(x)\mathrm{d}\mathcal{H}^1(y) \\
&\quad \le \max_{y\in\Omega}\|\nabla^2 G_\delta(\cdot-y)\|_{L^1(\Omega)}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \max_{y\in\partial\Omega}\|G_\delta(\cdot-y)\|_{L^1(\partial\Omega)}\|\boldsymbol{u}\|^2_{L^2(\partial\Omega)} \\
&\qquad +2\max_{y\in\partial\Omega}\|\nabla G_\delta(\cdot-y)\|^{1/2}_{L^1(\Omega)}\max_{y\in\Omega}\|\nabla G_\delta(\cdot-y)\|^{1/2}_{L^1(\partial\Omega)}\|\boldsymbol{u}\|_{L^2(\Omega)}\|\boldsymbol{u}\|_{L^2(\partial\Omega)} \\
&\quad \le C\delta^{-1}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + C|\ln\delta|\|\boldsymbol{u}\|^2_{L^2(\partial\Omega)} + C\delta^{-1/2}|\ln\delta|^{1/2}\|\boldsymbol{u}\|_{L^2(\Omega)}\|\boldsymbol{u}\|_{L^2(\partial\Omega)},
\end{aligned} \tag{4.43}
$$

for some $C > 0$ depending only on Ω. The conclusion follows by combining (4.42) and (4.43), after an application of Young's inequality. □

As a corollary to this lemma, we have the following result for vector fields that are uniformly bounded.

Lemma 4.7. *Let $\delta > 0$, and for $\varepsilon > 0$ let $\boldsymbol{u}_\varepsilon \in H^1(\Omega;\mathbb{R}^2)$ with $|\boldsymbol{u}_\varepsilon| \le 1$ in Ω. Then there exists a constant $C > 0$ depending only on Ω such that*

$$
\int_\Omega \int_\Omega \frac{\operatorname{div}\boldsymbol{u}_\varepsilon(x)\operatorname{div}\boldsymbol{u}_\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le \delta\|\nabla\boldsymbol{u}_\varepsilon\|^2_{L^2(\Omega)} + \frac{C}{\delta}. \tag{4.44}
$$

Moreover, if $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$ then

$$
\int_\Omega \int_\Omega \frac{\operatorname{div}\boldsymbol{u}_\varepsilon(x)\operatorname{div}\boldsymbol{u}_\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \xrightarrow{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{\operatorname{div}\boldsymbol{u}(x)\operatorname{div}\boldsymbol{u}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \tag{4.45}
$$

and

$$
\begin{aligned}
&\int_{\partial\Omega} \int_\Omega \frac{\operatorname{div}\boldsymbol{u}_\varepsilon(x)(\boldsymbol{u}_\varepsilon(y)\cdot\boldsymbol{n}(y))}{|x-y|}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^1(y) \\
&\qquad \xrightarrow{\varepsilon\to 0} \int_{\partial\Omega} \int_\Omega \frac{\operatorname{div}\boldsymbol{u}(x)(\boldsymbol{u}(y)\cdot\boldsymbol{n}(y))}{|x-y|}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^1(y).
\end{aligned} \tag{4.46}
$$

Proof. The estimate in (4.44) is an immediate corollary to Lemma 4.6. To prove (4.45), we first note that since $|x|^{-1} * \operatorname{div}\boldsymbol{u}$ belongs to $L^2(\Omega)$ by Young's inequality for convolutions, it is enough to prove that the left-hand side of (4.45) goes to zero when $\boldsymbol{u}_\varepsilon \rightharpoonup 0$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$. However, the latter is true by (4.40) in view of boundedness of $\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$, strong convergence of $\boldsymbol{u}$ to zero in $L^2(\Omega)$ and $L^2(\partial\Omega)$

by compact Sobolev and trace embeddings, and arbitrariness of $\delta > 0$. Similarly, since the integral over $\partial\Omega$ on the right-hand side of (4.46) defines a function of x that belongs to $L^2(\Omega)$ by the last inequality in (4.27), it is enough to show (4.46) when $\boldsymbol{u}_\varepsilon \rightharpoonup 0$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$. The latter follows via an application of the Cauchy–Schwarz inequality:

$$\begin{aligned}
&\left| \int_{\partial\Omega} \int_\Omega \frac{\operatorname{div} \boldsymbol{u}_\varepsilon(x)(\boldsymbol{u}_\varepsilon(y) \cdot \boldsymbol{n}(y))}{|x-y|} \, \mathrm{d}x \, \mathrm{d}\mathcal{H}^1(y) \right| \\
&\le \left(\int_{\partial\Omega} \int_\Omega \frac{(\boldsymbol{u}_\varepsilon(y) \cdot \boldsymbol{n}(y))^2}{|x-y|^{3/2}} \, \mathrm{d}x \, \mathrm{d}\mathcal{H}^1(y) \right)^{1/2} \\
&\quad \times \left(\int_{\partial\Omega} \int_\Omega \frac{|\operatorname{div} \boldsymbol{u}_\varepsilon(x)|^2}{|x-y|^{1/2}} \, \mathrm{d}x \, \mathrm{d}\mathcal{H}^1(y) \right)^{1/2} \\
&\le C \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega)} \|\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}\|_{L^2(\partial\Omega)} \to 0,
\end{aligned} \tag{4.47}$$

as $\varepsilon \to 0$, by the compact trace embedding, where $C > 0$ depends only on Ω. □

5. Analysis of the Magnetostatic Energy

In this section, we carry out an analysis of the magnetostatic energy which contains two propositions describing the behavior of the nonlocal terms in the stray field energy $\mathcal{W}_\varepsilon$ (cf. (3.13)), corresponding to the in-plane and out-of-plane magnetization components. We will then use these results to prove our main theorems formulated in Sec. 3.

We start by proving the following proposition for the non-local term due to the in-plane component of the magnetization. Note that here we need a stronger result than that of the type proved for the three-dimensional micromagnetic energy by Kohn and Slastikov in Ref. 43 in order to go beyond the regime studied there (see Theorems 3.3 and 3.4).

Proposition 5.1. *There exist $\bar{\varepsilon}, C > 0$ such that if $0 < \varepsilon < \bar{\varepsilon}$ and $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2; \mathbb{S}^2)$, then the magnetostatic energy for the in-plane component* (*cf.* (3.14))

$$\mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\operatorname{div}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon)(x) \operatorname{div}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon)(y)}{|x-y|} \mathrm{d}x \mathrm{d}y \tag{5.1}$$

satisfies

$$\begin{aligned}
&|\mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) - \mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) + 2\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) - 2|\ln\varepsilon| \, \|\boldsymbol{m}_\perp^\varepsilon \cdot \boldsymbol{n}\|^2_{L^2(\partial\Omega)}| \\
&\quad \le C_\varepsilon (1 + \|\eta_\varepsilon \nabla \boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}) + C \|\boldsymbol{m}_\perp^\varepsilon \cdot \boldsymbol{n}\|_{L^2(\partial\Omega)} (1 + \|\eta_\varepsilon \nabla \boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon)}),
\end{aligned} \tag{5.2}$$

where $C_\varepsilon \to 0$ as $\varepsilon \to 0$, and

$$\mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) := \int_\Omega \int_\Omega \frac{\operatorname{div} \boldsymbol{m}_\perp^\varepsilon(x) \operatorname{div} \boldsymbol{m}_\perp^\varepsilon(y)}{|x-y|} \mathrm{d}y \, \mathrm{d}x, \tag{5.3}$$

$$\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) := \int_{\partial\Omega} \int_\Omega \frac{(\boldsymbol{n} \cdot \boldsymbol{m}_\perp^\varepsilon)(\sigma) \operatorname{div} \boldsymbol{m}_\perp^\varepsilon(y)}{|\sigma - y|} \mathrm{d}y \, \mathrm{d}\mathcal{H}^1(\sigma). \tag{5.4}$$

Proof. As is common in the analysis of the limiting behaviors of non-local energy functionals, we decompose $\mathcal{V}$ into the sum of several terms and estimate each term separately.

