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Edge Domain Walls in Ultrathin Exchange-Biased Films

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Abstract

We present an analysis of edge domain walls in exchange-biased ferromagnetic films appearing as a result of a competition between the stray field at the film edges and the exchange bias field in the bulk. We introduce an effective two-dimensional micromagnetic energy that governs the magnetization behavior in exchange-biased materials and investigate its energy minimizers in the strip geometry. In a periodic setting, we provide a complete characterization of global energy minimizers corresponding to edge domain walls. In particular, we show that energy minimizers are one-dimensional and do not exhibit winding. We then consider a particular thin-film regime for large samples and relatively strong exchange bias and derive a simple and comprehensive algebraic model describing the limiting magnetization behavior in the interior and at the boundary of the sample. Finally, we demonstrate that the asymptotic results obtained in the periodic setting remain true in the case of finite rectangular samples.

Keywords Micromagnetics · Energy minimizers · Charged domain walls · Asymptotics

Mathematics Subject Classification 82D40 · 49S05

1 Introduction

Ferromagnetic films and multilayers are fundamental nanostructures widely used in present-day magnetoelectronics devices (Prinz 1998). As such, they have been the subject of intensive investigations over the last two decades in the engineering, physics

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and applied mathematics communities (Hubert and Schäfer 1998; Bader and Parkin 2010; Dennis et al. 2002; Fidler and Schrefl 2000; DeSimone et al. 2006). Some of the highlights of these activities include the discoveries of giant magnetoresistance, spin-transfer torque, spin-orbit coupling and the spin-Hall effect (Bader and Parkin 2010; Brataas et al. 2012; Soumyanarayanan et al. 2016; Hellman 2017). These new physical phenomena have led to the design of such technological applications as magnetic sensors, actuators, high-density magnetic storage devices and nonvolatile computer memory.

Surface and interfacial effects play a dominant role and are responsible for determining many properties of the nanostructured ferromagnetic materials (Hubert and Schäfer 1998; Dennis et al. 2002; Hellman 2017). These phenomena become increasingly important in the case of ultrathin films and multilayers. One basic example of such nanostructures is given by exchange-biased materials, which consist of a ferromagnetic film on top of an antiferromagnetic layer (Nogues et al. 2005). As a consequence of an exchange coupling between the two layers, the magnetization in the ferromagnetic film experiences a net bias induced by the magnetization at the interlayer interface, which furnishes the free layer with an effective unidirectional anisotropy. Additionally, nanostructure edges may also drastically change the equilibrium and the dynamic behaviors of the magnetization. For instance, the nanostructure edges often determine the mechanism of the magnetization reversal process (Hubert and Schäfer 1998; E et al. 2003; Muratov and Osipov 2008). However, despite the importance of edge effects, there exist just a handful of rigorous analytical studies characterizing the magnetization behavior near the film edges (Kohn and Slastikov 2005a; Kurzke 2006; Moser 2004; Lund et al. 2018; Muratov and Slastikov 2016).

Formation of *edge domain walls* is an important manifestation of edge effects observed in ferromagnetic films, double layers and exchange-biased materials (Hornreich 1963, 1964; Wade 1964; Rührig et al. 1990; Mattheis et al. 1997; Cho et al. 1999; Hubert and Schäfer 1998; Dennis et al. 2002; Rebouças et al. 2009). Edge domain walls appear as the result of a competition between magnetostatic energy dominating near the edges and the anisotropy or bias field effects in the bulk, leading to a mismatch in the preferred magnetization directions near and far from the film edges. It is well known that in ultrathin ferromagnetic films without perpendicular magnetic anisotropy the magnetization prefers to stay almost entirely in the film plane. At the same time, the magnetization tends to stay parallel to the film edge even if the magnetocrystalline anisotropy or the bias field favors a different magnetization direction in the interior. This effect is due to the stray field energy which produces a significant contribution near the sample edges (Kohn and Slastikov 2005a). Inside the sample, the bias field and/or magnetocrystalline anisotropy dominate the micromagnetic energy, favoring a single domain state. When these effects are sufficiently strong, they may also influence the magnetization behavior close to the sample boundary. As a result of the competition between the stray field and anisotropy/exchange bias energies, also taking into account the exchange energy, a transition layer near the edge, called edge domain wall, is formed. Although this simple phenomenological explanation gives an intuitive picture, apart from a few ansatz-based studies in the physics literature (Hornreich 1963, 1964; Nonaka et al. 1985; Hirono et al. 1986) there is currently little quantitative understanding of this phenomenon.

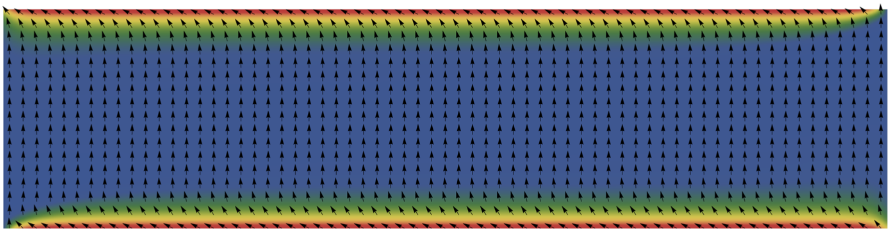

Fig. 1 A remanent magnetization in an exchange-biased permalloy film (exchange constant $A = 1.3 \times 10^{-11}$ J/m, saturation magnetization $M_s = 8 \times 10^5$ A/m, exchange bias field $H = 8.91 \times 10^3$ A/m) with dimensions 3.46μm $\times$ 0.87μm $\times$ 6nm. Result of a micromagnetic simulation, using the code developed in Muratov and Osipov (2006). The bias field is pointing up. Edge domain walls exhibiting partial alignment of the magnetization with the sample edges may be seen at the top and the bottom boundary

The goal of this paper is to understand the formation of edge domain walls in exchange-biased materials, viewed as minimizers of the micromagnetic energy. We are interested in soft ultrathin ferromagnetic films in the presence of a strong exchange bias field. Our analysis is based on a reduced two-dimensional micromagnetic energy with magnetization vector constrained to lie in the film plane, which is well known to adequately describe the magnetization behavior in ultrathin ferromagnetic films (Muratov and Osipov 2006; Kohn and Slastikov 2005a; DeSimone et al. 2006). Since we are concerned with the magnetization behavior near the edges, we consider one of the simplest and yet application-relevant geometries, namely that of a ferromagnetic strip. As described earlier, in this geometry the magnetization inside the strip aligns with the direction of the bias field, but at the edges it tends to align along the fixed edge direction. Typically, there is a misalignment between these two directions which, with the help of the exchange energy, results in the formation of a boundary layer near the edge (see Fig. 1). Let us stress that the situation considered here is very different from the case treated in Kohn and Slastikov (2005a), where the magnetization behavior at the boundary is controlled by the magnetization in the interior through the trace theorem. In larger ferromagnetic samples considered here, the exchange energy does not impose enough control over magnetization variation. This results in the detachment of the trace of the interior magnetization profile from the magnetization at the sample boundary. In particular, the actual magnetization behavior at the boundary is determined in a non-trivial way through the competition of exchange bias, stray field and bulk exchange energies.

Our analysis of the above problem in nanomagnetism proceeds as follows. First, we introduce a two-dimensional model, see (2.4), which governs the magnetization behavior in exchange-biased ultrathin nanostructures and accounts for the presence of nanostructure edges. This model is an extension of a reduced thin-film model introduced in the context of Ginzburg–Landau systems with dipolar repulsion that provides matching upper and lower bounds on the full three-dimensional energy for vanishing film thickness, together with universal error estimates (Muratov 2019). Instead of treating the magnetization as a discontinuous vector field having length one inside and zero outside a three-dimensional sample, we consider a two-dimensional domain occupied by the film in the plane (viewed from the top) and introduce a narrow band near the film edge, comparable in size to the film thickness. In this band, the magnetization is regularized for the stray field calculation, using a smooth cutoff function, see (2.3). Note that the magnetization behavior is asymptotically independent of the choice of the cut-off. We then proceed to analyze global energy minimizers associated with the energy in (2.4) in the presence of strong exchange bias in the direction normal to the strip edge.

We point out that the obtained non-convex, nonlocal, vectorial variational problem in full generality poses a formidable challenge to analysis. In particular, the system under consideration is known to exhibit winding magnetization configurations (Cho et al. 1999), which further complicates the situation. Nevertheless, within a periodic setting we are able to provide a complete characterization of global energy minimizers of the energy in (2.4). We first show that the energy minimizing configurations are one-dimensional, i.e., in those configurations the magnetization depends only on the distance to the edges. Furthermore, the magnetization vector does not exhibit winding and may rotate by at most 90 degrees away from the bias field direction. Thus, in the periodic setting the task of globally minimizing the energy (2.4) reduces to a particular one-dimensional variational problem. For the latter, we prove that there exist at most three minimizers, which are smooth solutions to a nonlocal Euler–Lagrange equation and possess C^2 regularity up to boundary, see Theorem 3.1.

We then consider a particular thin-film regime, in which the sample lateral dimensions also go to infinity with an appropriate rate, while the exchange bias, bulk exchange and magnetostatic energies all balance near the strip edge, see (2.9), (2.11) and (2.12). Still within the periodic setting, we then derive a simple and comprehensive *algebraic* model describing the magnetization behavior in the interior and at the boundary of the ferromagnet in the regime of strong exchange bias in the limit as the film thickness goes to zero, see Theorem 3.2. This reduced model uniquely determines the magnetization trace at the film edge for the minimizers, see Theorem 3.3. We also show that after a blowup the magnetization profile near the edge converges uniformly to an explicit profile in (3.9). Finally, we demonstrate that the asymptotic results for the limit behavior of the energy and the average trace of the magnetization on the sample edges obtained in the periodic setting remain true in the case of rectangular domains, see Theorem 3.4.

Our proofs in the periodic setting rely on a sharp, strict lower bound for the energy in (2.4) of a two-dimensional magnetization configuration in terms of the energy in (4.26) evaluated on the averages along the direction of the strip of the component of the magnetization normal to the strip edge. For the magnetostatic part of the energy, the corresponding lower bound is obtained, using Fourier techniques. For the local part of the energy, we use its convexity as a function of that component in the absence of winding. The latter is ensured by the choice of the reconstruction of the magnetization vector from the average of its component in the direction normal to the edge. We note that this argument crucially uses the specific form of the exchange bias energy and does not apply in the case of the uniaxial anisotropy considered by us in Lund et al. (2018). Once the one-dimensional nature of the minimizers has been established, the derivation of the Euler–Lagrange equation and the regularity still requires a delicate analysis due to the fact that nonlocality remains intertwined with the rest of the terms, producing an integro-differential equation. Additionally, under our Lipschitz assumption on the cutoff function, which also allows to mimic films with tapering edges, the nonlocal term may produce singularities near the sample boundaries, limiting the regularity of the minimizers up to the boundary. Finally, using the Euler–Lagrange equation we are able to show that the tangential component of the magnetization in a minimizer does not change sign. This allows us to take advantage of the convexity of the one-dimensional energy as a function of the normal component under this condition to establish the precise multiplicity of the minimizers.

For our asymptotic analysis, we first remark that in our problem it is necessary to go beyond the magnetostatics contribution at the sample edges considered in Kohn and Slastikov (2005a). Indeed, since the magnetization in the sample interior converges to a constant vector, the net magnetic line charge density at the strip edges is constant to the leading order. Therefore, one needs to perform an asymptotic expansion to extract the leading-order non-trivial contribution associated with the charge distribution between the strip edge and the strip interior in the boundary layer near the edge. After subtracting the leading-order constant, we deduce the asymptotic behavior of the minimal energy and the energy minimizers by establishing matching asymptotic upper and lower bounds on the energy. The lower bounds are a combination of the Modica–Mortola type bounds for the local part of the energy, while for the magnetostatic energy we use carefully chosen test potentials in a duality formulation that goes back to Brown (1963). In turn, the upper bounds rely on explicit Modica–Mortola transition layer profiles with an optimized boundary trace. Finally, we show that the presence of the additional edges parallel to the bias direction does not affect the asymptotic behavior of the energy for rectangular samples.

Our paper is organized as follows. In Sect. 2, we present the two-dimensional model analyzed throughout the paper and discuss the relevant scaling regime. In Sect. 3, we state our main results. In Sect. 4, we present the proof of Theorem 3.1 that characterizes the energy minimizers in the periodic setting. In Sect. 5, we present the proofs of Theorems 3.2 and 3.3 about the asymptotic behavior of the minimizers in the periodic setting in the considered regime. Finally, in Sect. 6, we present the proof of Theorem 3.4 about the asymptotics of the minimizers on a rectangular domain.

2 Model

In this paper, we investigate ultrathin ferromagnetic films with negligible magnetocrystalline anisotropy and in the presence of an exchange bias, which manifests itself as a Zeeman-like term in the energy. As our films of interest are only a few atomic layers thin, it is appropriate to model them using a two-dimensional micromagnetic framework. Furthermore, in the absence of perpendicular magnetic anisotropy the equilibrium magnetization vector is constrained to lie almost entirely in the film plane (DeSimone et al. 2000; Kohn and Slastikov 2005b; Garcia-Cervera and E 2001; Muratov and Osipov 2006). Therefore, in the case of an extended film the magnetization state may be described by a map $m : \mathbb{R}^2 \to \mathbb{S}^1$, with the associated energy (after a suitable rescaling) given by

$$
\begin{aligned}
E(m) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla m|^2 + h|m - e_2|^2 \right) \mathrm{d}x \\
+ \frac{\delta}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \cdot m(x) \nabla \cdot m(y)}{|x - y|} \, \mathrm{d}x \, \mathrm{d}y. \qquad (2.1)
\end{aligned}
$$

Here, the terms in the order of appearance are: the exchange energy, the Zeeman-like exchange bias energy due to an adjacent fixed magnetic layer and the stray field energy, respectively (Hubert and Schäfer 1998; DeSimone et al. 2000; Nogues et al. 2005). In writing (2.1), we measured lengths in the units of the exchange length and introduced the *effective* dimensionless film thickness $\delta > 0$ that plays the role of the strength of the magnetostatic interaction. Also, we have introduced the dimensionless constant $h > 0$ that characterizes the strength of the exchange bias along the vector e_2, the unit vector in the direction of the second coordinate axis. Note that due to rotational symmetry of the exchange and magnetostatic energies, the choice of the direction in the exchange bias term is arbitrary. Observe that by positive definiteness of the stray field term the unique global minimizer for the energy in (2.1) is given by the monodomain state $m(x) = e_2$.

2.1 Energy of a Finite Sample

We now turn our attention to films of finite extent, i.e., when the ferromagnetic material occupies a bounded domain in the plane, $D \subset \mathbb{R}^2$. One would naturally expect that the above model can be easily modified to describe the finite sample case by restricting the domains of integration to D. However, this is not the case as such a model would miss the contribution of the edge charges to the magnetostatic energy (Kohn and Slastikov 2005a). On the other hand, a simple extension of the magnetization m from D to the whole of $\mathbb{R}^2$ by zero and treating $\nabla \cdot m$ distributionally would not work in general, as in this case the magnetostatic energy becomes infinite unless the magnetization is tangential to the boundary ∂D of the sample (for further discussion, see Lund et al. 2018). This is due to the fact that a discontinuity in the normal component of the magnetization at the sample edge produces a divergent contribution to the magnetostatic energy. Physically, however, the thickness-averaged magnetization goes smoothly to zero on the atomic scale around the film edge, which for ultrathin films is comparable to the film thickness δ. Therefore, the magnetization profile appearing in the expression for the magnetostatic energy needs to be regularized:

$$m_\delta(x) := \eta_\delta(x) m(x) \qquad x \in D, \tag{2.2}$$

where $\eta_\delta(x)$ is a cutoff at scale δ that is determined by the detailed structure of the sample near the edges (e.g., compositional changes, elastic strain, edge roughness, tapering ends in as-grown film, etc.). Here, we take for simplicity

$$\eta_\delta(x) := \eta\left(\frac{\operatorname{dist}(x, \partial D)}{\delta}\right), \tag{2.3}$$

where $\eta \in C^\infty(\overline{\mathbb{R}}^+)$ satisfies $\eta'(t) > 0$ for all $0 < t < 1$, $\eta(0) = 0$ and $\eta(t) = 1$ for all $t \geq 1$. This defines a Lipschitz cutoff at scale δ near ∂D to smear the film edge on the scale of its thickness. The two-dimensional micromagnetic energy modeling the ultrathin ferromagnetic film of finite extent is now defined as

$$
\begin{aligned}
E(m) = {} & \frac{1}{2}\int_D \left(|\nabla m|^2 + h\,|m - e_2|^2\right) \mathrm{d}x \\
& + \frac{\delta}{8\pi}\int_D\int_D \frac{\nabla\cdot m_\delta(x)\nabla\cdot m_\delta(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{2.4}
$$

This energy is the starting point of our investigation. We note that the precise choice of the cutoff function will become unimportant in the asymptotic limit considered in Theorems 3.2–3.4, indicating that the detailed physics of the film edges does not affect the magnetization patterns in the considered regime. This is one of the main physical findings of our paper.

