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Work in progress!! This booklet is updated daily. For the sake of the forests, please do not print it! Greta watches you.

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To do

- Manca tutto, quindi continuare a scrivere in tutte le sezioni ...

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Prefazione

[Farsi venire in mente qualcosa da scrivere qui]

Buon lavoro!

Part I

Schede [Nice translation needed]

Chapter 1

Preliminaries

Metric spaces 1 – Basic definitions

1. **Distance**. Let $\mathbb{X}$ be a set. A *distance* (or *metric*) on $\mathbb{X}$ is a function $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ satisfying the following properties:

   (D1) (nonnegative) $d(x, y) \geq 0$ for every $(x, y) \in \mathbb{X}^2$,

   (D2) (identity of indiscernibles) $d(x, y) = 0$ if and only if $x = y$,

   (D3) (symmetry) $d(x, y) = d(y, x)$ for every $(x, y) \in \mathbb{X}^2$,

   (D4) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for every $(x, y, z) \in \mathbb{X}^3$.

2. **Metric space**. A *metric space* is an ordered pair $(\mathbb{X}, d)$, where $\mathbb{X}$ is a set and d is a distance on $\mathbb{X}$.

3. **Isometry**. An *isometry* between two metric spaces $(\mathbb{X}, d)$ and $(\mathbb{Y}, \delta)$ is a function $f : \mathbb{X} \to \mathbb{Y}$ such that

   $$\delta(f(x_1), f(x_2)) = d(x_1, x_2) \qquad \forall (x_1, x_2) \in \mathbb{X}^2.$$

   An isometry is always injective and 1-Lipschitz continuous, but not necessarily surjective.

Metric spaces n – Boundedness

1. **Bounded metric space**. A metric space $(\mathbb{X}, d)$ is called *bounded* if there exists a real number M such that $d(x, y) \leq M$ for every x and y in $\mathbb{X}$.

2. **Diameter**. The *diameter* of a metric space $(\mathbb{X}, d)$ is defined as

$$\operatorname{diam}(X) := \sup \left\{ d(x, y) : (x, y) \in \mathbb{X}^2 \right\}.$$

   It is finite if and only if the metric space is bounded.

3. **Totally bounded metric space**. A metric space $(\mathbb{X}, d)$ is called *totally bounded* if for every $r > 0$ there exists a finite subset $C_r \subseteq \mathbb{X}$ such that

$$\mathbb{X} = \bigcup_{x \in C_r} B(x, r).$$

   In other words, every point $x \in \mathbb{X}$ has distance less than r from at least one element of the finite set C_r.

4. **Epsilon-net**. Let $(\mathbb{X}, d)$ be a metric space, and let $\varepsilon > 0$. An ε-net is any subset $C \subseteq \mathbb{X}$ with the property that

$$\mathbb{X} = \bigcup_{x \in C} B(x, r).$$

   In other words, the ε-neighborhood of C is the whole $\mathbb{X}$. Roughly speaking, we can think C as the set of locations of gas stations, and each point of $\mathbb{X}$ has a gas station within a distance less than or equal to r.

   Therefore, we can rephrase the previous definition as follows: a metric space $(\mathbb{X}, d)$ is totally bounded if and only if for every $r > 0$ there exists a *finite* r-net C_r.

5. **Characterization of total boundedness**. A metric space $(\mathbb{X}, d)$ is totally bounded if and only if for every $r > 0$ there exists a subset $B_r \subseteq \mathbb{X}$ such that

   - B_r is totally bounded (with respect to the restriction of d),
   - every point of $\mathbb{X}$ has distance less than or equal to r from some element of B_r, namely for every $x \in \mathbb{X}$ there exists $y \in B_r$ such that $d(x, y) \leq r$.

   In other words, for every $r > 0$ the exists a (possibly infinite) r-net B_r which is totally bounded.

   Since compact metric spaces are totally bounded (see ...), a usual way to prove total boundedness is to show that for every $r > 0$ the elements of $\mathbb{X}$ lie in a neighborhood of radius r of some compact subset B_r.

Metric spaces n – Completeness

1. **Cauchy sequence**. Let $(\mathbb{X}, d)$ be a metric space. A sequence $\{x_n\} \subseteq \mathbb{X}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon$ for every $m \geq n_0$ and every $n \geq n_0$. In symbols:

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall m \geq n_0 \quad \forall n \geq n_0 \qquad d(x_n, x_m) \leq \varepsilon.$$

2. **Properties of Cauchy sequences**. The following properties can be easily established.
   - If a sequence is convergent, then it is a Cauchy sequence.
   - A Cauchy sequence x_n is always bounded (namely there exists $M \in \mathbb{R}$ such that $d(x_n, x_m) \leq M$ for every pair of indices m and n).
   - If a Cauchy sequence x_n has a subsequence converging to some x_∞, then the whole sequence converges to x_∞.

3. **Complete metric space**. A metric space $(\mathbb{X}, d)$ is called *complete* if every Cauchy sequence $\{x_n\} \subseteq \mathbb{X}$ admits a converging subsequence, namely there exists $x_\infty \in \mathbb{X}$ such that $x_n \to x_\infty$.

4. **Completion of a metric space**. The competion (or better a completion) of a metric space is a compete metric space which contains the given one as a dense subset.

   More precisely, a completion of $(\mathbb{X}, d)$ is a triple $(\widehat{\mathbb{X}}, \widehat{d}, i)$ where
   - $(\widehat{\mathbb{X}}, \widehat{d})$ is a complete metric space,
   - $i : \mathbb{X} \to \widehat{\mathbb{X}}$ is an isometry whose image $i(\mathbb{X})$ is dense in $\widehat{\mathbb{X}}$.

5. **Extension theorem**. Let $(\mathbb{X}, d)$ be a metric space, let $(\widehat{\mathbb{X}}, \widehat{d}, i)$ be a completion of $(\mathbb{X}, d)$, let $(\mathbb{Y}, \delta)$ be a complete metric space, and let $f : \mathbb{X} \to \mathbb{Y}$ be a uniformly continuous function.

   Then there exists a unique function $\widehat{f} : \widehat{\mathbb{X}} \to \mathbb{Y}$ which is continuous and extends f, namely $\widehat{f}(i(x)) = f(x)$ for every $x \in \mathbb{X}$. In addition, $\widehat{f}$ is uniformly continuous.

6. **Uniqueness of the completion**. The completion of a metric space is unique up to isometry.

   More precisely, let $(\widehat{\mathbb{X}}_1, \widehat{d}_1, i_1)$ and $(\widehat{\mathbb{X}}_2, \widehat{d}_2, i_2)$ be two completions of the same metric space $(\mathbb{X}, d)$. Then there exists a bijective isometry $f : \widehat{\mathbb{X}}_1 \to \widehat{\mathbb{X}}_2$ such that $f(i_1(x)) = i_2(x)$ for every $x \in \mathbb{X}$.

7. **Existence of the completion**. The existence of a completion of a metric space $(\mathbb{X}, d)$ can be proved through the following procedure.
   - We define $\mathcal{X}$ as the set of all Cauchy sequences in $\mathbb{X}$.
   - If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $\mathbb{X}$, then $\{d(x_n, y_n)\}$ turns out to be a Cauchy sequence in $[0, +\infty)$, hence it admits a finite limit, which we denote by $\delta(\{x_n\}, \{y_n\})$.
   - It turns out that $\delta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ has all the properties of a distance, but for the fact that it can be 0 even if the arguments are different. Moreover, an equivalence relation on $\mathcal{X}$ can be introduced by saying that $\{x_n\} \sim \{y_n\}$ if and only if $\delta(\{x_n\}, \{y_n\}) = 0$.

- We define $\widehat{\mathbb{X}}$ as the quotient of $\mathcal{X}$ with respect to the equivalence relation, we prove that δ passes to the quotient defining a metric $\widehat{d}$ on $\widehat{\mathbb{X}}$, and finally we define $i : \mathbb{X} \to \widehat{\mathbb{X}}$ by identifying every element of $\mathbb{X}$ with a constant Cauchy sequence. A lot of details have to be checked.
- If $\{x_n\}$ is a Cauchy sequence in $\mathbb{X}$ and $f : \mathbb{X} \to \mathbb{Y}$ is uniformly continuous, then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{Y}$. This defines an extension of f to $\mathcal{X}$. This extension passes to the quotient in $\widehat{\mathbb{X}}$, providing the extension of f required by the extension theorem. Once again, several standard details have to be checked.

Metric spaces n – Compactness

1. **Sequential compactness**. A metric space $(\mathbb{X}, d)$ is *sequentially compact* if every sequence in $\mathbb{X}$ admits a converging subsequence. More precisely, for every sequence $\{x_n\} \subseteq \mathbb{X}$ there exist an increasing sequence n_k of positive integers and $x_\infty \in \mathbb{X}$ such that $x_{n_k} \to x_\infty$.

2. **Covering compactness**. A metric space $(\mathbb{X}, d)$ is *covering compact* if every open covering of $\mathbb{X}$ has a finite subcovering. More precisely, for every family $\{U_i\}_{i \in I}$ of open subsets of $\mathbb{X}$ such that $\bigcup_{i \in I} U_i = \mathbb{X}$ there exists a finite subset $J \subseteq I$ such that $\bigcup_{j \in J} U_j = \mathbb{X}$.

3. **Compactness in metric spaces (Heine-Borel Theorem)**. For a metric space $(\mathbb{X}, d)$ the following three facts are equivalent:

   (i) $(\mathbb{X}, d)$ is complete and totally bounded,

   (ii) $(\mathbb{X}, d)$ is sequentially compact,

   (iii) $(\mathbb{X}, d)$ is covering compact.

   As a consequence, a metric space is called just compact (without any further specification) when it satisfies any of the three equivalent properties.

4. **Lebesgue number lemma**. Let $\{U_i\}_{i \in I}$ be an open covering of a compact metric space $(\mathbb{X}, d)$. Then there exists $r > 0$ (called the Lebesgue number of the covering) with the following property: for each $x \in \mathbb{X}$ there exists at least one index $i \in I$ such that $B(x, r) \subseteq U_i$.

5. **Proof of Heine-Borel Theorem**. The strategy of the proof is the following.

   - $\boxed{\text{(iii)} \Rightarrow \text{(ii)}}$ If a metric space $(\mathbb{X}, d)$ is covering compact then the following lemma holds true. For every sequence $\{x_n\} \subseteq \mathbb{X}$ there exists at least one point $x_\infty \in \mathbb{X}$ such that for every $r > 0$ it turns out that $x_n \in B(x_\infty, r)$ for infinitely many indices n. At this point, it is not difficult to show that x_n has a subsequence converging to x_∞.
   - $\boxed{\text{(ii)} \Rightarrow \text{(i)}}$ Completeness follows from the following lemma: a Cauchy sequence is convergent if and only if it admits a converging subsequence. Total boundedness follows by contradiction owing to the following lemma: if for some $r_0 > 0$ an r_0-net does not exist, then there exists a sequence x_n in $\mathbb{X}$ such that $d(x_i, x_j) \geq r_0$ for every $i \neq j$. Such a sequence admits no converging subsequence.
   - $\boxed{\text{(i)} \Rightarrow \text{(ii)}}$ The key lemma is the following: in a totally bounded metric space any sequence admits a Cauchy subsequence.
   - $\boxed{\text{(ii)} \Rightarrow \text{(iii)}}$ The key lemma is that sequential compactness implies that every open covering admits a positive Lebesgue number r_0. Since we already know that sequential compactness implies total boundedness, we can take a finite r_0-net and consider a finite subcovering in such a way that any ball centerd in the r_0-net is contained in some element of the subcovering.

Metric spaces n – Contractions

1. **Contractions**. Let $(\mathbb{X}, d)$ be a metric space. A function $f : \mathbb{X} \to \mathbb{X}$ is called a *contraction* if there exists a constant $\nu < 1$ such that

$$d(f(x), f(y)) \leq \nu d(x, y) \qquad \forall (x, y) \in \mathbb{X}^2.$$

This is equivalent to saying that f is Lipschitz continuous with Lipschitz constant less than 1.

2. **Fixed point theorem for contractions**. Let $(\mathbb{X}, d)$ be a *complete* metric space, and let $f : \mathbb{X} \to \mathbb{X}$ be a contraction.

Then f admits a *unique fixed point*, namely there exists a unique $x \in \mathbb{X}$ such that $f(x) = x$.

3. **Proof of the fixed point theorem**. The strategy of the proof is the following.

- Uniqueness follows trivially from the contraction property.
- As for existence, we choose any $x_0 \in \mathbb{X}$ and we define the sequence x_n through the recurrence $x_{n+1} = f(x_n)$.
- Exploiting once again the contraction property, we prove by induction that

$$d(x_{n+1}, x_n) \leq \nu^n d(x_1, x_0) \qquad \forall n \in \mathbb{N}.$$

- By triangle inequality we deduce that

$$d(x_m, x_n) \leq \frac{\nu^{n_0}}{1 - \nu} \qquad \text{whenever } m \geq n \geq n_0,$$

from which we deduce that x_n is a Cauchy sequence.

- Thanks to completeness, we deduce that x_n has a limit x_∞ in $\mathbb{X}$. Passing to the limit in the recursion we finally conclude that x_∞ is a fixed point of f.

Ascoli-Arzelà Theorem

1. **Theorem (Ascoli-Arzelà).** Let $\mathbb{X}$ and $\mathbb{Y}$ be two metric spaces, and let $f_n : \mathbb{X} \to \mathbb{Y}$ be a sequence of functions. Let us assume that

   (i) (compactness of ambient space) $\mathbb{X}$ is compact,

   (ii) (compactness of the images of every point) for every $x \in \mathbb{X}$ there exists a compact set $K \subseteq \mathbb{Y}$ such that
   $$f_n(x) \in K \qquad \forall n \in \mathbb{N},$$

   (iii) (equi-continuity) for every $\varepsilon > 0$ and every $x \in \mathbb{X}$, there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ and every $y \in B_{\mathbb{X}}(x, \delta)$ it turns out that $f_n(y) \in B_{\mathbb{Y}}(f(x), \varepsilon)$. In symbols
   $$\forall \varepsilon > 0 \quad \forall x \in \mathbb{X} \quad \exists \delta > 0 \quad \forall n \in \mathbb{N} \quad \forall y \in B_{\mathbb{X}}(x, \delta) \qquad f_n(y) \in B_{\mathbb{Y}}(f(x), \varepsilon).$$

   Then $\{f_n\}$ admits a subsequence that converges uniformly in $\mathbb{X}$, namely there exists an increasing sequence $\{n_k\}$ of positive integers and a function $f_\infty : \mathbb{X} \to \mathbb{Y}$ such that $f_{n_k} \to f_\infty$ uniformly in $\mathbb{X}$ as $k \to +\infty$.

2. **Variant for non-compact spaces.** Let us assume that $\mathbb{X}$ is not compact, but it admits an *exhaustion by compact sets*, namely a sequence $\{K_n\}$ of compact subsets such that
   $$K_n \subseteq \operatorname{Int}(k_{n+1}) \quad \forall n \in \mathbb{N} \qquad \text{and} \qquad \bigcup_{n \in \mathbb{N}} K_n = \mathbb{X}.$$

   Then assumption (ii) and (iii) are enough to guarantee that $\{f_n\}$ admits a subsequence that converges uniformly on compact subsets of $\mathbb{X}$.

3. **Comments on the assumptions.**

   - In the equicontinuity assumption (iii) the value of δ is allowed to depend both on ε and on $\mathbb{X}$. Nevertheless, if $\mathbb{X}$ is compact, then it can be shown that equicontinuity implies equi-uniform-continuity, namely (in symbols)
   $$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in \mathbb{X} \quad \forall n \in \mathbb{N} \quad \forall y \in B_{\mathbb{X}}(x, \delta) \qquad f_n(y) \in B_{\mathbb{Y}}(f(x), \varepsilon).$$

     In other words, the value of δ can be assumed to depend on ε only (and not on x).

   - In assumption (ii) the compact set K might depend on x. Nevertheless, it can be shown that assumptions (i), (ii) and (iii) imply the existence of a universal compact set $K \subseteq \mathbb{Y}$ such that
   $$f_n(x) \in K \qquad \forall x \in \mathbb{X} \quad \forall n \in \mathbb{N}.$$

Mollifiers

1. **Definition (mollifier).** Let d be a positive integer. A *mollifier* in $\mathbb{R}^d$ is a function $\rho \in C_c^\infty(\mathbb{R}^d)$ such that

   - $\rho(x) = 0$ for every $x \in \mathbb{R}^d$ with $|x| \geq 1$,
   - $\rho(x) > 0$ for every $x \in \mathbb{R}^d$ with $|x| < 1$,
   - it turns out that
   $$\int_{\mathbb{R}^d} \rho(x)\, dx = 1.$$

   In many applications we can limit ourselves to mollifiers with radial symmetry or with less regularity, for example of class C^k.

2. **Definition (regularization by convolution).** Let d be a positive integer, let $\rho \in C_c^\infty(\mathbb{R}^d)$ be a mollifier, and let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$.

   For every $\varepsilon > 0$ we define the *regularization* of u by *convolution* as
   $$(u * \rho_\varepsilon)(x) := \int_{\mathbb{R}^d} u(x + \varepsilon y) \cdot \rho(y)\, dy \qquad \forall x \in \mathbb{R}^d.$$

   We observe that, since ρ vanishes outside the ball $B(0,1)$, the integral is well-defined, and can be written in the equivalent forms
   $$(u * \rho_\varepsilon)(x) := \int_{B(0,1)} u(x + \varepsilon y) \cdot \rho(y)\, dy = \frac{(-1)^d}{\varepsilon^d} \int_{B(0,\varepsilon)} u(z) \cdot \rho\left(\frac{x-z}{\varepsilon}\right) dz.$$

3. **Basic regularity properties.** Let d be a positive integer, let $\rho \in C_c^\infty(\mathbb{R}^d)$ be a mollifier, and let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$. Let us set $u_\varepsilon := u * \rho_\varepsilon$.

   Then the following statements hold true.