First, expanding the divergence and exploiting the symmetry in the x, y variables, we write $\mathcal{V}(\boldsymbol{u}_\varepsilon) =: I_1 + 2I_2 + I_3$, where

$$I_1 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\eta_\varepsilon(x)\cdot \boldsymbol{m}_\perp^\varepsilon(x)\nabla\eta_\varepsilon(y)\cdot \boldsymbol{m}_\perp^\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{5.5}$$

$$I_2 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\eta_\varepsilon(y)\cdot \boldsymbol{m}_\perp^\varepsilon(y)\eta_\varepsilon(x)\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon(x)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{5.6}$$

$$I_3 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\eta_\varepsilon(x)\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon(x)\eta_\varepsilon(y)\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{5.7}$$

Note that I_1, I_2, I_3 depend on ε but we suppress this for ease of notation. We then have

$$\text{left-hand side of (5.2)} \le |I_3 - \mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon)| + 2|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) + I_2| \tag{5.8}$$

$$+ |I_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot \boldsymbol{n}\|^2_{L^2(\partial\Omega)}|, \tag{5.9}$$

and we proceed by estimating the terms on the right-hand side of the previous relation.

STEP 1. ESTIMATE OF $I_3 - \mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon)$. We split this term as $I_3 - \mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) = 2L_2 + L_3$, where

$$L_2 := \int_{\Omega}\int_{\mathcal{O}_\varepsilon^+} \frac{\eta_\varepsilon(x)\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon(x)\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{5.10}$$

$$L_3 := \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{\eta_\varepsilon(x)\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon(x)\eta_\varepsilon(y)\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{5.11}$$

To estimate L_2 and L_3, we use Young's inequality for convolutions

$$\int_{\mathbb{R}^2} |f(x)||(g*K)(x)|\mathrm{d}x \le \|f\|_p\|g\|_s\|K\|_r \tag{5.12}$$

with $p = r = \frac{4}{3}$ and $s = 2$. Observing that since Ω is bounded, there exists a ball U centered at the origin such that $x - y \in U$ for every $x, y \in \overline{\Omega}_\varepsilon$ and $\||\cdot|^{-1}\|_{L^r(U)} \le C$ we obtain that $L_2 \le C\|\eta_\varepsilon\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon\|_{L^p(\mathcal{O}_\varepsilon^+)}\|\eta_\varepsilon\mathrm{div}\, \boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon),}$. But by Hölder's inequality there holds $\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^p(\mathcal{O}_\varepsilon^+)} \le \|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon)}|\mathcal{O}_\varepsilon^+|^{1/4} \le C\varepsilon^{1/4}\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon)}$, for some constant $C > 0$, provided ε is small enough. Hence

$$L_2 \le C\varepsilon^{1/4}\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}. \tag{5.13}$$

The very same estimate is true for $|L_3|$ and, therefore, we conclude that for every ε sufficiently small there holds

$$|I_3 - \mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon)| \le 2|L_2| + |L_3| \le C\varepsilon^{1/4}\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}, \tag{5.14}$$

for some positive constant $C > 0$ and all ε small enough.

STEP 2. ESTIMATE OF $\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) + I_2$. Our aim here is to show that

$$|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) + I_2| \le C_\varepsilon(1 + \|\eta_\varepsilon \nabla \boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}), \tag{5.15}$$

for some $C_\varepsilon > 0$ such that $C_\varepsilon \to 0$ as $\varepsilon \to 0$. To estimate $\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) + I_2$, we use the fact that $\mathrm{supp}_{\mathbb{R}^2} \nabla\eta_\varepsilon \subseteq \overline{\mathcal{O}_\varepsilon^+}$ and $|\boldsymbol{m}_\perp^\varepsilon| \le 1$ in Ω_ε to obtain (cf. (3.9))

$$\begin{aligned} -I_2 &= \int_{\mathcal{O}_\varepsilon^+}\int_{\Omega_\varepsilon} \frac{|\nabla\eta_\varepsilon(y)|(\boldsymbol{n}(y)\cdot\boldsymbol{m}_\perp^\varepsilon(y))\eta_\varepsilon(x)\mathrm{div}\,\boldsymbol{m}_\perp^\varepsilon(x)}{|x-y|}\mathrm{d}x\mathrm{d}y & (5.16)\\ &=: M_1 + M_2, & (5.17)\end{aligned}$$

with

$$M_1 := \int_{\mathcal{O}_\varepsilon^+}\int_{\Omega_\varepsilon} \frac{|\nabla\eta_\varepsilon(y)|(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\boldsymbol{\pi}(y))\eta_\varepsilon(x)\mathrm{div}\,\boldsymbol{m}_\perp^\varepsilon(x)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{5.18}$$

$$M_2 := \int_{\mathcal{O}_\varepsilon^+}\int_{\Omega_\varepsilon} \frac{|\nabla\eta_\varepsilon(y)|[(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(y) - (\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\boldsymbol{\pi}(y))]\eta_\varepsilon(x)\mathrm{div}\,\boldsymbol{m}_\perp^\varepsilon(x)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{5.19}$$

Clearly, we have

$$|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) + I_2| \le |\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) - M_1| + |M_2|, \tag{5.20}$$

and we want to show that

$$\begin{aligned} |M_2| &\le C_\varepsilon(1 + \|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}) \quad \text{and} \\ |\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) - M_1| &\le C_\varepsilon(1 + \|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}), \end{aligned} \tag{5.21}$$

with $C_\varepsilon \to 0$ as $\varepsilon \to 0$. To estimate M_2, we use Young's inequality for convolutions (5.12) to obtain

$$\begin{aligned} M_2 \le{}& \||\cdot|^{-1}\|_{L^s(U)} \||\nabla\eta_\varepsilon|[(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\cdot) - (\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\boldsymbol{\pi}(\cdot))]\|_{L^p(\mathcal{O}_\varepsilon^+)} \\ &\times \|\eta_\varepsilon \mathrm{div}\,\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon)}, \end{aligned} \tag{5.22}$$

for some $p, s \ge 1$ such that $\frac{1}{p} + \frac{1}{s} = \frac{3}{2}$. We take $p := 1+\alpha$ and $s := 2\frac{1+\alpha}{1+3\alpha}$ with $\alpha > 0$ sufficiently small so that $1 < s < 2$. Then

$$M_2 \le C\||\nabla\eta_\varepsilon|[(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\cdot) - (\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\boldsymbol{\pi}(\cdot))]\|_{L^{1+\alpha}(\mathcal{O}_\varepsilon^+)}\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{5.23}$$

with some $C > 0$ depending only on Ω and s such that $\||\cdot|^{-1}\|_{L^s(U)} \le C$.

We conclude by showing that

$$A_\varepsilon := \||\nabla\eta_\varepsilon|[(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\cdot) - (\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\boldsymbol{\pi}(\cdot))]\|_{L^{1+\alpha}(\mathcal{O}_\varepsilon^+)} \tag{5.24}$$

is small, which guarantees that the limit relation in (5.21) holds. Indeed, using interpolation inequality $\|f\|_{L^{1+\alpha}} \le \|f\|^\theta_{L^1}\|f\|^{1-\theta}_{L^{1+2\alpha}}$ with $\alpha > 0$, $\theta = \frac{1}{2(1+\alpha)}$ and $1-\theta = \frac{1+2\alpha}{2(1+\alpha)}$, we immediately obtain

$$A_\varepsilon \le \||\nabla\eta_\varepsilon|[(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\cdot) - (\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\boldsymbol{\pi}(\cdot))]\|^\theta_{L^1(\mathcal{O}_\varepsilon^+)}\|2\nabla\eta_\varepsilon\|^{1-\theta}_{L^{1+2\alpha}(\mathcal{O}_\varepsilon^+)}. \tag{5.25}$$

Now, recalling that $\eta' \in L^{1+2\alpha}(0,1)$ for some α small enough depending on q, we get that

$$\|\nabla \eta_\varepsilon\|^{1-\theta}_{L^{1+2\alpha}(\mathcal{O}^+_\varepsilon)} \le C\varepsilon^{2\theta-1}. \tag{5.26}$$

Also, using Lemma 4.5 with $\mu = \varepsilon$ and Cauchy–Schwarz inequality we obtain

$$\begin{aligned} &\| |\nabla \eta_\varepsilon| [(\boldsymbol{n}\cdot \boldsymbol{m}^\varepsilon_\perp)(\cdot) - (\boldsymbol{n}\cdot \boldsymbol{m}^\varepsilon_\perp)(\boldsymbol{\pi}(\cdot))]\|_{L^1(\mathcal{O}^+_\varepsilon)} \\ &\quad \le C(\varepsilon + |\ln\varepsilon| |\mathcal{O}^+_\varepsilon|^{1/2} \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_\perp\|_{L^2(\Omega_\varepsilon)}) \\ &\quad \le C'\varepsilon^{1/2}|\ln\varepsilon|(1 + \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_\perp\|_{L^2(\Omega_\varepsilon)}), \end{aligned} \tag{5.27}$$

for some $C, C' > 0$ and all ε small enough. Now, combining the two previous estimates we obtain that

$$\begin{aligned} A_\varepsilon &\le C\varepsilon^{\frac{5\theta}{2}-1}|\ln\varepsilon|^\theta (1 + \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_\perp\|_{L^2(\Omega_\varepsilon)})^\theta \\ &\le C'\varepsilon^{1/8}(1 + \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_\perp\|_{L^2(\Omega_\varepsilon)}), \end{aligned} \tag{5.28}$$

for α sufficiently small, recalling that $\theta \to \frac{1}{2}$ as $\alpha \to 0$. Thus by Young's inequality we have

$$M_2 \le C\varepsilon^{1/8}(1 + \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_\perp\|^2_{L^2(\Omega_\varepsilon)}), \tag{5.29}$$

which implies the first estimate in (5.21).