2.2 Energy in a Periodic Setting

We are also interested in a particular situation in which the domain has the shape of an infinite strip along the x_1-direction, of width $b > 0$; this situation is not immediately covered by the previous discussion. We assume periodicity in x_1 with period $a > 0$ and define the energy per period:

$$
\begin{aligned}
E^{\#}(m) = {} & \frac{1}{2}\int_D \left(|\nabla m|^2 + h\,|m - e_2|^2\right) \mathrm{d}x \\
& + \frac{\delta}{8\pi}\int_D\int_{\mathbb{R}\times(0,b)} \frac{\nabla\cdot m_\delta(x)\nabla\cdot m_\delta(y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned} \tag{2.5}
$$

where $D = (0,a)\times(0,b)$. Note that this energy is translationally invariant in the x_1-direction. In particular, one-dimensional magnetization configurations independent of x_1 are natural candidates for minimizers of $E^{\#}$. We point out that choosing the strip axis to lie along the direction e_1 (perpendicular to e_2) creates a competition between the exchange bias favoring m to lie along e_2 and the shape anisotropy forcing m to lie along e_1, which makes this configuration the most interesting one.

2.3 Connection to Three-Dimensional Micromagnetics

Let us point out that the energy in (2.4) may also be justified in some regimes by considering suitable thin-film limits of the full three-dimensional micromagnetic energy:

$$
\begin{aligned}
\mathcal{E}(\mathbf{m}) = {} & \frac{1}{2}\int_\Omega \left(|\nabla \mathbf{m}|^2 + h|\mathbf{m} - e_2|^2\right) \mathrm{d}x \\
& + \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla\cdot \mathbf{m}(x)\nabla\cdot \mathbf{m}(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{2.6}
$$

where $\Omega \subset \mathbb{R}^3$ is the domain occupied by the material and $\mathbf{m} : \Omega \to \mathbb{S}^2$, with $\mathbf{m}$ extended by zero outside Ω and $\nabla \cdot m$ understood distributionally. Typically when considering thin films, the domain Ω is taken to be a cylinder $\Omega = D \times (0, \delta)$, where $D \subset \mathbb{R}^2$ is the base of the film and δ is the film thickness (DeSimone et al. 2000). In reality, the film edges are never straight, but vary on the scale of the film thickness δ, and averaging over the thickness, we recover an analog of the regularized magnetization m_δ introduced in (2.2) (for further discussion in a related context, see Muratov 2019). Indeed, when $0 < \delta \lesssim 1$, the out-of-plane component of the magnetization $\mathbf{m}(x) \in \mathbb{S}^2$ is strongly penalized, forcing the magnetization to be restricted to the equator of $\mathbb{S}^2$, identified with $\mathbb{S}^1$. Furthermore, the magnetization vector will be effectively constant on the length scale δ. Therefore, to the leading order in δ we will have

$$\mathbf{m}(x_1, x_2, x_3) = (m(x_1, x_2), 0) \qquad m : \mathbb{R}^2 \to \mathbb{S}^1, \tag{2.7}$$

and $\mathcal{E}(\mathbf{m}) \simeq E(m)\delta$, where m_δ in (2.4) is defined, using a cutoff function η_δ related to the shape of the film edge (see also Slastikov 2005).

2.4 Thin-Film Regime

We now introduce a particular asymptotic regime in which edge domain walls bifurcate from the monodomain state $m = e_2$ as global energy minimizers when the effective film thickness $\delta \to 0$. We note that for all other parameters fixed the minimizer of the two-dimensional energy in (2.4) or the three-dimensional energy in (2.6) would converge to the monodomain state (for a closely related result, see Muratov 2019). Therefore, in order to observe non-trivial minimizers in the thin-film limit the lateral size of the ferromagnetic sample must diverge with an appropriate rate simultaneously with $\delta \to 0$. To capture this balance, we introduce a small parameter $\varepsilon > 0$ corresponding to the inverse lateral size of the ferromagnetic sample, i.e., $\operatorname{diam}(D_\varepsilon) = O(\varepsilon^{-1})$, and set $\delta = \delta_\varepsilon \to 0$ as $\varepsilon \to 0$. We also allow $h = h_\varepsilon$ to depend on ε. We then have a one-parameter family of functionals, parametrized by ε and given by $E(m) = E^0_\varepsilon(m)$, where

$$\begin{aligned} E^0_\varepsilon(m) = \frac{1}{2}\int_{D_\varepsilon} \left(|\nabla m|^2 + h_\varepsilon\, |m - e_2|^2\right) \mathrm{d}x \\ + \frac{\delta_\varepsilon}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\cdot m_{\delta_\varepsilon}(x) \nabla\cdot m_{\delta_\varepsilon}(y)}{|x-y|}\, \mathrm{d}x\, \mathrm{d}y, \end{aligned} \tag{2.8}$$

with a slight abuse of notation, assuming the cutoff function in (2.3) is defined, using D_ε instead of D. If we then rescale D_ε to work on an $O(1)$ domain D, we obtain that $E^0_\varepsilon(m) = \varepsilon^{-1} E_\varepsilon(m(\cdot/\varepsilon))$, where

$$\begin{aligned} E_\varepsilon(m) := \frac{1}{2}\int_{D} \left(\varepsilon\, |\nabla m|^2 + \frac{h_\varepsilon}{\varepsilon}\, |m - e_2|^2\right) \mathrm{d}x \\ + \frac{\delta_\varepsilon}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\nabla\cdot m_{\varepsilon\delta_\varepsilon}(x) \nabla\cdot m_{\varepsilon\delta_\varepsilon}(y)}{|x-y|}\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{2.9}$$

To proceed, we take, once again, the domain D to be a rectangle, $D = (0, a) \times (0, b)$, and consider two magnetization configurations as competitors. The first one is the monodomain state $m^{(1)} = e_2$ and the second one is a profile $m^{(2)}$ in which the magnetization rotates smoothly from e_2 in $(0, a) \times (\varepsilon L_\varepsilon, b - \varepsilon L_\varepsilon)$ to e_1 at $x_2 = 0$

and $x_2 = b$ within layers of width $\varepsilon L_\varepsilon$ near the top and bottom edges of D such that $\varepsilon\delta_\varepsilon \ll \varepsilon L_\varepsilon \ll 1$. Note that while in the former the edge magnetic charges are concentrated within layers of thickness δ_ε (in the original, unscaled variables), in the latter the edge magnetic charges are spread within layers of width L_ε (again, before rescaling).

It is not difficult to see that as $\varepsilon \to 0$ we have

$$\begin{aligned} E_\varepsilon(m^{(1)}) &\simeq \frac{a\delta_\varepsilon}{2\pi}|\ln \varepsilon\delta_\varepsilon|, \\ E_\varepsilon(m^{(2)}) &\simeq a\left(\frac{c_1}{L_\varepsilon} + c_2 h_\varepsilon L_\varepsilon\right) + \frac{a\delta_\varepsilon}{2\pi}|\ln \varepsilon L_\varepsilon|, \end{aligned} \tag{2.10}$$

for some $c_{1,2} > 0$ depending on the choice of the transition profile. Clearly, when the exchange bias field $h_\varepsilon = O(1)$, the first two terms give an $O(1)$ contribution to the energy $E_\varepsilon(m^{(2)})$. Therefore, in order for the energy of the edge charges $E(m^{(1)})$ in a monodomain state to be comparable with the local contributions to the energy of edge domain walls, one needs to choose

$$\delta_\varepsilon = \frac{\lambda}{|\ln \varepsilon|}, \tag{2.11}$$

for some $\lambda > 0$ playing the role of the renormalized effective film thickness. Notice that this scaling has recently appeared in a different context in the studies of thin ferromagnetic films with perpendicular magnetic anisotropy (Knüpfer et al. 2019). At the same time, according to (2.10) the leading-order contribution to the magnetostatic energy of the edge charges for the optimal choice of the edge domain wall width $L_\varepsilon = O(1)$ turns out to be the same as the energy of the monodomain state. Therefore, for $h_\varepsilon = O(1)$ it is not energetically advantageous to form edge domain walls. These walls would thus form at lower values of the exchange bias field h_ε.

In order to balance the energies of the two configurations above for δ_ε given by (2.11) and $h_\varepsilon \ll 1$, we need to evaluate the difference between the two at optimal wall width $L_\varepsilon = O(h_\varepsilon^{-1/2})$. Matching the wall energy $O(h_\varepsilon^{1/2})$ with the energy difference $O(\delta_\varepsilon \ln(L_\varepsilon/\delta_\varepsilon))$ then yields that one needs to choose

$$h_\varepsilon = \beta\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right)^2, \tag{2.12}$$

for some $\beta > 0$ playing the role of the renormalized field strength. The corresponding optimal choice of L_ε is $L_\varepsilon = O(|\ln\varepsilon|/(\ln|\ln\varepsilon|))$. Furthermore, under (2.11) and (2.12) one would expect that a transition from the monodomain state to states containing edge domain walls takes place at some critical value of β for fixed value of λ as $\varepsilon \to 0$. Below, we will show that this is indeed the case and identify the critical value of β.

3 Statement of Results

We now proceed to formulate the main results of this paper. Throughout the rest of the paper, the functions m_δ and η_δ are defined in (2.2) and (2.3), respectively, and the function η satisfies the assumptions specified in Sect. 2.1. We begin with the simplest setting, namely that of a periodic magnetization on a strip oriented normally to the direction of the bias field as described in Sect. 2.2. Our main result here is the identification of one-dimensional edge domain wall profiles as unique global energy minimizers of the energy $E^\#$ irrespectively of the relationship between a, b, δ and h. Throughout the rest of this paper, we always assume that $\delta < b/2$.

We start by defining the admissible class in which we will seek the minimizers of $E^\#$:

$$\mathcal{A}^\# := \{m \in H^1_{loc}(\mathbb{R} \times [0, b]; \mathbb{S}^1) : m(x_1 + a, x_2) = m(x_1, x_2)\}, \tag{3.1}$$

and introduce the representation of the magnetization in $\mathcal{A}^\#$ in terms of the angle that m makes with respect to the x_2-axis:

$$m = (-\sin\theta, \cos\theta). \tag{3.2}$$

We also define, for $\alpha \in (0, 1)$, the one-dimensional half-Laplacian acting on $u \in C^{1,\alpha}([0, b])$ that vanishes at the endpoints, extended by zero to the rest of $\mathbb{R}$:

$$\left(-\frac{d^2}{dx^2}\right)^{1/2} u(x) := \frac{1}{\pi} \fint_0^b \frac{u(x) - u(y)}{(x - y)^2}\, \mathrm{d}y + \frac{b u(x)}{\pi x(b - x)} \qquad x \in (0, b). \tag{3.3}$$

Finally, with a slight abuse of notation we will use $\eta_\delta(x)$ to define the cutoff as a function of one variable, $x = x_2$, and extend it by zero outside $(0, b)$.

We have the following basic characterization of the minimizers of $E^\#$ over $\mathcal{A}^\#$.

Theorem 3.1 *There exist at most three minimizers m of $E^\#$ over $\mathcal{A}^\#$. Each minimizer is one-dimensional, i.e., $m = m(x_2)$, and symmetric with respect to the midline, i.e., $m(x_2) = m(b - x_2)$. Furthermore, $m_2(x_2) \geq 0$ and $m_1(x_2)$ is either identically zero or does not change sign. In addition, if θ is such that m satisfies (3.2), then $\theta \in C^\infty(0, b) \cap C^2([0, b])$ and satisfies*

$$0 = \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - h\sin\theta + \frac{\delta}{2}\eta_\delta \sin\theta \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} \eta_\delta \cos\theta \qquad x \in (0, b), \tag{3.4}$$

together with $\theta'(0) = \theta'(b) = 0$.

It is clear that $m = e_2$ is one possibility for a minimizer in Theorem 3.1, which corresponds to the monodomain state. Note that by (3.4) the state $m = e_2$ is always a critical point of the energy $E^\#$. Furthermore, it is easy to see that $m = e_2$ is a local minimizer of $E^\#$ if the Schrödinger-type operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \qquad V(x) := h - \frac{\delta}{2}\eta_\delta(x)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2}\eta_\delta(x) \tag{3.5}$$

has only positive eigenvalues when $x \in (0, b)$. The monodomain state competes with a profile having $\theta = \theta(x_2) \in (0, \frac{1}{2}\pi]$ and another, symmetric profile obtained by replacing θ with $-\theta$, both corresponding to the edge domain walls.

Remark 3.1 Observe that by Theorem 3.1 the minimizers of $E^\#$ do not exhibit winding, i.e., the size of the range of θ associated with the minimizer does not reach or exceed 2π. Notice that a priori winding cannot be excluded, since the nonlocal term in the energy may favor oscillations of m. In fact, winding will be required if the minimization of $E^\#$ is carried out over an admissible class with a prescribed nonzero winding number across the period along x_1 (for a related study, see Ignat and Moser 2017).

We now turn to the regime described in Sect. 2.4, in which edge domain walls emerge as minimizers of $E^\#$. We begin by introducing a periodic version of the rescaled energy in (2.9):

$$\begin{aligned} E_\varepsilon^\#(m) := \frac{1}{2}\int_D \left(\varepsilon\,|\nabla m|^2 + \frac{h_\varepsilon}{\varepsilon}\,|m - e_2|^2\right)\mathrm{d}x + \\ \frac{\delta_\varepsilon}{8\pi}\int_D \int_{\mathbb{R}\times(0,b)} \frac{\nabla\cdot m_{\varepsilon\delta_\varepsilon}(x)\nabla\cdot m_{\varepsilon\delta_\varepsilon}(y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

This energy is still well defined on the admissible class $\mathcal{A}^\#$ for $D = (0, a) \times (0, b)$. We are going to completely characterize the minimizers of $E_\varepsilon^\#$ under the scaling assumptions in (2.11) and (2.12) as $\varepsilon \to 0$. In particular, we will show that for small enough β the minimizers asymptotically consist of edge domain walls of width of order $\varepsilon L_\varepsilon$, where

$$L_\varepsilon := \frac{|\ln\varepsilon|}{\ln|\ln\varepsilon|}. \tag{3.7}$$

To see this, let us drop the nonlocal term in (3.6) for the moment and consider a magnetization profile m given by (3.2) with $\theta = \theta(x_2)$ satisfying $\theta(0) = \theta_0 \in (0, \frac{\pi}{2}]$. Then, after the rescaling of x_2 by $\varepsilon L_\varepsilon$ and formally passing to the limit $\varepsilon \to 0$ we obtain the following local one-dimensional energy:

$$E_{1d}^\infty(\theta) := \int_0^\infty \left(\frac{1}{2}|\theta'|^2 + \beta(1-\cos\theta)\right)\mathrm{d}x. \tag{3.8}$$

For θ_0 fixed, this energy is explicitly minimized by

$$\theta_\infty(x) = 4\arctan\left(\mathrm{e}^{2\sqrt{\beta}(x_0-x)}\right), \qquad x_0 = \frac{1}{2\sqrt{\beta}}\ln\tan\left(\frac{\theta_0}{4}\right), \tag{3.9}$$

and the corresponding minimal energy is given by

$$E_{1d}^{\infty}(\theta_\infty) = 8\sqrt{\beta}\sin^2\left(\frac{\theta_0}{4}\right). \tag{3.10}$$

Indeed, using the Modica–Mortola trick (Modica 1987) we find that

$$\begin{aligned} E_{1d}^{\infty}(\theta) &\geq -2\sqrt{\beta}\int_0^\infty \sin\left(\frac{\theta}{2}\right)\theta'\,\mathrm{d}x \\ &+ \frac{1}{2}\int_0^\infty \left[\theta' + 2\sqrt{\beta}\sin\left(\frac{\theta}{2}\right)\right]^2 \mathrm{d}x \geq E_{1d}^{\infty}(\theta_\infty), \end{aligned} \tag{3.11}$$

and equality holds if and only if $\theta = \theta_\infty$.