   - *Regularity.* It turns out that $u_\varepsilon \in C^\infty(\mathbb{R}^d)$ for every $\varepsilon > 0$, and its partial derivatives are given by
   $$D^\alpha u_\varepsilon(x) = \frac{(-1)^d}{\varepsilon^{d+|\alpha|}} \int_{B(0,\varepsilon)} u(z) \cdot D^\alpha \rho\left(\frac{x-z}{\varepsilon}\right) dz = \frac{1}{\varepsilon^{|\alpha|}} \int_{B(0,1)} u(x + \varepsilon y) \cdot D^\alpha \rho(y)\, dy$$
   for every $x \in \mathbb{R}^d$, every $\varepsilon > 0$, and every multi-index α.
   More precisely, we obtain that $u_\varepsilon \in C^{k+1}(\mathbb{R}^d)$ if we assume only that $\rho \in C_c^k(\mathbb{R}^d)$.
   - *Support.* The support of u_ε is contained in a neighborhood of radius ε of the support of u, and in particular the support of u_ε is compact when the support of u is compact.

4. **Convergence properties.** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $\rho \in C_c^\infty(\mathbb{R}^d)$ be a mollifier, and let u be a function defined only in Ω.

   Let us set $u_\varepsilon := \widehat{u} * \rho_\varepsilon$, where $\widehat{u} : \mathbb{R}^d \to \mathbb{R}$ is the extension by zero of u, defined by
   $$\widehat{u} = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

   Then the following statements hold true.

- *Estimate.* If $u \in L^p(\Omega)$ for some $p \in [1, +\infty]$, then it turns out that $u_\varepsilon \in L^p(\Omega)$ for every $\varepsilon > 0$, and
$$\|u_\varepsilon\|_{L^p(\Omega} \leq \|u\|_{L^p(\Omega)} \qquad \forall \varepsilon > 0.$$
- *Convergence.* If $u \in L^p(\Omega)$ for some $p \in [1, +\infty)$, then it turns out that $u_\varepsilon \to u$ in $L^p(\Omega)$ as $\varepsilon \to 0^+$.

We point out that the case $p = +\infty$ is included in the first statement (estimate), but excluded in the second one (convergence).

Partitions of the unity

1. **Open set covered by compactly contained open subsets**. Let d be a positive integer, let $A \subseteq \mathbb{R}^d$ be an open set, and let $\{A_k\}$ be a sequence of open sets such that

    - $A_k \subset\subset A$ for every $k \in \mathbb{N}$,
    - they form a *covering* of A, namely
    $$\bigcup_{k\in\mathbb{N}} A_k = A,$$
    - the covering is *locally finite*, namely for every compact set $K \subseteq A$ the set
    $$\{k \in \mathbb{N} : A_k \cap K \neq \emptyset\}$$
    is finite.

    Then there exists a *partition of the unit* relative to the covering $\{A_k\}$, namely a sequence of functions $\psi_k : \mathbb{R}^d \to \mathbb{R}$ such that

    - $0 \leq \psi_k(x) \leq 1$ for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}^d$,
    - $\psi_k \in C_c^\infty(\mathbb{R}^d)$ for every $k \in \mathbb{N}$,
    - the support of ψ_k is contained in A_k for every $k \in \mathbb{N}$,
    - it turns out that
    $$\sum_{k\in\mathbb{N}} \psi_k(x) = 1 \qquad \forall x \in A.$$

2. **Covering of an open set with compact boundary**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let n be a positive integer, and let $A_1, \ldots, A_n$ be open sets.

    Let us assume that $\partial\Omega$ is compact, and
    $$\partial\Omega \subseteq \bigcup_{i=1}^{n} A_i.$$

    Then there exist $n+1$ functions $\psi_i : \mathbb{R}^d \to \mathbb{R}$, with $i \in \{0, 1, \ldots, n\}$, such that

    - $0 \leq \psi_i(x) \leq 1$ for every $i \in \{0, 1, \ldots, n\}$ and every $x \in \mathbb{R}^d$,
    - $\psi_i \in C^\infty(\mathbb{R}^d)$ for every $i \in \{0, 1, \ldots, n\}$,
    - the support of ψ_0 (which is not necessarily compact) is contained in Ω,
    - the support of ψ_i is contained in A_i for every $i \in \{1, \ldots, n\}$,
    - it turns out that
    $$\sum_{i=0}^{n} \psi_i(x) = 1 \qquad \forall x \in \Omega.$$

Chapter 2

Normed and Banach Spaces

Linear and continuous operators

1. **Characterization of continuous linear functions**. Let V and W be two normed spaces, and let $f : V \to W$ be a linear function.

   Then the following facts are equivalent:

   (i) f is continuous in V in the ε/δ sense,

   (ii) f is sequentially continuous in V,

   (iii) the image of every bounded subset of V is bounded in W,

   (i') f is continuous in 0 in the ε/δ sense,

   (ii') f is sequentially continuous in 0,

   (iii') the image of the ball $B_V(0,1)$ is bounded in W,

   (iv) f is Lipschitz continuous.

2. **The space of continuous linear functions**. Let V and W be normed spaces. The set of all continuous and linear functions $f : V \to W$ is denoted by $\mathcal{L}(V,W)$.

   It turns out that

   - $\mathcal{L}(V,W)$ is a vector space,
   - $\mathcal{L}(V,W)$ is a normed space with respect to the norm

   $$\|f|_{\mathcal{L}(V,W)} := \sup\left\{\|f(v)\|_W : v \in V \text{ and } \|v\|_V \leq 1\right\},$$

   - if W is a Banach space, then $\mathcal{L}(V,W)$ is a Banach space as well.

   Note that the norm of f in $\mathcal{L}(V,W)$ is actually the Lipschitz constant of f.

3. **Topological dual space**. Let V be a normed space.

   The *(topological) dual* of V is the space V' of all continuous linear functions from V to $\mathbb{R}$, namely

   $$V' := \mathcal{L}(V,\mathbb{R}).$$

   The topological dual is always a Banach space, even if V is just a normed space, and the norm is given by

   $$\|f\|_{V'} = \sup\left\{|f(v)| : v \in V \text{ and } \|v\|_V \leq 1\right\}.$$

Hahn-Banach Theorem – Analytic form

1. **Definition (pseudo-norm).** Let V be a vector space. A *pseudo-norm* on V is a function $p : V \to \mathbb{R}$ satisfying the following two properties:

   - (sub-additivity) $p(x + y) \leq p(x) + p(y)$ for every $(x, y) \in V^2$,
   - (positive homogeneity) $p(\lambda x) = \lambda p(x)$ for every $x \in V$ and every real number $\lambda > 0$.

2. **Hahn-Banach theorem.** Let V be a vector space, and let p be a pseudo-norm on V. Let $E \subseteq V$ be a vector subspace, and let $f : E \to \mathbb{R}$ be a linear function such that

   $$f(x) \leq p(x) \qquad \forall x \in E.$$

   Then there exists a linear function $\widehat{f} : V \to \mathbb{R}$ that extends f, namely

   $$\widehat{f}(x) = f(x) \qquad \forall x \in E,$$

   and satisfies the estimate

   $$\widehat{f}(x) \leq p(x) \qquad \forall x \in V.$$

3. **Aligned functional.** Let V be a normed space, and let $x_0 \in V$.

   Then there exists $f \in V'$ such that

   $$\|f\|_{V'} \leq 1 \qquad \text{and} \qquad f(x_0) = \|x_0\|_V.$$

4. **Dual characterization of the norm.** Let V be a normed space, and let $x_0 \in V$.

   Then it turns out that

   $$\|x_0\| = \max\left\{f(x_0) : f \in V' \text{ and } \|f\|_{V'} \leq 1\right\},$$

   and in particular the maximum in the right-hand side exists.

Hahn-Banach Theorem – Geometric forms

1. **Definition (weak/strict separation of subsets).** Let V be a normed space, let V' be its topological dual, and let $A \subseteq V$ and $B \subseteq V$ be two disjoint subsets.

   - We say that A and B can be *weakly separated* if there exists $f \in V'$ such that
   $$f(b) < f(a) \qquad \forall a \in A, \quad \forall b \in B.$$
   - We say that A and B can be *strictly separated* if there exist $f \in V'$ and two real numbers $\delta_1 > \delta_2$ such that
   $$f(a) \geq \delta_1 \quad \forall a \in A \qquad \text{and} \qquad f(b) \leq \delta_2 \quad \forall b \in B.$$

2. **Convex-convex weak separation.** Let V be a normed space, and let V' be its topological dual. Let $A \subseteq V$ and $B \subseteq V$ be two nonempty subsets.

   Let us assume that

   - $A \cap B = \emptyset$,
   - A is convex and *open*,
   - B is convex.

   Then A and B can be *weakly* separated.

3. **Convex-convex strict separation.** Let V be a normed space, and let V' be its topological dual. Let $A \subseteq V$ and $B \subseteq V$ be two nonempty subsets.

   Let us assume that

   - $A \cap B = \emptyset$,
   - A is convex and *compact*,
   - B is convex and *closed*.

   Then A and B can be *strictly* separated.

4. **Gauge of a convex set.** Let V be a normed space, and let $C \subseteq V$ be a subset.

   Let us assume that

   - C is convex,
   - C is open,
   - C contains the origin.

   Let us define the function $p : V \to \mathbb{R}$ as
   $$p(x) := \inf \left\{ \alpha > 0 : \frac{x}{\alpha} \in C \right\} \qquad \forall x \in V.$$

   Then the following statements hold true.

- The function p is a pseudo-norm on V.
- There exists a real constant M such that

$$0 \leq p(x) \leq M\|x\|_V \qquad \forall x \in V.$$

- We can characterize the set C as the "open unit ball" with respect to p, namely

$$C = \{x \in V : p(x) < 1\}.$$

Weak and weak* convergence

1. **Definition (weak and weak* convergence).** Let V be a normed space, and let V' be its (topological) dual.

   - We say that a sequence $\{v_n\} \subseteq V$ converges weakly to some $v_\infty \in V$, and we write $v_n \rightharpoonup v_\infty$, if
   $$\lim_{n\to+\infty} f(v_n) = f(v_\infty) \qquad \forall f \in V'.$$
   - We say that a sequence $\{f_n\} \subseteq V'$ converges weakly* to some $f_\infty \in V'$, and we write $f_n \overset{*}{\rightharpoonup} f_\infty$, if
   $$\lim_{n\to+\infty} f_n(v) = f_\infty(v) \qquad \forall v \in V.$$

2. **Lower semi-continuity of the norm.** Let V be a normed space, and let V' be its topological dual.

   Then the following statements hold true.

   - If $v_n \rightharpoonup v_\infty$ is a weakly convergent sequence in V, then
   $$\liminf_{n\to+\infty} \|v_n\|_V \geq \|v_\infty\|_V.$$
   - If $f_n \overset{*}{\rightharpoonup} f_\infty$ is a weakly* convergent sequence in V', then
   $$\liminf_{n\to+\infty} \|f_n\|_{V'} \geq \|f_\infty\|_{V'}.$$

3. **Weak* compactness of balls.** Let V be a Banach space, let V' be its topological dual, and let $\{f_n\} \subseteq V'$ be a sequence.

   Let us assume that

   - the sequence $\{f_n\}$ is *bounded* in V',
   - V is *separable*.

   Then the sequence $\{f_n\}$ admits a weakly* convergent subsequence, namely there exist an increasing sequence $\{n_k\}$ of positive integers and $f_\infty \in V'$ such that $f_{n_k} \overset{*}{\rightharpoonup} f_\infty$.

Reflexive Spaces

1. **Canonical injection in the bidual**. Let V be a normed space, and let $(V')'$ be its topological bidual, namely the topological dual of the topological dual of V. The canonical injection is the map $J : V \to (V')'$ defined by

$$[Jv](f) := f(v) \qquad \forall v \in V, \quad \forall f \in V'.$$

   It turns out that

   - J is linear,
   - J is an isometry (and in particular it is continuous and injective).

2. **Reflexive spaces**. A normed space V is called *reflexive* if the canonical injection in the bidual $J : V \to (V')'$ is surjective.

   This implies in particular that V is a Banach space.

3. **Achtung!**. There are examples of Banach spaces V such that

   - V is not reflexive,
   - there exists a map $\Phi : V \to (V')'$ that is linear, bijective and an isometry (of course this map is not the canonical injection).

4. **Weak compactness of balls in reflexive spaces**. Let V be a Banach space, and let $\{v_n\} \subseteq V$ be a sequence.

   Let us assume that

   - the sequence $\{v_n\}$ is *bounded* in V,
   - V is *separable*,
   - V is *reflexive*.

   Then the sequence $\{v_n\}$ admits a weakly convergent subsequence, namely there exist an increasing sequence $\{n_k\}$ of positive integers and $v_\infty \in V$ such that $v_{n_k} \rightharpoonup v_\infty$.

5. **Separability vs separability of the dual**. Let V be a normed space, and let V' be its topological dual.

   - Without any further assumption it turns out that

$$V \text{ separable} \qquad \Longleftarrow \qquad V' \text{ separable},$$

     but the other implication is in general false.
   - If in addition V is reflexive, then it turns out that

$$V \text{ separable} \qquad \Longleftrightarrow \qquad V' \text{ separable}.$$

Baire spaces

1. **Definition (Baire spaces).** A topological space X is a *Baire space* if it satisfies any of the following three equivalent conditions.

   - For every sequence $\{A_n\}$ of open subsets of X, the following implication holds true
   $$A_n \text{ is dense in } X \text{ for every } n \in \mathbb{N} \qquad \Longrightarrow \qquad \bigcap_{n\in\mathbb{N}} A_n \text{ is dense in } X.$$
   - For every sequence $\{C_n\}$ of closed subsets of X, the following implication holds true
   $$\operatorname{Int}(C_n) = \emptyset \text{ for every } n \in \mathbb{N} \qquad \Longrightarrow \qquad \operatorname{Int}\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \emptyset.$$
   - For every sequence $\{C_n\}$ of closed subsets of X, the following implication holds true
   $$\operatorname{Int}\left(\bigcup_{n\in\mathbb{N}} C_n\right) \neq \emptyset \qquad \Longrightarrow \qquad \exists n_0 \in \mathbb{N} \text{ such that } \operatorname{Int}(C_{n_0}) \neq \emptyset.$$

2. **Large classes of Baire spaces.** The following are the classical examples of Baire spaces.

   - Every complete metric space is a Baire space.
   - Every locally compact topological space is a Baire space.
   - Every open subset of a Baire space is a Baire space (with respect to the topology inherited from the ambient space).

3. **F-sigma and G-delta sets.** Let X be a topological space. A subset $Y \subseteq X$ is called

   - a F_σ set if it can be written as the union of countably many closed subsets of X,
   - a G_δ set if it can be written as the intersection of countably many open subsets of X.

4. **Stability of F-sigma and G-delta sets.** In every topological space X it turns out that

   - the union of countably many F_σ sets is again a F_σ set,
   - the intersection of countably many G_δ sets is again a G_δ set.

5. **Nowhere dense, meager and residual sets.** Let X be a topological space. A subset $Y \subseteq X$ is called

   - *nowhere dense* if its *closure has empty interior*,
   - *meager* if it can be written as the union of countably many subsets of X that are nowhere dense,
   - *residual* if its complement $X \setminus Y$ is meager.

   A residual set can be written as the intersection of countably many open subsets of X whose interior part is dense in X.

6. **Discontinuity sets are F-sigma sets**. Let X and Y be two metric spaces, and let $f : X \to Y$ be any function.

   Then the set of discontinuity points of f, namely the set of all points $x \in X$ such that f is not continuous in x, is a F_σ set.

   Indeed, it can be written in the form

$$\bigcup_{n\in\mathbb{N}} \left\{ x \in X : \forall r > 0 \quad \exists (y,z) \in [B_X(x,r)]^2 \quad d_Y(f(y), f(z)) \geq 1/n \right\}.$$

7. **Pointwise limits of continuous functions have meager discontinuity set**. Let X be a Baire space, let Y be a metric space, let $f_n : X \to Y$ be a sequence of functions, and let $f_\infty : X \to Y$ be a function.

   Let us assume that

   - f_n is continuous for every $n \in \mathbb{N}$,
   - f_∞ is the pointwise limit of f_n, namely

$$\lim_{n\to+\infty} f_n(x) = f_\infty(x) \qquad \forall x \in X,$$

     where the limit is of course intended in Y.

   Then the set of discontinuity points of f_∞ is meager.

Banach-Steinhaus Theorem

1. **Banach-Steinhaus theorem (as an equivalence).** Let V be a Banach space, let W be a normed space, let I be an index set, and let $\{L_i\}_{i \in I}$ be a family of linear continuous operators $L_i : V \to W$.

   Then the following two conditions are equivalent.

   - *Qualitative boundedness.* It turns out that
   $$\sup_{i \in I} \|L_i(v)\|_W < +\infty \qquad \forall v \in V.$$
   - *Quantitative boundedness.* There exists $M \in \mathbb{R}$ such that
   $$\|L_i(v)\|_W \leq M\|v\|_V \qquad \forall v \in V, \quad \forall i \in I,$$
   which in turn is equivalent to saying that
   $$\sup_{i \in I} \|L_i\|_{\mathcal{L}(V,W)} < +\infty.$$

2. **Banach-Steinhaus theorem (as an alternative).** Let V be a Banach space, let W be a normed space, let I be an index set, and let $\{L_i\}_{i \in I}$ be a family of linear continuous operators $L_i : V \to W$.

   Then *one and only one* of the following two conditions holds true.

   - *Quantitative boundedness.* There exists $M \in \mathbb{R}$ such that
   $$\|L_i(v)\|_W \leq M\|v\|_V \qquad \forall v \in V, \quad \forall i \in I.$$
   - *Unboundedness on a residual set.* There exists a residual subset $A \subseteq V$ such that
   $$\sup_{i \in I} \|L_i(v)\|_W = +\infty \qquad \forall v \in A.$$

Open Mapping Theorem 1 – Statement

1. **Characterization of surjective functions**. Let X and Y be two sets, and let $f : X \to Y$ be a function.

   Then the following two conditions are equivalent:

   - f is surjective,
   - there exists a function $S : Y \to X$ such that

   $$f(S(y)) = y \qquad \forall y \in Y.$$

   The function S is sometimes called a *right inverse* of f, or a *qualitative solver* for the equation $f(x) = y$.

2. **Characterization of open mappings**. Let X and Y be two normed spaces, and let $f : X \to Y$ be a linear mapping (not necessarily continuous).