It remains to estimate the quantity $|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}^\varepsilon_\perp) - M_1|$. For that we split it further as $M_1 =: N_1 + N_2$, where

$$N_1 := \int_{\mathcal{O}^+_\varepsilon} \int_{\Omega_\varepsilon} \frac{|\nabla \eta_\varepsilon(y)| (\boldsymbol{n}\cdot \boldsymbol{m}^\varepsilon_\perp)(\boldsymbol{\pi}(y)) \operatorname{div} \boldsymbol{m}^\varepsilon_\perp(x)}{|x-y|} \mathrm{d}x\mathrm{d}y, \tag{5.30}$$

$$N_2 := \int_{\mathcal{O}^+_\varepsilon} \int_{\mathcal{O}^+_\varepsilon} \frac{|\nabla \eta_\varepsilon(y)| (\boldsymbol{n}\cdot \boldsymbol{m}^\varepsilon_\perp)(\boldsymbol{\pi}(y)) \eta_\varepsilon(x) \operatorname{div} \boldsymbol{m}^\varepsilon_\perp(x)}{|x-y|} \mathrm{d}x\mathrm{d}y, \tag{5.31}$$

and given that $|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}^\varepsilon_\perp) - M_1| \le |\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}^\varepsilon_\perp) - N_1| + |N_2|$ we aim at showing that

$$\begin{aligned} |\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}^\varepsilon_\perp) - N_1| &\le C_\varepsilon(1 + \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_\perp\|^2_{L^2(\Omega_\varepsilon)}) \quad \text{and} \\ |N_2| &\le C_\varepsilon(1 + \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon_\perp\|^2_{L^2(\Omega_\varepsilon)}), \end{aligned} \tag{5.32}$$

with some $C_\varepsilon \to 0$ as $\varepsilon \to 0$. The bound (5.32) will prove the second estimate in (5.21).

We can estimate N_2 as

$$|N_2| \le \int_{\mathcal{O}^+_\varepsilon} f_\varepsilon(x)\eta_\varepsilon(x) |\operatorname{div} \boldsymbol{m}^\varepsilon_\perp(x)| \mathrm{d}x, \tag{5.33}$$

where f_ε is defined in (4.1). Now we apply Lemma 4.1, the Cauchy–Schwarz and Young's inequalities to obtain that for ε sufficiently small we have

$$|N_2| \le (2|\ln\varepsilon| + C) \int_{\mathcal{O}^+_\varepsilon} \eta_\varepsilon(x)\, |\operatorname{div} \boldsymbol{m}^\varepsilon_\perp(x)| \mathrm{d}x$$

$$\leq C'|\ln\varepsilon|\,|\mathcal{O}_\varepsilon^+|^{1/2}\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon)}$$
$$\leq C''\varepsilon^{1/2}|\ln\varepsilon|(1+\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}), \tag{5.34}$$

for some $C, C', C'' > 0$, yielding the second relation in (5.32).

To estimate $|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) - N_1|$, we observe that

$$|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) - N_1| = \left|\int_\Omega \operatorname{div}\boldsymbol{m}_\perp^\varepsilon(x)\rho_\varepsilon(x)\mathrm{d}x\right| \tag{5.35}$$

with

$$\rho_\varepsilon(x) := \int_{\mathcal{O}_\varepsilon^+}\frac{|\nabla\eta_\varepsilon(y)|(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\boldsymbol{\pi}(y))}{|x-y|}\mathrm{d}y - \int_{\partial\Omega}\frac{(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\sigma)}{|x-\sigma|}\mathrm{d}\mathcal{H}^1(\sigma). \tag{5.36}$$

Using the coarea formula, we infer that for any $x\in\Omega$ there holds

$$\rho_\varepsilon(x) = \int_{\partial\Omega}\int_0^1(\boldsymbol{n}\cdot\boldsymbol{m}_\perp^\varepsilon)(\sigma)|\eta'(t)|\left(\frac{1+\varepsilon t\kappa(\sigma)}{|x-\sigma-\varepsilon t\boldsymbol{n}(\sigma)|} - \frac{1}{|x-\sigma|}\right)\mathrm{d}t\,\mathrm{d}\mathcal{H}^1(\sigma), \tag{5.37}$$

and by Lebesgue's dominated convergence theorem we have $\rho_\varepsilon(x)\to 0$ as $\varepsilon\to 0$ for every $x\in\Omega$, Furthermore, by $|\boldsymbol{m}_\perp^\varepsilon|\leq 1$ and Lemma 4.3 we have

$$|\rho_\varepsilon(x)| \leq f_\varepsilon(x) + \int_{\partial\Omega}\frac{1}{|x-\sigma|}\mathrm{d}\mathcal{H}^1(\sigma) \leq C\big(1+|\ln(\operatorname{dist}(x,\partial\Omega))|\big), \tag{5.38}$$

for some $C > 0$ and all ε small enough.

From (5.38) and the pointwise convergence of ρ_ε to zero as $\varepsilon\to 0$, one can conclude by Lebesgue's dominated convergence theorem that $\|\rho_\varepsilon\|_{L^2(\Omega)}\to 0$ when $\varepsilon\to 0$. Therefore, by the Cauchy–Schwarz and Young's inequalities we obtain

$$|\mathcal{V}_{\partial\Omega\times\Omega}(\boldsymbol{m}_\perp^\varepsilon) - N_1| \leq C_\varepsilon(1+\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon),}) \tag{5.39}$$

for some $C_\varepsilon > 0$ such that $C_\varepsilon\to 0$ when $\varepsilon\to 0$. This concludes the proof of (5.32) and, therefore, of (5.15).

STEP 3. ESTIMATE OF $|I_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}|$. By adding and subtracting $\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))$ to $\boldsymbol{m}_\perp^\varepsilon(x)$, we rewrite I_1 as $I_1 = J_1 + 2J_2 - J_3$, where

$$J_1 := \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+}\frac{\nabla\eta_\varepsilon(x)\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))\nabla\eta_\varepsilon(y)\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(y))}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{5.40}$$

$$J_2 := \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+}\frac{\nabla\eta_\varepsilon(x)\cdot[\boldsymbol{m}_\perp^\varepsilon(x)-\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))]\nabla\eta_\varepsilon(y)\cdot\boldsymbol{m}_\perp^\varepsilon(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \tag{5.41}$$

$$J_3 := \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+}\frac{\nabla\eta_\varepsilon(x)\cdot[\boldsymbol{m}_\perp^\varepsilon(x)-\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))]\nabla\eta_\varepsilon(y)\cdot[\boldsymbol{m}_\perp^\varepsilon(y)-\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(y))]}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{5.42}$$

In writing the previous relations, we exploited that $\mathrm{supp}\nabla\eta_\varepsilon \subseteq \overline{\mathcal{O}_\varepsilon^+}$. Also, to avoid cumbersome notations we use the same symbol to denote both $\boldsymbol{m}_\perp^\varepsilon$ and its trace $\boldsymbol{m}^\varepsilon_{\perp|\partial\Omega}$ on $\partial\Omega$. When we write $\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))$ we mean $\boldsymbol{m}^\varepsilon_{\perp|\partial\Omega}(\boldsymbol{\pi}(x))$.