We now define the function

$$F_0(n) := 4\sqrt{\beta}\left(1 - \sqrt{\frac{1+n}{2}}\right) + \frac{\lambda}{4\pi}(2n^2 - 1) \qquad n \in [0, 1], \tag{3.12}$$

and observe that $F_0(\cos\theta_0) = E_{1d}^{\infty}(\theta_\infty)$ when $\lambda = 0$. In the following, we will show that, up to an additive constant, the minimum of $E_\varepsilon^\#$ may be bounded below as $\varepsilon \to 0$ by a multiple of $F_0(n_\varepsilon)$, where n_ε is the trace of the second component of the minimizer on the edge. Moreover, this lower bound turns out to be sharp in the limit, allowing to characterize the global energy minimizers of $E_\varepsilon^\#$ in terms of those of F_0. The latter can in principle be computed as roots of a cubic polynomial, resulting in a cumbersome explicit formula. Taking advantage of the fact that $F_0(n)$ is a strictly convex function of n, however, one can conclude that F_0 admits a unique minimizer for every $\lambda > 0$ and $\beta > 0$. We have the following result regarding the minimizers of F_0, whose proof is a simple calculus exercise.

Lemma 3.1 *Let $F_0(n)$ be defined by (3.12), and let $n_0 = n_0(\beta, \lambda)$ be a minimizer of F_0 on $[0, 1]$. Then, n_0 is unique, and if*

$$\beta_c := \frac{\lambda^2}{\pi^2}, \tag{3.13}$$

we have $n_0 = 1$ and $F_0(n_0) = \frac{\lambda}{4\pi}$ for all $\beta \geq \beta_c$, while $0 < n_0 < 1$ and $F_0(n_0) < \frac{\lambda}{4\pi}$ for all $\beta < \beta_c$.

We also remark that the bifurcation at $\beta = \beta_c$ can be seen to be transcritical, and that $n_0(\beta, \lambda)$ is monotone increasing in β and goes to zero as $\beta \to 0$ with $\lambda > 0$ fixed. The latter is consistent with the fact that the magnetization wants to align tangentially to the film edge when the energy at the edge is dominated by the stray field (see also Hornreich 1963, 1964; Wade 1964; Nonaka et al. 1985; Hirono et al. 1986; DeSimone et al. 2000; Kohn and Slastikov 2005a; Lund et al. 2018).

Our next result gives an asymptotic relation between the energy of the minimizers of $E_\varepsilon^\#$ and that of the minimizers of F_0.

Theorem 3.2 *Let* $\lambda > 0$ *and* $\beta > 0$. *Assume* δ_ε *and* h_ε *are given by* (2.11) *and* (2.12). *Then, as* $\varepsilon \to 0$ *we have*

$$\frac{|\ln \varepsilon|}{\ln|\ln \varepsilon|}\left(\min_{m\in\mathcal{A}^\#} E^\#_\varepsilon(m) - \frac{a\lambda}{2\pi}\right) \to 2a \min_{n\in[0,1]} F_0(n). \tag{3.14}$$

We note that since $\min_{m\in\mathcal{A}^\#} E^\#_\varepsilon$ is bounded in the limit as $\varepsilon \to 0$ and since the energy in (3.6) consists of the sum of three positive terms, we also get that $m_\varepsilon \to e_2$ in $L^2(D;\mathbb{R}^2)$ for any minimizer m_ε of $E^\#_\varepsilon$ (or even for any configuration with finite energy). However, much more can be said about the minimizers of $E^\#_\varepsilon$ in the limit $\varepsilon \to 0$, which is the content of our next theorem. Let $m_\varepsilon = (m_{\varepsilon,1}, m_{\varepsilon,2})$ be a minimizer, which by Theorem 3.1 is one-dimensional, and define

$$\theta_\varepsilon(x) := -\arcsin m_{\varepsilon,1}(0, \varepsilon L_\varepsilon x) \qquad x \in (0, \varepsilon^{-1}L_\varepsilon^{-1}b), \tag{3.15}$$

where L_ε is defined in (3.7). Then, we have the following result.

Theorem 3.3 *Let* $\lambda > 0$ *and* $\beta > 0$. *Assume* δ_ε *and* h_ε *are given by* (2.11) *and* (2.12), *let* m_ε *be a minimizer of* $E^\#$ *over* $\mathcal{A}^\#$, *and let* θ_ε *be defined in* (3.15). *Then, as* $\varepsilon \to 0$ *we have*

$$|\theta_\varepsilon| \to \theta_\infty \text{ in } H^1_{loc}(\overline{\mathbb{R}^+}), \tag{3.16}$$

where θ_∞ *is given by* (3.9) *with* $\theta_0 = \arccos n_0$ *and* n_0 *is as in Lemma* 3.1. *In particular,* $m_{2,\varepsilon}(\cdot, 0) \to n_0$. *Moreover, convergence in* (3.16) *is uniform on* $[0, \frac{1}{2}\varepsilon^{-1}L_\varepsilon^{-1}b]$.

We remark that in view of the reflection symmetry of the minimizers guaranteed by Theorem 3.1, the same conclusions hold in the vicinity of the top edge as well. We also note that by Theorem 3.3 and Lemma 3.1, there is a bifurcation from the monodomain state to a state containing edge domain walls as the energy minimizers at $\beta = \beta_c$ in the limit as $\varepsilon \to 0$, with $\theta_\infty = 0$ for all $\beta \geq \beta_c$ and $\theta_\infty \neq 0$ for all $\beta < \beta_c$.

We now go to the original problem on the rectangular domain described by the energy in (2.9). In our final theorem, we establish that both the energy of the minimizers and their average trace on the top and the bottom edges of the rectangle approach the same values as in the case of the minimizers in the periodic setting as $\varepsilon \to 0$.

Theorem 3.4 *Let* $\lambda > 0$ *and* $\beta > 0$. *Assume* δ_ε *and* h_ε *are given by* (2.11) *and* (2.12), *and let* m_ε *be a minimizer of* E_ε *from* (2.9) *over* $H^1(D;\mathbb{S}^1)$. *Then, as* $\varepsilon \to 0$ *we have*

$$\frac{|\ln \varepsilon|}{\ln|\ln \varepsilon|}\left(E_\varepsilon(m_\varepsilon) - \frac{a\lambda}{2\pi}\right) \to 2aF_0(n_0), \tag{3.17}$$

where $n_0 \in [0, 1]$ *is the unique minimizer of* F_0 *in* (3.12). *Furthermore,* $m_\varepsilon(x) \to e_2$ *for a.e.* $x \in D$, *and we have*

$$\frac{1}{a}\int_0^a m_{2,\varepsilon}(t, 0)\,\mathrm{d}t \to n_0 \quad \text{and} \quad \frac{1}{a}\int_0^a m_{2,\varepsilon}(t, b)\,\mathrm{d}t \to n_0. \tag{3.18}$$

The statement of the above theorem implies that when D is a rectangle aligned with the direction of the preferred magnetization, the minimal energy behaves asymptotically as twice the horizontal edge length times the energy of the one-dimensional edge domain wall, while the average trace of the minimizer at the top and bottom edges agrees with that in the one-dimensional edge domain wall. At the same time, the magnetization in the bulk tends to its preferred value $m = e_2$. This is consistent with the expectation that a one-dimensional boundary layer should form near the charged edges.

4 Proof of Theorem 3.1

First of all, existence of a minimizer $m \in \mathcal{A}^\#$ follows from the direct method of calculus of variations, using standard arguments. To prove that the minimizer is one-dimensional, for any admissible m we define a competitor $\overline{m} = (\overline{m}_1, \overline{m}_2)$, where

$$\overline{m}_2(x_1, x_2) := \frac{1}{a}\int_0^a m_2(t, x_2)\,\mathrm{d}t, \qquad \overline{m}_1(x_1, x_2) := \sqrt{1 - \overline{m}_2^2(x_1, x_2)}. \tag{4.1}$$

We are now going to establish several useful results concerning $\overline{m}$, using some ideas related to those in Sandier and Shafrir (1993).

Lemma 4.1 *Let $m \in \mathcal{A}^\#$, and let $\overline{m}$ be defined by (4.1). Then, $\overline{m} \in \mathcal{A}^\#$,*

$$\int_D |\nabla \overline{m}|^2\,\mathrm{d}x \le \int_D |\nabla m|^2\,\mathrm{d}x, \tag{4.2}$$

and equality in the above expression holds if and only if m is independent of x_1.

Proof Since $m(x) = (m_1(x), m_2(x)) \in \mathbb{S}^1$ for a.e. $x \in D$, we have

$$m_1^2(x) + m_2^2(x) = 1 \quad \text{for a.e. } x \in D. \tag{4.3}$$

Therefore, applying weak chain rule (Lieb and Loss 2010, Theorem 6.16) to the above expression yields

$$m_1 \nabla m_1 = -m_2 \nabla m_2 \qquad \text{a.e. in } D. \tag{4.4}$$

Combining (4.3) and (4.4), and using the fact that $\nabla m_1(x) = 0$ for a.e. $x \in A \subseteq D$ whenever $m_1 = 0$ on A and $|A| > 0$ (Lieb and Loss 2010, Theorem 6.19), we have

$$|\nabla m(x)| = \begin{cases} \frac{|\nabla m_2(x)|}{\sqrt{1-m_2^2}}, & |m_2(x)| < 1, \\ 0, & |m_2(x)| = 1, \end{cases} \qquad \text{for a.e. } x \in D. \tag{4.5}$$

Note that this implies $\nabla m_2 = 0$ on the set A as well. Then, by monotone convergence theorem we can write

$$\int_D |\nabla m|^2 \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_D \frac{|\nabla m_2|^2}{1+\varepsilon - m_2^2} \, \mathrm{d}x. \tag{4.6}$$

Now for $\varepsilon > 0$, consider the function:

$$F_\varepsilon(u, v) := \frac{v^2}{1+\varepsilon - u^2} \qquad (u, v) \in [-1, 1] \times \mathbb{R}. \tag{4.7}$$

By direct computation, this function is convex for all $\varepsilon > 0$. Therefore,

$$\begin{aligned} \int_D \frac{|\nabla m_2|^2}{1+\varepsilon - m_2^2} \, \mathrm{d}x &= \int_D \frac{|\partial_1 m_2|^2}{1+\varepsilon - m_2^2} \, \mathrm{d}x + \int_D F_\varepsilon(m_2, \partial_2 m_2) \, \mathrm{d}x \\ &\geq \int_D F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2) \, \mathrm{d}x + \int_D \partial_u F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2)(m_2 - \overline{m}_2) \, \mathrm{d}x \\ &\quad + \int_D \partial_v F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2)(\partial_2 m_2 - \partial_2 \overline{m}_2) \, \mathrm{d}x. \end{aligned} \tag{4.8}$$

At the same time, by Fubini's theorem and the definition of $\overline{m}_2$ we have

$$\begin{aligned} &\int_D \partial_u F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2)(m_2 - \overline{m}_2) \, \mathrm{d}x \\ &\quad = \int_0^b \left(\partial_u F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2) \int_0^a (m_2 - \overline{m}_2) \, \mathrm{d}x_1 \right) \mathrm{d}x_2 = 0, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\int_D \partial_v F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2)(\partial_2 m_2 - \partial_2 \overline{m}_2) \, \mathrm{d}x \\ &\quad = \int_0^b \left(\partial_v F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2) \int_0^a (\partial_2 m_2 - \partial_2 \overline{m}_2) \, \mathrm{d}x_1 \right) \mathrm{d}x_2 \\ &\quad = \int_0^b \left(\partial_v F_\varepsilon(\overline{m}_2, \partial_2 \overline{m}_2) \, \partial_2 \int_0^a (m_2 - \overline{m}_2) \, \mathrm{d}x_1 \right) \mathrm{d}x_2 = 0. \end{aligned} \tag{4.10}$$

This yields

$$\int_D \frac{|\nabla m_2|^2}{1+\varepsilon - m_2^2} \, \mathrm{d}x \geq \int_D \frac{|\nabla \overline{m}_2|^2}{1+\varepsilon - \overline{m}_2^2} \, \mathrm{d}x. \tag{4.11}$$

We now argue by approximation and take $m^\delta \in C^\infty(\mathbb{R} \times [0, b]; \mathbb{S}^1)$ such that $m^\delta \to m$ in $H^1_{loc}(\mathbb{R} \times [0, b]; \mathbb{R}^2)$ as $\delta \to 0$ (Bethuel and Zheng 1988; Bourgain et al. 2000). Then, we have $\overline{m}_2^\delta \in C^\infty(\mathbb{R} \times [0, b])$ as well. Turning to $\overline{m}_1^\delta$ defined in (4.1), observe that $\overline{m}_1^\delta \in C(\mathbb{R} \times [0, b])$. Furthermore, since $\overline{m}_1$ is a composition of a smooth nonnegative function with the square root, we also have that $\overline{m}_1^\delta \in W^{1,\infty}(\mathbb{R} \times (0, b))$. Thus, $\overline{m}^\delta \in H^1_{loc}(\mathbb{R} \times [0, b])$, and by the arguments at the beginning of the proof, we have

$$\int_D |\nabla \overline{m}^\delta|^2 \,\mathrm{d}x = \lim_{\varepsilon \to 0} \int_D \frac{|\nabla \overline{m}_2^\delta|^2}{1 + \varepsilon - |\overline{m}_2^\delta|^2} \,\mathrm{d}x. \tag{4.12}$$

Combining this equality with (4.6) and (4.11), we arrive at (4.2) for m^δ and $\overline{m}^\delta$. Passing to the limit $\delta \to 0$, by lower semicontinuity of $\int_D |\nabla \overline{m}^\delta|^2 \,\mathrm{d}x$ we obtain that $\overline{m}_1 \in H^1_{loc}(\mathbb{R} \times [0, b])$ and (4.2) holds. Furthermore, by construction $|\overline{m}| = 1$, and $\overline{m}$ is independent of x_1, hence, $\overline{m} \in \mathcal{A}^\#$. Finally, if equality holds in (4.2) then we have $\int_D |\partial_1 m_2|^2 \,\mathrm{d}x = 0$, yielding the rest of the claim. □

With a slight abuse of notation, from now we will frequently refer to $\overline{m}$ as a function of one variable, i.e., $\overline{m} = \overline{m}(x_2)$, and extend it by zero for all $x_2 \notin (0, b)$. Similarly, we treat η_δ in (2.3) as a function of one variable, i.e., $\eta_\delta = \eta_\delta(x_2)$, and extended it by zero for all $x_2 \notin (0, b)$ as well.

Lemma 4.2 *Let $m \in \mathcal{A}^\#$. Then,*

$$\begin{aligned} &\int_D \int_{\mathbb{R}\times(0,b)} \frac{\nabla \cdot m_\delta(x) \nabla \cdot m_\delta(y)}{|x - y|} \,\mathrm{d}x \,\mathrm{d}y \\ &\quad \geq a \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\overline{m}_2(x)\eta_\delta(x) - \overline{m}_2(y)\eta_\delta(y))^2}{(x - y)^2} \,\mathrm{d}x \,\mathrm{d}y, \end{aligned} \tag{4.13}$$

where m_δ is defined in (2.2) and $\overline{m}$ is given by (4.1). Moreover, equality holds if and only if $m_2(x) = \overline{m}_2(x)$ for a.e. $x \in D$.