   Then the following three conditions are equivalent.

   - The function f is open, namely for every open set $A \subseteq X$ it turns out that $f(A)$ is an open subset of Y,
   - There exists $R > 0$ such that $F(B_X(0, R)) \supseteq B_Y(0, 1)$,

   $$f(S(y)) = y \qquad \forall y \in Y.$$

   - There exists a *quantitative solver*, namely a function $S : Y \to X$ and a constant $M \in \mathbb{R}$ such that

   $$f(S(y)) = y \qquad \text{and} \qquad \|S(y)\|_X \le M\|y\|_Y$$

   for all $y \in Y$.

3. **Open mapping theorem**. Let X and Y be two Banach spaces, and let $f : X \to Y$ be a linear and continuous mapping.

   Then the following two conditions are equivalent.

   - The mapping f is surjective (and hence equation $f(x) = y$ admits a *qualitative solver*).
   - The mapping f is open (and hence equation $f(x) = y$ admits a *quantitative solver*).

Open Mapping Theorem 2 – Quantitative solvers

1. **Approximation lemma**. Let V be a normed space, let $D \subseteq V$ be a dense subset, and let $v \in V$ be any vector.

   Then there exists a sequence $\{v_n\} \subseteq D$ such that
   $$\sum_{n=1}^{\infty} \|v_n\| \leq 2\|v\|,$$
   and
   $$v = \sum_{n=1}^{\infty} v_n,$$
   and in particular v is the sum of a normally convergent series of elements of D.

2. **Testing quantitative solvability on a dense subset**. Let X be a Banach space, let Y be a normed spaces, and let $f : X \to Y$ be a linear and continuous mapping.

   Then the following two conditions are equivalent.

   - There exists a quantitative solver, namely a function $S : Y \to X$ and a constant $M \in \mathbb{R}$ such that
     $$f(S(y)) = y \qquad \text{and} \qquad \|S(y)\|_X \leq M\|y\|_Y$$
     for all $y \in Y$.
   - There exists a quantitative solver on a dense subset, namely a dense subset $D \subseteq Y$, a function $\widehat{S} : D \to X$, and a constant $\widehat{M} \in \mathbb{R}$ such that
     $$f(\widehat{S}(y)) = y \qquad \text{and} \qquad \|\widehat{S}(y)\|_X \leq \widehat{M}\|y\|_Y$$
     for all $y \in D$.

3. **Existence of linear quantitative solvers**. Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective function.

   Then the following facts hold true.

   - If f is linear, then there exists a linear solver.
   - If f is open, then there exists a quantitative solver.
   - If f is linear and continuous, then it is open, and therefore there exists a quantitative solver.
   - If f is linear and continuous, in general there does not exist a linear quantitative solver.
   - If f is linear and continuous, then there exists a linear quantitative solver if and only if the kernel of f admits a topological complement in X, namely a closed subspace $V \subseteq X$ such that $X = \ker(f) \oplus V$.
   - A topological complement always exists if X is a Hilbert space.

Open Mapping Theorem 3 – Corollaries

1. **Continuity of the inverse**. Let X and Y be Banach spaces, and let $f : X \to Y$ be a linear and continuous mapping.

   Then the following two conditions are equivalent.

   - *Qualitative invertibility.* The function f is bijective.
   - *Quantitative invertibility.* The function f admits an inverse function $f^{-1} : Y \to X$ that is continuous (and linear).

2. **Equivalence of norms**. Let V be a vector space, and let $\|v\|_1$ and $\|v\|_2$ denote two norms on V.

   Let us assume that

   - the space V is complete with respect to both norms,
   - there exists a constant $M \in \mathbb{R}$ such that

   $$\|v\|_1 \leq M\|v\|_2 \qquad \forall v \in V.$$

   Then there exists a real constant $m > 0$ such that

   $$\|v\|_1 \geq m\|v\|_2 \qquad \forall v \in V,$$

   and hence the two norm are equivalent.

3. **Closed graph theorem**. Let X and Y be Banach spaces, and let $L : X \to Y$ be a linear mapping.

   Then the following two conditions are equivalent.

   - The mapping L is continuous.
   - The graph of L is closed in the product space $X \times Y$.

Weak convergence in Lebesgue spaces

1. **Definition**. Let $(\mathbb{X}, \mathcal{M}, \mu)$ be a measure space, let $p \geq 1$ be a real number, and let q be the conjugate exponent satisfying
$$\frac{1}{p} + \frac{1}{q} = 1.$$
Let $\{f_n\} \subseteq L^p(\mathbb{X}, \mathcal{M}, \mu)$ be a sequence of functions, and let $f_\infty \in L^p(\mathbb{X}, \mathcal{M}, \mu)$.

   - We say that f_n converges *weakly* to f_∞ in $L^p(\mathbb{X}, \mathcal{M}, \mu)$, and we write
   $$f_n \rightharpoonup f_\infty \qquad \text{in } L^p(\mathbb{X}, \mathcal{M}, \mu)$$
   if
   $$\int_{\mathbb{X}} f_n(x)g(x)\,d\mu = \int_{\mathbb{X}} f_\infty(x)g(x)\,d\mu \qquad \forall g \in L^q(\mathbb{X}, \mathcal{M}, \mu).$$
   - We say that f_n converges *weakly star* to f_∞ in $L^\infty(\mathbb{X}, \mathcal{M}, \mu)$, and we write
   $$f_n \overset{*}{\rightharpoonup} f_\infty \qquad \text{in } L^\infty(\mathbb{X}, \mathcal{M}, \mu)$$
   if
   $$\int_{\mathbb{X}} f_n(x)g(x)\,d\mu = \int_{\mathbb{X}} f_\infty(x)g(x)\,d\mu \qquad \forall g \in L^1(\mathbb{X}, \mathcal{M}, \mu).$$

2. **Reduced testing of weak (and weak star) convergence**.

Chapter 3

Hilbert Spaces

Scalar products

1. **Definition (scalar product)**. Let V be a vector space. A *scalar product* in V is a function $p : V^2 \to \mathbb{R}$ (usually denoted as $p(x, y) = \langle x, y\rangle$) with the following properties.

   - It is *symmetric* in the sense that
   $$\langle x, y\rangle = \langle y, x\rangle \qquad \forall (x, y) \in V^2.$$
   - It is *(bi)-linear* in the sense that
   $$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \qquad \forall (x, y, z) \in V^3,$$
   and
   $$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle \qquad \forall (x, y) \in V^2, \quad \forall \lambda \in \mathbb{R}.$$
   - It is *positive definite* in the sense that
   $$\langle x, x\rangle > 0 \qquad \forall x \in V \setminus \{0\}.$$

2. **Norm, distance, notion of convergence induced by a scalar product**. Let V be a vector space, and let $\langle x, y\rangle$ be a scalar product in V.

   Then in V we can introduce

   - a *norm* defined by
   $$\|x\| := \sqrt{\langle x, x\rangle} \qquad \forall x \in V,$$
   - a *distance* defined by
   $$d(x, y) := \|x - y\| \qquad \forall (x, y) \in V^2,$$
   - a *notion of convergence* defined by
   $$x_n \to x_\infty \qquad \Longleftrightarrow \qquad \|x_\infty - x_n\| \to 0.$$

3. **Cauchy-Schwarz inequality**. Let V be a vector space, let $\langle x, y\rangle$ be a scalar product in V, and let $\|x\|$ denote the norm in V induced by the scalar product.

   Then it turns out that
   $$|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \qquad \forall (x, y) \in V^2,$$
   with equality if and only if x and y are *linearly dependent* (namely if either $x = 0$ or $y = \lambda x$ for some real number λ).

4. **Continuity of the norm and the scalar product**. Let V be a vector space with a scalar product.

   Lets us assume that
   $$x_n \to x_\infty \qquad \text{and} \qquad y_n \to y_\infty.$$
   Then it turns out that
   $$\|x_n\| \to \|x_\infty\| \qquad \text{and} \qquad \langle x_n, y_n\rangle \to \langle x_\infty, y_\infty\rangle.$$

5. **Definition (Hilbert space)**. A Hilbert space is a vector space H with a scalar product that induces a distance with respect to which H turns out to be a *complete* metric space.

Orthonormal bases 1 – Definition and existence

1. **Definition (orthonormal basis)**. Let H be a Hilbert space with infinite dimension. A *orthonormal basis* (sometimes also called *Hilbert basis* or *complete orthonormal system*) is a sequence $\{e_n\} \subseteq H$ such that

   - the elements of the sequence are orthonormal vectors, namely

   $$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

   - $\text{Span}(\{e_n\})$ is dense in H.

2. **Theorem (existence of a countable orthonormal basis)**. Let H be a Hilbert space such that

   - the (algebraic) dimension of H is infinite,
   - H is separable (namely there exists a countable subset that is dense in H).

   Then there exists an orthonormal basis in H.

3. **Remark**. The previous statement is actually an "if and only if" result. Indeed, if a Hilbert space H admits a (countable) orthonormal basis $\{e_n\}$, then necessarily H is separable and has infinite dimension.

Orthonormal bases 2 – Components of vectors

1. **Components of a vector with respect to an orthonormal basis**. Let H be a Hilbert space. Let us assume that there exists an orthonormal basis $\{e_n\}$ of H.

   For every $x \in H$ and every $n \in \mathbb{N}$, the real number

   $$x_n := \langle x, e_n \rangle$$

   is called the *component* of x with respect to e_n. The sequence $\{x_n\}$ is called the *sequence of components* of x with respect to e_n.

2. **Properties of components with respect to an orthonormal basis**. Let H be a Hilbert space. Let us assume that there exists an orthonormal basis $\{e_n\}$ of H. Let $x \in H$, and let $\{x_n\}$ be the sequence of its components.

   Then the following statements hold true.

   - *Representation of the vector.* It turns out that

     $$x = \sum_{n=1}^{\infty} x_n e_n = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n.$$

   - *Representation of the norm.* It turns out that

     $$\|x\|^2 = \sum_{n=1}^{\infty} x_n^2 = \sum_{n=1}^{\infty} \langle x, e_n \rangle^2.$$

   - *Representation of the scalar product.* For every $y \in H$ with components $\{y_n\}$ it turns out that

     $$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \langle x, e_n \rangle \langle y, e_n \rangle.$$

Orthonormal bases 3 – Basic properties

1. **Characterization of convergent series**. Let H be a Hilbert space, let $\{e_n\} \subseteq H$ be any sequence of orthonormal vectors, and let $\{a_n\} \subseteq \mathbb{R}$ be any sequence of real numbers.

   Then it turns out that

   $$\sum_{n=1}^{\infty} a_n e_n \quad \text{converges} \qquad \Longleftrightarrow \qquad \sum_{n=1}^{\infty} a_n^2 \quad \text{converges.}$$

   Note that the series on the left is a series of vectors in H, while the series on the right is a series of nonnegative real numbers.

2. **Norm and components of the sum of a series**. Let H be a Hilbert space, let $\{e_n\} \subseteq H$ be any sequence of orthonormal vectors, and let $\{a_n\} \subseteq \mathbb{R}$ be any sequence of real numbers such that

   $$\sum_{n=1}^{\infty} a_n^2 < +\infty.$$

   Let us consider the vector

   $$v := \sum_{n=1}^{\infty} a_n e_n \in H,$$

   which is well-defined because of the characterization of convergent series.

   Then it turns out that

   $$\|v\|^2 = \sum_{n=1}^{\infty} a_n^2,$$

   and

   $$\langle v, e_n \rangle = a_n \qquad \forall n \in \mathbb{N}.$$

3. **Characterization of the null vector**. Let H be a Hilbert space, let $V \subseteq H$ be any subset such that $\operatorname{Span}(V)$ is dense in H.

   Let $x \in H$ be a vector such that

   $$\langle x, v \rangle = 0 \qquad \forall v \in V.$$

   Then we can conclude that $x = 0$.

Orthonormal bases 4 – The general case

1. **Definition (orthonormal basis – general case).** Let H be any Hilbert space. An *orthonormal basis* is a subset $\{e_i\}_{i\in I} \subseteq H$ (where I is an index set that can be finite, countable, or uncountale) such that

   - the elements of the subset are orthonormal vectors, namely for every i and j in I it turns out that
   $$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$
   - $\text{Span}(\{e_i : i \in I\})$ is dense in H.

2. **Theorem (existence of orthonormal bases – general case).** Every Hilbert space H admits a (finite, countable, or uncountable) orthonormal basis.

3. **Components with respect to an orthonormal basis – general case.** Let H be a Hilbert space, and let $\{e_i\}_{i\in I}$ be an orthonormal basis of H.

   Then for every $x \in H$ the following statements hold true.

   - *Non-zero components are at most countable.* The set
   $$I(x) := \{i \in I : \langle x, e_i\rangle \neq 0\}$$
   is finite or countable.
   - *Representation of the vector and its norm.* It turns out that
   $$x = \sum_{i\in I(x)} \langle x, e_i\rangle e_i$$
   and
   $$\|x\|^2 = \sum_{i\in I(x)} \langle x, e_i\rangle^2.$$
   - *Representation of the scalar product.* For every $y \in H$ it turns out that
   $$\langle x, y\rangle = \sum_{i\in I(x)\cap I(y)} \langle x, e_i\rangle\langle y, e_i\rangle.$$

   We point out that all summations involve only a finite or countable number of terms (and all series are convergent when the number of terms is countable).

Weak convergence 1 – Definition and basic properties

1. **Definition (weak convergence).** Let H be a Hilbert space, let $\{v_n\} \subseteq H$ be a sequence, and let $v_\infty \in H$.

   We say that v_n converges *weakly* to v_∞, and we write

   $$v_n \rightharpoonup v_\infty$$

   if it happens that

   $$\langle v_n, v \rangle \to \langle v_\infty, v \rangle \qquad \forall v \in H.$$

2. **Basic properties of weak convergence.** Let H be a Hilbert space.

   - *Uniqueness.* The weak limit, when it exists, is unique.
   - *Strong implies weak.* If $x_n \to x_\infty$, then it turns out that also $x_n \rightharpoonup x_\infty$.
   - *Linearity.* If $x_n \rightharpoonup x_\infty$ and $y_n \rightharpoonup y_\infty$, then it turns out that

     $$ax_n + by_n \rightharpoonup ax_\infty + bx_\infty \qquad \forall (a, b) \in \mathbb{R}^2.$$

   - *Subsequences.* If $x_n \rightharpoonup x_\infty$, then every subsequence x_{n_k} converges weakly to x_∞ as well.
   - *Equivalence in finite dimension.* If the dimension of H is finite, then $x_n \to x_\infty$ if and only if $x_n \rightharpoonup x_\infty$.
   - *Non-equivalence in infinite dimension.* If the dimension of H is infinite, then there exists a sequence $\{x_n\} \subseteq H$ such that $x_n \rightharpoonup x_\infty$ for some $x_\infty \in H$, while $\{x_n\}$ is not strongly convergent (and actually it has no strongly convergent subsequence).

     For example, any orthonormal sequence $\{e_n\}$ (not necessarily with dense span) has this property.

3. **Passing to the limit in scalar products.** Let H be a Hilbert space, and let $\{v_n\} \subseteq H$ and $\{w_n\} \subseteq H$ be two sequences such that

   - $v_n \to v_\infty$ (strong convergence),
   - $w_n \rightharpoonup w_\infty$ (weak convergence),
   - w_n is bounded (this assumption actually follows from weak convergence).

   Then we can pass to the limit in scalar products, in the sense that

   $$\langle v_n, w_n \rangle \to \langle v_\infty, w_\infty \rangle.$$

Weak convergence 2 – Convergence of components

1. **Reduced testing of weak convergence**. Let H be a Hilbert space, let $\{x_n\} \subseteq H$ be a sequence, let $x_\infty \in H$, and let $V \subseteq H$ be a subset.

   Let us assume that

   - the sequence $\{x_n\}$ is *bounded*,
   - $\langle x_n, v\rangle \to \langle x_\infty, v\rangle$ for every $v \in V$,
   - $\mathrm{Span}(V)$ is dense in H.

   Then it turns out that $x_n \rightharpoonup x_\infty$.

   In other words this means that, in the case of *bounded sequences*, we can limit ourselves to testing weak convergence against a set V that spans a dense subset of H.

2. **Weak convergence and convergence of components**. Let H be a *separable* Hilbert space, let $\{e_n\} \subseteq H$ be an orthonormal basis, let $\{v_n\} \subseteq H$ be a sequence, and let $v_\infty \in H$.

   Let us assume that the sequence $\{v_n\}$ is *bounded*.

   Then it turns out that

   $$v_n \rightharpoonup v_\infty \qquad \Longleftrightarrow \qquad \langle v_n, e_i\rangle \to \langle v_\infty, e_i\rangle \quad \forall i \in \mathbb{N}.$$

   In other words this means that, in the case of *bounded sequences* in *separable* Hilbert spaces, weak convergence is equivalent to convergence of components.

Weak convergence 3 – The core business

1. **Lower semicontinuity of the norm**. Let H be a Hilbert space, and let $v_n \to v_\infty$ be a weakly convergent sequence.

   Then it turns out that

   $$\liminf_{n\to+\infty} \|v_n\| \geq \|v_\infty\|.$$

   This amounts to saying that the norm is lower semicontinuous with respect to weak convergence.

2. **Weak compactness of balls**. Let H be a Hilbert space, and let $\{v_n\} \subseteq H$ be a sequence.

   Let us assume that

   - H is *separable*,
   - the sequence $\{v_n\}$ is *bounded*.

   Then there exists a weakly convergent subsequence, namely there exist an increasing sequence $\{n_k\}$ of positive integers and $v_\infty \in H$ such that $v_{n_k} \rightharpoonup v_\infty$.

Parallelogram identity

1. **Parallelogram law**. Let V be a vector space with a scalar product, and let $\|x\|$ be the norm induced by the scalar product.

   Then it turns out that

   $$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 \qquad \forall (x,y) \in V^2.$$

   This relation is known as *parallelogram law* or *parallelogram identity*.

   Geometrically, it means that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

2. **Theorem (Jordan-Fréchet-von Neumann)**. Let V be a vector space, and let $\|v\|$ be a norm on V.

   Then the norm originates from a scalar product if and only if the parallelogram identity holds true for every $(x,y) \in V^2$.