Observe that

$$|I_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}| \leq |J_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}| + |J_2| + |J_3|, \tag{5.43}$$

and we first want to estimate J_2 and J_3. Using the estimate in Lemma 4.1, we obtain that as soon as ε is small enough, there holds

$$\begin{aligned} |J_2| &\leq \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{|\nabla\eta_\varepsilon(x)||\nabla\eta_\varepsilon(y)||\boldsymbol{m}_\perp^\varepsilon(x) - \boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))|}{|x-y|}\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathcal{O}_\varepsilon^+} f_\varepsilon(x)|\nabla\eta_\varepsilon(x)||\boldsymbol{m}_\perp^\varepsilon(x) - \boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))|\mathrm{d}x \\ &\leq 3|\ln\varepsilon|\left(\int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)||\boldsymbol{m}_\perp^\varepsilon(x) - \boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))|\mathrm{d}x\right). \end{aligned} \tag{5.44}$$

Applying the L^1-type estimate in Lemma 4.5 with $\mu = \varepsilon$, we infer that

$$|J_2| \leq C|\ln\varepsilon|(\varepsilon + |\ln\varepsilon|\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^1(\mathcal{O}_\varepsilon^+)}), \tag{5.45}$$

for some $C > 0$. Using the Cauchy–Schwarz and Young's inequalities, we then obtain

$$\begin{aligned} |J_2| &\leq C|\ln\varepsilon|(\varepsilon + |\ln\varepsilon|\,|\mathcal{O}_\varepsilon^+|^{1/2}\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon),}) \\ &\leq C'\varepsilon^{1/2}|\ln\varepsilon|^2(1 + \|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}), \end{aligned} \tag{5.46}$$

for some $C' > 0$ and all ε small enough. In the same way, we obtain

$$|J_3| \leq 2C'\varepsilon^{1/2}|\ln\varepsilon|^2(1 + \|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}), \tag{5.47}$$

for all ε small enough. Hence, from (5.43), (5.46) and (5.47) we get that

$$\begin{aligned} &|I_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}| \\ &\quad\leq |J_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}| + C\varepsilon^{1/2}|\ln\varepsilon|^2(1 + \|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}), \end{aligned} \tag{5.48}$$

for all ε small enough.

It remains to estimate $|J_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}|$. We proceed by decomposing J_1 as $J_1 := K_1 + K_2$ with

$$K_1 := \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{|\boldsymbol{n}(\boldsymbol{\pi}(x))\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))|^2}{|x-y|}|\nabla\eta_\varepsilon(x)||\nabla\eta_\varepsilon(y)|\mathrm{d}x\mathrm{d}y, \tag{5.49}$$

$$\begin{aligned} K_2 := &\int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)|\boldsymbol{n}(\boldsymbol{\pi}(x))\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x)) \\ &\times\int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(y)|\frac{\boldsymbol{n}(\boldsymbol{\pi}(y))\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(y)) - \boldsymbol{n}(\boldsymbol{\pi}(x))\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))}{|x-y|}\mathrm{d}y\mathrm{d}x, \end{aligned} \tag{5.50}$$

and we show that

$$|J_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}| \le |K_1 - 2|\ln\varepsilon|\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)}| + |K_2|$$
$$\le C(1+\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega)})\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|_{L^2(\partial\Omega)}, \qquad (5.51)$$

for some $C>0$ and all ε small enough.

We estimate K_2 to obtain

$$|K_2| \le \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{|\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(y))\cdot\boldsymbol{n}(\boldsymbol{\pi}(y)) - \boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))\cdot\boldsymbol{n}(\boldsymbol{\pi}(x))|}{|\boldsymbol{\pi}(x)-\boldsymbol{\pi}(y)|}$$
$$\times |\boldsymbol{n}(\boldsymbol{\pi}(x))\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))|\frac{|\boldsymbol{\pi}(x)-\boldsymbol{\pi}(y)|}{|x-y|}|\nabla\eta_\varepsilon(x)||\nabla\eta_\varepsilon(y)|\mathrm{d}x\mathrm{d}y. \qquad (5.52)$$

Since $\partial\Omega$ is of class C^2 and compact, the projection map $\boldsymbol{\pi}:\mathcal{O}_{\bar\varepsilon}\to\partial\Omega$ is uniformly Lipschitz for sufficiently small $\bar\varepsilon$. Thus, there exists a constant $C_{\boldsymbol{\pi}}>0$ such that

$$|\boldsymbol{\pi}(x)-\boldsymbol{\pi}(y)| \le C_{\boldsymbol{\pi}}|x-y| \quad \forall\, x,\ \ y\in\mathcal{O}_{\bar\varepsilon}, \qquad (5.53)$$

and passing to the curvilinear coordinates we obtain

$$|K_2| \le C_{\boldsymbol{\pi}}\int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{|\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(y))\cdot\boldsymbol{n}(\boldsymbol{\pi}(y)) - \boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))\cdot\boldsymbol{n}(\boldsymbol{\pi}(x))|}{|\boldsymbol{\pi}(x)-\boldsymbol{\pi}(y)|}\cdot$$
$$\cdot|\boldsymbol{n}(\boldsymbol{\pi}(x))\cdot\boldsymbol{m}_\perp^\varepsilon(\boldsymbol{\pi}(x))||\nabla\eta_\varepsilon(x)||\nabla\eta_\varepsilon(y)|\mathrm{d}x\mathrm{d}y \qquad (5.54)$$
$$\le 2C_{\boldsymbol{\pi}}\int_0^1\int_0^1|\eta'(s)||\eta'(t)|\int_{\partial\Omega}\int_{\partial\Omega}\frac{|\boldsymbol{m}_\perp^\varepsilon(\mu)\cdot\boldsymbol{n}(\mu)-\boldsymbol{m}_\perp^\varepsilon(\sigma)\cdot\boldsymbol{n}(\sigma)|}{|\mu-\sigma|}\cdot$$
$$\cdot|\boldsymbol{n}(\sigma)\cdot\boldsymbol{m}_\perp^\varepsilon(\sigma)|\mathrm{d}\mathcal{H}^1(\mu)\mathrm{d}\mathcal{H}^1(\sigma)\mathrm{d}s\mathrm{d}t, \qquad (5.55)$$

provided $\bar\varepsilon$ is small enough.

Since $\|\eta'\|_{L^1(0,1)}=1$, using the Cauchy–Schwarz inequality we obtain

$$|K_2| \le C\int_{\partial\Omega}\int_{\partial\Omega}\frac{|\boldsymbol{m}_\perp^\varepsilon(\mu)\cdot\boldsymbol{n}(\mu)-\boldsymbol{m}_\perp^\varepsilon(\sigma)\cdot\boldsymbol{n}(\sigma)|}{|\mu-\sigma|}|\boldsymbol{n}(\sigma)\cdot\boldsymbol{m}_\perp^\varepsilon(\sigma)|\mathrm{d}\mathcal{H}^1(\mu)\mathrm{d}\mathcal{H}^1(\sigma)$$
$$\le C'\left(\int_{\partial\Omega}|\boldsymbol{n}(\sigma)\cdot\boldsymbol{m}_\perp^\varepsilon(\sigma)|^2\mathrm{d}\mathcal{H}^1(\sigma)\right)^{\frac12}$$
$$\times\left(\int_{\partial\Omega}\int_{\partial\Omega}\frac{|\boldsymbol{m}_\perp^\varepsilon(\mu)\cdot\boldsymbol{n}(\mu)-\boldsymbol{m}_\perp^\varepsilon(\sigma)\cdot\boldsymbol{n}(\sigma)|^2}{|\mu-\sigma|^2}\mathrm{d}\mathcal{H}^1(\mu)\mathrm{d}\mathcal{H}^1(\sigma)\right)^{\frac12}$$
$$\le C''\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|_{H^{1/2}(\partial\Omega)}\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|_{L^2(\partial\Omega)}, \qquad (5.56)$$

for some $C,C',C''>0$ and all ε small enough. Finally, using $|\boldsymbol{m}_\perp^\varepsilon|\le 1$ and the trace embedding theorem, we obtain

$$|K_2| \le C(1+\|\eta_\varepsilon\nabla\boldsymbol{m}_\perp^\varepsilon\|_{L^2(\Omega_\varepsilon)})\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|_{L^2(\partial\Omega)}. \qquad (5.57)$$

Lastly, we show that

$$|K_1 - 2|\ln\varepsilon\left|\int_{\partial\Omega}(\boldsymbol{m}_\perp^\varepsilon(\sigma)\cdot\boldsymbol{n}(\sigma))^2\mathrm{d}\mathcal{H}^1(\sigma)\right| \le C\|\boldsymbol{m}_\perp^\varepsilon\cdot\boldsymbol{n}\|_{L^2(\partial\Omega)}. \qquad (5.58)$$

