Università di Pisa - Corso di Laurea in Matematica

Istituzioni di Analisi Matematica – A.A. 2020/21

Programma per argomenti

Aggiornato al 26 settembre 2020

Preliminaries/Prerequisites

- Real analysis for one-variable and multi-variable functions: liminf/limsup, differential calculus, minimum problems with or without constraints, convex functions, sequences and series of functions (pointwise/uniform/normal/absolute convergence), integral calculus, ordinary differential equations, curve and surface integrals, Gauss-Green theorem.
- Linear algebra: vector spaces and linear applications, spectral theorem, scalar products, norms, quadratic forms.
- Basic measure theory: different notions of measure (outer measure, measure defined on a sigma-algebra, signed measure), Lebesgue measure and integral, monotone convergence theorem, dominated convergence theorem, continuity and differentiability of parametric integrals, L^p spaces (completeness and separability), mollifiers and convolution, approximation of continuous or integrable functions by piecewise constant or smooth functions, L^2 theory of Fourier series, Radon-Nikodym theorem for signed measures.
- Basic general topology: terminology (open/closed sets, boundary, neighborhoods, accumulation points), compactness, connectedness.
- Metric spaces: characterization of compact sets, Banach fixed point theorem, completion of a metric space, extension theorem, Ascoli-Arzelà theorem.

Normed and Banach spaces

- Norms, scalar products, normed spaces, Banach spaces, Hilbert spaces.
- Normal convergence implies convergence for series in Banach spaces.
- Characterization of linear and continuous operators between normed spaces. Norm in the space of linear continuous operators.
- Pseudo-norms and analytic form of Hahn-Banach theorem (as an extension theorem).
- Topological dual of a normed space.
- Aligned functional and dual characterization of the norm.
- Weak and weak* convergence. Lower semicontinuity of the norm with respect to weak and weak* convergence. Weak* compactness of closed balls.
- Pseudo-norm associated to a convex set. Geometric forms of Hahn-Banach theorem (separation of convex sets by hyperplanes in normed spaces).
- Sets that are strongly closed and convex are weakly closed. Functions that are strongly lower semicontinuous and convex are weakly lower semicontinuous (in normed spaces).

- Classical normed sequence spaces: ℓ^p, c_{00}, c_0, c. Characterization of their duals.
- Characterization of the dual of classical L^p spaces. Weak convergence and weak compactness in classical L^p spaces.
- Bidual of a space. Reflexive spaces. Weak compactness of closed balls in reflexive spaces.
- Baire spaces, G_δ and F_σ sets. Residual sets. Classical examples of Baire spaces: complete metric spaces, locally compact spaces, open subsets of Baire spaces.
- Classical applications of Baire spaces: existence of nowhere differentiable continuous functions, the pointwise limit of continuous functions is continuous in a residual set, nonexistence of Banach spaces with a countable algebraic basis, boundedness of weakly convergent sequences.
- Banach-Steinhaus theorem. Classical application: existence of a residual set of continuous periodic functions whose Fourier series does not converge on a residual set.
- Characterization of open functions in terms of quantitative solvers. Open mapping theorem and closed graph theorem. Continuity of the inverse of a linear function.
- Existence of the complement of a subspace vs existence of a linear quantitative solver.
- Nonlinear projections and approximation of compact operators by operators with finite dimensional range.
- Fixed point theorems from Brouwer to Schauder. Proof of Peano theorem for ODEs through Schauder fixed point theorem.

Hilbert spaces

- Basic examples of Hilbert spaces (ℓ^2 and L^2 spaces with respect to a measure). Example of a non-separable Hilbert space.
- Orthonormal bases (or Hilbert bases). Every separable Hilbert spaces admits a (finite or countable) orthonormal basis.
- Components of a vector with respect to an orthonormal basis. Representation of vectors, norms and scalar products in terms of components.
- Weak convergence vs strong convergence in separable Hilbert spaces. Weak convergence and convergence of components.
- Weak compactness of balls in separable Hilbert spaces. Lower semicontinuity of the norm with respect to weak convergence.
- Parallelogram identity. Characterization of norms originating from a scalar product (Jordan-Fréchet-von Neumann theorem).
- Projection onto a closed convex set: existence, uniqueness, Lipschitz continuity, characterization through negative scalar products.
- Projection onto a closed vector subspace: characterization and linearity. Orthogonal space of a closed subspace. Direct orthogonal sums.