   In this case the scalar product is given by the so-called *polarization identity*

   $$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \qquad \forall (x,y) \in V^2.$$

Projection onto a closed convex set

1. **Projection onto a closed convex set**. Let H be a Hilbert space, let $K \subseteq H$ be a nonempty *closed convex* subset, and let $x_0 \in H$.

   Then there exists a unique point in K that minimizes the distance from x_0. More precisely, the following statements hold true.

   - *Existence and uniqueness.* There exists a unique point $P_K x_0 \in K$ such that
   $$\|x_0 - P_K x_0\| \leq \|x_0 - y\| \qquad \forall y \in K.$$
   - *1-Lipschitz continuity.* The function $P_K : H \to K$ is 1-lipschitz continuous, namely
   $$\|P_K y - P_K x\| \leq \|y - x\| \qquad \forall (x, y) \in H^2.$$
   - *Characterization.* The point $P_K x_0$ is the *unique* point in K such that
   $$\langle x_0 - P_K x_0, y - P_K x_0 \rangle \leq 0 \qquad \forall y \in K.$$

2. **Projection onto a closed subspace**. Let H be a Hilbert space, let $V \subseteq H$ be a *closed* subspace, and let $P_V : H \to V$ be the projection onto V.

   Then the following statements hold true.

   - For every $x \in H$, it turns out that $P_V x$ is the unique point in V such that
   $$\langle x - P_V x, v \rangle = 0 \qquad \forall v \in V.$$
   - The projection P_V is linear.
   - Setting
   $$V^\perp := \{x \in H : \langle x, v \rangle = 0 \ \forall v \in V\},$$
   it turns out that $V^\perp$ is a closed subspace of H and
   $$H = V \oplus V^\perp.$$

3. **Projection onto a closed ball**. Let H be a Hilbert space, and let $B \subseteq H$ be the *closed* ball with center in the origin and radius 1.

   Then for every $x \in H$ the projection $P_B x$ is given by
   $$P_B x := \begin{cases} x & \text{if } \|x\| \leq 1, \\ \dfrac{x}{\|x\|} & \text{if } \|x\| \geq 1. \end{cases}$$

Separation of convex sets

1. **Point-convex strong separation**. Let H be a Hilbert space, let $K \subseteq H$ be a nonempty closed convex subset, and let $x_0 \in H \setminus K$.

   Then there exist $w \in H$, and two real numbers $\delta_1 > \delta_2$ such that

   $$\langle w, x_0 \rangle = \delta_1 \qquad \text{and} \qquad \langle w, y \rangle \leq \delta_2 \quad \forall y \in K.$$

   More precisely, we can take

   $$w := x_0 - P_k x_0, \qquad \delta_1 := \langle x_0, w \rangle, \qquad \delta_2 := \langle P_K x_0, w \rangle.$$

2. **Convex-convex strict separation**. Let H be a Hilbert space, and let $A \subseteq H$ and $B \subseteq H$ be two nonempty subsets.

   Let us assume that

   - $A \cap B = \emptyset$,
   - A is convex and *compact*,
   - B is convex and *closed*.

   Then the following statements hold true.

   - *Existence and uniqueness of closest points.* There exists a unique pair $(a_0, b_0) \in A \times B$ such that
     $$\|a - b\| \geq \|a_0 - b_0\| \qquad \forall (a, b) \in A \times B,$$
     and it turns out that
     $$b_0 = P_B a_0 \qquad \text{and} \qquad a_0 = P_A b_0.$$
   - *Strong separation.* There exist $w \in H$, and two real numbers $\delta_1 > \delta_2$ such that
     $$\langle w, a \rangle \geq \delta_1 \quad \forall a \in A \qquad \text{and} \qquad \langle w, y \rangle \leq \delta_2 \quad \forall b \in B.$$
     More precisely, we can take
     $$w := a_0 - b_0, \qquad \delta_1 := \langle a_0, w \rangle, \qquad \delta_2 := \langle b_0, w \rangle.$$

Strong and convex implies weak

1. **Strongly closed and convex implies weakly closed**. Let H be a Hilbert space, and let $K \subseteq H$ be a nonempty subset.

   Let us assume that

   - K is *strongly closed*, namely for every strongly convergent sequence $x_n \to x_\infty$, with $x_n \in K$ for every $n \in \mathbb{N}$, it turns out that $x_\infty \in K$,
   - K is convex.

   Then K is *weakly closed*, namely for every weakly convergent sequence $x_n \rightharpoonup x_\infty$, with $x_n \in K$ for every $n \in \mathbb{N}$, it turns out that $x_\infty \in K$

2. **Strongly LSC and convex implies weakly LSC**. Let H be a Hilbert space, and let $F : H \to \mathbb{R}$ be a function.

   Let us assume that

   - F is *strongly lower semicontinuous*, namely for every strongly convergent sequence $x_n \to x_\infty$ it turns out that
     $$\liminf_{n \to +\infty} F(x_n) \geq F(x_\infty), \tag{LSC}$$
   - F is convex.

   Then F is *weakly lower semicontinuous*, namely (LSC) holds true for every weakly convergent sequence $x_n \rightharpoonup x_\infty$.

Compact operators 1 – Basic properties

1. **Definition (compact operator).** Let X be a normed space, and let Y be a metric space. A function $f : X \to Y$ is called *compact* if the image of every *bounded* subset of X is a *relatively compact* subset of Y.

   This is equivalent to saying that, for every bounded sequence $\{x_n\} \subseteq X$, the sequence $\{f(x_n)\}$ of the images admits a subsequence that is convergent in Y.

2. **Strong-to-strong continuity of linear compact operators.** Let X and Y be normed spaces, and let $f : X \to Y$ be a function.

   Let us assume that

   - f is linear,
   - f is compact.

   Then f is continuous with respect to strong convergences, namely for every convergent sequence $x_n \to x_\infty$ in X it turns out that $f(x_n) \to f(x_\infty)$ in Y.

3. **Definition (symmetric operator in a Hilbert space).** Let H be a Hilbert space. A linear operator $A : H \to H$ is called *symmetric* if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \qquad \forall (x, y) \in H^2.$$

4. **Basic properties of symmetric operators.** Let H be a Hilbert space, and let $A : H \to H$ be a linear and symmetric operator.

   Then the following statements hold true.

   - Eigenvectors corresponding to distinct eigenvalues are necessarily orthogonal.
   - If $V \subseteq H$ is a subspace that is A-invariant (namely $Av \in V$ for every $v \in V$), then $V^\perp$ is A-invariant as well.

5. **Weak-to-strong continuity of symmetric compact operators.** Let H be a Hilbert space, and let $A : H \to H$ be an operator.

   Let us assume that

   - f is linear and symmetric,
   - f is compact.

   Then f is continuous from the weak convergence to the strong convergence, namely for every weakly convergent (and therefore bounded) sequence $x_n \rightharpoonup x_\infty$ in H it turns out that $f(x_n) \to f(x_\infty)$ strongly in H.

Compact operators 2 – Spectral Theorem

1. **Spectral theorem for compact operators**. Let H be a Hilbert space, and let $A : H \to H$ be an operator.

   Let us assume that

   (i) A is linear and symmetric,

   (ii) A is compact,

   (iii) H is separable with infinite dimension.

   Then the following statements hold true.

   - There exists an orthonormal basis of H made by eigenvectors of A.
   - The eigenspace of every non-zero eigenvalue has finite dimension.
   - Non-zero eigenvalues are either a finite set, or a sequence $\{\lambda_n\}$ with $|\lambda_n| \to 0$.
   - The set of eigenvalues is a bounded subset of $\mathbb{R}$, and the only possible accumulation point is the origin.

Compact operators 3 – Rayleigh quotient

1. **Definition (Rayleigh quotient).** Let H be a Hilbert space, and let $A : H \to H$ be an operator.

   The *Rayleigh quotient* is the function $q : H \setminus \{0\} \to \mathbb{R}$ defined by

$$q(x) := \frac{\langle Ax, x\rangle}{\|x\|^2} \qquad \forall x \in H \setminus \{0\}.$$

2. **Properties of the Rayleigh quotient.** Let H be a Hilbert space, let $A : H \to H$ be an operator, let $q : H \setminus \{0\} \to \mathbb{R}$ be the Rayleigh quotient, and let $V \subseteq H$ be a closed subspace with $V \neq \{0\}$.

   Let us assume that

   (i) A is linear and symmetric,

   (ii) A is compact,

   (iii) V is A-invariant, namely $Av \in V$ for every $v \in V$.

   Then the following statements hold true.

   - There exists
$$\lambda_0 := \max\{|q(x)| : x \in V \setminus \{0\}\},$$
     and it can be characterized as follows:
$$\lambda_0 = \max\{|\langle Ax, x\rangle| : x \in V \text{ and } \|x\| = 1\} = \max\{|\langle Ax, x\rangle| : x \in V \text{ and } \|x\| \leq 1\}.$$

   - If $\lambda_0 = |q(x_0)|$ for some $v_0 \in V$, then $q(v_0)$ (without the absolute value) is an eigenvalue of A, and all non-zero vectors $v \in V$ with $q(v) = q(v_0)$ are eigenvectors of A with eigenvalue $q(v_0)$.

Compact operators 4 – Linear approximation

1. **Uniform limit of compact operators is compact**. Let X be a normed space, let Y be a metric space, let $f_n : X \to Y$ be a sequence of operators, and let $f_\infty : X \to Y$.

   Let us assume that

   - Y is complete,
   - f_n is a compact operator for every $n \in \mathbb{N}$,
   - $f_n \to f_\infty$ uniformly on bounded subsets of X.

   Then f_∞ is a compact operator.

2. **Linear approximation of compact operators in Hilbert spaces**. Let X be a normed space, let H be a separable Hilbert space, and let $f : X \to H$ be a compact operator (not necessarily linear or continuous).

   Let $\{e_n\}$ be an orthonormal basis, let $H_n := \mathrm{Span}(e_1, \ldots, e_n)$ be the finite dimensional subspace spanned by the first n vectors of the basis, let $P_n : H \to H_n$ denote the orthogonal projection onto H_n, and let $f_n : X \to H_n$ be defined by

   $$f_n(x) := P_n(f(x)) \qquad \forall x \in X.$$

   Then $f_n \to f$ uniformly on bounded subsets of X.

   In addition

   - if f is linear, then also f_n is linear for every $n \geq 1$,
   - if f is continuous, then also f_n is continuous for every $n \geq 1$.

3. **Characterization of compact operators**. Let X be a normed space, let H be a separable Hilbert space, and let $f : X \to H$ be any operator.

   Then the following statements hold true.

   - The operator f is compact if and only if it can be approximated, uniformly on bounded subsets of X, by a sequence of compact operators with finite dimensional range (namely whose image is contained in a finite dimensional subspace of H, of course depending on n).
   - The operator f is continuous and compact if and only if it can be approximated, uniformly on bounded subsets of X, by a sequence of continuous operators with finite dimensional range.
   - The operator f is linear, continuous and compact if and only if it can be approximated, uniformly on bounded subsets of X, by a sequence of linear continuous operators with finite dimensional range. We point out that, in the case of linear continuous operators, uniform convergence on bounded sets is equivalent to convergence in the space $\mathcal{L}(X, H)$ (namely convergence in norm as operators).

Compact operators 5 – Nonlinear approximation

1. **Nonlinear projection in normed spaces**. Let Y be a normed space, let $K \subseteq Y$ be a *compact* set, and let $\varepsilon > 0$.

   Then there exist a subset $K_\varepsilon \subseteq Y$, and a function $P_\varepsilon : K \to K_\varepsilon$ such that

   - K_ε is the convex hull of a *finite* subset of K (and in particular K_ε is *convex* and *compact*),
   - P_ε is continuous and satisfies
   $$\|P_\varepsilon y - y\| \leq \varepsilon \qquad \forall y \in K.$$

2. **Nonlinear approximation of functions with compact range**. Let S be a set, let Y be a normed space, let $K \subseteq Y$ be a compact set, and let $f : S \to K$ be any function.

   Then there exist a sequence of subsets $K_n \subseteq Y$, and a sequence of functions $f_n : S \to K_n$ such that

   - for every $n \in \mathbb{N}$ the set K_n is the convex hull of a finite subset of K (and in particular K_n is convex and compact),
   - $f_n \to f$ uniformly in S.

   If in addition S is a metric space and f is continuous, then also the functions f_n can be chosen to be continuous for every $n \geq 1$.

3. **Nonlinear approximation of compact operators in normed spaces**. Let X and Y be normed spaces, and let $f : X \to Y$ be a compact operator (not necessarily linear or continuous).

   Then there exist a sequence of subspaces $Y_n \subseteq Y$, and a sequence of functions $f_n : X \to Y_n$ such that

   - Y_n is a finite dimensional subspace of Y for every $n \in \mathbb{N}$,
   - $f_n \to f$ uniformly on bounded subsets of X.

   If in addition f is continuous, then also the functions f_n can be chosen to be continuous for every $n \geq 1$ (on the contrary, linearity cannot be preserved in general).

4. **Characterization of compact operators**. Let X and Y be normed spaces, and let $f : X \to Y$ be any operator.

   Then the following statements hold true.

   - The operator f is compact if and only if it can be approximated, uniformly on bounded subsets of X, by a sequence of compact operators with finite dimensional range (namely whose image is contained in a finite dimensional subspace of Y, of course depending on n).
   - The operator f is continuous and compact if and only if it can be approximated, uniformly on bounded subsets of X, by a sequence of continuous operators with finite dimensional range.

Schauder fixed point Theorem

1. **Brouwer fixed point theorem (classical statement)**. Let d be a positive integer, let $\overline{B}_d(0,1)$ be the closed unit ball in $\mathbb{R}^d$ with center in the origin, and let $f : \overline{B}_d(0,1) \to \overline{B}_d(0,1)$ be a continuous function.

   Then f admits *at least* one fixed point.

2. **Brouwer fixed point theorem (more general statement)**. Let d be a positive integer, let $D \subseteq \mathbb{R}^d$ be a subset, and let $f : D \to D$ be a function.

   Let us assume that

   - D is *convex* and *compact*,
   - f is continuous.

   Then f admits *at least* one fixed point.

3. **Schauder fixed point theorem**. Let Y be a normed space, let $C \subseteq Y$ be a *convex* set, let $K \subseteq C$ be a *compact* set, and let $f : C \to K$ be a continuous function.

   Then f admits *at least* one fixed point.

Chapter 4

Sobolev Spaces

Weak derivatives – Definition W – 1D

1. **Definition (Weak derivative – Definition W)**. Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $u \in L^1_{\mathrm{loc}}((a,b))$.

   We say that u has a W-weak derivative in (a,b) if there exists $v \in L^1_{\mathrm{loc}}((a,b))$ such that

$$\int_a^b u(x)\varphi'(x)\,dx = -\int_a^b v(x)\varphi(x)\,dx \qquad \forall \varphi \in C_c^\infty((a,b)).$$

   In this case v is called the W-weak derivative of u.

2. **Basic properties**. Let $(a,b) \subseteq \mathbb{R}$ be an interval.

   - *Uniqueness.* The W-weak derivative of a function $u \in L^1_{\mathrm{loc}}((a,b))$, if it exists, is unique.
   - *Linearity.* Let $W^{1,1}_{\mathrm{loc}}((a,b))$ denote the set of all functions $u \in L^1_{\mathrm{loc}}((a,b))$ that admit a W-weak derivative.

     Then $W^{1,1}_{\mathrm{loc}}((a,b))$ is a vector space, and the weak derivative is a linear application

$$W^{1,1}_{\mathrm{loc}}((a,b)) \to L^1_{\mathrm{loc}}((a,b)).$$

   - *Compatibility with the classical notion (trivial fact).* If $u \in C^1((a,b))$, then $u'(x)$ (the classical derivative) is also the W-weak derivative of u.
   - *Compatibility with the classical notion (less trivial fact).* If $u \in L^1_{\mathrm{loc}}((a,b))$, and its W-weak derivative v belongs to $C^0((a,b))$, then actually $u \in C^1((a,b))$ (in the sense that it coincides almost everywhere in (a,b) with a function of class C^1).

3. **Stability when passing to the limit**. Let $\{u_n\}$ and $\{v_n\}$ be two sequences of functions in $L^1_{\mathrm{loc}}((a,b))$, and let u_∞ and v_∞ be in $L^1_{\mathrm{loc}}((a,b))$ as well.

   Let us assume that

   (i) v_n is the W-weak derivative of u_n for every $n \in \mathbb{N}$,

   (ii) $u_n \to u_\infty$ in $L^1_{\mathrm{loc}}((a,b))$,

   (iii) $v_n \to v_\infty$ in $L^1_{\mathrm{loc}}((a,b))$.

   Then v_∞ is the W-weak derivative of u_∞.

4. **Stronger stability result**. In the previous result we can weaken the second and third assumption by just asking that

$$u_n \rightharpoonup u_\infty \quad \text{weakly in } L^1(\Omega') \qquad \text{and} \qquad v_n \rightharpoonup v_\infty \quad \text{weakly in } L^1(\Omega')$$

   for every open set $\Omega' \subset\subset (a,b)$.

   Actually even weaker notions of convergence guarantee the same result.

Sobolev Spaces – Definition W – 1D

1. **Definition (Sobolev spaces – Definition W).** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $p \in [1, +\infty]$ (including endpoints).

   The Sobolev space $W^{1,p}((a, b))$ is the set of all functions $u \in L^p((a, b))$ that admit a W-weak derivative $v \in L^p((a, b))$.

2. **Sobolev functions are antiderivatives of their W-weak derivatives.** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $p \in [1, +\infty]$ (including endpoints), let $u \in W^{1,p}((a, b))$, and let v be its W-weak derivative. Let us consider the integral function $V : [a, b] \to \mathbb{R}$ defined by

$$V(x) := \int_a^x v(t)\, dt \qquad \forall x \in [a, b].$$

   Then there exists a constant $c \in \mathbb{R}$ such that $u(x) = V(x) + c$ for almost every $x \in (a, b)$.

3. **Continuity and pointwise values of Sobolev functions.** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $p \in [1, +\infty]$ (including endpoints). The following facts are a corollary of the previous statement.