Indeed, using the coarea formula and recalling the definition of f_ε in (4.1), we have

$$K_1 = \int_{\partial\Omega} |\boldsymbol{n}(\sigma)\cdot\boldsymbol{m}_\perp^\varepsilon(\sigma)|^2 \left(\int_0^1 |\eta'(t)| f_\varepsilon(\sigma+\varepsilon t\boldsymbol{n}(\sigma))(1+\varepsilon t\kappa(\sigma))\mathrm{d}t\right)\mathrm{d}\mathcal{H}^1(\sigma). \tag{5.59}$$

Therefore, using the asymptotics of $f_\varepsilon(\sigma+\varepsilon t\boldsymbol{n}(\sigma))$ given in Lemma 4.1 and the fact that $|\boldsymbol{m}_\perp^\varepsilon| \leq 1$, we infer (5.58). Combining (5.57) and (5.58), we get (5.51). Finally, combining it with (5.48), (5.9) (5.14), and (5.15) we get the desired estimate (5.2). This concludes the proof. □

Proposition 5.2. *There exist $\bar\varepsilon, C > 0$ such that if $0 < \varepsilon < \bar\varepsilon$ and $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$, then the magnetostatic energy for the out-of-plane component (cf. (3.15))*

$$\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla(\eta_\varepsilon m_\parallel^\varepsilon)(x)\cdot\nabla(\eta_\varepsilon m_\parallel^\varepsilon)(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \tag{5.60}$$

satisfies

$$\begin{aligned}&\left|\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) - \tilde{\mathcal{V}}_{\Omega\times\Omega}(m_\parallel^\varepsilon) + 2\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m_\parallel^\varepsilon) - D_\varepsilon + 2|\ln\varepsilon|\int_{\partial\Omega}(1-|m_\parallel^\varepsilon|^2)\mathrm{d}\mathcal{H}^1(\sigma)\right|\\ &\quad\leq C_\varepsilon(1+\|\eta_\varepsilon\nabla m_\parallel^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2) + C\|m_\parallel^\varepsilon+1\|_{L^2(\partial\Omega)}^{1/2}\|m_\parallel^\varepsilon-1\|_{L^2(\partial\Omega)}^{1/2}\\ &\quad\quad\times(1+\|\eta_\varepsilon\nabla m_\parallel^\varepsilon\|_{L^2(\Omega_\varepsilon)}),\end{aligned} \tag{5.61}$$

where $C_\varepsilon \to 0$ as $\varepsilon \to 0$, D_ε is defined in (3.27), and

$$\tilde{\mathcal{V}}_{\Omega\times\Omega}(m_\parallel^\varepsilon) := \int_\Omega\int_\Omega \frac{\nabla m_\parallel^\varepsilon(x)\cdot\nabla m_\parallel^\varepsilon(y)}{|y-x|}\mathrm{d}y\mathrm{d}x, \tag{5.62}$$

$$\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m_\parallel^\varepsilon) := \int_\Omega\int_{\partial\Omega} \frac{\nabla m_\parallel^\varepsilon(x)\cdot\boldsymbol{n}(\sigma)m_\parallel^\varepsilon(\sigma)}{|\sigma-x|}\mathrm{d}\mathcal{H}^1(\sigma)\mathrm{d}x. \tag{5.63}$$

Proof. We begin by writing $\tilde{\mathcal{V}}$ in the form similar to that of the non-local term in Proposition 5.1:

$$\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) = \sum_{i=1}^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\operatorname{div}(\eta_\varepsilon m_\parallel^\varepsilon \boldsymbol{e}_i)(x)\operatorname{div}(\eta_\varepsilon m_\parallel^\varepsilon \boldsymbol{e}_i)(y)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{5.64}$$

Proceeding exactly as in Steps 1–3 in the proof of Proposition 5.1, we obtain

$$|\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) - \tilde{J}_1 - \tilde{\mathcal{V}}_{\Omega\times\Omega}(m_\parallel^\varepsilon) + 2\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m_\parallel^\varepsilon)| \leq C_\varepsilon(1+\|\eta_\varepsilon\nabla m_\parallel^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2), \tag{5.65}$$

where $C_\varepsilon \to 0$ as $\varepsilon \to 0$ and

$$\tilde{J}_1 := \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{\nabla\eta_\varepsilon(x)\cdot\nabla\eta_\varepsilon(y)m_\parallel^\varepsilon(\boldsymbol{\pi}(x))m_\parallel^\varepsilon(\boldsymbol{\pi}(y))}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{5.66}$$

To account for the nonzero limiting boundary data for $m_\parallel^\varepsilon$, we represent $\tilde{J}_1$ in the following way:

$$\tilde{J}_1 = \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{\nabla\eta_\varepsilon(x)\cdot\nabla\eta_\varepsilon(y)(m_\parallel^\varepsilon(\boldsymbol{\pi}(x))-1)(m_\parallel^\varepsilon(\boldsymbol{\pi}(y))+1)}{|x-y|}\mathrm{d}x\mathrm{d}y + D_\varepsilon, \tag{5.67}$$

where we recalled the definition of D_ε from (3.27) and noted that the terms linear in $m_\parallel^\varepsilon$ cancel upon expansion of the integral due to the symmetry of the kernel. We denote by $\bar{J}_1$ the first integral in the above expression (i.e. $\bar{J}_1 := \tilde{J}_1 - D_\varepsilon$) and split it in a similar way to what we did for J_1 in Proposition 5.1. Specifically, we set $\bar{J}_1 := \tilde{K}_1 + \tilde{K}_2$, where

$$\tilde{K}_1 := \int_{\mathcal{O}_\varepsilon^+}\int_{\mathcal{O}_\varepsilon^+} \frac{|m_\parallel^\varepsilon(\boldsymbol{\pi}(x))|^2 - 1}{|x-y|} |\nabla\eta_\varepsilon(x)||\nabla\eta_\varepsilon(y)| \mathrm{d}x\mathrm{d}y, \tag{5.68}$$

$$\tilde{K}_2 := \int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)|(m_\parallel^\varepsilon(\boldsymbol{\pi}(x)) - 1)\boldsymbol{n}(\boldsymbol{\pi}(x)) \cdot \int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(y)| \frac{\boldsymbol{n}(\boldsymbol{\pi}(y))(m_\parallel^\varepsilon(\boldsymbol{\pi}(y)) + 1) - \boldsymbol{n}(\boldsymbol{\pi}(x))(m_\parallel^\varepsilon(\boldsymbol{\pi}(x)) + 1)}{|x-y|} \mathrm{d}y\mathrm{d}x. \tag{5.69}$$

By the same arguments used in the proof of STEP 3 in Proposition 5.1 to estimate K_2, we then obtain the estimate

$$\tilde{K}_2 \le C(1 + \|\eta_\varepsilon \nabla m_\parallel^\varepsilon\|_{L^2(\Omega_\varepsilon)})\|m_\parallel^\varepsilon - 1\|_{L^2(\partial\Omega)}, \tag{5.70}$$

for some $C > 0$ and all ε small enough.

Alternatively, writing $\tilde{K}_2$ in the following equivalent way:

$$\tilde{K}_2 = \int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(x)|(m_\parallel^\varepsilon(\boldsymbol{\pi}(x)) + 1)\boldsymbol{n}(\boldsymbol{\pi}(x)) \cdot \int_{\mathcal{O}_\varepsilon^+} |\nabla\eta_\varepsilon(y)| \frac{\boldsymbol{n}(\boldsymbol{\pi}(y))(m_\parallel^\varepsilon(\boldsymbol{\pi}(y)) - 1) - \boldsymbol{n}(\boldsymbol{\pi}(x))(m_\parallel^\varepsilon(\boldsymbol{\pi}(x)) - 1)}{|x-y|} \mathrm{d}y\mathrm{d}x, \tag{5.71}$$

we infer

$$\tilde{K}_2 \le C(1 + \|\eta_\varepsilon \nabla m_\parallel^\varepsilon\|_{L^2(\Omega_\varepsilon)})\|m_\parallel^\varepsilon + 1\|_{L^2(\partial\Omega)}, \tag{5.72}$$

and taking the geometric mean of (5.70) and (5.72), we obtain

$$\tilde{K}_2 \le C(1 + \|\eta_\varepsilon \nabla m_\parallel^\varepsilon\|_{L^2(\Omega_\varepsilon)})\|m_\parallel^\varepsilon + 1\|_{L^2(\partial\Omega)}^{1/2} \|m_\parallel^\varepsilon - 1\|_{L^2(\partial\Omega)}^{1/2}, \tag{5.73}$$

for some $C > 0$ and all ε small enough.