Proof The proof proceeds via passing to Fourier space. For $n \in \mathbb{Z}$ and $\xi \in \mathbb{R}$, we define Fourier coefficients $c(n, \xi) \in \mathbb{R}^2$ as

$$c(n, \xi) := \int_D \mathrm{e}^{-iq(n,\xi)\cdot x} m_\delta(x) \,\mathrm{d}x, \tag{4.14}$$

where $q(n, \xi) := (2\pi a^{-1} n, \xi) \in \mathbb{R}^2$. Then, the inversion formula reads (see, e.g., Milisic and Razafison 2013, Sect. 4):

$$m_\delta(x) = \frac{1}{2\pi a} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{e}^{iq(n,\xi)\cdot x} c(n, \xi) \,\mathrm{d}\xi. \tag{4.15}$$

In terms of $c(n, \xi)$, the left-hand side of (4.13) may be written as

$$\int_D \int_{\mathbb{R}\times(0,b)} \frac{\nabla \cdot m_\delta(x) \nabla \cdot m_\delta(y)}{|x - y|} \,\mathrm{d}x \,\mathrm{d}y$$

$$= \frac{1}{a} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{|q(n,\xi) \cdot c(n,\xi)|^2}{|q(n,\xi)|} \, d\xi. \tag{4.16}$$

Keeping only the $n = 0$ contribution in the right-hand side, we, therefore, have

$$\begin{aligned} &\int_D \int_{\mathbb{R}\times(0,b)} \frac{\nabla \cdot m_\delta(x) \nabla \cdot m_\delta(y)}{|x-y|} \, dx \, dy \\ &\geq \frac{1}{a} \int_{\mathbb{R}} |\xi| \, |c_2(0,\xi)|^2 \, d\xi. \end{aligned} \tag{4.17}$$

Passing back to real space, with the help of the integral formula for the $\mathring{H}^{1/2}(\mathbb{R})$ norm (Di Nezza et al. 2012) we obtain (4.13). Finally, by (4.15) and (4.16) the inequality in (4.13) is strict, unless $m_2 = \overline{m}_2$ almost everywhere. □

Having obtained the above auxiliary results for $\overline{m}$, we now proceed to the proof of our first theorem.

Proof of Theorem 3.1 Let $m \in \mathcal{A}^\#$ be a minimizer of $E^\#$. By Lemmas 4.1 and 4.2, we have $E^\#(m) \geq E^\#(\overline{m})$, where $\overline{m}$ is defined in (4.1). In particular, this inequality is in fact an equality, and by Lemma 4.1, we have $m = m(x_2)$. Moreover, by Lemma 4.2 we have $E^\#(m) = a E^\#_{1d}(m)$, where

$$\begin{aligned} E^\#_{1d}(m) := &\frac{1}{2} \int_0^b \left(|m'|^2 + h|m - e_2|^2 \right) dx \\ &+ \frac{\delta}{8\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m_2(x)\eta_\delta(x) - m_2(y)\eta_\delta(y))^2}{(x-y)^2} \, dx \, dy, \end{aligned} \tag{4.18}$$

with the usual abuse of notation that m and η_δ are treated as functions of one variable in the right-hand side of (4.18), and $m_2\eta_\delta$ has been extended by zero outside $(0, b)$.

We now claim that $m_2(x_2) \geq 0$ for all $x_2 \in (0, b)$. Indeed, taking $\widetilde{m} := (m_1, |m_2|) \in \mathcal{A}^\#$ as a competitor, we have $|\nabla \widetilde{m}| = |\nabla m|$ and

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m_2(x)\eta_\delta(x) - m_2(y)\eta_\delta(y))^2}{(x-y)^2} \, dx \, dy \\ &\geq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\widetilde{m}_2(x)\eta_\delta(x) - \widetilde{m}_2(y)\eta_\delta(y))^2}{(x-y)^2} \, dx \, dy, \end{aligned} \tag{4.19}$$

where the last inequality follows from the fact that the integrand in the right-hand side of (4.19) is pointwise no greater than that in its left-hand side. On the other hand, since $|m - e_2|^2 = 2 - 2m_2$, we have $E^\#_{1d}(m) > E^\#_{1d}(\widetilde{m})$, unless $\widetilde{m}(x_2) = m(x_2)$ for all $x_2 \in (0, b)$.

Now that we established that $m_2 \geq 0$, we may define $\theta(x_2) := -\arcsin m_1(x_2) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so that m satisfies (3.2). Then, we can rewrite the energy of the minimizer as

$$E_{1d}^{\#}(m) = \int_0^b \left(\frac{1}{2}|\theta'|^2 + h(1 - \cos\theta) \right) \mathrm{d}x \\ + \frac{\delta}{8\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\cos\theta(x)\eta_\delta(x) - \cos\theta(y)\eta_\delta(y))^2}{(x-y)^2} \mathrm{d}x\, \mathrm{d}y, \tag{4.20}$$

where in the exchange energy we approximated θ by functions bounded away from $\pm\frac{\pi}{2}$ and passed to the limit with the help of monotone convergence theorem. In particular, from boundedness of the right-hand side of (4.20) it follows that $\theta \in H^1(0, b)$. Therefore, θ satisfies the weak form of (3.4) (for further details, see Capella et al. 2007; Chermisi and Muratov 2013). At the same time, since $\eta_\delta \cos\theta \in H^1(\mathbb{R})$ by weak product and chain rules (Brezis 2011, Corollaries 8.10 and 8.11), and the operator $(-d^2/dx^2)^{1/2}$ is a bounded linear operator from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$, we also have $\theta'' \in L^2(0, b)$, and, hence, $\theta \in C^{1,1/2}([0, b])$. In particular, we can use the formula in (3.3) to compute the nonlocal term in (3.4).

We now apply a bootstrap argument to establish further interior regularity of θ. Note that this result is not immediate, since the function η_δ extended by zero to the whole real line is only Lipschitz continuous. Nevertheless, for every $x \in I$ where $I \Subset (0, b)$ is open we can introduce a partition of unity whereby we have

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} \eta_\delta(x)\cos\theta(x) = \frac{1}{\pi} \fint_{\mathbb{R}} \frac{\eta_\delta(x)\cos\theta(x) - \eta_\delta(y)\cos\theta(y)\chi(y)}{(x-y)^2} \mathrm{d}y \\ - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\eta_\delta(y)\cos\theta(y)(1-\chi(y))}{(x-y)^2} \mathrm{d}y, \tag{4.21}$$

where $\chi \in C_c^\infty(\mathbb{R})$ is such that $\chi \equiv 1$ in I and $\mathrm{supp}(\chi) \subset (0, b)$. Taking the distributional derivative of the right-hand side in (4.21) and using the fact that now $\eta_\delta \chi \cos\theta \in H^2(\mathbb{R})$, we get that the left-hand side of (4.21) is in $H^1(I)$. Applying the bootstrap argument locally, we thus obtain that $\theta \in H^3_{loc}(0, b)$ and, hence, $\theta \in C^\infty(0, b)$, and (3.4) holds classically for all $x \in (0, b)$. Once the latter is established, we obtain the boundary condition $\theta'(0) = \theta'(b) = 0$ via integration by parts.

To establish higher regularity of θ near the boundary, we estimate the nonlocal term, using the fact that $\eta_\delta \in C^\infty([0, b])$ and $\theta' \in C^{1/2}([0, b])$. For $x \in (0, b)$, let $u(x) := \eta_\delta(x)\cos\theta(x)$. Notice that

$$|u(x)| \leq Cx(b - x), \tag{4.22}$$

for some $C > 0$. Focusing on the first term in the right-hand side of (3.3), with the help of Taylor formula we can write for $x \in (0, \frac{1}{2}b)$:

$$\left| \fint_0^b \frac{u(x) - u(y)}{(x-y)^2} \mathrm{d}y \right| \leq \left| \fint_0^{2x} \frac{u(x) - u(y)}{(x-y)^2} \mathrm{d}y \right| + \left| \int_{2x}^b \frac{u(x) - u(y)}{(x-y)^2} \mathrm{d}y \right|$$

$$
\begin{aligned}
&\leq \int_0^{2x} \frac{|u'(\xi_1) - u'(x)|}{|x-y|}\, \mathrm{d}y + \int_{2x}^{b} \frac{|u'(\xi_2)|}{|x-y|}\, \mathrm{d}y \\
&\leq Cx^{1/2} + C \ln(2b/x),
\end{aligned} \tag{4.23}
$$

for some $C > 0$, where $|\xi_1 - x| < |x - y|$ and $\xi_2 \in (x, y)$. Combining this with (4.22) yields

$$
\left| \eta_\delta(x) \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} \eta_\delta(x) \cos\theta(x) \right| \leq Cx(1 + x^{1/2} + \ln x^{-1}), \tag{4.24}
$$

for some $C > 0$ and all x sufficiently small. Thus, the expression in the left-hand side of (4.24) is continuous and vanishes at $x = 0$. By the same argument, the same holds true near $x = b$. Using this fact, from (3.4) we conclude that $\theta \in C^2([0, b])$.

We now prove that there are at most three minimizers of $E^{\#}$ in $\mathcal{A}^{\#}$. Let m be a minimizer associated with $\theta \in H^1(0, b)$. Then, by (4.20) the function $\widetilde{m} \in \mathcal{A}^{\#}$ associated with $\tilde{\theta} = |\theta|$ is also a minimizer. In particular, $\tilde{\theta} \in C^2([0, b])$ and solves (3.4) classically. Now, suppose that there exists a point $x_0 \in [0, b]$ such that $\tilde{\theta}(x_0) = 0$. By regularity of $\tilde{\theta}$ in the interior or homogeneous Neumann boundary conditions, we then also have $\tilde{\theta}'(x_0) = 0$. We now apply a maximum principle- type argument that goes back to Capella et al. (2007), based on the uniqueness of the solution of the initial value problem for (3.4) considered as an ordinary differential equation with the nonlocal term treated as a given function of $x \in [0, b]$:

$$
\begin{aligned}
\theta''(x) = c(x) \sin\theta(x), \quad c(x) &:= h - \frac{\delta}{2} \eta_\delta(x) \\
&\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{1/2} \eta_\delta(x) \cos\theta(x).
\end{aligned} \tag{4.25}
$$

Indeed, by the argument in the preceding paragraph the function $c(x)$ is continuous on $[0, b]$. Therefore, if $\tilde{\theta}(x)$ vanishes for some $x_0 \in [0, b]$, we have $\tilde{\theta} \equiv 0$ on $[0, b]$. Alternatively, $\tilde{\theta} > 0$ for all $x \in [0, b]$, which means that θ does not change sign.

To conclude the proof of the multiplicity of the minimizers, observe that in view of the above we need to show that there is at most one minimizer $\theta \in (0, \frac{\pi}{2}]$ of the right-hand side of (4.20). In this case, we can rewrite the energy in terms of $m_2 < 1$:

$$
\begin{aligned}
E^{\#}_{1d}(m) = \frac{1}{2} \int_0^b &\left(\frac{|m_2'|^2}{1 - m_2^2} + 2h(1 - m_2) \right) \mathrm{d}x \\
&+ \frac{\delta}{8\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m_2(x)\eta_\delta(x) - m_2(y)\eta_\delta(y))^2}{(x-y)^2}\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned} \tag{4.26}
$$

By inspection, this energy is convex. Furthermore, the last term in (4.26) is strictly convex in view of the fact that $m_2\eta_\delta$ vanishes identically outside $(0, b)$. Thus, there is at most one minimizer with $m_1 > 0$. If such a minimizer exists, then by reflection symmetry the function $\widetilde{m} := (-m_1, m_2)$ is also a minimizer, which is the only minimizer with $\widetilde{m}_1 < 0$. Finally, the symmetry of the minimizer with respect to reflections $x_2 \to b - x_2$ follows from the invariance of the energy in (4.26) with respect to such reflections. □

5 Proof of Theorems 3.2 and 3.3

In view of the result in Theorem 3.1, it suffices to consider the minimizers of a suitably rescaled version of the one-dimensional energy in (4.26) when $m_2 < 1$:

$$E^{\#}_{\varepsilon,1d}(m) := \frac{1}{2}\int_0^b \left(\frac{\varepsilon|m_2'|^2}{1-m_2^2} + \frac{2h_\varepsilon}{\varepsilon}(1-m_2)\right) \mathrm{d}x + \frac{\delta_\varepsilon}{8\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(m_2(x)\eta_{\varepsilon\delta_\varepsilon}(x) - m_2(y)\eta_{\varepsilon\delta_\varepsilon}(y))^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.1}$$

Let us also define a rescaled version of this energy, up to an additive constant:

$$F_{\varepsilon,1d}(m) := \frac{1}{2}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b} \left(\frac{|m_2'|^2}{1-m_2^2} + 2\beta(1-m_2)\right) \mathrm{d}x - \frac{\lambda|\ln\varepsilon|}{2\pi\ln|\ln\varepsilon|} + \frac{\lambda}{8\pi\ln|\ln\varepsilon|}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(m_2(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x) - m_2(y)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(y))^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y, \tag{5.2}$$

where $\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x) := \eta(L_\varepsilon \min(x, \varepsilon^{-1}L_\varepsilon^{-1}b - x)/\delta_\varepsilon)$. Using these definitions, we have

$$E^{\#}_{\varepsilon,1d}(m) = \frac{\lambda}{2\pi} + \frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|} F_{\varepsilon,1d}(m(\cdot/(\varepsilon L_\varepsilon))). \tag{5.3}$$

With these notations, proving Theorem 3.2 is equivalent to showing that $\min F_{\varepsilon,1d}(m(\cdot/(\varepsilon L_\varepsilon)))$ converges to $2F_0(n_0)$ as $\varepsilon \to 0$, where the minimization is done over

$$\mathcal{A}^{1d}_\varepsilon := H^1\big((0, \varepsilon^{-1}L_\varepsilon^{-1}b); \mathbb{S}^1\big). \tag{5.4}$$

Below, we show that this is indeed the case by establishing the matching upper and lower bounds for $\min F_{\varepsilon,1d}$.

To proceed, we separate the energy $F_{\varepsilon,1d}$ into the local and the nonlocal parts:

$$F_{\varepsilon,1d}(m) = F^{MM}_{\varepsilon,1d}(m) + F^{S}_{\varepsilon,1d}(m), \tag{5.5}$$

where

$$F^{MM}_{\varepsilon,1d}(m) := \frac{1}{2}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b} \left(\frac{|m_2'|^2}{1-m_2^2} + 2\beta(1-m_2)\right) \mathrm{d}x \tag{5.6}$$

is the Modica–Mortola-type energy and

$$F^S_{\varepsilon,1d}(m) := \frac{\lambda}{8\pi \ln|\ln\varepsilon|}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\big(m_2(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x) - m_2(y)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(y)\big)^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda|\ln\varepsilon|}{2\pi \ln|\ln\varepsilon|} \tag{5.7}$$

is the stray field energy, up to an additive constant. Note that using the standard Modica–Mortola trick (Modica 1987), one obtains a lower bound for $F^{MM}_{\varepsilon,1d}$.

Lemma 5.1 *Let $m = (m_1, m_2) \in \mathcal{A}^{1d}_\varepsilon$ with $0 \le m_2 < 1$. Then, for every $R \in (0, \varepsilon^{-1}L_\varepsilon^{-1}b/2]$ and every $r \in [0, R]$ we have*

$$F^{MM}_{\varepsilon,1d}(m) \ge 4\sqrt{\beta}\left(1-\sqrt{\frac{1+m_2(r)}{2}}\right) + 4\sqrt{\beta}\left(1-\sqrt{\frac{1+m_2\big(\varepsilon^{-1}L_\varepsilon^{-1}b-r\big)}{2}}\right) - 8\sqrt{\beta}\left(1-\sqrt{\frac{1+m_2(R)}{2}}\right). \tag{5.8}$$

In order to obtain the upper and lower bounds on the stray field energy, we prove the following lemma that offers two characterizations of the one-dimensional fractional homogeneous Sobolev norm. Here, by $\mathring{H}^1(\mathbb{R}^2)$ we understand the space of functions in $L^2_{loc}(\mathbb{R}^2)$ whose distributional gradient is in $L^2(\mathbb{R}^2;\mathbb{R}^2)$.

Lemma 5.2 *Let $u \in H^1(\mathbb{R})$ and have compact support. Then,*

(i)

$$\frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(u(x)-u(y))^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \ln|x-y|^{-1}u'(x)u'(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{5.9}$$

(ii)

$$\frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(u(x)-u(y))^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y = -\min_{v\in\mathring{H}^1(\mathbb{R}^2)}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\nabla v(x,z)|^2\,\mathrm{d}x\,\mathrm{d}z + 2\int_{\mathbb{R}} v(x,0)u'(x)\,\mathrm{d}x\right). \tag{5.10}$$

Proof For the proof of (5.9), we refer to the Appendix in Lund et al. (2018). To obtain (5.10), we first note that the minimum in the right-hand side of (5.10) is attained.