   - It turns out that
$$W^{1,p}((a, b)) \subseteq C^0([a, b]),$$
     in the sense that every Sobolev function coincides almost everywhere with a continuous function defined up to the boundary.
   - The existence of a continuous function that coincides almost everywhere with u allows to consider pointwise values of u, up to the boundary (so that also $u(a)$ and $u(b)$ are well-defined).

4. **Hölder continuity of Sobolev functions.** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $p \in (1, +\infty]$ (note that $p = 1$ is not included, while $p = +\infty$ is allowed).

   Let p' denote the conjugate exponent of p, defined through the relation (with obvious meaning in the case $p = +\infty$)
$$\frac{1}{p} + \frac{1}{p'} = 1.$$

   Then u is $(1/p')$-Hölder continuous in $[a, b]$, and more precisely

$$|u(y) - u(x)| \leq \|u'\|_{p,(a,b)} \cdot |y - x|^{1/p'} \qquad \forall (x, y) \in [a, b]^2,$$

   where $\|u'\|_{p,(a,b)}$ denotes the norm in $L^p((a, b))$ of the W-weak derivative u' of u.

   As always, this statement has to be interpreted in the sense that u coincides almost everywhere with a function satisfying the above relation.

5. **Achtung!**. Functions in $W^{1,1}((a, b))$ are continuous as well, but they are not necessarily Hölder continuous of any order.

Weak derivatives – Definition H – 1D

1. **Definition (Weak derivative – Definition H)**. Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $u \in L^1_{\text{loc}}((a,b))$.

   We say that u has a H-weak derivative in (a,b) if there exist $v \in L^1_{\text{loc}}((a,b))$ and a sequence $\{u_n\} \subseteq C^\infty((a,b))$ (not necessarily with compact support) such that

   - $u_n \to u$ in $L^1_{\text{loc}}((a,b))$,
   - $u_n' \to v$ in $L^1_{\text{loc}}((a,b))$.

   In this case v is called the H-weak derivative of u.

   We recall that convergence in $L^1_{\text{loc}}((a,b))$ means convergence in $L^1(\Omega')$ for every open set $\Omega' \subset\subset (a,b)$.

2. **Regularity of approximating functions**. We stated the definition of H-weak derivatives using an approximating sequence in $C^\infty((a,b))$. Nevertheless, we obtain the same notion even if we start from $C^1((a,b))$, or any space in between.

   On the contrary, we obtain a different notion if we start from functions with compact support.

3. **H-weak derivatives = W-weak derivatives**. Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let u and v be in $L^1_{\text{loc}}((a,b))$.

   Then v is the W-weak derivative of u if and only if v is the H-weak derivative of u.

   In particular, H-weak derivatives have all the properties of W-derivatives (uniqueness, linearity, compatibility with the classical notion, stability when passing to the limit).

4. **The convergence of functions is actually uniform**. Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let u, v, and $\{u_n\}$ be as in the definition of H-weak derivatives.

   Then actually $u_n \to u$ uniformly on compact subsets of (a,b) (in the usual sense that u_n converges uniformly in (a,b) to a function that coincides almost everywhere with u).

Sobolev Spaces – Definition H – 1D

1. **Definition (Sobolev Spaces – Definition H).** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $p \in [1, +\infty]$ (including endpoints). Let us set

$$C^{1,p}((a,b)) := \{u \in C^\infty((a,b)) : u \in L^p((a,b)) \text{ and } u' \in L^p((a,b))\}.$$

   It turns out that $C^{1,p}((a,b))$ is a vector space, and

$$\|u\|_{1,p,(a,b)} := \|u\|_{L^p((a,b))} + \|u'\|_{L^p((a,b))}$$

   is a norm on this space.

   The Sobolev space $H^{1,p}((a,b))$ is the completion of $C^{1,p}((a,b))$ with respect to this norm.

2. **Equivalent definition.** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $p \in [1, +\infty]$ (including endpoints).

   The Sobolev space $H^{1,p}((a,b))$ is the set of all functions $u \in L^p((a,b))$ for which there exist $v \in L^p((a,b))$ and a sequence $\{u_n\} \subseteq C^\infty((a,b))$ (not necessarily with compact support) such that

   - $u_n \to u$ in $L^p((a,b))$,
   - $u'_n \to v$ in $L^p((a,b))$.

3. **Regularity of approximating functions.** We stated the definition H of Sobolev spaces starting from functions in $C^\infty((a,b))$. Nevertheless, we obtain the same notion even if we start from $C^1((a,b))$, or any space in between.

   On the contrary, we obtain a different notion if we start from functions with compact support.

4. **The convergence of functions is actually uniform.** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let p, u, v, and $\{u_n\}$ be as in the definition of H-weak derivatives.

   Then the functions u_n can be extended as continuous functions up to the boundary of (a, b), and actually $u_n \to u$ uniformly in $[a, b]$ (in the usual sense).

5. **Big theorem (H=W in an interval).** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $p \in [1, +\infty)$ (note that $p = +\infty$ is not allowed).

   Then it turns out that

$$W^{1,p}((a,b)) = H^{1,p}((a,b)).$$

Weak derivatives – Definition W – Any dimension

1. **Definition (W-Weak derivative)**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $\alpha \in \mathbb{N}^d$ be a multi-index, and let $u \in L^1_{\text{loc}}(\Omega)$.

   We say that u has a W-weak derivative of order α in Ω if there exists $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega u(x) \cdot D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v(x) \cdot \varphi(x)\, dx \qquad \forall \varphi \in C_c^\infty(\Omega).$$

   In this case v is called the W-weak derivative of order α of u, and we set $D^\alpha u := v$.

2. **Basic properties**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $\alpha \in \mathbb{N}^d$ be a multi-index.

   - *Uniqueness.* The W-weak derivative of order α of a function $u \in L^1_{\text{loc}}(\Omega)$, if it exists, is unique.
   - *Linearity.* Let $V_\alpha(\Omega)$ (there in no official name for this set) denote the set of all functions $u \in L^1_{\text{loc}}(\Omega)$ that admit a W-weak derivative of order α.

     Then $V_\alpha(\Omega)$ is a vector space, and the weak derivative of order α is a linear application

$$V_\alpha(\Omega) \to L^1_{\text{loc}}((a, b)).$$

   - *Compatibility with the classical notion.* If $u \in C^{|\alpha|}(\Omega)$, then $D^\alpha u(x)$ (the classical partial derivative) is also the W-weak derivative of order α of u.

3. **Stability when passing to the limit**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $\alpha \in \mathbb{N}^d$ be a multi-index. Let $\{u_n\}$ and $\{v_n\}$ be two sequences of functions in $L^1_{\text{loc}}(\Omega)$, and let u_∞ and v_∞ be in $L^1_{\text{loc}}(\Omega)$ as well.

   Let us assume that

   (i) v_n is the W-weak derivative of order α of u_n for every $n \in \mathbb{N}$,

   (ii) $u_n \rightharpoonup u_\infty$ weakly in $L^1_{\text{loc}}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$,

   (iii) $v_n \rightharpoonup v_\infty$ weakly in $L^1_{\text{loc}}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

   Then v_∞ is the W-weak derivative of order α of u_∞.

Weak derivatives – Definition H – Any dimension

1. **Definition (H-Weak derivative)**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $\alpha \in \mathbb{N}^d$ be a multi-index, and let $u \in L^1_{\text{loc}}(\Omega)$.

   We say that u has a H-weak derivative of order α in Ω if there exist $v \in L^1_{\text{loc}}(\Omega)$ and a sequence $\{u_n\} \subseteq C^\infty(\Omega)$ (not necessarily with compact support) such that

   - $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$,
   - $D^\alpha u_n \to v$ in $L^1_{\text{loc}}(\Omega)$.

   In this case v is called the H-weak derivative of order α of u.

   We recall that convergence in $L^1_{\text{loc}}(\Omega)$ means convergence in $L^1(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

2. **Regularity of approximating functions**. We stated the definition of H-weak derivatives of order α by using an approximating sequence in $C^\infty(\Omega)$. Nevertheless, we obtain the same notion even if we start from functions in $C^{|\alpha|}(\Omega)$, or any space in between.

   On the contrary, we obtain a different notion if we start from functions with compact support.

3. **Equivalence between W-weak and H-weak derivatives**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $\alpha \in \mathbb{N}^d$ be a multi-index, and let u and v be functions in $L^1_{\text{loc}}(\Omega)$.

   Then v is the W-weak derivative of order α of u if and only if v is the H-weak derivative of order α of u.

   In particular, H-weak derivatives have all the properties of W-derivatives (uniqueness, linearity, compatibility with the classical notion, stability when passing to the limit).

4. **Approximation result**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $\alpha \in \mathbb{N}^d$ be a multi-index, and let $u \in L^1_{\text{loc}}(\Omega)$ be a function that admits a weak derivative $D^\alpha u$ of order α.

   Then there exists a sequence $\{u_n\} \subseteq C^\infty(\Omega)$ such that

   - $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$,
   - $D^\alpha u_n \to D^\alpha u$ in $L^1_{\text{loc}}(\Omega)$.

   If in addition $u \in L^p_{\text{loc}}(\Omega)$ and $D^\alpha u \in L^p_{\text{loc}}(\Omega)$ for some $p \in [1, +\infty)$ (note that $p = +\infty$ is excluded), then we can assume that the convergence is in $L^p_{\text{loc}}(\Omega)$ as well.

Sobolev Spaces – Definition W and H – Any dimension

1. **Sobolev spaces – Definition W**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty]$ (including endpoints).

   The Sobolev space $W^{m,p}(\Omega)$ is the set of all functions $u \in L^p(\Omega)$ that admit a W-weak derivative of order α in $L^p(\Omega)$ for every multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \le m$. More concisely we can write

   $$W^{m,p}(\Omega) := \left\{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \forall |\alpha| \le m \right\}.$$

2. **Sobolev Spaces – Definition H (abstract version)**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty]$ (including endpoints).

   Let us set

   $$C^{m,p}(\Omega) := \left\{ u \in C^\infty(\Omega) : u \in L^p(\Omega) \text{ and } D^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^d \text{ with } |\alpha| \le m \right\}.$$

   It turns out that $C^{m,p}(\Omega)$ is a vector space, and

   $$\|u\|_{m,p,(a,b)} := \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p(\Omega)}$$

   is a norm on this space.

   The Sobolev space $H^{m,p}(\Omega)$ is the completion of $C^{m,p}(\Omega)$ with respect to this norm.

3. **Sobolev Spaces – Definition H (operative version)**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty]$ (including endpoints).

   The Sobolev space $H^{m,p}(\Omega)$ is the set of all functions $u \in L^p(\Omega)$ for which there exists a sequence $\{u_n\} \subseteq C^\infty(\Omega)$ (not necessarily with compact support) such that

   - $u_n \to u$ in $L^p(\Omega)$,
   - $D^\alpha u_n$ has a limit in $L^p(\Omega)$ for every multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \le m$.

   As we know, the limit of $D^\alpha u_n$ is necessarily the H-weak derivative of u of order α.

Sobolev Spaces – Approximation results

1. **First approximation result (low cost approximation)**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty)$ (note that $p = +\infty$ is not allowed).

   Then for every $u \in W^{m,p}(\Omega)$ there exists a sequence $\{u_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ such that

   - $u_n \to u$ in $L^p(\Omega)$,
   - $D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega')$ for every open set $\Omega' \subset\subset \Omega$ and every multi-index with $|\alpha| \leq m$.

2. **Second approximation result (H=W, by Meyers and Serrin 1964)**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty)$ (note that $p = +\infty$ is not allowed).

   Then for every $u \in W^{m,p}(\Omega)$ there exists a sequence $\{u_n\} \subseteq C^\infty(\Omega)$ such that

   - $u_n \to u$ in $L^p(\Omega)$,
   - $D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega)$ for every multi-index with $|\alpha| \leq m$.

3. **Third approximation result (deluxe approximation)**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty)$ (note that $p = +\infty$ is not allowed).

   Let us assume that Ω is *regular enough*.

   Then for every $u \in W^{m,p}(\Omega)$ there exists a sequence $\{u_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ such that

   - $u_n \to u$ in $L^p(\Omega)$,
   - $D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega)$ for every multi-index with $|\alpha| \leq m$.

4. **Comparison of approximation results**. Let us compare assumptions and conclusions of the approximation results.

   - In the low cost statement, the approximating sequence is very regular (C^∞ with compact support on the whole space $\mathbb{R}^d$), but the convergence of derivatives is quite poor (only on compact subsets of Ω).
   - In the second statement, the convergence is optimal, but the approximating sequence has minimal regularity. Indeed, it is of class C^∞ only in Ω, without compact support, and (more important) there is no guarantee that the functions u_n and/or their derivatives can be extended at least to the closure of Ω.

     Nevertheless, this second result is exactly what is needed in order to establish that
     $$W^{m,p}(\Omega) = H^{m,p}(\Omega)$$
     for *all* open sets Ω, without any regularity assumption.
   - In the deluxe result, the convergence is good as in the second result, and the approximating sequence is good as in the first one, and in particular all approximating functions can be extended with all their derivatives to the closure of Ω.

     The price to pay is that this result requires Ω to be enough regular.

Sobolev Spaces – Approximation tools???

1. **W-weak derivatives commute with convolutions**. Let d be a positive integer, let $\alpha \in \mathbb{N}^d$ be a multi-index, and let $A \subset\subset B \subseteq \mathbb{R}^d$ be two open sets.

   Let $u \in L^1(B)$ be a function that admits a W-weak derivative $D^\alpha u$ of order α in $L^1(B)$, and let us consider the positive real number

   $$\delta := \min\{d(x, \partial B) : x \in A\}.$$

   Then it turns out that

   $$\left[D^\alpha(u * \rho_\varepsilon)\right](x) = \left[(D^\alpha u) * \rho_\varepsilon\right](x) \qquad \forall x \in A, \quad \forall \varepsilon \in (0, \delta),$$

   namely in the smaller open set A the α-derivative of the convolution coincides with the convolution of the α-derivative.

2. **Sobolev times smooth yields Sobolev**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty]$ (endpoints included).

   Let us assume that $u \in W^{m,p}(\Omega)$ and $\psi \in C_c^\infty(\Omega)$.

   The the product $u(x) \cdot \psi(x)$ belongs to $W^{m,p}(\Omega)$, and actually also to $W^{m,p}(\mathbb{R}^d)$. In addition, it turns out that

   $$D^\alpha(u \cdot \psi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u \cdot D^{\alpha-\beta}\psi.$$

Products and compositions of Sobolev functions

1. **Product of Sobolev functions.** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $p \in [1, +\infty]$ (endpoints included).

   Let us consider two functions

   $$u \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \qquad \text{and} \qquad v \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

   Then the product $u \cdot v$ belongs to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and its weak derivatives are

   $$D_{x_i}(u \cdot v) = D_{x_i}u \cdot v + u \cdot D_{x_i}v.$$

2. **Smooth external composition.** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $p \in [1, +\infty]$ (endpoints included).

   Let $u \in W^{1,p}(\Omega)$, and let $g : \mathbb{R} \to \mathbb{R}$ be a function such that

   - $g(0) = 0$ (this assumption is not needed if the measure of Ω is finite),
   - g is of class C^1,
   - the derivative of g is bounded, namely there exists a constant M such that $|g'(s)| \leq M$ for every $s \in \mathbb{R}$.

   Then the composition $g(u(x))$ belongs to $W^{1,p}(\Omega)$, and its weak derivatives are

   $$D_{x_i}g(u(x)) = g'(u(x)) \cdot D_{x_i}u(x).$$

3. **Piecewise smooth external composition.** In the previous result we can weaken the assumption that g is of class C^1 by asking only that g is piecewise C^1.

   In particular, in the case $g(s) = |s|$, we obtain that $|u| \in W^{1,p}(\Omega)$ and

   $$D_{x_i}|u(x)| = \operatorname{sign}(u(x)) \cdot D_{x_i}u(x).$$

4. **Smooth internal composition.** Let $d \geq 1$ be a positive integer, let $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^d$ be two pen sets, and let $p \in [1, +\infty]$ (endpoints included).

   Let $\Phi : A \to B$ be a function such that

   - Φ is invertible,
   - $\Phi \in C^1(A)$ and $\Phi^{-1} \in C^1(B)$,
   - $J\Phi(x)$ (the Jacobian matrix of Φ) is bounded in A, and $J\Phi^{-1}(x)$ is bounded in B.

   For every $v : B \to \mathbb{R}$, let us define the function $u : A \to \mathbb{R}$ as $u(x) := v(\Phi(x))$.

   Then it turns out that

   $$u \in W^{1,p}(A) \qquad \Longleftrightarrow \qquad v \in W^{1,p}(B),$$

   and the weak derivatives of u and v satisfy the expected relation

   $$\frac{\partial u}{\partial x_i}(x) = \sum_{j=1}^{d} \frac{\partial v}{\partial x_j}(\Phi(x)) \cdot \frac{\partial \Phi_j}{\partial x_i}(x).$$

Sobolev Imbedding – Order one – Whole space

1. **Sobolev Imbedding (first order derivatives).** Let d be a positive integer, let $p \in [1, +\infty]$ (endpoints included), and let $u \in W^{1,p}(\mathbb{R}^d)$.

   Then the following statements hold true.

   - If $p < d$, then it turns out that $u \in L^{p^*}(\mathbb{R}^d)$, where $p*$ satisfies

     $$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{d} \qquad \left(\text{namely } p^* = \frac{dp}{d-p}\right),$$

     and

     $$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq c(p, d) \cdot \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

     As a consequence, it turns out also that $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, p^*]$, with (we point out that in this case the right-hand side involves the p-norm of both u and ∇u)

     $$\|u\|_{L^q(\mathbb{R}^d)} \leq c(p, d, q) \cdot \left(\|u\|_{L^p(\mathbb{R}^d)} + \|\nabla u\|_{L^p(\mathbb{R}^d)}\right).$$

   - If $p = d$, then it turns out that $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, +\infty)$ (but not necessarily for $q = +\infty$), and

     $$\|u\|_{L^q(\mathbb{R}^d)} \leq c(p, d, q) \cdot \left(\|u\|_{L^p(\mathbb{R}^d)} + \|\nabla u\|_{L^p(\mathbb{R}^d)}\right) \qquad \forall q \geq p.$$

   - If $p > d$, then it turns out that $u \in L^\infty(\mathbb{R}^d)$ and (again the right-hand side involves the p-norm of both u and ∇u)

     $$\|u\|_{L^\infty(\mathbb{R}^d)} \leq c(p, d) \cdot \left(\|u\|_{L^p(\mathbb{R}^d)} + \|\nabla u\|_{L^p(\mathbb{R}^d)}\right).$$

     Moreover it turns out that $u \in C^{0,\alpha}(\mathbb{R}^d)$ with Hölder exponent

     $$\alpha := 1 - \frac{d}{p}$$

     (which means Lipschitz continuity in the case $p = +\infty$) and

     $$|u(y) - u(x)| \leq c(p, d) \cdot \|\nabla u\|_{L^p(\mathbb{R}^d)} \cdot |y - x|^\alpha \qquad \forall (x, y) \in (\mathbb{R}^d)^2.$$

Sobolev Imbedding – Any order – Whole space

1. **Sobolev Imbedding (higher order derivatives)**. Let d and m be positive integers, let $p \in [1, +\infty]$ (endpoints included), and let $u \in W^{m,p}(\mathbb{R}^d)$.