Finally, the estimate for $\tilde{K}_1$ can be obtained in the same way we derived the estimate for K_1 in STEP 3 of the proof of Proposition 5.1, together with a Cauchy–Schwarz inequality, to obtain

$$\left|\tilde{K}_1 - 2|\ln\varepsilon| \int_{\partial\Omega} (|m_\parallel^\varepsilon|^2 - 1)\mathrm{d}\mathcal{H}^1(\sigma)\right| \le C\|m_\parallel^\varepsilon + 1\|_{L^2(\partial\Omega)}^{1/2}\|m_\parallel^\varepsilon - 1\|_{L^2(\partial\Omega)}^{1/2}. \tag{5.74}$$

Combining all of the above estimates, we obtain the result. □

6. Proof of Γ-convergence

In this section, we provide the proof of our main theorems. Since the proofs of Theorems 3.1 and 3.3 follow essentially verbatim those of Theorems 3.2 and 3.4, respectively, we only give the proofs of the latter. Theorems 3.1 and 3.3 may in fact be thought of as the limiting cases of Theorems 3.2 and 3.4.

Proof of Theorem 3.2. Without loss of generality we may suppose that $\gamma_\varepsilon = \gamma$.

(i) (Compactness) We first prove the compactness result. Let us assume that $\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) \le C$ for some constant $C > 0$ independent of ε. We recall that (cf. (3.17))

$$\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) = \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \lambda \mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\gamma}{2|\ln\varepsilon|}\mathcal{V}(\eta_\varepsilon \boldsymbol{m}^\varepsilon_\perp) - \frac{\gamma}{2|\ln\varepsilon|}\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel). \tag{6.1}$$

First, note that, up to a constant term, we can absorb the DMI energy $\lambda\mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon)$ into the Dirichlet energy. Indeed, since $|\boldsymbol{m}| = 1$ a.e. in Ω_ε and $|\eta_\varepsilon| \le 1$, by the Cauchy–Schwarz and Young's inequalities for every $0 < \varepsilon < \bar\varepsilon$ and every $\delta > 0$ there holds

$$\begin{aligned}|\mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon)| &\le \int_{\Omega_\varepsilon} |m^\varepsilon_\parallel \operatorname{div} \boldsymbol{m}^\varepsilon_\perp - \boldsymbol{m}^\varepsilon_\perp \cdot \nabla m^\varepsilon_\parallel||\eta_\varepsilon| \mathrm{d}x \\ &\le C\|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|_{L^2(\Omega_\varepsilon)} \le \frac{\delta}{2}\|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{C^2}{2\delta},\end{aligned} \tag{6.2}$$

for some $\bar\varepsilon, C > 0$ that depend only on Ω. Therefore, without loss of generality, we can assume from the very beginning that

$$\|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{\gamma}{|\ln\varepsilon|}\mathcal{V}(\eta_\varepsilon \boldsymbol{m}^\varepsilon_\perp) - \frac{\gamma}{|\ln\varepsilon|}\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel) \le C \tag{6.3}$$

for some constant $C > 0$ independent of ε.

By positivity of $\mathcal{V}(\eta_\varepsilon \boldsymbol{m}^\varepsilon_\perp)$, we may, furthermore, drop this term from (6.3). On the other hand, from Proposition 5.2, Lemma 4.4 and the estimates

$$\tilde{\mathcal{V}}_{\Omega\times\Omega}(m^\varepsilon_\parallel) \le \int_\Omega\int_\Omega \frac{|\nabla m^\varepsilon_\parallel(x)|^2}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x \le C'\|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}, \tag{6.4}$$

$$\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m^\varepsilon_\parallel) \le \int_\Omega\int_{\partial\Omega} \frac{|\nabla m^\varepsilon_\parallel(x)|}{|x-\sigma|}\,\mathrm{d}\mathcal{H}^1(\sigma)\,\mathrm{d}x \le C''\|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{6.5}$$

for some $C', C'' > 0$ depending only on Ω that follow from the Cauchy–Schwarz inequalities, we immediately obtain the existence of a positive constant $C > 0$ such that for all sufficiently small ε there holds

$$(1 - \gamma C_\varepsilon)\,\|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le C, \tag{6.6}$$

for some $C_\varepsilon > 0$ such that $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

From (6.6) we conclude that for ε sufficiently small we can uniformly bound $\boldsymbol{m}^\varepsilon$ in $H^1(\Omega; \mathbb{S}^2)$ and, therefore, up to a subsequence, there exists $\boldsymbol{m} \in H^1(\Omega; \mathbb{S}^2)$ such that

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{m} \quad \text{strongly in } L^2(\Omega; \mathbb{R}^3), \tag{6.7}$$

$$\nabla \boldsymbol{m}^\varepsilon \rightharpoonup \nabla \boldsymbol{m} \quad \text{weakly in } L^2(\Omega; \mathbb{R}^{2\times 3}), \tag{6.8}$$

$$m_{\parallel}^{\varepsilon} \to m_{\parallel} \quad \text{strongly in } L^p(\partial\Omega) \text{ for } p \geq 1, \tag{6.9}$$

$$\boldsymbol{m}_{\perp}^{\varepsilon} \to \boldsymbol{m}_{\perp} \quad \text{strongly in } L^p(\partial\Omega;\mathbb{R}^2) \text{ for } p \geq 1, \tag{6.10}$$

as $\varepsilon \to 0$. Moreover, since $|\boldsymbol{m}^\varepsilon| = 1$ on Ω_ε we also have $\eta_\varepsilon \boldsymbol{m}^\varepsilon \to 0$ strongly in $L^2(\mathbb{R}^2\backslash\overline{\Omega};\mathbb{R}^3)$. Hence, we infer that if $\boldsymbol{m}$ is extended by zero outside Ω then

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{m} \quad \text{strongly in } L^2(\mathbb{R}^2;\mathbb{R}^3). \tag{6.11}$$

In particular, we have $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$.

(ii) (Γ-*liminf* inequality) Let $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ and $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ be such that $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ as $\varepsilon \to 0$. We may further assume that $\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) < +\infty$, since otherwise the statement is trivially true. Hence, using the compactness statement (maybe passing to a subsequence) we have $\boldsymbol{m}_\varepsilon \rightharpoonup \boldsymbol{m}$ weakly in $H^1(\Omega;\mathbb{R}^3)$, and using Propositions 5.1 and 5.2, together with Lemma 4.4 and the lower semicontinuity of the Dirichlet energy on Ω and the compactness of trace embedding of functions in $H^1(\Omega)$ into $L^2(\partial\Omega)$ we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) \geq \int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x + \lambda \int_\Omega (m_{\parallel} \operatorname{div} \boldsymbol{m}_{\perp} - \boldsymbol{m}_{\perp} \cdot \nabla m_{\parallel}) \mathrm{d}x \\ + \gamma \int_{\partial\Omega} ((\boldsymbol{m}_{\perp} \cdot \boldsymbol{n})^2 - m_{\parallel}^2) \mathrm{d}\mathcal{H}^1(\sigma) = \mathcal{G}_0(\boldsymbol{m}).\end{aligned} \tag{6.12}$$

(iii) (Γ-*limsup* inequality) Let $\boldsymbol{m} \in H_0(\mathbb{R}^2;\mathbb{S}^2)$ be such that $\mathcal{G}_0(\boldsymbol{m}) < +\infty$. Take $\bar{\varepsilon} > 0$ sufficiently small and extend $\boldsymbol{m}$ to $\widetilde{\boldsymbol{m}} \in H^1(\Omega_{\bar{\varepsilon}}, \mathbb{S}^2)$, e.g. by setting $\widetilde{\boldsymbol{m}}(x) := \boldsymbol{m}(x - 2d_{\partial\Omega}(x)\boldsymbol{n}(\boldsymbol{\pi}(x)))$. For every $\varepsilon < \bar{\varepsilon}$ we now define $\boldsymbol{m}^\varepsilon = \widetilde{\boldsymbol{m}}$ in Ω_ε and $\boldsymbol{m}^\varepsilon = 0$ outside Ω_ε. It is clear that $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2;\mathbb{S}^2)$ and $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ as $\varepsilon \to 0$. Moreover, using Propositions 5.1 and 5.2, Lemma 4.4 and the strong convergence of $\boldsymbol{m}^\varepsilon$ to $\boldsymbol{m}$ in $H^1(\Omega;\mathbb{R}^3)$, we have $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ in $L^2(\partial\Omega;\mathbb{R}^3)$ and can pass to the limit in the magnetostatic energy term. Finally, using the fact that

$$\int_{\Omega_\varepsilon} \eta_\varepsilon^2 |\nabla \boldsymbol{m}^\varepsilon|^2 \mathrm{d}x = \int_{\mathcal{O}_\varepsilon^+} \eta_\varepsilon^2 |\nabla \widetilde{\boldsymbol{m}}|^2 \mathrm{d}x + \int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x \xrightarrow{\varepsilon\to 0} \int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x, \tag{6.13}$$

we obtain

$$\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) = \mathcal{G}_0(\boldsymbol{m}). \tag{6.14}$$