Indeed, considering the elements of the homogeneous Sobolev space $\mathring{H}^1(\mathbb{R}^2)$ as equivalence classes of functions modulo additive constants makes this space into a Banach space (Ortner and Süli 2012), and by coercivity and strict convexity of the expression in the brackets, we hence get existence of a unique minimizer (up to an additive constant). Note that the integrals in the right-hand side of (5.10) are unchanged when an arbitrary constant is added to v, and that $v(\cdot, 0) \in L^2_{loc}(\mathbb{R})$ is well defined as the trace of a Sobolev function.

The minimizer $v_0 \in \mathring{H}^1(\mathbb{R}^2)$ of the expression in the right-hand side of (5.10) solves the following Poisson-type equation:

$$\Delta v_0 = u'(x)\delta(z) \qquad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{5.11}$$

where $\delta(\cdot)$ is the one-dimensional Dirac delta function. Therefore, v_0 is easily seen to be (again, up to an additive constant)

$$v_0(x, z) = \frac{1}{2\pi}\int_{\mathbb{R}} u'(y) \ln\sqrt{(x-y)^2 + z^2}\, \mathrm{d}y. \tag{5.12}$$

In particular, since u' has compact support and, therefore, integrates to zero over $\mathbb{R}$, we have an estimate for the function v_0 in (5.12):

$$|v_0(x, z)| \le \frac{C}{\sqrt{x^2+z^2}} \qquad |\nabla v_0(x, z)| \le \frac{C}{x^2+z^2}, \tag{5.13}$$

for some $C > 0$ and all $x^2 + z^2$ large enough. Furthermore, it is not difficult to see that $v_0 \in C^{1/2}(\mathbb{R}^2)$:

$$|v(x_1, z_1) - v(x_2, z_2)|^2 \le \frac{1}{16\pi^2}\int_{\mathbb{R}} |u'(y)|^2\, \mathrm{d}y \int_{\mathbb{R}} \ln^2\left\{\frac{(y-x_1)^2 + z_1^2}{(y-x_2)^2 + z_2^2}\right\} \mathrm{d}y, \tag{5.14}$$

where we used Cauchy–Schwarz inequality, and the last integral may be dominated by $C(|x_1 - x_2| + |z_1 - z_2|)$ for some universal $C > 0$.

We now multiply both parts of (5.11) by v_0 and integrate over $\mathbb{R}^2$. After integrating by parts and taking into account (5.13), we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla v_0(x, z)|^2\, \mathrm{d}x\, \mathrm{d}z = -\int_{\mathbb{R}} v_0(x, 0)u'(x)\, \mathrm{d}x. \tag{5.15}$$

From this, we get

$$\min_{v\in \mathring{H}^1(\mathbb{R}^2)} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla v(x, z)|^2\, \mathrm{d}x\, \mathrm{d}z + 2\int_{\mathbb{R}} v(x, 0)u'(x)\, \mathrm{d}x\right) = \int_{\mathbb{R}} v_0(x, 0)u'(x)\, \mathrm{d}x. \tag{5.16}$$

Finally, combining (5.9), (5.12) and (5.16), we obtain (5.10). □

Using the definition of $F^S_{\varepsilon,1d}(m)$ and Lemma 5.2, we arrive at the following lower bound for the stray field energy.

Lemma 5.3 *Let $m \in \mathcal{A}^{1d}_\varepsilon$. Then,*

$$F^S_{\varepsilon,1d}(m) \geq -\frac{\lambda|\ln\varepsilon|}{2\pi\ln|\ln\varepsilon|} - \frac{\lambda}{2\ln|\ln\varepsilon|}\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla v(x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z$$
$$-\frac{\lambda}{\ln|\ln\varepsilon|}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b} v(x,0)\left(m_2(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x, \tag{5.17}$$

for every $v \in \mathring{H}^1(\mathbb{R}^2)$, where $v(\cdot,0)$ is understood in the sense of trace.

We will also find useful the following basic upper bound for the minimum energy.

Lemma 5.4 *There exists $C > 0$ such that*

$$\min_{m\in\mathcal{A}^{1d}_\varepsilon} F_{\varepsilon,1d}(m) \leq C, \tag{5.18}$$

for all ε sufficiently small. Furthermore, if $F_{\varepsilon,1d}$ is minimized by $m = e_2$, then the reverse inequality also holds.

Proof The proof is obtained by testing the energy with $m = e_2$. Then, $F^{MM}_{\varepsilon,1d}(m) = 0$, and by Lemma 5.2, we have

$$\begin{aligned}
&4\pi\lambda^{-1}\ln|\ln\varepsilon|F^S_{\varepsilon,1d}(m) + 2|\ln\varepsilon| \\
&= \int_0^{\frac{b}{\varepsilon L_\varepsilon}}\int_0^{\frac{b}{\varepsilon L_\varepsilon}} \ln|x-y|^{-1}\widetilde{\eta}'_{\delta_\varepsilon/L_\varepsilon}(x)\widetilde{\eta}'_{\delta_\varepsilon/L_\varepsilon}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_0^{\frac{b}{\varepsilon\delta_\varepsilon}}\int_0^{\frac{b}{\varepsilon\delta_\varepsilon}} \ln|x-y|^{-1}\eta'(x)\eta'(y)\,\mathrm{d}x\,\mathrm{d}y \\
&- 2\int_0^{\frac{b}{\varepsilon\delta_\varepsilon}}\int_0^{\frac{b}{\varepsilon\delta_\varepsilon}} \ln|x-y|^{-1}\eta'(x)\eta'(\varepsilon^{-1}\delta_\varepsilon^{-1}b-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq C + 2\ln(\varepsilon^{-1}\delta_\varepsilon^{-1}b) \leq 2|\ln\varepsilon| + 2\ln|\ln\varepsilon| + C',
\end{aligned} \tag{5.19}$$

for some $C, C' > 0$ and all $\varepsilon \ll 1$, where we took into account (2.11). This inequality is equivalent to (5.18). Finally, if $m = e_2$ is the minimizer, the matching asymptotic lower bound then follows. □

Proof of Theorem 3.2 Let m_ε be a minimizer of $F_{\varepsilon,1d}$ over $\mathcal{A}^{1d}_\varepsilon$. Note that in view of Lemma 5.4 and Theorem 3.1 we may assume that $m_{2,\varepsilon} < 1$. With the help of the rescalings introduced earlier, proving Theorem 3.2 amounts to establishing that

$$2F_0(n_0) \leq \liminf_{\varepsilon\to 0} F_{\varepsilon,1d}(m_\varepsilon) \leq \limsup_{\varepsilon\to 0} F_{\varepsilon,1d}(m_\varepsilon) \leq 2F_0(n_0), \tag{5.20}$$

where $n_0 \in [0,1]$ is the minimizer of F_0 from Lemma 3.1. The proof proceeds in four steps.

Step 1 Construction of a test potential We first establish the liminf inequality in (5.20). Focusing on the stray field energy, we use Lemma 5.3 with the test function $v \in \mathring{H}^1(\mathbb{R}^2)$ constructed as follows. For $n_\varepsilon := m_{\varepsilon,2}(\delta_\varepsilon/L_\varepsilon)$, define

$$v_1(\rho) := \begin{cases} -\frac{n_\varepsilon}{2\pi} \ln\left(\frac{b}{2\varepsilon\delta_\varepsilon}\right), & 0 \le \rho \le \delta_\varepsilon/L_\varepsilon, \\ -\frac{n_\varepsilon}{2\pi} \ln\left(\frac{b}{2\varepsilon L_\varepsilon \rho}\right), & \delta_\varepsilon/L_\varepsilon \le \rho \le b/(2\varepsilon L_\varepsilon), \\ 0, & \rho \ge b/(2\varepsilon L_\varepsilon), \end{cases} \tag{5.21}$$

and

$$v_2(\rho) := \begin{cases} \frac{n_\varepsilon - 1}{2\pi} \ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right), & 0 \le \rho \le 1, \\ \frac{n_\varepsilon - 1}{2\pi} \ln\left(\frac{b}{2\varepsilon L_\varepsilon \rho}\right), & 1 \le \rho \le b/(2\varepsilon L_\varepsilon), \\ 0, & \rho \ge b/(2\varepsilon L_\varepsilon), \end{cases} \tag{5.22}$$

We then define, for all $(x, z) \in \mathbb{R}^2$, the test potential

$$\begin{aligned} v(x,z) &:= v_1\left(\sqrt{x^2+z^2}\right) + v_2\left(\sqrt{x^2+z^2}\right) \\ &-v_1\left(\sqrt{(\varepsilon^{-1}L_\varepsilon^{-1}b - x)^2 + z^2}\right) - v_2\left(\sqrt{(\varepsilon^{-1}L_\varepsilon^{-1}b - x)^2 + z^2}\right). \end{aligned} \tag{5.23}$$

Clearly, v is admissible. Furthermore, in view of the symmetry of m_ε guaranteed by Theorem 3.1 we have

$$\begin{aligned} &\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b} v(x,0)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)' \,\mathrm{d}x \\ &\quad = 2\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v(x,0)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)' \,\mathrm{d}x. \end{aligned} \tag{5.24}$$

Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla v(x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z &= 2\int_{\mathbb{R}}\int_{-\infty}^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} |\nabla v(x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z \\ &= 4\pi \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} |\nabla v_1(\rho) + \nabla v_2(\rho)|^2 \rho \,\mathrm{d}\rho. \end{aligned} \tag{5.25}$$

Carrying out the integration in polar coordinates yields

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla v(x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z &= 4\pi \int_{\delta_\varepsilon/L_\varepsilon}^1 |\nabla v_1(\rho)|^2 \rho \,\mathrm{d}\rho \\ &\quad + 4\pi \int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} |\nabla v_1(\rho) + \nabla v_2(\rho)|^2 \rho \,\mathrm{d}\rho \\ &= \frac{n_\varepsilon^2}{\pi} \ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right) + \frac{1}{\pi} \ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right). \end{aligned} \tag{5.26}$$

Step 2 Computation of the potential energy We now write, using the definition of the potential v in (5.23):

$$\begin{aligned}&\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v(x,0)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x\\ &\quad=\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_1(x)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x\\ &\qquad+\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_2(x)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x.\end{aligned}\tag{5.27}$$

With the help of the definition of v_1 in (5.21), we have for the first term in the right-hand side of (5.27):

$$\begin{aligned}&\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_1(x)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x\\ &=v_1(0)n_\varepsilon+\int_{\delta_\varepsilon/L_\varepsilon}^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_1(x)m'_{\varepsilon,2}(x)\,\mathrm{d}x\\ &=v_1(0)n_\varepsilon+\int_{\delta_\varepsilon/L_\varepsilon}^{1} v_1(x)m'_{\varepsilon,2}(x)\,\mathrm{d}x+\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_1(x)m'_{\varepsilon,2}(x)\,\mathrm{d}x\\ &=(v_1(0)-v_1(1))n_\varepsilon+v_1(1)m_{\varepsilon,2}(1)+\int_{\delta_\varepsilon/L_\varepsilon}^{1}(v_1(x)-v_1(1))m'_{\varepsilon,2}(x)\,\mathrm{d}x\\ &\quad+\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_1(x)m'_{\varepsilon,2}(x)\,\mathrm{d}x\\ &=(v_1(0)-v_1(1))n_\varepsilon+v_1(1)+\int_{\delta_\varepsilon/L_\varepsilon}^{1}(v_1(x)-v_1(1))m'_{\varepsilon,2}(x)\,\mathrm{d}x\\ &\quad+\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v'_1(x)(1-m_{\varepsilon,2}(x))\,\mathrm{d}x,\end{aligned}\tag{5.28}$$

where in the last line, we used integration by parts. Similarly, with the help of the definition of v_2 in (5.22) we have for the second term in the right-hand side of (5.27):

$$\begin{aligned}&\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_2(x)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x\\ &\quad=v_2(1)m_{\varepsilon,2}(1)+\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v_2(x)m'_{\varepsilon,2}(x)\,\mathrm{d}x\\ &\quad=v_2(1)+\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v'_2(x)(1-m_{\varepsilon,2}(x))\,\mathrm{d}x,\end{aligned}\tag{5.29}$$

again, using integration by parts. Combining the two formulas above yields

$$
\begin{aligned}
&\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v(x,0)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x\\
&= v_1(1)+v_2(1)+(v_1(0)-v_1(1))n_\varepsilon\\
&\quad+\int_{\delta_\varepsilon/L_\varepsilon}^1 (v_1(x)-v_1(1))m_{\varepsilon,2}'(x)\,\mathrm{d}x\\
&\quad+\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} (v_1'(x)+v_2'(x))(1-m_{\varepsilon,2}(x))\,\mathrm{d}x. \qquad (5.30)
\end{aligned}
$$

Finally, recalling the precise definitions of v_1 and v_2, we obtain

$$
\begin{aligned}
&\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v(x,0)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x\\
&= -\frac{1}{2\pi}\ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right)-\frac{n_\varepsilon^2}{2\pi}\ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)\\
&\quad+\frac{n_\varepsilon}{2\pi}\int_{\delta_\varepsilon/L_\varepsilon}^1 m_{\varepsilon,2}'(x)\ln x\,\mathrm{d}x\\
&\quad+\frac{1}{2\pi}\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2}\frac{1-m_{\varepsilon,2}(x)}{x}\,\mathrm{d}x. \qquad (5.31)
\end{aligned}
$$

Step 3 Lower bound We now estimate the left-hand side of (5.31), using Young's inequality:

$$
\begin{aligned}
&\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v(x,0)\left(m_{\varepsilon,2}(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\,\mathrm{d}x\\
&\le -\frac{1}{2\pi}\ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right)-\frac{n_\varepsilon^2}{2\pi}\ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)\\
&\quad+\frac{1}{4\pi}\int_{\delta_\varepsilon/L_\varepsilon}^1\left(\ln^2 x+|m_{2,\varepsilon}'(x)|^2\right)\mathrm{d}x\\
&\quad+\frac{1}{2\pi}\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2}(1-m_{\varepsilon,2}(x))\,\mathrm{d}x\\
&\le -\frac{1}{2\pi}\ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right)-\frac{n_\varepsilon^2}{2\pi}\ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)\\
&\quad+C\left(1+F_{\varepsilon,1d}^{MM}(m_\varepsilon)\right), \qquad (5.32)
\end{aligned}
$$

for some $C>0$ independent of $\varepsilon\ll 1$. Thus, according to Lemma 5.3 and (5.26), we have

$$
F_{\varepsilon,1d}^{S}(m_\varepsilon)\ge-\frac{\lambda|\ln\varepsilon|}{2\pi\ln|\ln\varepsilon|}+\frac{\lambda}{2\pi\ln|\ln\varepsilon|}
$$

$$\times \ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right) + \frac{\lambda n_\varepsilon^2}{2\pi \ln|\ln\varepsilon|} \ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)$$
$$- \frac{C}{\ln|\ln\varepsilon|}\left(1 + F^{MM}_{\varepsilon,1d}(m_\varepsilon)\right), \tag{5.33}$$

again, for some $C > 0$ independent of $\varepsilon \ll 1$. Recalling (2.11) and (3.7), this translates into

$$F^S_{\varepsilon,1d}(m_\varepsilon) \geq -\frac{\lambda}{2\pi} + \frac{\lambda n_\varepsilon^2}{\pi} - \frac{C}{\ln|\ln\varepsilon|}\left(1 + F^{MM}_{\varepsilon,1d}(m_\varepsilon)\right). \tag{5.34}$$

Therefore, for any $\alpha \in (0, \frac{1}{2})$ and all ε small enough we can write

$$\begin{aligned} F_{\varepsilon,1d}(m_\varepsilon) &= F^{MM}_{\varepsilon,1d}(m_\varepsilon) + F^S_{\varepsilon,1d}(m_\varepsilon) \\ &\geq (1-\alpha)\left[F^{MM}_{\varepsilon,1d}(m_\varepsilon) + \frac{\lambda}{2\pi}(2n_\varepsilon^2 - 1)\right] - C\alpha, \end{aligned} \tag{5.35}$$

for some $C > 0$ independent of ε and α.