   Then the following statements hold true.

   - If $mp < d$, then it turns out that $u \in L^q(\mathbb{R}^d)$, where q satisfies

     $$\frac{1}{p} - \frac{1}{q} = \frac{m}{d} \qquad \left(\text{namely } q = \frac{dp}{d - mp}\right),$$

     and

     $$\|u\|_{L^q(\mathbb{R}^d)} \leq c(p, d, m) \cdot \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)}.$$

   - If $p = md$, then it turns out that $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, +\infty)$ (but not necessarily for $q = +\infty$), and

     $$\|u\|_{L^q(\mathbb{R}^d)} \leq c(p, d, q, m) \cdot \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)} \qquad \forall q \geq p.$$

   - If $p > md$, then it turns out that $u \in L^\infty(\mathbb{R}^d)$ and (we point out that in this case the right-hand side involves the p-norm of all derivatives of u up to order m)

     $$\|u\|_{L^\infty(\mathbb{R}^d)} \leq c(p, d, m) \cdot \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)}.$$

     Moreover, if h denotes the smallest integer such that $hp > d$, then it turns out that $u \in C^{k-h,\alpha}(\mathbb{R}^d)$ with Hölder exponent

     $$\alpha := 1 - \frac{d}{hp}$$

     (which means Lipschitz continuity in the case $p = +\infty$) and for every multi-index β with $|\beta| = k - h$ it turns out that

     $$|D^\beta u(y) - D^\beta u(x)| \leq c(p, d, m) \cdot \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)} \cdot |y - x|^\alpha \qquad \forall (x, y) \in (\mathbb{R}^d)^2.$$

Sobolev Imbedding – Any order – Regular open set

1. **Sobolev Imbedding (higher order derivatives)**. Let d and m be positive integers, let $p \in [1, +\infty]$ (endpoints included), let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $u \in W^{m,p}(\mathbb{R}^d)$.

   Let us assume that Ω is *regular enough*.

   Then the following statements hold true.

   - If $mp < d$, then it turns out that $u \in L^q(\Omega)$, where q satisfies

     $$\frac{1}{p} - \frac{1}{q} = \frac{m}{d} \qquad \left(\text{namely } q = \frac{dp}{d - mp}\right),$$

     and

     $$\|u\|_{0,q,\Omega} \leq c(p, d, m, \Omega) \cdot \|u\|_{m,p,\Omega}.$$

   - If $p = md$, then it turns out that $u \in L^q(\Omega)$ for every $q \in [p, +\infty)$ (but not necessarily for $q = +\infty$), and

     $$\|u\|_{0,q,\Omega} \leq c(p, d, q, m, \Omega) \cdot \|u\|_{m,p,\Omega} \qquad \forall q \geq p.$$

   - If $p > d$, then it turns out that $u \in L^\infty(\Omega)$ and

     $$\|u\|_{0,\infty,\Omega} \leq c(p, d, m, \Omega) \cdot \|u\|_{m,p,\Omega}.$$

     Moreover, if h denotes the smallest integer such that $hp > d$, then it turns out that $u \in C^{k-h,\alpha}(\Omega)$ with Hölder exponent

     $$\alpha := 1 - \frac{d}{hp}$$

     (which means Lipschitz continuity in the case $p = +\infty$) and

     $$|u(y) - u(x)| \leq c(p, d, m, \Omega) \cdot \|u\|_{m,p,\Omega} \cdot |y - x|^\alpha \qquad \forall (x, y) \in \Omega^2.$$

2. **Achtung!**. The values of the exponents and the conclusions are analogous to the case where $\Omega = \mathbb{R}^d$. The remarkable differences are that

   - here some regularity of $\partial\Omega$ is required (a sufficient condition is the existence of a $(1, p)$-extension operator, which is enough to guarantee the result also for $W^{m,p}(\Omega)$),
   - in all the estimates the constants do depend also on Ω,
   - in all the estimates the right-hand sides involve always the full norm of u in $W^{m,p}(\Omega)$, and not just the norm of the higher order derivatives.

Gagliardo-Brascamp-Lieb Inequality

1. **Gagliardo inequality (1958)**. Let $d \geq 2$ be an integer. For every $i \in \{1, \ldots, d\}$, let $\varphi_i \in C_c^\infty(\mathbb{R}^{d-1})$ be a *nonnegative* function, and let $P_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection that "eliminates the i-th coordinate".

   Let us consider the function $\varphi : \mathbb{R}^d \to [0, +\infty)$ defined by

   $$\varphi(x) := \prod_{i=1}^{d} \varphi_i(P_i x) \qquad \forall x \in \mathbb{R}^d.$$

   Then it turns out that

   $$\|\varphi\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^{d} \|\varphi_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

2. **Brascamp-Lieb inequality**. Let $d \geq 2$ and $n \geq 1$ be integers. For every $i \in \{1, \ldots, n\}$ let us choose

   - an integer $d_i \in \{1, \ldots, d\}$,
   - an exponent $p_i \in [1, +\infty]$,
   - a *surjective* linear function $L_i : \mathbb{R}^d \to \mathbb{R}^{d_i}$,
   - a *nonnegative* function $\varphi_i \in C_c^\infty(\mathbb{R}^{d_i})$.

   Finally, let us define $\varphi : \mathbb{R}^d \to [0, +\infty)$ as

   $$\varphi(x) := \prod_{i=1}^{n} \varphi_i(L_i x) \qquad \forall x \in \mathbb{R}^d.$$

   Let us assume that for every vector subspace $V \subseteq \mathbb{R}^d$ it turns out that

   $$\dim(V) \leq \sum_{i=1}^{n} \frac{\dim(L_i V)}{p_i},$$

   with equality in the case $V = \mathbb{R}^d$.

   Then there exists a constant D, independent of the functions φ_i, such that

   $$\|\varphi\|_{L^1(\mathbb{R}^d)} \leq D \cdot \prod_{i=1}^{n} \|\varphi_i\|_{L^{p_i}(\mathbb{R}^{d_i})}.$$

Compact Imbedding

1. **Compact Imbedding**. Let d be a positive integer, let $p \in [1, +\infty]$ (endpoints included), and let $\Omega \subseteq \mathbb{R}^d$ be any open set for which the Sobolev imbedding theorem hods true.

   Let us assume in addition that Ω is *bounded*.

   Then the following statements hold true.

   - If $p < d$, then for every $q \in [1, p^*)$ (but not for $q = p^*$) the imbedding
   $$W^{1,p}(\Omega) \to L^q(\Omega)$$
   is compact.
   - If $p = d$, then for every $q \geq 1$ the imbedding
   $$W^{1,p}(\Omega) \to L^q(\Omega)$$
   is compact.
   - If $p > d$, then the imbedding
   $$W^{1,p}(\Omega) \to C^0(\mathrm{Clos}(\Omega))$$
   is compact.

2. **Relative compactness in Lebesgue spaces**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $p \in [1, +\infty)$ (the case $p = +\infty$ is excluded).

   Let $\mathcal{F} \subseteq L^p(\Omega)$ be a family of functions.

   Let us assume that

   - Ω is bounded,
   - $\mathcal{F}$ is bounded, namely there exist $M \in \mathbb{R}$ such that
   $$\|u\|_{L^p(\Omega)} \leq M \qquad \forall u \in \mathcal{F},$$
   - $\mathcal{F}$ is equicontinuous in a integral sense, namely for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in \mathbb{R}^d$ with $|h| < \delta$ it turns out that
   $$\|\tau_h u - u\|_{L^p(\Omega)} \leq \varepsilon \qquad \forall u \in \mathcal{F}.$$

   Then $\mathcal{F}$ is relatively compact in $L^p(\Omega)$.

Extension operators 1 – Definitions

1. **Definition (extension operators).** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, let m be a positive integer, and let $p \in [1, +\infty]$ (endpoints included).

   - A (m, p)-extension operator is a *linear* function

     $$E_{m,p} : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$$

     such that

     – for every $u \in W^{m,p}(\Omega)$ it turns out that $(E_{m,p}u)(x) = u(x)$ for almost every $x \in \Omega$,

     – there exists a constant $c(m, p, d)$ such that

     $$\|E_{m,p}u\|_{m,p,\mathbb{R}^d} \leq c(m, p, d) \cdot \|u\|_{m,p,\Omega}.$$

   - A strong m-extension operator is a *linear* function

     $$E_m : L^1_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\mathbb{R}^d)$$

     such that the restriction of E_m to $W^{k,p}(\Omega)$ is a (k, p)-extension operator for every positive integer $k \leq m$ and every $p \in [1, +\infty)$ (including $+\infty$?).

   - A universal extension operator is a *linear* function

     $$E : L^1_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\mathbb{R}^d)$$

     which turns out to be a strong m-extension for every positive integer m.

   More generally, in an analogous way we can define extension operators for every pair of open sets $A \subseteq B \subseteq \mathbb{R}^d$.

2. **Extension implies approximation and imbedding.** Let d be a positive integer, and let $\Omega \subseteq \mathbb{R}^d$ be an open set.

   Then the following statements are true.

   - If Ω admits a (m, p)-extension operator for some positive integer m and some exponent $p \in [1, +\infty)$, then the *deluxe* approximation theorem holds true for $W^{m,p}(\Omega)$. In turn, the deluxe approximation result is a fundamental tool in order to define *boundary values* (traces) of Sobolev functions.

   - If Ω admits a $(1, p)$-extension operator for some exponent $p \in [1, +\infty)$, then the immersion theorem for $W^{m,p}(\Omega)$ holds true in Ω for every positive integer m.

     We stress that we obtain the imbedding for every m by assuming the existence of the extension operator just for $m = 1$.

Extension operators 2 – Basic configurations

1. **Definition (cylinders).** Let $d \geq 2$ be an integer, let $A \subseteq \mathbb{R}^{d-1}$ be an open set. Let us consider the open sets (cylinders with base A) in $\mathbb{R}^d$ defined by

$$C_A := A \times (-1, 1) \qquad \text{and} \qquad C_A^+ := A \times (0, 1).$$

In the special case where A is the ball $B_{d-1}(0, 1)$ in $\mathbb{R}^{d-1}$ with center in the origin and radius 1, we set

$$Q := B_{d-1}(0, 1) \times (-1, 1) \qquad \text{and} \qquad Q^+ := B_{d-1}(0, 1) \times (0, 1).$$

2. **Definition (reflection and anti-reflection operators).** Let $d \geq 2$ be an integer, let $A \subseteq \mathbb{R}^{d-1}$ be an open set, and let C_A and C_A^+ be the two cylinders defined as above.

The reflection operator $E_1 : L^1_{\text{loc}}(C_A^+) \to L^1_{\text{loc}}(C_A)$ is defined by

$$[E_1 u](x, y) := \begin{cases} u(x, y) & \text{if } (x, y) \in A \times (0, 1), \\ u(x, -y) & \text{if } (x, y) \in A \times (-1, 0). \end{cases}$$

The anti-reflection operator $\widehat{E}_1 : L^1_{\text{loc}}(C_A^+) \to L^1_{\text{loc}}(C_A)$ is defined by

$$[\widehat{E}_1 u](x, y) := \begin{cases} u(x, y) & \text{if } (x, y) \in A \times (0, 1), \\ -u(x, -y) & \text{if } (x, y) \in A \times (-1, 0). \end{cases}$$

We note that in both cases there is no need to define the extended function also for $y = 0$.

3. **Extension by reflection.** Let $d \geq 2$ be an integer, let $A \subseteq \mathbb{R}^{d-1}$ be an open set, and let C_A and C_A^+ be the two cylinders defined as above. Then the reflection operator E_1 defined above is a strong 1-extension operator.

Moreover, for every $u \in W^{1,p}(C_A^+)$ it turns out that

$$\frac{\partial Eu}{\partial x_i} = E \frac{\partial u}{\partial x_i} \qquad \forall i \in \{1, \ldots, d-1\},$$

and

$$\frac{\partial Eu}{\partial y} = \widehat{E} \frac{\partial u}{\partial y}.$$

Extension operators 3 – Regular domains

1. **Open sets with smooth boundary**. Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let m be a positive integer.

   We say that $\partial\Omega$ is of class C^m if for every $x \in \partial\Omega$ there exist an open neighborhood U of x, and a function $\Phi : U \to Q$ such that

   - Φ is invertible,
   - $\Phi \in C^m(U)$ and $\Phi^{-1} \in C^m(Q)$,
   - all derivatives of Φ and Φ^{-1} up to order m are bounded (in U and Q, respectively),
   - $\Phi(U \cap \Omega) = Q^+$,
   - $\Phi(U \cap \partial\Omega) = Q_0$.

2. **Extension operators in smooth domains**. Let d be a positive integer, and let $\Omega \subseteq \mathbb{R}^d$ be an open set.

   Let us assume that

   - $\partial\Omega$ is compact,
   - $\partial\Omega$ is of class C^1.

   Then Ω admits a strong 1-extension operator.

Regularity

1. **Model case in the whole space**. Let d be a positive integer, let $f \in L^2(\mathbb{R}^d)$, and let $u \in H^1(\mathbb{R}^d)$ be a weak solution to equation

$$\Delta u = f \qquad \text{in } \mathbb{R}^d.$$

Then it turns out that $u \in H^2(\mathbb{R}^d)$, and

$$\|D^2 u_{x_k}\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)},$$

where the left-hand side has to be intended as

$$\left\{ \int_{\mathbb{R}^d} \sum_{j=1}^{d} \left| \frac{\partial^2 u}{\partial x_i \partial x_k}(x) \right|^2 dx \right\}^{1/2}.$$

2. **Model case in the half-space with DBC**. Let d be a positive integer, let $f \in L^2(\mathbb{R}^d_+)$, and let $u \in H^1_0(\mathbb{R}^d_+)$ be a weak solution to equation

$$\Delta u = f \qquad \text{in } \mathbb{R}^d_+.$$

Then it turns out that $u \in H^2(\mathbb{R}^d_+)$, and

$$\|D u_{x_k}\|_{L^2(\mathbb{R}^d_+)} \leq \|f\|_{L^2(\mathbb{R}^d_+)}.$$

3. **Interior regularity**. Let $d \geq 1$ be a positive integer, let m be a nonnegative integer, and let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let us consider equation

$$\operatorname{div}(A(x)Du) = f \qquad \text{in } \Omega,$$

and let us assume that

- $A(x)$ satisfies the uniform ellipticity assumption in Ω,
- $A(x)$ belongs to $C^{m+1}(\Omega)$,
- $f \in H^m_{\text{loc}}(\Omega)$,
- $u \in H^1_{\text{loc}}(\Omega)$ is a weak solution of the equation.

Then $u \in H^{m+2}_{\text{loc}}(\Omega)$, and for every pair of open sets $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ it turns out that

$$\|u\|_{m+2,2,\Omega'} \leq c\left(\nu, m, \|A\|_{C^{m+1}(\Omega'')}, \Omega', \Omega''\right) \cdot \left\{\|f\|_{m,2,\Omega''} + \|u\|_{1,2,\Omega''}\right\}.$$

4. **Regularity up to the boundary for the Dirichlet problem**. Let $d \geq 1$ be a positive integer, let m be a nonnegative integer, and let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let us consider equation

$$\operatorname{div}(A(x)Du) = f \qquad \text{in } \Omega,$$

and let us assume that

- $A(x)$ satisfies the uniform ellipticity assumption in Ω,
- $A(x)$ belongs to $C^{m+1}(\mathrm{Clos}(\Omega))$,
- $f \in H^m(\Omega)$,
- $u \in H^1_0(\Omega)$ is a weak solution of the equation.

Then $u \in H^{m+2}(\Omega)$, and it turns out that

$$\|u\|_{m+2,2,\Omega} \leq c\left(\nu, m, \|A\|_{C^{m+1}(\mathrm{Clos}(\Omega))}\right) \cdot \left\{\|f\|_{m,2,\Omega} + \|u\|_{1,2,\Omega}\right\}.$$

Chapter 5

Indirect method in the Calculus of Variations

Fundamental Lemma in the Calculus of Variations

1. **Statement of FLCV (classic setting).** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $f : \Omega \to \mathbb{R}$ be a *continuous* function. Let us assume that

$$\int_\Omega f(x)v(x)\,dx = 0 \qquad \forall v \in C_c^\infty(\Omega). \tag{5.1}$$

Then $f(x) = 0$ for every $x \in \Omega$.

2. **Statement of FLCV (Lebesgue setting).** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $f \in L^1_{\text{loc}}(\Omega)$. Let us assume that

$$\int_\Omega f(x)v(x)\,dx = 0 \qquad \forall v \in C_c^\infty(\Omega).$$

Then $f(x) = 0$ for almost every $x \in \Omega$, namely there exists a subset $E \subseteq \Omega$ with $\text{meas}(E) = 0$ such that $f(x) = 0$ for every $x \in \Omega \setminus E$.

3. **More general test functions.** In both settings, we can limit ourselves to considering smaller classes of *test functions*, instead of $C_c^\infty(\Omega)$. More precisely, we can conclude that $f(x) = 0$ for almost every $x \in \Omega$ if we assume that

$$\int_\Omega f(x)v(x)\,dx = 0 \qquad \forall v \in V,$$

where V is a class of functions such that $\text{Span}(V)$ is "dense" in the space of bounded measurable functions with compact support.