This completes the proof. □

Proof of Theorem 3.4. (i) (Compactness) We first prove the compactness result. Let us assume that $\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2} D_\varepsilon \leq C$ for some constant $C > 0$ independent of ε. We recall that now $\mathcal{G}_\varepsilon$ reads as (cf. (3.17) with $\gamma_\varepsilon = \nu|\ln\varepsilon|$)

$$\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) = \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \lambda \mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2}\mathcal{V}(\eta_\varepsilon \boldsymbol{m}_{\perp}^\varepsilon) - \frac{\nu}{2}\tilde{\mathcal{V}}(\eta_\varepsilon m_{\parallel}^\varepsilon). \tag{6.15}$$

As in the proof of Theorem 3.2, up to a constant term, we can absorb the DMI energy $\lambda\mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon)$ into the Dirichlet energy and drop the $\mathcal{V}(\eta_\varepsilon \boldsymbol{m}_{\perp}^\varepsilon)$ term due to its positivity. Therefore, without loss of generality, we can assume from the very beginning that

$$\|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - \nu\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel) + \nu D_\varepsilon \leq C, \tag{6.16}$$

for some constant $C > 0$ independent of ε.

Next, aiming to invoke Proposition 5.2, we write

$$\begin{aligned} -\nu(\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel) - D_\varepsilon) &= \nu(2\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m^\varepsilon_\parallel) - \tilde{\mathcal{V}}_{\Omega\times\Omega}(m^\varepsilon_\parallel) + 2|\ln\varepsilon| \int_{\partial\Omega}(1 - |m^\varepsilon_\parallel|^2)\mathrm{d}\mathcal{H}^1(\sigma)) \\ &\quad - \nu(\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel) - D_\varepsilon - \tilde{\mathcal{V}}_{\Omega\times\Omega}(m^\varepsilon_\parallel) + 2\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m^\varepsilon_\parallel) \\ &\quad + 2|\ln\varepsilon| \int_{\partial\Omega}(1 - |m^\varepsilon_\parallel|^2)\mathrm{d}\mathcal{H}^1(\sigma)), \end{aligned} \tag{6.17}$$

from which by Proposition 5.2 it follows that

$$\begin{aligned} -\nu(\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel) - D_\varepsilon) \geq -C - C'\|\eta_\varepsilon \nabla m^\varepsilon_\parallel\|_{L^2(\Omega_\varepsilon)} - C_\varepsilon \|\eta_\varepsilon \nabla m^\varepsilon_\parallel\|^2_{L^2(\Omega_\varepsilon)} \\ + 2\nu|\ln\varepsilon| \int_{\partial\Omega}(1 - |m^\varepsilon_\parallel|^2)\mathrm{d}\sigma + \nu(2\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m^\varepsilon_\parallel) \\ - \tilde{\mathcal{V}}_{\Omega\times\Omega}(m^\varepsilon_\parallel)), \end{aligned} \tag{6.18}$$

where $C_\varepsilon \to 0$ as $\varepsilon \to 0$ and $C, C' > 0$ are independent of ε. Now, from (6.5) and Young's inequality it is clear that for any $\delta > 0$ we have

$$\nu|\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m^\varepsilon_\parallel)| \leq \delta\|\eta_\varepsilon \nabla m^\varepsilon_\parallel\|^2_{L^2(\Omega_\varepsilon)} + C\nu^2\delta^{-1}, \tag{6.19}$$

for some $C > 0$ depending only on Ω. Also, applying Lemma 4.7 to $\boldsymbol{u}_\varepsilon = m^\varepsilon_\parallel \boldsymbol{e}_i$, $i = 1, 2$, one obtains that for any $\delta > 0$ there holds

$$\nu|\tilde{\mathcal{V}}_{\Omega\times\Omega}(m^\varepsilon_\parallel)| \leq \delta\|\eta_\varepsilon \nabla m^\varepsilon_\parallel\|^2_{L^2(\Omega_\varepsilon),} + C\nu^2\delta^{-1}, \tag{6.20}$$

again, for some $C > 0$ depending only on Ω. Based on the above estimates and another application of Young's inequality in (6.18) we deduce that for any $0 < \varepsilon < \bar{\varepsilon}$ and any $\delta > 0$ there holds

$$\begin{aligned} -\nu(\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel) - D_\varepsilon) \geq -(4\delta + C_\varepsilon)\|\eta_\varepsilon \nabla m^\varepsilon_\parallel\|^2_{L^2(\Omega_\varepsilon)} - C(1 + \nu^2\delta^{-1}) \\ + 2\nu|\ln\varepsilon| \int_{\partial\Omega}(1 - |m^\varepsilon_\parallel|^2)\mathrm{d}\mathcal{H}^1(\sigma), \end{aligned} \tag{6.21}$$

for some $C > 0$ depending only on Ω. Therefore, we can absorb the term $-\nu(\tilde{\mathcal{V}}(\eta_\varepsilon m^\varepsilon_\parallel) - D_\varepsilon)$ into the Dirichlet energy by choosing δ sufficiently small universal, and for any $0 < \varepsilon < \bar{\varepsilon}$ there holds

$$\frac{1}{2}\|\eta_\varepsilon \nabla m^\varepsilon_\parallel\|^2_{L^2(\Omega_\varepsilon)} + 2\nu|\ln\varepsilon| \int_{\partial\Omega}(1 - |m^\varepsilon_\parallel|^2)\mathrm{d}\mathcal{H}^1(\sigma) \leq C, \tag{6.22}$$

for some $C > 0$ independent of ε. This gives us (as in the proof of Theorem 3.2) the existence of $\boldsymbol{m} \in H_0(\mathbb{R}^2; \mathbb{S}^2)$ and a subsequence such that

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{m} \quad \text{strongly in } L^2(\mathbb{R}^2; \mathbb{R}^3), \tag{6.23}$$

$$\nabla \boldsymbol{m}^\varepsilon \rightharpoonup \nabla \boldsymbol{m} \quad \text{weakly in } L^2(\Omega; \mathbb{R}^{2\times 3}), \tag{6.24}$$

$$|m_\parallel^\varepsilon| \to 1 \quad \text{strongly in } L^p(\partial\Omega) \text{ for any } p \geq 1, \tag{6.25}$$

$$\boldsymbol{m}_\perp^\varepsilon \to 0 \quad \text{strongly in } L^p(\partial\Omega; \mathbb{R}^2) \text{ for any } p \geq 1. \tag{6.26}$$

Hence, upon a further subsequence, we have $|m_\parallel^\varepsilon| \to 1$ a.e. in $\partial\Omega$. In fact, since the trace of the limit belongs to $\mathrm{VMO}(\partial\Omega)$ and takes only values ± 1, it is in fact constant a.e. on $\partial\Omega$.[14, 15]

In what follows, without loss of generality, we assume that $m_\parallel^\varepsilon \to 1$ strongly in $L^p(\partial\Omega)$, $p \geq 1$, i.e. that the limit configuration $\boldsymbol{m}$ satisfies the boundary condition $\boldsymbol{m} = \boldsymbol{e}_3$ a.e. on $\partial\Omega$.