Now, applying Lemma 5.1 we arrive at

$$F_{\varepsilon,1d}(m_\varepsilon) \geq 2(1-\alpha)F_0(n_\varepsilon) - 8\sqrt{\beta}\left(1 - \sqrt{\frac{1 + m_{2,\varepsilon}(R)}{2}}\right) - C\alpha, \tag{5.36}$$

for any $R \in (0, \varepsilon^{-1}L_\varepsilon^{-1}b/2]$ and $C > 0$ independent of $\varepsilon \ll 1$, α and R. At the same time, using Lemma 5.4 and (5.36) we obtain

$$\beta \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b} (1 - m_{2,\varepsilon})\,\mathrm{d}x \leq F^{MM}_{\varepsilon,1d}(m_\varepsilon) \leq C, \tag{5.37}$$

for some $C > 0$ and all $\varepsilon \ll 1$. Therefore, there exists $R_\varepsilon \in [\varepsilon^{-1}L_\varepsilon^{-1}b/4, \varepsilon^{-1}L_\varepsilon^{-1}b/2]$ such that choosing $R = R_\varepsilon$ we have $m_{\varepsilon,2}(R) \geq 1 - 4C\varepsilon L_\varepsilon/(\beta b)$, so that the next-to-last term in (5.36) can be absorbed into the last term. Thus, we have

$$F_{\varepsilon,1d}(m_\varepsilon) \geq 2(1-\alpha) \min_{n\in[0,1]} F_0(n) - C\alpha, \tag{5.38}$$

for some $C > 0$ independent of α and ε, for all ε small enough, and the liminf inequality follows by sending $\alpha \to 0$.

Step 4 Upper bound Finally, we derive an asymptotically matching upper bound for the energy. We use the truncated optimal Modica–Mortola profile at the edges as a test configuration. More precisely, for $K > 1$ and ε sufficiently small, we define $m \in \mathcal{A}_\varepsilon^{1d}$ satisfying (3.2) with $\theta(x) = \bar{\theta}(\min(x, b/(\varepsilon L_\varepsilon) - x))$, where

$$\bar{\theta}(x) := \begin{cases} \theta_0, & 0 \le x \le \frac{\delta_\varepsilon}{L_\varepsilon}, \\ 4\arctan\left(\mathrm{e}^{-2\sqrt{\beta}\left(x - \frac{\delta_\varepsilon}{L_\varepsilon}\right)} \tan\frac{\theta_0}{4}\right), & \frac{\delta_\varepsilon}{L_\varepsilon} \le x \le K + \frac{\delta_\varepsilon}{L_\varepsilon}, \\ 4\arctan\left(\mathrm{e}^{-2K\sqrt{\beta}} \tan\frac{\theta_0}{4}\right)\left[1 - \eta\left(\frac{x}{K} - 1 - \frac{\delta_\varepsilon}{KL_\varepsilon}\right)\right], & K + \frac{\delta_\varepsilon}{L_\varepsilon} \le x \le 2K + \frac{\delta_\varepsilon}{L_\varepsilon}, \\ 0, & 2K + \frac{\delta_\varepsilon}{L_\varepsilon} \le x \le \frac{b}{2\varepsilon L_\varepsilon}, \end{cases} \tag{5.39}$$

and $\theta_0 = \arccos n_0$, where n_0 is the unique minimizer of $F_0(n)$ in Lemma 3.1. By the argument leading to the case of equality in (3.11), we obtain

$$F_{\varepsilon,1d}^{MM}(m) = 8\sqrt{\beta}\left(1 - \sqrt{\frac{1+n_0}{2}}\right) - 8\sqrt{\beta}\left(1 - \sqrt{\frac{1 + \cos\left(\bar{\theta}\left(K + \frac{\delta_\varepsilon}{L_\varepsilon}\right)\right)}{2}}\right) + 2\beta(1-n_0)\frac{\delta_\varepsilon}{L_\varepsilon} + \int_{\frac{\delta_\varepsilon}{L_\varepsilon}}^{K+\frac{\delta_\varepsilon}{L_\varepsilon}} \left(|\bar{\theta}'|^2 + 2\beta(1-\cos\bar{\theta})\right) \mathrm{d}x. \tag{5.40}$$

Thus, we have

$$F_{\varepsilon,1d}^{MM}(m) \le 8\sqrt{\beta}\left(1 - \sqrt{\frac{1+n_0}{2}}\right) + \frac{C\ln|\ln\varepsilon|}{|\ln\varepsilon|^2} + CKe^{-4K\sqrt{\beta}}, \tag{5.41}$$

for some $C > 0$ independent of ε and K and all ε sufficiently small.

Turning now to the stray field energy, with the help of Lemma 5.2 we can write

$$\begin{aligned} F_{\varepsilon,1d}^{S}(m) = &-\frac{\lambda|\ln\varepsilon|}{2\pi\ln|\ln\varepsilon|} \\ &+ \frac{\lambda}{4\pi\ln|\ln\varepsilon|}\int_0^{\frac{b}{\varepsilon L_\varepsilon}}\int_0^{\frac{b}{\varepsilon L_\varepsilon}} \ln\frac{|x-y|^{-1}}{\varepsilon L_\varepsilon} \\ &\times \left(m_2(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\left(m_2(y)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(y)\right)' \,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{5.42}$$

where we took into account that inserting a constant factor to the argument of the logarithm does not change the stray field energy. With the help of the definition of m, this is equivalent to

$$\begin{aligned} F_{\varepsilon,1d}^{S}(m) = &-\frac{\lambda|\ln\varepsilon|}{2\pi\ln|\ln\varepsilon|} \\ &+ \frac{\lambda}{2\pi\ln|\ln\varepsilon|}\int_0^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}}\int_0^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \ln\frac{|x-y|^{-1}}{\varepsilon L_\varepsilon} \\ &\times \left(m_2(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)'\left(m_2(y)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(y)\right)' \,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

$$
+ \frac{\lambda}{2\pi \ln|\ln\varepsilon|} \int_0^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \int_{\frac{b}{\varepsilon L_\varepsilon}-2K-\frac{\delta_\varepsilon}{L_\varepsilon}}^{\frac{b}{\varepsilon L_\varepsilon}} \ln \frac{|x-y|^{-1}}{\varepsilon L_\varepsilon} \times \left(m_2(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)' \left(m_2(y)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(y)\right)' \,\mathrm{d}x\,\mathrm{d}y. \tag{5.43}
$$

Observe that the integral in the last line of (5.43) is bounded above by a constant independent of ε and K for all ε sufficiently small. Therefore, we now concentrate on estimating the remaining terms in (5.43).

We can write the integral in the second line in (5.43) as follows:

$$
\begin{aligned}
&\int_0^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \int_0^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \ln \frac{|x-y|^{-1}}{\varepsilon L_\varepsilon} \left(m_2(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)\right)' \left(m_2(y)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(y)\right)' \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = n_0^2 \int_0^{\frac{\delta_\varepsilon}{L_\varepsilon}} \int_0^{\frac{\delta_\varepsilon}{L_\varepsilon}} \ln \frac{|x-y|^{-1}}{\varepsilon L_\varepsilon} \widetilde{\eta}'_{\delta_\varepsilon/L_\varepsilon}(x)\widetilde{\eta}'_{\delta_\varepsilon/L_\varepsilon}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2n_0 \int_{\frac{\delta_\varepsilon}{L_\varepsilon}}^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \int_0^{\frac{\delta_\varepsilon}{L_\varepsilon}} \ln \frac{|x-y|^{-1}}{\varepsilon L_\varepsilon} \widetilde{\eta}'_{\delta_\varepsilon/L_\varepsilon}(x) m_2'(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\frac{\delta_\varepsilon}{L_\varepsilon}}^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \int_{\frac{\delta_\varepsilon}{L_\varepsilon}}^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \ln \frac{|x-y|^{-1}}{\varepsilon L_\varepsilon} m_2'(x)m_2'(y)\,\mathrm{d}x\,\mathrm{d}y =: I_1 + I_2 + I_3.
\end{aligned} \tag{5.44}
$$

For the first integral, we have

$$
I_1 = n_0^2 \ln \frac{1}{\varepsilon\delta_\varepsilon} + n_0^2 \int_0^1 \int_0^1 \ln|x-y|^{-1}\eta'(x)\eta'(y)\,\mathrm{d}x\,\mathrm{d}y \le n_0^2 \ln\frac{1}{\varepsilon\delta_\varepsilon} + C, \tag{5.45}
$$

for some $C > 0$ independent of ε and K and all ε sufficiently small. At the same time, noting that $m_2'(x + \delta_\varepsilon L_\varepsilon^{-1}) \ge 0$ for all $0 < x < 2K$, we get

$$
\begin{aligned}
I_2 &\le 2n_0(1-n_0)\ln\frac{1}{\varepsilon L_\varepsilon} + 2n_0 \int_0^{2K} \ln|y|^{-1} m_2'(y+\delta_\varepsilon L_\varepsilon^{-1})\,\mathrm{d}y \\
&\le 2n_0(1-n_0)\ln\frac{1}{\varepsilon L_\varepsilon} + C,
\end{aligned} \tag{5.46}
$$

again, for some $C > 0$ independent of ε and K and all ε sufficiently small. Finally, for the third integral we have

$$
\begin{aligned}
I_3 &= (1-n_0)^2 \ln\frac{1}{\varepsilon L_\varepsilon} \\
&\quad + \int_{\frac{\delta_\varepsilon}{L_\varepsilon}}^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \int_{\frac{\delta_\varepsilon}{L_\varepsilon}}^{2K+\frac{\delta_\varepsilon}{L_\varepsilon}} \ln|x-y|^{-1} m_2'(x)m_2'(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \le (1-n_0)^2 \ln\frac{1}{\varepsilon L_\varepsilon} + CK,
\end{aligned} \tag{5.47}
$$

once again, for some $C > 0$ independent of ε and K and all ε sufficiently small.

We now put all the obtained estimates together:

$$I_1 + I_2 + I_3 \le \ln \frac{1}{\varepsilon L_\varepsilon} + n_0^2 \ln \frac{L_\varepsilon}{\delta_\varepsilon} + CK. \tag{5.48}$$

Then, recalling the definitions in (2.11) and (3.7) and combining the estimate in (5.48) with the one in (5.41), we arrive at

$$\begin{aligned} F_{\varepsilon,1d}(m) \le\ & 8\sqrt{\beta}\left(1 - \sqrt{\frac{1+n_0}{2}}\right) + \frac{\lambda}{2\pi}(2n_0^2 - 1) \\ & + \frac{C \ln|\ln\varepsilon|}{|\ln\varepsilon|^2} + CKe^{-4K\sqrt{\beta}} + \frac{CK}{\ln|\ln\varepsilon|}, \end{aligned} \tag{5.49}$$

for some $C > 0$ independent of ε and K and all ε sufficiently small. Taking the limsup as $\varepsilon \to 0$, therefore, yields

$$\limsup_{\varepsilon \to 0} F_{\varepsilon,1d}(m) \le 2F_0(n_0) + CKe^{-4K\sqrt{\beta}}. \tag{5.50}$$

Finally, the result follows by sending $K \to \infty$. □

Proof of Theorem 3.3 As in the proof of Theorem 3.2, we consider minimizers m_ε of $F_{\varepsilon,1d}$ and write $F^{MM}_{\varepsilon,1d}(m_\varepsilon)$ in the form:

$$F^{MM}_{\varepsilon,1d}(m_\varepsilon) = \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b} \left(\frac{1}{2}|\theta_\varepsilon'|^2 + \beta(1 - \cos\theta_\varepsilon)\right) \mathrm{d}x. \tag{5.51}$$

Also, without loss of generality we may assume that $\theta_\varepsilon \ge 0$. Then, with the help of the estimate in (5.34) we can write

$$F_{\varepsilon,1d}(m_\varepsilon) \ge \frac{1}{2} F^{MM}_{\varepsilon,1d}(m_\varepsilon) - C, \tag{5.52}$$

for some $C > 0$ independent of $\varepsilon \ll 1$. On the other hand, by (5.20) we know that the left-hand side of (5.52) is bounded independently of $\varepsilon \ll 1$, which, in turn, implies that

$$\|\theta_\varepsilon'\|_{L^2(0,\varepsilon^{-1}L_\varepsilon^{-1}b)} \le C. \tag{5.53}$$

Now, pick a sequence of $\varepsilon_k \to 0$ as $k \to \infty$. Then, up to a subsequence (not relabeled) we have $\theta_{\varepsilon_k} \rightharpoonup \bar\theta$ in $H^1_{loc}(\overline{\mathbb{R}}^+)$ and locally uniformly by the estimate in (5.53). At the same time, using (5.35) and the Modica–Mortola trick (Modica 1987), we have

$$F_{\varepsilon_k,1d}(m_{\varepsilon_k}) \ge (1-\alpha)\Bigg[4\sqrt{\beta}\int_0^{\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b/2} \sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)|\theta_{\varepsilon_k}'|\,\mathrm{d}x$$

$$+\int_0^{\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b/2}\left[|\theta'_{\varepsilon_k}|-2\sqrt{\beta}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)\right]^2\mathrm{d}x+\frac{\lambda}{2\pi}(2n_{\varepsilon_k}^2-1)\Bigg]-C\alpha, \tag{5.54}$$

for some $C>0$ and any $\alpha\in(0,\frac{1}{2})$, for all k sufficiently large. Here, we used the reflection symmetry of the minimizers and defined $n_{\varepsilon_k}:=m_{\varepsilon_k,2}(\delta_{\varepsilon_k}/L_{\varepsilon_k})$. As in the proof of Theorem 3.2, we can find $R_k\in(\alpha^{-3},2\alpha^{-3})$ such that $\theta_{\varepsilon_k}(R_k)<\alpha$ for all α sufficiently small. Then, from (5.54) we obtain

$$\begin{aligned}F_{\varepsilon_k,1d}(m_{\varepsilon_k})\geq(1-\alpha)\Bigg[&-4\sqrt{\beta}\int_0^{R_k}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)\theta'_{\varepsilon_k}\,\mathrm{d}x\\&+4\sqrt{\beta}\int_{R_k}^{\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b/2}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)|\theta'_{\varepsilon_k}|\,\mathrm{d}x\\&+\int_0^{R_k}\left[\theta'_{\varepsilon_k}+2\sqrt{\beta}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)\right]^2\mathrm{d}x+\frac{\lambda}{2\pi}(2n_{\varepsilon_k}^2-1)\Bigg]-C\alpha\\=(1-\alpha)\Bigg[&2F_0[n_{\varepsilon_k}]-8\sqrt{\beta}\left(1+\sqrt{\frac{1+\cos\theta_{\varepsilon_k}(R_k)}{2}}\right)\\&+8\sqrt{\beta}\left(1+\sqrt{\frac{1+\cos\theta_{\varepsilon_k}(0)}{2}}\right)\\&-8\sqrt{\beta}\left(1+\sqrt{\frac{1+\cos\theta_{\varepsilon_k}(\delta_{\varepsilon_k}/L_{\varepsilon_k})}{2}}\right)\\&+4\sqrt{\beta}\int_{R_k}^{\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b/2}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)|\theta'_{\varepsilon_k}|\,\mathrm{d}x\\&+\int_0^{R_k}\left[\theta'_{\varepsilon_k}+2\sqrt{\beta}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)\right]^2\mathrm{d}x\Bigg]-C\alpha.\end{aligned} \tag{5.55}$$

In view of the definition of R_k, the term involving $\cos\theta_{\varepsilon_k}(R_k)$ in (5.55) may be absorbed into the last term for all α sufficiently small. Similarly, by (5.53) and Sobolev embedding the second line in the right-hand side of (5.55) may be bounded by $(\delta_{\varepsilon_k}/L_{\varepsilon_k})^{1/2}$ and, hence, absorbed into the last term as well for all k sufficiently large depending on α. Thus, taking into account that $F_{\varepsilon_k,1d}(m_{\varepsilon_k})\to 2F_0(n_0)$ as $k\to\infty$, we obtain for all k large enough:

$$\begin{aligned}(1-\alpha)^{-1}F_0(n_0)+C\alpha\geq F_0(n_{\varepsilon_k})&+2\sqrt{\beta}\int_{R_k}^{\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b/2}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)|\theta'_{\varepsilon_k}|\,\mathrm{d}x\\&+\frac{1}{2}\int_0^{R_k}\left[\theta'_{\varepsilon_k}+2\sqrt{\beta}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right)\right]^2\mathrm{d}x.\end{aligned} \tag{5.56}$$