Here "density" is enough in the following sense. For every open ball $B(x_0, r) \subset\subset \Omega$, and every measurable function $v : \Omega \to \{-1, 0, 1\}$ with $v(x) = 0$ for every $x \in \Omega \setminus B(x_0, r)$, there exist a sequence $\{v_n\} \subseteq \text{Span}(V)$ and a real number M such that

- (domination) $|v_n(x)| \le M$ for every $x \in \Omega$ and every $n \in \mathbb{N}$,
- (pointwise convergence) $v_n(x) \to v(x)$ for almost every $x \in \Omega$.

4. **Ideas for proofs of FLCV.** There are at least two possible strategies.

- In the classic setting
  - we assume by contradiction that $f(x_0) \neq 0$ for some $x_0 \in \Omega$,
  - we observe that $f(x)$ has constant sign in some ball $B(x_0, r)$,
  - we consider a test function with constant sign in the same ball, and zero elsewhere.
- In the Lebesgue setting (and hence a fortiori in the classic setting)
  - we consider *any* ball $B(x_0, r) \subset\subset \Omega$,
  - we consider a sequence of test functions converging to $\text{sign}(f(x))$ in $B(x_0, r)$, and to zero elsewhere, in the sense of dominated almost everywhere convergence,
  - we conclude by the arbitrariness of the ball.

FLCV with zero average

1. **Statement of FLCV with zero average (classic setting).** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $f : \Omega \to \mathbb{R}$ be a *continuous* function. Let us assume that
$$\int_\Omega f(x)v(x)\, dx = 0$$
for every $v \in C_c^\infty(\Omega)$ with *zero average*, namely such that
$$\int_\Omega v(x)\, dx = 0.$$
Then $f(x)$ is constant in Ω, namely there exists $c \in \mathbb{R}$ such that $f(x) = c$ for every $x \in \Omega$.

2. **Statement of FLCV with zero average (Lebesgue setting).** Let d be a positive integer, let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $f \in L^1_{\text{loc}}(\Omega)$. Let us assume that
$$\int_\Omega f(x)v(x)\, dx = 0$$
for every $v \in C_c^\infty(\Omega)$ with *zero average*.

   Then $f(x)$ is constant almost everywhere in Ω, namely there exists $c \in \mathbb{R}$, and there exists a subset $E \subseteq \Omega$ with $\text{meas}(E) = 0$, such that $f(x) = c$ for every $x \in \Omega \setminus E$.

3. **The key observation.** Both in the classic and in the Lebesgue setting, if a function f satisfies the assumption in the FLCV with zero average, then for every constant $c \in \mathbb{R}$ it turns out that also
$$\int_\Omega (f(x) - c)v(x)\, dx = 0$$
for every $v \in C_c^\infty(\Omega)$ with zero average.

4. **Ideas for proofs of FLCV.** There are at least two possible strategies, both based on the previous key observation.

   - In the classic setting
     - we assume by contradiction that $f(x_0) \neq f(y_0)$,
     - we choose $c \in \mathbb{R}$ and $r > 0$ such that $f(x) - c$ has different sign in the balls $B(x_0, r)$ and $B(y_0, r)$,
     - we consider a test function that has the same sign of $f(x) - c$ in the two balls (and vanishes elsewhere), and is "symmetric" in order to have zero average.
   - In the Lebesgue setting (and hence a fortiori in the classic setting)
     - we consider *any* open set $\Omega' \subset\subset \Omega$ (unions of two disjoint balls would be enough),
     - we choose $c \in \mathbb{R}$ such that $f(x) - c$ has zero average in Ω',
     - we choose $d \in \mathbb{R}$ such that $\text{sign}(f(x) - c) - d$ has zero average in Ω',
     - we consider a sequence of test functions with zero average converging in the usual sense to $\text{sign}(f(x) - c) - d$ in Ω', and to zero elsewhere,
     - we conclude by the arbitrariness of Ω' (the conclusion is not completely trivial because c depends a priori on Ω').

Du Bois-Reymond Lemma

1. **Du Bois-Reymond Lemma (classic setting)**. Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $f : (a,b) \to \mathbb{R}$ be a *continuous* function such that

$$\int_a^b f(x)v'(x)\,dx = 0 \qquad \forall v \in C_c^\infty((a,b)).$$

Then $f(x)$ is constant in (a,b).

2. **Du Bois-Reymond Lemma (Lebesgue setting)**. Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $f \in L^1_{\mathrm{loc}}((a,b))$ be a function such that

$$\int_a^b f(x)v'(x)\,dx = 0 \qquad \forall v \in C_c^\infty((a,b)).$$

Then $f(x)$ is constant almost everywhere in (a,b).

3. **Idea of the proof**. In the case of an interval, the key observation is that the set of derivatives of functions in $C_c^\infty((a,b))$ coincides with the set of functions in $C_c^\infty((a,b))$ with zero average, namely

$$\left\{v'(x) : v \in C_c^\infty((a,b))\right\} = \left\{v \in C_c^\infty((a,b)) : \int_a^b v(x)\,dx = 0\right\}.$$

Therefore, the assumption of the Du-Bois Reymond lemma coincides with the assumption of the FLCV with zero average.

4. **A matter of connectedness**. The same statement holds true if we replace the interval (a,b) with a half line of the form $(-\infty,a)$ or $(a,+\infty)$, or with $\mathbb{R}$. Again the proof is based on the same key observation.

Minimum and minimum point(s)

1. **Definition (Minimum).** Let $\mathbb{S}$ be a set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function. We say that $m \in \mathbb{R}$ is the (global) *minimum* of F in $\mathbb{S}$ if
   - $F(x) \geq m$ for every $x \in \mathbb{S}$,
   - there exists $x_0 \in \mathbb{S}$ such that $F(x_0) = m$.

   In this case we say that F attains its minimum on $\mathbb{S}$, and we write
   $$m = \min \{F(x) : x \in \mathbb{S}\},$$

2. **Definition (Minimum points).** Let $\mathbb{S}$ be a set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function. Let us assume that F attains the minimum on $\mathbb{S}$, namely
   $$m := \min \{F(x) : x \in \mathbb{S}\}$$
   exists.

   We call (global) *minimum points* all the elements $x \in \mathbb{S}$ such that $F(x) = m$, and we set
   $$\operatorname{argmin} \{F(x) : x \in \mathbb{S}\} := \{x \in \mathbb{S} : F(x) = m\}.$$

3. **Remarks (existence/uniqueness of min and argmin).** Let $\mathbb{S}$ and F be as in the definition of minimum and minimum points.
   - The function F does not necessarily attain its minimum in $\mathbb{S}$ (but it has always the infimum).
   - If the function F attains its minimum in $\mathbb{S}$, then the minimum is necessarily unique.
   - If the function F attains its minimum in $\mathbb{S}$, then there exists at least one minimum point, but the set of global minimum points might have more than one element.

First Variation along a curve

1. **Definition (Curve through a point)**. Let $\mathbb{S}$ be a set, and let $x_0 \in \mathbb{S}$.

   A *curve through* x_0 is any function $\gamma : (-r, r) \to \mathbb{S}$ (for some real number $r > 0$ that depends on the curve) such that $\gamma(0) = x_0$.

   We point out that, at this level of generality, we cannot ask any regularity property on γ.

2. **Definition (first variation of a function along a curve)**. Let $\mathbb{S}$ be a set, let $F : \mathbb{S} \to \mathbb{R}$ be a function, let $x_0 \in \mathbb{S}$, let $r > 0$ be a real number, and let $\gamma : (-r, r) \to \mathbb{S}$ be a curve through x_0.

   Let us consider the function $\varphi(t) : (-r, r) \to \mathbb{R}$ defined by

   $$\varphi(t) := F(\gamma(t)) \qquad \forall t \in (-r, r).$$

   If the first derivative of $\varphi(t)$ in $t = 0$ exists, then we set

   $$\delta F(x_0, \gamma) := \varphi'(0) = \lim_{t \to 0} \frac{F(\gamma(t)) - F(x_0)}{t},$$

   and we call $\delta F(x_0, \gamma)$ the *first variation* of F in x_0 along the curve γ.

3. **Theorem (necessary condition for minimality)**. Let $\mathbb{S}$ be a set, let $F : \mathbb{S} \to \mathbb{R}$ be a function, and let $x_0 \in \mathbb{S}$.

   Let us assume that x_0 is a (global) minimum point for F in $\mathbb{S}$, namely

   $$F(x) \geq F(x_0) \qquad \forall x \in \mathbb{S}.$$

   Let γ be a curve through x_0, and let us assume that $\delta F(x_0, \gamma)$ exists.

   Then necessarily it turns out that $\delta F(x_0, \gamma) = 0$.

4. **Necessary condition for minimality as an alternative**. Another way of stating the previous condition is the following. If x_0 is a (global) minimum point for F in $\mathbb{S}$, then for every curve γ through x_0 the limit

   $$\lim_{t \to 0} \frac{F(\gamma(t)) - F(x_0)}{t}$$

   has only two possibilities (and both behaviors are possible, depending on the curve):

   - either it does not exists,
   - or it is equal to 0.

First Variation along lines – Gateaux derivatives

1. **Definition (First Variation of a function in a direction)**. Let $\mathbb{X}$ be an *affine space* with associated vector space V, let $F : \mathbb{S} \to \mathbb{R}$ be a function, let $x_0 \in \mathbb{X}$ and $v \in V$.

   We observe that $x_0 + tv \in \mathbb{X}$ for every $t \in \mathbb{R}$, and therefore we can define $\varphi(t) : \mathbb{R} \to \mathbb{R}$ as

   $$\varphi(t) := F(x_0 + tv) \qquad \forall t \in \mathbb{R}.$$

   If the first derivative of $\varphi(t)$ in $t = 0$ exists, then we set

   $$\delta F(x_0, v) := \varphi'(0) = \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t},$$

   and we call $\delta F(x_0, v)$ the *first variation* of F in x_0 in the direction v.

   This is also called the *Gateaux derivative* (or the directional derivative) of F in x_0 in the direction v, and it coincides with the first variation of F in x_0 along the curve $\gamma(t) := x_0 + tv$.

2. **Definition (Directional Local Minimum point – DLM)**. Let $\mathbb{X}$, V, F, x_0, v be as in the definition of Gateaux derivative.

   The point x_0 is said to be a *direction local minimum point* for F in the direction v if there exists $r > 0$ such that

   $$F(x_0 + tv) \geq F(x_0) \qquad \forall t \in (-r, r).$$

   This is equivalent to saying that $t = 0$ is a local minimum point for the function $\varphi : \mathbb{R} \to \mathbb{R}$ introduced in the definition of Gateaux derivative.

3. **Achtung!**. A directional local minimum point is not necessarily a global (or local, when this makes sense) minimum point, not even in $\mathbb{R}^2$.

4. **Necessary condition for a DLM**. Let $\mathbb{X}$, V, F, x_0, v be as in the definition of Gateaux derivative.

   Let us assume that x_0 is a directional local minimum point for F in the direction v, and that $\delta F(x_0, v)$ exists.

   Then it turns out that $\delta F(x_0, v) = 0$.

5. **Alternative for a global minimum point**. Let $\mathbb{X}$, V, F, x_0 be as in the definition of Gateaux derivative. Let us assume that x_0 is a global minimum point for F.

   Then for every direction v the Gateaux derivative

   $$\lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t}$$

   has only two possibilities (and both behaviors are possible, depending on v):

   - either it does not exists,
   - or it is equal to 0.

   An analogous statement holds true for local minimum points, when this notion makes sense (it requires some notion of neighborhood compatible with the structure of affine space).

Integral functionals – First Variation

1. **Integral functionals (basic example).** Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let

$$L : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

be a function, usually called *Lagrangian* and denoted by $L(x, s, p)$.

An integral functional is a functional of the form

$$F(u) = \int_a^b L(x, u(x), u'(x))\, dx.$$

This functional is well defined in a classical sense if L is continuous in $[a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $u : [a, b] \to \mathbb{R}$ is of class C^1.

2. **First integral form of the first variation.** Let us consider the basic integral functional $F(u)$ defined as above. Let us assume that

   - the Lagrangian $L(x, s, p)$ is of class C^1 in $[a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,
   - the function $u : [a, b] \to \mathbb{R}$ is of class C^1,
   - the perturbation $v : [a, b] \to \mathbb{R}$ are of class C^1.

   Then it turns out that

$$\delta F(u, v) = \int_a^b \left\{ L_s(x, u(x), u'(x)) \cdot v(x) + L_p(x, u(x), u'(x)) \cdot v'(x) \right\} dx.$$

3. **Second integral form of the first variation.** Let us consider the basic integral functional $F(u)$ defined as above. Let us assume that

   - the Lagrangian $L(x, s, p)$ is of class C^2 in $[a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,
   - the function $u : [a, b] \to \mathbb{R}$ is of class C^2,
   - the perturbation $v : [a, b] \to \mathbb{R}$ are of class C^1.

   Then with an integration by parts we deduce that

$$\begin{aligned} \delta F(u, v) &= \int_a^b \left\{ -\frac{d}{dx} L_p(x, u(x), u'(x)) + L_s(x, u(x), u'(x)) \right\} \cdot v(x)\, dx \\ &\quad + L_p(b, u(b), u'(b)) \cdot v(b) - L_p(a, u(a), u'(a)) \cdot v(a). \end{aligned}$$

4. **Comment on the assumptions.** Concerning the assumptions on L, u, v, we observe that

   - for the first integral form of the first variation we just need the differentiability in $t = 0$ of the parametric integral $F(u + tv)$,
   - for the second integral form of the first variation we just need to integrate by parts the term with $v'(x)$ in the first integral form.

Integral functionals – Euler-Lagrange Equation

1. **Euler-Lagrange equation (ELE)**. Let us consider the basic integral functional $F(u)$ defined as always. Let us assume that

   - the Lagrangian $L(x, s, p)$ is of class C^2 in $[a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,
   - the function $u : [a, b] \to \mathbb{R}$ is of class C^2,
   - u is a directional local minimum (DLM) for F with respect to all directions $v \in C_c^\infty((a, b))$ (or in any class that triggers the FLCV).

   Then it turns out that

   $$\frac{d}{dx} L_p(x, u(x), u'(x)) = L_s(x, u(x), u'(x)) \qquad \forall x \in [a, b].$$

2. **Comments on ELE**. Let us consider ELE as written above.

   - It is a *second order* (ordinary) *differential equation*.
   - The derivative with respect to x in the left-hand side is a *total derivative*. When expanded, ELE takes the form (for the sake of shortness, we do not write explicitly the dependence on x of u and u')

     $$L_{px}(x, u, u') + L_{ps}(x, u, u')u' + L_{pp}(x, u, u')u'' = L_s(x, u, u').$$

   - If $L_{pp}(x, s, p) \neq 0$ for every admissible value of (x, s, p), then ELE can be written in normal form

     $$u''(x) = \Phi(x, u(x), u'(x))$$

     for a suitable function Φ.

3. **Neumann boundary conditions**. If u satisfies ELE, and it is a DLM also with respect to a direction v with $v(a) \neq 0$ and $v(b) = 0$, then it turns out that

   $$L_p(a, u(a), u'(a)) = 0.$$

   Similarly, if u satisfies ELE, and it is a DLM also with respect to a direction v with $v(a) = 0$ and $v(b) \neq 0$, then it turns out that

   $$L_p(b, u(b), u'(b)) = 0.$$

4. **More general boundary conditions**. When ELE holds true, every relation between nontrivial values of an admissible variation in the endpoints of the interval yields some relation between the values of $L_p(x, u(x), u'(x))$ at the same endpoints.

ELE in DBR form – Beltrami identity

1. **ELE in Du-Bois Reymond form**. Let us consider the basic integral functional $F(u)$ defined as always. Let us assume that

   - the Lagrangian $L(x, s, p)$ is of class C^1 in $[a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,
   - the function $u : [a, b] \to \mathbb{R}$ is of class C^1,
   - u is a directional local minimum for F with respect to all directions $v \in C_c^\infty((a, b))$ (or in any good class for the FLCV).

   Then there exists a real number c that

   $$L_p(x, u(x), u'(x)) = c + \int_a^x L_s(t, u(t), u'(t))\, dt \qquad \forall x \in [a, b].$$

2. **Comments on the DBR form**.

   - The DBR form above is equivalent to the classical differential form. The equivalence follows from the usual equivalence between a differential equation and an integral equation (also the value of c can be easily computed).
   - Despite of this equivalence, the derivation of ELE via DBR form requires less regularity assumptions on both the Lagrangian and u (one needs C^1 instead of C^2).
   - The DBR form follows from DBR lemma after writing the first variation in the form

     $$\delta F(u, v) = \int_a^b \big\{\Lambda(x) + L_p(x, u(x), u'(x))\big\} \cdot v'(x)\, dx.,$$

     where $\Lambda(x)$ is any antiderivative of $L_s(x, u(x), u'(x))$.

3. **Beltrami identity**. Let us consider the basic integral functional $F(u)$ defined as always. Let us assume that

   - the Lagrangian $L(s, p)$ does not depend on the variable x (*autonomous* Lagrangian),
   - the Lagrangian $L : \mathbb{R}^2 \to \mathbb{R}$ and the function $u : [a, b] \to \mathbb{R}$ are of class C^2,
   - u is a directional local minimum for F with respect to all directions $v \in C_c^\infty((a, b))$ (or in any good class for the FLCV).

   Then there exists a real number c such that

   $$u'(x)L_p(u(x), u'(x)) - L_s(u(x), u'(x)) = c \qquad \forall x \in [a, b].$$

   In other words, in the autonomous case the Euler-Lagrange equation admits a *first integral*. This *conservation law* is known as *Beltrami identity* or *Erdmann's equation*.

4. **Non equivalence between ELE and Beltrami identity**.

   - Every solution to ELE satisfies the Beltrami identity. The converse is in general false (for example, every constant function satisfies the Beltrami identity).
   - If u satisfies the Beltrami identity, then u satisfies ELE for every $x \in (a, b)$ such that $u'(x) \neq 0$ (and hence for every $x \in [a, b]$ if the set of zeroes of $u'(x)$ is discrete).