- Separation of convex sets by hyperplanes in Hilbert spaces. Sets that are strongly closed and convex are weakly closed. Functions that are strongly lower semicontinuous and convex are weakly lower semicontinuous (Hilbert spaces version).
- Dual of a Hilbert space (Riesz-Fréchet representation theorem).
- Existence of solutions to linear elliptic PDEs as an application of Riesz-Fréchet representation theorem (Lax-Milgram approach).
- Compact operators. Linear approximation by operators with finite dimensional range.
- Rayleigh quotient and variational characterization of eigenvalues and eigenvectors.
- Spectral theorem for linear symmetric compact operators in separable Hilbert spaces.
- Inverse of a compact operator as an unbounded operator.
- Powers of a positive symmetric unbounded diagonal operator and their domains.
- Laplacian with suitable boundary conditions as an unbounded operator. Fractional Sobolev spaces and distributions as domains of powers of the Laplacian.

Sobolev spaces

- Weak derivatives of order one and Sobolev spaces in an interval: definition W vs definition H. Compatibility with classical derivatives. Equivalence of the two definitions.
- Continuity and Hölder continuity of Sobolev functions in an interval.
- Weak derivatives of any order in any dimension: definition W vs definition H. Compatibility with classical derivatives. Equivalence of the two definitions.
- Stability of weak derivatives with respect to weak limits.
- Sobolev spaces of any order in any dimension: definition W vs definition H.
- Partitions of the unity (two cases: covering of an open set, and covering of an open set with compact boundary).
- Mollifiers and convolution of Sobolev functions.
- Approximation of Sobolev functions by smooth functions (low-cost approximation, H=W, deluxe approximation).
- Algebraic theorems for Sobolev functions: product of Sobolev functions, external and internal composition with smooth functions.
- Gagliardo inequality. Embedding theorems (Sobolev, Morrey) and corresponding inequalities (both in the whole space, and in suitable open subsets).
- Extension of Sobolev functions from an open set to the whole space. Applications to approximation and embedding theorems.
- Characterization of relatively compact subsets in metric spaces. Relatively compact subsets of L^p spaces. Compact embedding theorems for Sobolev spaces (Rellich-Kondrakov).

- Trace theorems in codimension one: existence and further regularity of the trace.
- Sobolev functions that are limits of smooth functions with compact support: characterization (through extension and traces) and embedding theorems.
- Inequalities à la Poincaré-Sobolev-Wirtinger.

Indirect method in the Calculus of Variations

- First variation of a functional along a curve. Gateaux derivative of a functional defined in an affine space. Necessary conditions for minimality.
- Inner/outer (horizontal/vertical) variations for integral functionals.
- Fundamental Lemma in the Calculus of Variations and Du Bois Reymond Lemma, both in the classical and in the Lebesgue setting.
- Genesis of boundary conditions (Dirichlet, Neumann, periodic) for integral functionals in dimension one.
- Different forms of the Euler-Lagrange equation (integral/differential/Erdmann).
- Directional local minima: definition and first order necessary conditions.
- Strategies in order to prove optimality: convexity and auxiliary functionals.
- First variation for integral functionals involving multiple integrals: Euler-Lagrange equation in divergence form, Laplacian, Neumann conditions in more space dimensions.

Direct method in the Calculus of Variations

- Compactness, lower semicontinuity, and Weierstrass theorem with respect to a notion of convergence. Coercivity and variants of Weierstrass theorem.
- Approximation results and different functional settings for a variational problem. Existence and non existence of minima.
- Road map of the direct method: weak formulation, compactness, lower semicontinuity, regularity.
- Compactness of sub-levels of integral functionals vs growth conditions on the Lagrangian and compactness results in functional spaces.
- Weak lower semicontinuity of integral functionals vs convexity of the Lagrangian with respect to derivatives.
- Variational formulation of boundary problems for ODEs and elliptic PDEs in divergence form. Variational formulation of different boundary conditions.
- Truncation arguments and qualitative properties of solutions.
- Regularity results in dimension one vs convexity assumptions on the Lagrangian (through weak formulation of the Euler-Lagrange equation and bootstrap arguments).
- Elliptic equations in divergence form: interior and boundary regularity (in H^k spaces): a priori estimates and method of translations.