We now observe that by minimality of $F_0(n_0)$ both integrals in the right-hand side of (5.56) are bounded above by $C\alpha$, for some $C>0$ independent of α and k. In particular, this implies that the total variation of $\cos(\theta_{\varepsilon_k}/2)$ on $(R_k, \frac{1}{2}\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b)$ is bounded by $C\alpha$, and in view of the fact that $\theta_{\varepsilon_k}(R_k) < \alpha$, we conclude that $\theta_{\varepsilon_k}(x) < C\alpha$ for all $x \in [2\alpha^{-3}, \frac{1}{2}\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b]$ for some $C > 0$ independent of α and k for all k sufficiently large. On the other hand, sending $\alpha \to 0$ on a sequence and extracting a further subsequence (not relabeled), we conclude that

$$\theta'_{\varepsilon_k} + 2\sqrt{\beta}\sin\left(\frac{\theta_{\varepsilon_k}}{2}\right) \to 0 \text{ in } L^2_{loc}(\overline{\mathbb{R}^+}), \tag{5.57}$$

as $k \to \infty$. Testing the left-hand side of (5.57) against $\phi \in C_c^\infty(\mathbb{R}^+)$ and passing to the limit, we then conclude that $\bar\theta$ satisfies

$$\frac{\mathrm{d}\bar\theta}{\mathrm{d}x} + 2\sqrt{\beta}\sin\left(\frac{\bar\theta}{2}\right) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^+). \tag{5.58}$$

In particular, since $\bar\theta \in C(\overline{\mathbb{R}^+})$, we have that $\bar\theta(x)$ also satisfies (5.58) classically for all $x > 0$. Finally, by strict convexity of F_0 we can also conclude that $n_{\varepsilon_k} \to n_0$ as $k \to \infty$. Therefore, we have

$$\arccos n_0 = \lim_{k\to\infty} \arccos n_{\varepsilon_k} = \lim_{k\to\infty} \theta_{\varepsilon_k}(\delta_{\varepsilon_k}/L_{\varepsilon_k}) = \bar\theta(0). \tag{5.59}$$

Thus, $\bar\theta = \theta_\infty$, where the latter is given by (3.9) with $\theta_0 = \arccos n_0$. Combining this with the uniform closeness of $\theta_{k_\varepsilon}(x)$ to zero far from $x = 0$ and the asymptotic decay of $\theta_\infty(x)$ for large $x > 0$ then yields uniform convergence of θ_{k_ε} to θ_∞ on $[0, \frac{1}{2}\varepsilon_k^{-1}L_{\varepsilon_k}^{-1}b]$. From (5.57), we conclude that this convergence is also strong in $H^1_{loc}(\overline{\mathbb{R}^+})$. Finally, in view of the uniqueness of $\bar\theta$ the limit does not depend on the choice of the subsequence and, hence, is a full limit. □

6 Proof of Theorem 3.4

The proof follows closely the arguments in Sect. 5, except that we can no longer reduce the problem to studying a one-dimensional profile due to lack of translational symmetry in the x_1-direction. Therefore, we need to incorporate the relevant corrections to the upper and lower bounds in the proof of Theorem 3.2 and show that they are indeed negligible in comparison with the limit energy F_0.

As in Sect. 5, for $\widetilde{D}_\varepsilon := \varepsilon^{-1}L_\varepsilon^{-1}D$ and $m \in \mathcal{A}_\varepsilon$, where

$$\mathcal{A}_\varepsilon := H^1(\widetilde{D}_\varepsilon; \mathbb{S}^1), \tag{6.1}$$

we introduce

$$F_\varepsilon(m) := \frac{1}{2}\int_{\widetilde{D}_\varepsilon\cap\{|m_2|<1\}} \frac{|\nabla m_2|^2}{1-m_2^2}\,\mathrm{d}x + \beta\int_{\widetilde{D}_\varepsilon}(1-m_2)\,\mathrm{d}x - \frac{\lambda a}{2\pi\varepsilon}$$

$$+\frac{\lambda}{8\pi \ln|\ln\varepsilon|}\int_{\widetilde{D}_\varepsilon}\int_{\widetilde{D}_\varepsilon}\frac{\nabla\cdot\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(x)\nabla\cdot\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y, \tag{6.2}$$

where $\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(x) := m(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)$, with $\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x) := \eta(\mathrm{dist}(x,\partial\widetilde{D}_\varepsilon)L_\varepsilon/\delta_\varepsilon)$. Then, for $m \in H^1(D;\mathbb{S}^1)$ the connection between F_ε and the original energy E_ε is given by

$$E_\varepsilon(m) = \frac{\lambda a}{2\pi} + \varepsilon F_\varepsilon(m(\cdot/(\varepsilon L_\varepsilon))), \tag{6.3}$$

which follows by a straightforward rescaling and applying the weak chain rule (Lieb and Loss 2010, Theorem 6.16) to the identity $|m|^2 = 1$. Therefore, the first statement of Theorem 3.4 is equivalent to

$$\frac{\varepsilon|\ln\varepsilon|}{\ln|\ln\varepsilon|}\min_{m\in\mathcal{A}_\varepsilon} F_\varepsilon(m) \to 2a \min_{n\in[0,1]} F_0(n) \quad \text{as} \quad \varepsilon\to 0. \tag{6.4}$$

As in the proof of Theorem 3.2, we split the rescaled energy into the local and the nonlocal parts:

$$F_\varepsilon(m) = F_\varepsilon^{MM}(m) + F_\varepsilon^{S}(m), \tag{6.5}$$

where

$$F_\varepsilon^{MM}(m) := \frac{1}{2}\int_{\widetilde{D}_\varepsilon\cap\{|m_2|<1\}}\frac{|\nabla m_2|^2}{1-m_2^2}\,\mathrm{d}x + \beta\int_{\widetilde{D}_\varepsilon}(1-m_2)\,\mathrm{d}x, \tag{6.6}$$

and

$$F_\varepsilon^{S}(m) := \frac{\lambda}{8\pi \ln|\ln\varepsilon|}\int_{\widetilde{D}_\varepsilon}\int_{\widetilde{D}_\varepsilon}\frac{\nabla\cdot\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(x)\nabla\cdot\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda a}{2\pi\varepsilon}. \tag{6.7}$$

We begin by stating an analog of Lemma 5.1 in the case of a rectangular domain.

Lemma 6.1 *Let $m = (m_1, m_2) \in \mathcal{A}_\varepsilon$, and let $\overline{m} = (\overline{m}_1, \overline{m}_2)$ be defined as*

$$\begin{aligned}\overline{m}_2(x_1,x_2) &:= \frac{\varepsilon L_\varepsilon}{a}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a} m_2(t,x_2)\,\mathrm{d}t,\\ \overline{m}_1(x_1,x_2) &:= \sqrt{1-\overline{m}_2^2(x_1,x_2)}.\end{aligned} \tag{6.8}$$

Then, $\overline{m} \in \mathcal{A}_\varepsilon \cap C(\widetilde{D}_\varepsilon)$, and for every $R \in (0, \varepsilon^{-1}L_\varepsilon^{-1}b/2]$ and every $r \in (0, R)$, there holds

$$\frac{\varepsilon L_\varepsilon F_\varepsilon^{MM}(m)}{a} \geq 4\sqrt{\beta}\left(1-\sqrt{\frac{1+\overline{m}_2}{2}}\right)\Bigg|_{x_2=r}$$

$$
+ 4\sqrt{\beta}\left(1 - \sqrt{\frac{1+\overline{m}_2}{2}}\right)\Bigg|_{x_2=\varepsilon^{-1}L_\varepsilon^{-1}b-r}
$$
$$
- 8\sqrt{\beta}\left(1 - \sqrt{\frac{1+\overline{m}_2}{2}}\right)\Bigg|_{x_2=R}. \tag{6.9}
$$

Proof Since $m \in H^1(\widetilde{D}_\varepsilon; \mathbb{S}^1)$, its trace on $\widetilde{D}_\varepsilon \cap \{x_2 = t\}$ is well defined for every $t \in (0, \varepsilon^{-1}L_\varepsilon^{-1}b)$. Arguing by approximation, we have $\overline{m}_2 \in H^1(\widetilde{D}_\varepsilon) \cap C(\widetilde{D}_\varepsilon)$, in view of the one-dimensional character of $\overline{m}_2$. Furthermore, arguing exactly as in the proof of Lemma 4.1, we also obtain that $\overline{m} \in \mathcal{A}_\varepsilon \cap C(\widetilde{D}_\varepsilon)$ and

$$
\int_{\widetilde{D}_\varepsilon} |\nabla m|^2 \,\mathrm{d}x \geq \int_{\widetilde{D}_\varepsilon} |\nabla \overline{m}|^2 \,\mathrm{d}x = \int_{\widetilde{D}_\varepsilon \cap \{|\overline{m}_2|<1\}} \frac{|\nabla \overline{m}_2|^2}{1-\overline{m}_2^2} \,\mathrm{d}x. \tag{6.10}
$$

In particular, since $\overline{m}$ is independent of x_1, it may be chosen to be continuous in $\widetilde{D}_\varepsilon$.

By (6.10), we have

$$
F_\varepsilon^{MM}(m) \geq \frac{1}{2}\int_{\widetilde{D}_\varepsilon \cap \{|\overline{m}_2|<1\}} \frac{|\nabla \overline{m}_2|^2}{1-\overline{m}_2^2} \,\mathrm{d}x + \beta \int_{\widetilde{D}_\varepsilon} (1-\overline{m}_2) \,\mathrm{d}x. \tag{6.11}
$$

Therefore, using the Modica–Mortola trick (Modica 1987), for every $\delta \in (0, 1)$ we obtain

$$
F_\varepsilon^{MM}(m) \geq \frac{1}{2}\int_{\widetilde{D}_\varepsilon \cap \{|\overline{m}_2|<1\}} \frac{|\nabla \overline{m}_2|^2}{1-\overline{m}_2^2} \,\mathrm{d}x + \beta \int_{\widetilde{D}_\varepsilon \cap \{|\overline{m}_2|<1\}} (1-\overline{m}_2) \,\mathrm{d}x
$$
$$
\geq \sqrt{2\beta}\int_{\widetilde{D}_\varepsilon \cap \{\overline{m}_2 > -1+\delta\}} \frac{|\nabla \overline{m}_2|}{\sqrt{1+\overline{m}_2}} \,\mathrm{d}x = \sqrt{8\beta}\int_{\widetilde{D}_\varepsilon} \left|\nabla\left(\sqrt{1+\overline{m}_2^\delta} - \sqrt{2}\right)\right| \,\mathrm{d}x, \tag{6.12}
$$

where $\overline{m}_2^\delta := \max(-1+\delta, \overline{m}_2) \in H^1(\widetilde{D}_\varepsilon)$ and we used weak chain rule (Lieb and Loss 2010, Theorem 6.16) and the fact that $\nabla \overline{m}_2^\delta = 0$ on $\{\overline{m}_2^\delta = -1+\delta\} \cup \{\overline{m}_2^\delta = 1\}$ (Lieb and Loss 2010, Theorem 6.19). Thus, in view of continuity of $\overline{m}_2^\delta$ we get (with a slight abuse of notation)

$$
\frac{\varepsilon L_\varepsilon F_\varepsilon^{MM}(m)}{a} \geq \sqrt{8\beta}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a} \left|\left(\sqrt{2} - \sqrt{1+\overline{m}_2^\delta(x_2)}\right)'\right| \,\mathrm{d}x_2
$$
$$
\geq \sqrt{8\beta}\int_r^R \left(\sqrt{2} - \sqrt{1+\overline{m}_2^\delta(x_2)}\right)' \,\mathrm{d}x_2
$$
$$
- \sqrt{8\beta}\int_R^{\varepsilon^{-1}L_\varepsilon^{-1}a-r} \left(\sqrt{2} - \sqrt{1+\overline{m}_2^\delta(x_2)}\right)' \,\mathrm{d}x_2, \tag{6.13}
$$

which yields the rest of the claim in view of arbitrariness of δ. □

Lower bound for the stray field In order to get the required estimates for the lower bound, we have to extend the definition of the test potential in a suitable way. Using the same arguments as in the periodic case, we have a similar lower bound for the stray field energy:

$$F_\varepsilon^S(m) \geq -\frac{\lambda}{2\ln|\ln\varepsilon|}\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x \\ -\frac{\lambda}{\ln|\ln\varepsilon|}\int_{\widetilde{D}_\varepsilon} v(x_1,x_2,0)\nabla\cdot\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}\,\mathrm{d}x - \frac{\lambda a}{2\pi\varepsilon} \tag{6.14}$$

for every $v \in \mathring{H}^1(\mathbb{R}^3)$. We also define

$$n_\varepsilon^- := \frac{\varepsilon L_\varepsilon}{a}\int_0^{a\varepsilon^{-1}L_\varepsilon^{-1}} \widetilde{m}_{\delta_\varepsilon/L_\varepsilon,2}(t,\delta_\varepsilon/L_\varepsilon)\,\mathrm{d}t, \tag{6.15}$$

$$n_\varepsilon^+ := \frac{\varepsilon L_\varepsilon}{a}\int_0^{a\varepsilon^{-1}L_\varepsilon^{-1}} \widetilde{m}_{\delta_\varepsilon/L_\varepsilon,2}(t,b/(\varepsilon L_\varepsilon)-\delta_\varepsilon/L_\varepsilon)\,\mathrm{d}t. \tag{6.16}$$

The construction of the potential is done in the same way as before with the only difference that we now do not have the reflection symmetry for $\widetilde{m}_{\delta_\varepsilon/L_\varepsilon,2}$ and have to consider different distributions of charges near the bottom and the top boundaries. We will carry out the calculation only near the bottom boundary; the other calculation is completely analogous. To avoid cumbersome notation, we will suppress the superscript "−" throughout the argument.