(ii) (Γ-*liminf* inequality) We consider the energy functional

$$\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2} D_\varepsilon = \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \lambda \mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2} D_\varepsilon + \frac{\nu}{2} \mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) - \frac{\nu}{2} \tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) \tag{6.27}$$

and will prove a lim inf inequality for this functional. Let $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2; \mathbb{S}^2)$ satisfy $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ strongly in $L^2(\mathbb{R}^2; \mathbb{R}^3)$ as $\varepsilon \to 0$. We may assume that $\liminf_{\varepsilon\to 0} (\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2} D_\varepsilon) < +\infty$, otherwise the statement is trivially true. Hence (maybe after passing to a subsequence) we may assume that

$$\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2} D_\varepsilon = \lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2} D_\varepsilon < +\infty. \tag{6.28}$$

Using the compactness result and the implied convergence, by the lower semicontinuity of the norm and the weak-strong argument we immediately obtain

$$\begin{aligned} \liminf_{\varepsilon\to 0} \|\eta_\varepsilon \nabla \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &\geq \|\nabla \boldsymbol{m}\|^2_{L^2(\Omega)}, \\ \lim_{\varepsilon\to 0} \lambda \mathcal{D}_\varepsilon(\boldsymbol{m}^\varepsilon) &= \lambda \int_\Omega (m_\parallel \operatorname{div} \boldsymbol{m}_\perp - \boldsymbol{m}_\perp \cdot \nabla m_\parallel) \mathrm{d}x. \end{aligned} \tag{6.29}$$

Therefore, the Γ-*liminf* inequality is proved once we show that

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \left(\frac{\nu}{2} D_\varepsilon + \frac{\nu}{2} \mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) - \frac{\nu}{2} \tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) \right) \\ &\qquad \geq \frac{\nu}{2} \mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp) - \frac{\nu}{2} \tilde{\mathcal{V}}_{\Omega\times\Omega}(m_\parallel) + \nu \tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m_\parallel). \end{aligned} \tag{6.30}$$

For that, we consider separately the convergence of the terms due to the in-plane and the out-of-plane components.

The in-plane magnetostatic contribution. As a direct consequence of Proposition 5.1, Lemma 4.7 and the convergence in (6.24), (6.26), we obtain

$$\frac{\nu}{2} \mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) - \nu |\ln \varepsilon| \int_{\partial\Omega} (\boldsymbol{m}_\perp^\varepsilon \cdot \boldsymbol{n})^2 \mathrm{d}\mathcal{H}^1(\sigma) \xrightarrow{\varepsilon\to 0} \frac{\nu}{2} \mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp). \tag{6.31}$$

The out-of-plane magnetostatic contribution. As a direct consequence of Proposition 5.2, Lemma 4.7 and the convergence in (6.24), (6.9), we obtain

$$\frac{\nu}{2}(\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) - D_\varepsilon) + \nu|\ln\varepsilon| \int_{\partial\Omega} (1 - |m_\parallel^\varepsilon|^2) \mathrm{d}\mathcal{H}^1(\sigma)$$
$$\xrightarrow{\varepsilon\to 0} \frac{\nu}{2}\tilde{\mathcal{V}}_{\Omega\times\Omega}(m_\parallel) - \nu\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m_\parallel). \tag{6.32}$$

Proof of (6.30). Combining (6.31) and (6.32) we get

$$\begin{aligned}
&\frac{\nu}{2}D_\varepsilon + \frac{\nu}{2}\mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) - \frac{\nu}{2}\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) \\
&\quad \geq \frac{\nu}{2}\mathcal{V}(\eta_\varepsilon \boldsymbol{m}_\perp^\varepsilon) - \nu|\ln\varepsilon| \int_{\partial\Omega} (\boldsymbol{m}_\perp^\varepsilon \cdot \boldsymbol{n})^2 \mathrm{d}\mathcal{H}^1(\sigma) \\
&\qquad - \frac{\nu}{2}(\tilde{\mathcal{V}}(\eta_\varepsilon m_\parallel^\varepsilon) - D_\varepsilon) - \nu|\ln\varepsilon| \int_{\partial\Omega} (1 - |m_\parallel^\varepsilon|^2) \mathrm{d}\mathcal{H}^1(\sigma) \\
&\quad \xrightarrow{\varepsilon\to 0} \frac{\nu}{2}\mathcal{V}_{\Omega\times\Omega}(\boldsymbol{m}_\perp) - \frac{\nu}{2}\tilde{\mathcal{V}}_{\Omega\times\Omega}(m_\parallel) + \nu\tilde{\mathcal{V}}_{\partial\Omega\times\Omega}(m_\parallel).
\end{aligned} \tag{6.33}$$

Therefore, using the definition of the vector field $\boldsymbol{b}(x)$ (see (2.23)) we obtain

$$\liminf_{\varepsilon\to 0} \left(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2}D_\varepsilon \right) \geq \tilde{\mathcal{G}}_0(\boldsymbol{m}). \tag{6.34}$$

(iii) (Γ-limsup inequality) We proceed as in the proof of Theorem 3.2. Let $\boldsymbol{m} \in H_0(\Omega; \mathbb{S}^2)$ be such that $\tilde{\mathcal{G}}_0(\boldsymbol{m}) < +\infty$ and, without loss of generality, $\boldsymbol{m} = \boldsymbol{e}_3$ on $\partial\Omega$. We take $\bar{\varepsilon} > 0$ and extend $\boldsymbol{m}$ to $\widetilde{\boldsymbol{m}} \in H^1(\Omega_{\bar{\varepsilon}}, \mathbb{S}^2)$ by setting $\widetilde{\boldsymbol{m}} = \boldsymbol{e}_3$ in $\Omega_{\bar{\varepsilon}} \backslash \Omega$. For every $\varepsilon < \bar{\varepsilon}$ we now define $\boldsymbol{m}^\varepsilon = \widetilde{\boldsymbol{m}}$ in Ω_ε and $\boldsymbol{m}^\varepsilon = 0$ outside Ω_ε. It is clear that $\boldsymbol{m}^\varepsilon \in H_\varepsilon(\mathbb{R}^2; \mathbb{S}^2)$ and satisfies $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ strongly in $L^2(\mathbb{R}^2; \mathbb{R}^3)$ as $\varepsilon \to 0$. Moreover, due to the fact that $\boldsymbol{m}^\varepsilon = \boldsymbol{m}$ in $H^1(\Omega; \mathbb{S}^2)$, we have $\boldsymbol{m}^\varepsilon = \boldsymbol{e}_3$ on $\partial\Omega$. Noting that in this case the inequality in (6.33) is actually an equality, and using the fact that

$$\int_{\Omega_\varepsilon} \eta_\varepsilon^2 |\nabla \boldsymbol{m}^\varepsilon|^2 \mathrm{d}x = \int_{\mathcal{O}_\varepsilon^+} \eta_\varepsilon^2 |\nabla \widetilde{\boldsymbol{m}}|^2 \mathrm{d}x + \int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x = \int_\Omega |\nabla \boldsymbol{m}|^2 \mathrm{d}x, \tag{6.35}$$

we obtain

$$\limsup_{\varepsilon\to 0} \left(\mathcal{G}_\varepsilon(\boldsymbol{m}^\varepsilon) + \frac{\nu}{2}D_\varepsilon \right) = \tilde{\mathcal{G}}_0(\boldsymbol{m}). \tag{6.36}$$

This completes the proof. □

Remark 6.1. An examination of the proof of Theorem 3.4 shows that

$$|\ln\varepsilon| \int_{\partial\Omega} (\boldsymbol{m}_\perp^\varepsilon \cdot \boldsymbol{n})^2 \mathrm{d}\mathcal{H}^1(\sigma) \to 0, \quad |\ln\varepsilon| \int_{\partial\Omega} (1 - |m_\parallel^\varepsilon|^2) \mathrm{d}\mathcal{H}^1(\sigma) \to 0, \tag{6.37}$$

as $\varepsilon \to 0$ for any sequence of minimizers $\boldsymbol{m}^\varepsilon$ of $\mathcal{G}_\varepsilon$.

Acknowledgments

G.DiF. and C.B.M. are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM. G.DiF. acknowledges support from the Austrian Science Fund (FWF) through the project *Analysis and Modeling of Magnetic Skyrmions* (grant 10.55776/P34609). G.DiF. also acknowledges support from the Italian Ministry of Education and Research through the PRIN2022 project *Variational Analysis of Complex Systems in Material Science, Physics and Biology* (No. 2022HKBF5C). The work of C.B.M. was supported, in part, by NSF via grant DMS-1908709. V.V.S. acknowledges support by Leverhulme grant RPG-2018-438. This study also received funding from the European Union — Next Generation EU — PRIN 2022 PNRR Project P2022WJW9H. C.B.M. acknowledges the MUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001. G.DiF. and C.B.M. thank the Hausdorff Research Institute for Mathematics in Bonn for its hospitality during the Trimester Program *Mathematics for Complex Materials* funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany Excellence Strategy — EXC-2047/1 — 390685813. G.DiF. also thanks TU Wien and MedUni Wien for their support and hospitality. Finally, G.DiF. and V.V.S. would like to thank the Max Planck Institute for Mathematics in the Sciences in Leipzig for support and hospitality.

ORCID

Giovanni Di Fratta https://orcid.org/0000-0003-0254-2957
Cyrill B. Muratov https://orcid.org/0000-0002-3337-6165
Valeriy V. Slastikov https://orcid.org/0000-0003-4550-4301

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