We would like to find a suitable test potential v that vanishes for $x_2 > b/(2\varepsilon L_\varepsilon)$ to obtain an appropriate asymptotic lower bound. Let us define v as follows: For $x_1 \in (0, \varepsilon^{-1}L_\varepsilon^{-1}a)$, we define

$$v(x_1,x_2,x_3) := v_1\left(\sqrt{x_2^2+x_3^2}\right) + v_2\left(\sqrt{x_2^2+x_3^2}\right), \tag{6.17}$$

where v_1 and v_2 are as in (5.21) and (5.22), respectively, while for $x_1 \in (-\infty, 0)$ we extend the definition of v as

$$v(x_1,x_2,x_3) := v_1\left(\sqrt{x_1^2+x_2^2+x_3^2}\right) + v_2\left(\sqrt{x_1^2+x_2^2+x_3^2}\right). \tag{6.18}$$

Finally, for $x_1 \in (\varepsilon^{-1}L_\varepsilon^{-1}a, +\infty)$ we extend the definition of v as

$$v(x_1,x_2,x_3) := v_1\left(\sqrt{\left(\varepsilon^{-1}L_\varepsilon^{-1}a - x_1\right)^2 + x_2^2+x_3^2}\right) \\ + v_2\left(\sqrt{\left(\varepsilon^{-1}L_\varepsilon^{-1}a - x_1\right)^2 + x_2^2+x_3^2}\right) \tag{6.19}$$

It is clear that $v \in \mathring{H}^1(\mathbb{R}^3)$, and we can compute $I := \int_{\mathbb{R}^3} |\nabla v|^2 \, \mathrm{d}x$ explicitly. First, we split this integral into three parts:

$$
\begin{aligned}
I = &\int_{(-\infty,0)\times\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}x + \int_{(0,\varepsilon^{-1}L_\varepsilon^{-1}a)\times\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}x \\
&+ \int_{(\varepsilon^{-1}L_\varepsilon^{-1}a,+\infty)\times\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}x.
\end{aligned}
\tag{6.20}
$$

It is clear that the first and the last integrals in the above expression coincide and the second integral was already essentially computed in (5.26). Due to the symmetry of v, it is not difficult to see that

$$
\begin{aligned}
&\int_{(-\infty,0)\times\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}^3x = 2\pi \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} \left(\frac{\partial v_1}{\partial r} + \frac{\partial v_2}{\partial r} \right)^2 r^2 \, \mathrm{d}r \\
&= \frac{1}{2\pi} \left(n_\varepsilon^2 - 1 + \frac{b}{2\varepsilon L_\varepsilon} - \frac{\delta_\varepsilon n_\varepsilon^2}{L_\varepsilon} \right).
\end{aligned}
\tag{6.21}
$$

Therefore, we obtain

$$
\begin{aligned}
I = &\frac{a n_\varepsilon^2}{2\pi\varepsilon L_\varepsilon} \ln\left(\frac{L_\varepsilon}{\delta_\varepsilon} \right) + \frac{a}{2\pi\varepsilon L_\varepsilon} \ln\left(\frac{b}{2\varepsilon L_\varepsilon} \right) \\
&+ \frac{1}{\pi} \left(n_\varepsilon^2 - 1 + \frac{b}{2\varepsilon L_\varepsilon} - \frac{\delta_\varepsilon n_\varepsilon^2}{L_\varepsilon} \right).
\end{aligned}
\tag{6.22}
$$

Next, we compute

$$
J := \int_{\widetilde{D}_\varepsilon} v(x_1, x_2, 0) \, \nabla \cdot \widetilde{m}_{\delta_\varepsilon/L_\varepsilon} \, \mathrm{d}x. \tag{6.23}
$$

Note that for $x_1 \in (0, \varepsilon^{-1}L_\varepsilon^{-1}a)$ our function $v(x_1, x_2, 0)$ depends only on x_2, and $\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}$ vanishes at the boundary of $\widetilde{D}_\varepsilon$. Therefore, with a slight abuse of notation we have

$$
\begin{aligned}
J &= \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a} \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v(x_2, 0) \left(\partial_1 \widetilde{m}_{\delta_\varepsilon/L_\varepsilon,1}(x_1, x_2) + \partial_2 \widetilde{m}_{\delta_\varepsilon/L_\varepsilon,1}(x_1, x_2) \right) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&= \frac{a}{\varepsilon L_\varepsilon} \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b} v(x_2, 0) \partial_2 \overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_2) \, \mathrm{d}x_2,
\end{aligned}
\tag{6.24}
$$

where $\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_2) := \frac{\varepsilon L_\varepsilon}{a} \int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a} \widetilde{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_1, x_2) \, \mathrm{d}x_1$. Using the same arguments as for the periodic case, we obtain a formula analogous to (5.31), with $m_{\varepsilon,2}$ replaced by $\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}$:

$$
\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} v(x_2, 0) \left(\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_2) \right)' \, \mathrm{d}x_2
$$

$$
\begin{aligned}
&= -\frac{1}{2\pi}\ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right) - \frac{n_\varepsilon^2}{2\pi}\ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right) \\
&\quad + \frac{n_\varepsilon}{2\pi}\int_{\delta_\varepsilon/L_\varepsilon}^{1} \left(\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_2)\right)' \ln x_2 \,\mathrm{d}x_2 \\
&\quad + \frac{1}{2\pi}\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} \frac{1-\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_2)}{x_2}\,\mathrm{d}x_2. \qquad (6.25)
\end{aligned}
$$

We now would like to obtain an analog of (5.32) and need to estimate the last two terms in the right-hand side of (6.25). The first term can be bounded as follows:

$$
\begin{aligned}
&\left|\frac{n_\varepsilon}{2\pi}\int_{\delta_\varepsilon/L_\varepsilon}^{1} \left(\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_2)\right)' \ln x_2 \,\mathrm{d}x_2\right| \le \frac{\varepsilon L_\varepsilon}{2\pi a} \\
&\quad \times \left|\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a}\int_{\delta_\varepsilon/L_\varepsilon}^{1} \partial_2 m_2(x_1,x_2)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1)\ln x_2\,\mathrm{d}x_2\,dx_1\right| \\
&\quad \le \frac{\varepsilon L_\varepsilon}{4\pi a}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a}\int_{\delta_\varepsilon/L_\varepsilon}^{1}\left(|\partial_2 m_2(x_1,x_2)|^2 + |\ln x_2|^2\right)\mathrm{d}x_2\,dx_1 \\
&\quad \le \frac{\varepsilon L_\varepsilon}{2\pi a} F_\varepsilon^{MM}(m) + C, \qquad (6.26)
\end{aligned}
$$

for some universal $C > 0$. Similarly, we can obtain

$$
\begin{aligned}
&\frac{1}{2\pi}\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} \frac{1-\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_2)}{x_2}\,\mathrm{d}x_2 \\
&\quad = \frac{\varepsilon L_\varepsilon}{2\pi a}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a}\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} \frac{1-m_2(x_1,x_2)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1)}{x_2}\,\mathrm{d}x_2\;\mathrm{d}x_1 \\
&\quad \le \frac{\varepsilon L_\varepsilon}{2\pi a}\int_{\delta_\varepsilon/L_\varepsilon}^{\varepsilon^{-1}L_\varepsilon^{-1}a-\delta_\varepsilon/L_\varepsilon}\int_1^{\varepsilon^{-1}L_\varepsilon^{-1}b/2} \frac{1-m_2(x_1,x_2)}{x_2}\,\mathrm{d}x_2\;\mathrm{d}x_1 \\
&\qquad + \frac{2\varepsilon\delta_\varepsilon}{\pi a}\ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right) \\
&\quad \le \frac{\varepsilon L_\varepsilon}{2\pi\beta a}F_\varepsilon^{MM}(m) + C, \qquad (6.27)
\end{aligned}
$$

for some universal $C > 0$, provided that ε is small enough independently of m. Thus, after some straightforward algebra we arrive at the following bound for J:

$$
J \le -\frac{a}{2\pi\varepsilon L_\varepsilon}\left[\ln\left(\frac{b}{2\varepsilon L_\varepsilon}\right) + n_\varepsilon^2\ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)\right] + C\left(\frac{1}{\varepsilon L_\varepsilon} + F_\varepsilon^{MM}(m)\right), \quad (6.28)
$$

for some $C > 0$ and all ε small enough independent of m.

Using the estimates for I and J above, and combining them with the estimates for the similarly defined potential that vanishes for $x_2 < b/(2\varepsilon L_\varepsilon)$, after some tedious algebra we obtain the following asymptotic lower bound for the stray field energy:

$$F_\varepsilon^S(m) \geq \frac{\lambda a \ln|\ln\varepsilon|}{2\pi\varepsilon|\ln\varepsilon|}\Big(|n_\varepsilon^-|^2 + |n_\varepsilon^+|^2 - 1\Big) - \frac{C}{\ln|\ln\varepsilon|}\left(F_\varepsilon^{MM}(m) + \frac{\ln|\ln\varepsilon|}{\varepsilon|\ln\varepsilon|}\right). \tag{6.29}$$

Upper bound for stray field To derive an asymptotically sharp upper bound for the nonlocal energy, we want to estimate from above the integral

$$W := \frac{\lambda}{8\pi\ln|\ln\varepsilon|}\int_{\widetilde{D}_\varepsilon}\int_{\widetilde{D}_\varepsilon}\frac{\nabla\cdot\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(x)\nabla\cdot\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y, \tag{6.30}$$

where $\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}(x) := m_\varepsilon(x)\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x)$, and choose the test sequence

$$m_\varepsilon(x_1,x_2) := \left(\sqrt{1-m_{\varepsilon,2}^2(x_2)},\, m_{\varepsilon,2}(x_2)\right), \tag{6.31}$$

in which $m_{\varepsilon,2}$ is as defined by the one-dimensional construction in Sect. 5 (see Fig. 2 for an illustration). We then obtain that $W = \frac{\lambda}{8\pi\ln|\ln\varepsilon|}I$, where

$$\begin{aligned} I = I_1 + I_2 + I_3 &:= \int_{\widetilde{D}_\varepsilon}\int_{\widetilde{D}_\varepsilon}\frac{\partial_1\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1,x_2)m_{\varepsilon,1}(x_2)\;\partial_1\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(\xi_1,\xi_2)m_{\varepsilon,1}(\xi_2)}{|x-\xi|}\,\mathrm{d}x\;\mathrm{d}\xi \\ &+\int_{\widetilde{D}_\varepsilon}\int_{\widetilde{D}_\varepsilon}\frac{\partial_2(\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1,x_2)m_{\varepsilon,2}(x_2))\partial_2(\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(\xi_1,\xi_2)m_{\varepsilon,2}(\xi_2))}{|x-\xi|}\,\mathrm{d}x\;\mathrm{d}\xi \\ &+2\int_{\widetilde{D}_\varepsilon}\int_{\widetilde{D}_\varepsilon}\frac{\partial_1\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1,x_2)m_{\varepsilon,1}(x_2)\partial_2(\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(\xi_1,\xi_2)m_{\varepsilon,2}(\xi_2))}{|x-\xi|}\,\mathrm{d}x\;\mathrm{d}\xi. \end{aligned} \tag{6.32}$$

We see that the middle integral I_2 is asymptotically equivalent to the one computed in the periodic case. Therefore, it is enough to estimate the first and the last integrals and show that they only give a negligible contribution into the stray field energy in the limit.

We now estimate the first integral I_1. Using the definition of $\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}$, we obtain that $\partial_1\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1,x_2) = 0$ for $\delta_\varepsilon L_\varepsilon^{-1} < x_1 < \varepsilon^{-1}L_\varepsilon^{-1}b - \delta_\varepsilon L_\varepsilon^{-1}$. Moreover, outside this interval $|\partial_1\widetilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1,x_2)| \leq L_\varepsilon/\delta_\varepsilon$. We also know that $m_{\varepsilon,1}(x_2) = 0$ for $x_2 \in (0,\delta_\varepsilon/L_\varepsilon)\cup(2K+\delta_\varepsilon/L_\varepsilon, \varepsilon^{-1}L_\varepsilon^{-1}b - 2K - \delta_\varepsilon/L_\varepsilon)\cup(\varepsilon^{-1}L_\varepsilon^{-1}b - \delta_\varepsilon/L_\varepsilon)$, where K is the same constant as in the one-dimensional construction. Therefore, by direct computation we can estimate for all ε sufficiently small:

$$\begin{aligned} I_1 &\leq C\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)^2\int_0^{2K+\delta_\varepsilon L_\varepsilon^{-1}}\int_0^{2K+\delta_\varepsilon L_\varepsilon^{-1}}\int_0^{\delta_\varepsilon L_\varepsilon^{-1}}\int_0^{\delta_\varepsilon L_\varepsilon^{-1}}\frac{\mathrm{d}x_1\,\mathrm{d}\xi_1\,\mathrm{d}x_2\,\mathrm{d}\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} \\ &\leq CK\ln\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right), \end{aligned} \tag{6.33}$$

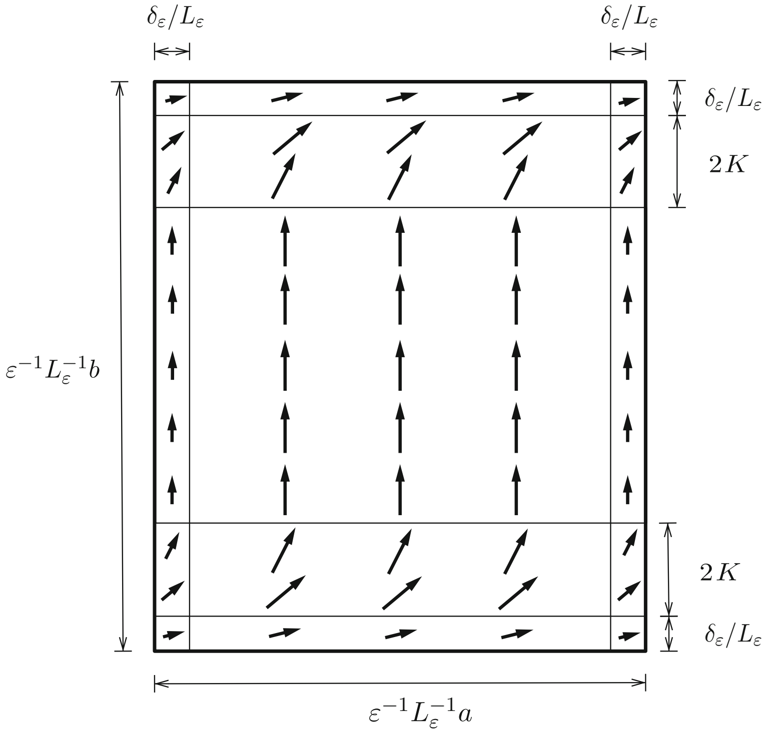

Fig. 2 Schematics of the truncated test magnetization configuration $\widetilde{m}_{\delta_\varepsilon/L_\varepsilon}$ used in the upper bound construction of Sect. 6

for some universal $C > 0$. Similarly, the last integral I_3 can be estimated as

$$\begin{aligned} I_3 &\le C\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)\int_{\delta_\varepsilon L_\varepsilon^{-1}}^{2K+\delta_\varepsilon L_\varepsilon^{-1}}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a}\int_0^{2K+\delta_\varepsilon L_\varepsilon^{-1}}\int_0^{\delta_\varepsilon L_\varepsilon^{-1}}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} \\ &\quad+\left(\frac{L_\varepsilon}{\delta_\varepsilon}\right)^2\int_0^{\delta_\varepsilon L_\varepsilon^{-1}}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a}\int_0^{2K+\delta_\varepsilon L_\varepsilon^{-1}}\int_0^{\delta_\varepsilon L_\varepsilon^{-1}}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} \\ &\le CK\ln\left(\frac{1}{\varepsilon}\right), \end{aligned} \tag{6.34}$$

again, for some universal $C > 0$ and all ε small enough.

Proof of Theorem 3.4 We can combine the lower bounds for F_ε^{MM} and F_ε^S and proceed in the same way as in the one-dimensional case. There is a slight mismatch, as the definition of $n_\varepsilon^\pm$ uses the average of $\widetilde{m}_{\delta_\varepsilon/L_\varepsilon,2}$, while the lower bound (6.9) for F_ε^{MM} uses $\overline{m}_{\varepsilon,2}$. However, we observe that

$$\left|\frac{\varepsilon L_\varepsilon}{a}\int_0^{\varepsilon^{-1}L_\varepsilon^{-1}a}\widetilde{m}_{\delta_\varepsilon/L_\varepsilon,2}(x_1,x_2)\,\mathrm{d}x_1-\overline{m}_{\varepsilon,2}(x_2)\right|$$

$$\leq \frac{\varepsilon L_\varepsilon}{a} \int_0^{\varepsilon^{-1} L_\varepsilon^{-1} a} |m_{\varepsilon,2}(x_1, x_2)|(1 - \tilde{\eta}_{\delta_\varepsilon/L_\varepsilon}(x_1, x_2))\, \mathrm{d}x_1 \leq C\varepsilon\delta_\varepsilon, \quad (6.35)$$

for some $C > 0$ independent of ε, and, therefore, asymptotically we can interchange the average of $\widetilde{m}_{\delta_\varepsilon/L_\varepsilon,2}$ with $\overline{m}_{\varepsilon,2}$ in the formula in (6.9) and arrive at the full lower bound as in the one-dimensional case. Using in addition the upper bound construction, the proof of (3.17) follows exactly as in the proof of Theorem 3.2 with the help of Lemma 6.1. Convergence of m_ε to e_2 trivially follows from positivity of the stray field energy and boundedness of $E_\varepsilon(m_\varepsilon)$ as $\varepsilon \to 0$.

Assuming m_ε is a minimizer of E_ε, in the same way as in the proof of the Theorem 3.2 it follows that $n_\varepsilon^- \to n_0$ and $n_\varepsilon^+ \to n_0$, therefore we have

$$\overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(\delta_\varepsilon/L_\varepsilon) \to n_0 \quad \text{and} \quad \overline{m}_{\delta_\varepsilon/L_\varepsilon,2}(b/(\varepsilon L_\varepsilon) - \delta_\varepsilon/L_\varepsilon) \to n_0. \qquad (6.36)$$

Using the inequality in (6.35) and recalling (6.4), we obtain the desired result. □

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