Integral functionals with higher order derivatives

1. **Setting**. Let $(a, b) \subseteq \mathbb{R}$ be an interval, let m be a positive integer, and let $L : [a, b] \times \mathbb{R} \times \mathbb{R}^{m+1} \to \mathbb{R}$ be a function, again called *Lagrangian*.

   We consider integral functionals of the form

$$F(u) = \int_a^b L(x, u(x), u'(x), \ldots, u^{(m)}(x))\, dx,$$

   where $u^{(i)}(x)$ denotes the i-th derivative of u with respect to the variable x.

2. **Integral forms of the first variation**. Under natural regularity assumptions on the Lagrangian L, the function u, and the perturbation v, one can write the first variation in the form

$$\delta F(u, v) = \int_a^b \left\{ L_s(x, u(x), \ldots, u^{(m)}(x)) \cdot v(x) + \sum_{i=1}^{m} L_{p_i}(x, u(x), \ldots, u^{(m)}(x)) \cdot v^{(m)}(x) \right\} dx.$$

   Under more restrictive regularity assumptions on L and u, after a suitable number of integrations by parts one can write the first variation in the form

$$\delta F(u, v) = \int_a^b \left\{ \sum_{i=1}^{m} (-1)^i \frac{d^i}{dx^i} L_{p_i}(x, u(x), \ldots, u^{(m)}(x)) + L_s(x, u(x), \ldots, u^{(m)}(x)) \right\} \cdot v(x)\, dx$$

$$+ \left[\sum_{i=1}^{m} \sum_{j=i}^{m} (-1)^{j-i} v^{(i-1)}(x) \cdot \frac{d^{j-i}}{dx^{j-i}} L_{p_j}(x, u(x), \ldots, u^{(m)}(x)) \right]_{x=a}^{x=b}.$$

3. **Euler-Lagrange equation**. If L and u are regular enough, and u is a directional local minimum for F with respect to all directions $v \in C_c^\infty((a, b))$ (or in any good class for the FLCV), then it turns out that

$$\sum_{i=1}^{m} (-1)^{i+1} \frac{d^i}{dx^i} L_{p_i}(x, u(x), \ldots, u^{(m)}(x)) = L_s(x, u(x), \ldots, u^{(m)}(x)) \qquad \forall x \in [a, b].$$

   This is an ordinary differential equation of order $2m$. As in the basic case, all derivatives with respect to x in the left-hand side are total derivatives.

4. **Genesis of boundary conditions**. When u satisfies ELE, and is DML also with respect to variations that do not vanish with their derivatives at the boundary, one can always deduce the correct number of boundary conditions from the boundary terms that appear in the second integral form of the first variation.

Integral functionals with multiple integrals

1. **Integral functionals with multiple integrals**. Let d be a positive integer, and let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let us consider the Lagrangian

$$L : \mathrm{Clos}(\Omega) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},$$

and the functional

$$F(u) = \int_\Omega L(x, u(x), \nabla u(x))\, dx.$$

This functional is well defined in a classical sense if L is continuous in $\mathrm{Clos}(\Omega) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and $u : \mathrm{Clos}(\Omega) \to \mathbb{R}$ is of class C^1.

2. **First integral form of the first variation (multiple integrals)**. Let us consider the basic integral functional $F(u)$ defined as above. Let us assume that

   - the Lagrangian $L(x, s, p)$ is of class C^1 in $\mathrm{Clos}(\Omega) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$,
   - the function $u : \mathrm{Clos}(\Omega) \to \mathbb{R}$ is of class C^1,
   - the perturbation $v : \mathrm{Clos}(\Omega) \to \mathbb{R}$ are of class C^1.

Then it turns out that

$$\delta F(u, v) = \int_\Omega \left\{ L_s(x, u(x), \nabla u(x)) \cdot v(x) + \langle \nabla_p L(x, u(x), \nabla u(x)), \nabla v(x) \rangle \right\} dx.$$

We point out that $\nabla_p L$ is the gradient of L with respect to the last d variables, so that

$$\langle \nabla_p L(x, u(x), \nabla u(x)), \nabla v(x) \rangle = \sum_{i=1}^{d} \frac{\partial L}{\partial p_i}(x, u(x), \nabla u(x)) \cdot \frac{\partial v}{\partial x_i}(x).$$

3. **Second integral form of the first variation (multiple integrals)**. Let us consider the basic integral functional $F(u)$ defined as above. Let us assume that

   - the Lagrangian $L(x, s, p)$ is of class C^2 in $\mathrm{Clos}(\Omega) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$,
   - the function $u : \mathrm{Clos}(\Omega) \to \mathbb{R}$ is of class C^2,
   - the perturbation $v : \mathrm{Clos}(\Omega) \to \mathbb{R}$ are of class C^1,
   - the boundary $\partial\Omega$ is regular enough for the Gauss-Green theorem to be true.

Then the first variation can be rewritten as

$$\begin{aligned} \delta F(u, v) &= \int_a^b \left\{ -\operatorname{div}\left(\nabla L_p(x, u(x), \nabla u(x)) \right) + L_s(x, u(x), \nabla u(x)) \right\} \cdot v(x)\, dx \\ &\quad + \int_{\partial\Omega} \langle \nabla L_p(x, u(x), \nabla u(x)), \vec{n} \rangle \cdot v(x)\, d\sigma, \end{aligned}$$

where "div" denotes the divergence with respect to space variables, and the last integral is a flux integral ($\vec{n}$ denotes the external normal versor to $\partial\Omega$) with respect to the $d - 1$ dimensional measure $d\sigma$.

Euler Lagrange equation for multiple integrals

1. **Euler-Lagrange equation (ELE) in divergence form**. Let us consider the basic integral functional $F(u)$ defined as above. Let us assume that

   - the Lagrangian $L(x, s, p)$ is of class C^2 in $\mathrm{Clos}(\Omega) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$,
   - the function $u : \mathrm{Clos}(\Omega) \to \mathbb{R}$ is of class C^2,
   - the boundary $\partial\Omega$ is regular enough for the Gauss-Green theorem to be true.
   - u is a directional local minimum (DLM) for F with respect to all directions $v \in C_c^\infty(\Omega)$ (or in any class that triggers the fundamental lemma in the calculus of variazions).

   Then it turns out that

   $$\mathrm{div}\left(\nabla L_p(x, u(x), \nabla u(x))\right) = L_s(x, u(x), \nabla u(x)) \qquad \forall x \in \mathrm{Clos}(\Omega).$$

   We point out that this is a *second order partial differential equation*, and all the x-derivatives in the divergence of the left-hand side are total derivatives.

   When expanded, the left-hand side of ELE takes the form (for the sake of shortness, we do not write explicitly the dependence of u and ∇u on x)

   $$\sum_{i=1}^{d} \frac{\partial^2 L}{\partial p_i \partial x_i}(x, u, \nabla u) + \sum_{i=1}^{d} \frac{\partial^2 L}{\partial p_i \partial s}(x, u, \nabla u)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2 L}{\partial p_i \partial p_j}(x, u, \nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

2. **Dirichlet functional, Laplacian, normal derivative**. The simplest example of integral functionals with multiple integrals is the *Dirichlet functional*

   $$F(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx.$$

   The second integral form of its first variation is

   $$\delta F(u, v) = -\int_\Omega \Delta u(x) \cdot v(x)\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}}(x) \cdot v(x)\, d\sigma,$$

   where Δu is the *Laplacian* of u defined by

   $$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(x),$$

   and $\partial u/\partial \vec{n}$ denotes the *directional derivative* of u in the direction perpendicular to $\partial\Omega$.

3. **Neumann boundary conditions**.

How to prove minimality

1. **Definition (Convex function in an affine space)**. Let $\mathbb{X}$ be an affine space.

   A function $F : \mathbb{X} \to \mathbb{R}$ is called *convex* if
   $$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \qquad \forall x \in \mathbb{X}, \quad \forall y \in \mathbb{X}, \quad \forall \lambda \in [0,1].$$
   The function F is called *strictly convex* if the inequality is strict whenever $x \neq y$ and $\lambda \in (0,1)$.

2. **Convexity is actually a one dimensional notion**. Let $\mathbb{X}$ be an affine space with associated vector space V, and let $F : \mathbb{X} \to \mathbb{R}$ be a function.

   For every $x \in \mathbb{X}$ and every $v \in V$, let us consider the restriction
   $$\mathbb{R} \ni t \to F(x+tv) \in \mathbb{R}.$$
   Then it turns out that

   - F is convex in $\mathbb{X}$ if and only if all its restrictions $F(x+tv)$ are convex in $\mathbb{R}$,
   - F is strictly convex in $\mathbb{X}$ if and only if all its restrictions $F(x+tv)$ are strictly convex in $\mathbb{R}$.

3. **Theorem (Minimality through convexity)**. Let $\mathbb{X}$ be an affine space with associated vector space V, let $F : \mathbb{X} \to \mathbb{R}$ be a function, and let $x_0 \in \mathbb{X}$.

   Let us assume that

   - F is convex,
   - $\delta F(x_0, v) = 0$ for every $v \in V$ (namely x_0 is a directional local minimum of F with respect to every direction $v \in V$).

   Then x_0 is a global minimum point for F.

   If in addition F is strictly convex, then x_0 is the unique minimum point.

4. **Convexity of integral functionals**. The usual integral functional $F(u)$ is convex if and only if, for every $x \in [a,b]$, the function $(s,p) \to L(x,s,p)$ is convex in $\mathbb{R}^2$. Analogous statements hold true for functionals involving higher order derivatives and/or multiple integrals.

5. **Theorem (Minimality through auxiliary functional)**. Let $\mathbb{S}$ be a set, let $F : \mathbb{S} \to \mathbb{R}$ be a function, and let $x_0 \in \mathbb{S}$.

   Let us assume that there exists a function $G : \mathbb{S} \to \mathbb{R}$ such that

   (i) $F(x) \geq G(x)$ for every $x \in \mathbb{S}$,

   (ii) $G(x) \geq G(x_0)$ for every $x \in \mathbb{S}$ (namely x_0 is a global minimum point for G),

   (iii) $F(x_0) = G(x_0)$.

   Then x_0 is a global minimum point for F.

Chapter 6

Direct method in the Calculus of Variations

Notion of convergence and Weierstrass Theorem

1. **Definition (Notion of convergence)**. Let S be a set, and let Seq($\mathbb{S}$) denote the set of sequences with values in $\mathbb{S}$, namely

$$\text{Seq}(\mathbb{S}) := \{f : \mathbb{N} \to \mathbb{S}\}.$$

A notion of convergence in $\mathbb{S}$ is any subset of Seq($\mathbb{S}$) $\times$ $\mathbb{S}$.

Roughly speaking, a notion of convergence is a list of all converging sequences in $\mathbb{S}$ with their limits.

2. **Remark**. At this level of generality, we do not ask any reasonable property on a notion of convergence. For example, we do not even ask

   - that a constant sequence is convergent, or that it converges to a unique value,
   - that a subsequence of a converging sequence is convergent, or that it converges to the same limit of the original sequence.

3. **Definition (Compactness with respect to a notion of convergence)**. Let $\mathbb{S}$ be a set with a notion of convergence. A subset $K \subseteq \mathbb{S}$ is called *compact* if every sequence with values in $\mathbb{S}$ admits a converging subsequence.

More formally, for every $\{x_n\} \subseteq K$ there exist an increasing sequence $\{n_k\}$ of positive integers and an element $x_\infty \in K$ such that $x_{n_k} \to x_\infty$.

4. **Definition (Lower semicontinuity with respect to a notion of convergence)**. Let $\mathbb{S}$ be a set with a notion of convergence. A function $F : \mathbb{S} \to \mathbb{R}$ is called *lower semicontinuous* if for every converging sequence sequence $x_n \to x_\infty$ in $\mathbb{S}$ it turns out that

$$\liminf_{n\to+\infty} F(x_n) \geq F(x_\infty).$$

5. **Weierstrass theorem with respect to a notion of convergence**. Let $\mathbb{S}$ be a *nonempty* set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function.

Let us assume that there exists a notion of convergence is $\mathbb{S}$ with respect to which

   (i) $\mathbb{S}$ is compact,

   (ii) F is lower semicontinuous.

Then F attains its minimum in $\mathbb{S}$, namely there exists $x_0 \in \mathbb{S}$ such that

$$F(x) \geq F(x_0) \qquad \forall x \in \mathbb{S}.$$

6. **A short blanket**. We observe that, in the previous theorem, the compactness of $\mathbb{S}$ and the lower semicontinuity of F are two "competing properties", in the sense that

   - if there are many converging sequences, than compactness is easier but lower semicontinuity becomes harder,
   - if there are few converging sequences, than compactness is harder but lower semicontinuity becomes easier.

Coercive functions and generalized Weierstrass Theorem

1. **Definition (coercive functions).** Let $\mathbb{S}$ be a set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function.

   The function F is called *coercive* if there exists a nonempty compact set $K \subseteq \mathbb{S}$ such that

   $$\inf \{F(x) : x \in K\} = \inf \{F(x) : x \in \mathbb{S}\}.$$

2. **Weierstrass theorem for coercive functions.** Let $\mathbb{S}$ be a *nonempty* set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function.

   Let us assume that there exists a notion of convergence in $\mathbb{S}$ with the property that

   (i) F is lower semicontinuous,

   (ii) F is coercive.

   Then F attains its minimum in $\mathbb{S}$.

3. **Corollary (nonempty sub-level contained in a compact set).** Let $\mathbb{S}$ be a set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function.

   Let us assume that there exists a notion of convergence in $\mathbb{S}$ with the property that

   (i) F is lower semicontinuous,

   (ii) there exist $M \in \mathbb{R}$ and a compact set $K \subseteq \mathbb{S}$ such that

   $$\emptyset \neq \{x \in \mathbb{S} : F(x) \leq M\} \subseteq K.$$

   Then F attains its minimum in $\mathbb{S}$.

4. **Corollary (Weierstrass theorem in Euclidean spaces).** Let d be a positive integer, and let $F : \mathbb{R}^d \to \mathbb{R}$.

   Let us assume that F is lower semicontinuous (with respect to the usual notion of convergence), and

   $$\lim_{|x| \to +\infty} F(x) = +\infty.$$

   Then F attains its minimum in $\mathbb{R}^n$.

5. **Generalizations.** The theory can be extended with minimal changes in order to include

   - upper semi-continuous functions, and therefore also maximum problems,
   - functions with values in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$.

Road map of the direct method

1. **Setting and goal**. Let $\mathbb{S}$ be a set, and let $F : \mathbb{S} \to \mathbb{R}$ (or $F : \mathbb{S} \to \mathbb{R} \cup \{+\infty\}$) be a function. The goal is proving the existence of
$$\min\{F(x) : x \in \mathbb{S}\}.$$

2. **The four steps of the road map**. The strategy for proving the existence of the minimum involves four main steps.

   - *Weak formulation.* The idea is to extend the problem to a more general setting. This amounts to finding a set $\widehat{\mathbb{S}}$ and a function $\widehat{F} : \widehat{\mathbb{S}} \to \mathbb{R}$ (or $\widehat{F} : \widehat{\mathbb{S}} \to \mathbb{R} \cup \{+\infty\}$) that extends F in the sense that
   $$\widehat{F}(x) \leq F(x) \qquad \forall x \in \mathbb{S}.$$
   - *Compactness.* The idea is proving that sublevels of $\widehat{F}$ are relatively compact in $\widehat{\mathbb{S}}$. This amounts to finding a notion of convergence in $\widehat{\mathbb{S}}$ such that every sequence $\{x_n\} \subseteq \widehat{\mathbb{S}}$ for which there exists $M \in \mathbb{R}$ with
   $$\widehat{F}(x_n) \leq M \qquad \forall n \in \mathbb{N}$$
   admits a converging subsequence.
   - *Lower semicontinuity.* This amounts to proving that $\widehat{F}$ is lower semicontinuous in $\widehat{\mathbb{S}}$ with respect to the *same notion of convergence* used in the compactness step.
   - *Regularity.* At the end of the first three steps we can already deduce the existence of
   $$\min\left\{\widehat{F}(x) : x \in \widehat{\mathbb{S}}\right\}.$$
   Now the idea is to conclude that the minimum is achieved also in $\mathbb{S}$.
   This amounts to proving that there exists $x_0 \in \mathbb{S}$, namely in the original set, such that
   $$F(x_0) = \widehat{F}(x_0) = \min\left\{\widehat{F}(x) : x \in \widehat{\mathbb{S}}\right\}.$$
   This is enough to conclude that x_0 is a minimum point for F in $\mathbb{S}$.

Part II

Exercises

Chapter 7

Functional analysis

Hilbert spaces

Subject: ... **Difficulty**: too easy

Prerequisites: ...

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1. Let H be a Hilbert space that admits a *countable* orthonormal basis.
   (a) Prove that the algebraic dimension of H is infinite.
   (b) Prove that H is separable.

2. Let V be a vector space with scalar product, and let $\{v_n\} \subset V$ be an orthonormal system (not necessarily complete).
   (a) Prove that
$$\sum_{n=0}^{\infty} \langle v, e_n \rangle^2 \leq \|v\|^2 \qquad \forall v \in V.$$
   (b) Characterize the set of all vectors $v \in V$ for which the equality holds true.

3. (Weak sequential closure of the unit sphere) Let H be a Hilbert space.

   Determine the set of all vectors $v \in H$ for which there exists a sequence $\{v_n\} \subseteq H$ such that
$$\|v_n\| = 1 \qquad \forall n \in \mathbb{N} \qquad \text{and} \qquad v_n \rightharpoonup v.$$

4. For every $n \in \mathbb{N}$, let us consider the function
$$f_n(x) := \begin{cases} 1 & \text{if } x \in [n, n+1], \\ 0 & \text{otherwise} \end{cases}$$

   Prove that $f_n(x) \rightharpoonup 0$ in $L^2(\mathbb{R})$.

Baire spaces

Subject: ... **Difficulty**: too easy

Prerequisites: ...

1. Determine which of the following subsets of $\mathbb{R}$ are Baire spaces (with respect to the topology inherited from $\mathbb{R}$):

$$[0,1], \qquad (0,1), \qquad \{0,1\}, \qquad [0,+\infty), \qquad \mathbb{Z}, \qquad \mathbb{Q}, \qquad \mathbb{R}\setminus\mathbb{Q},$$
$$\left\{a+b\sqrt{2} : (a,b)\in\mathbb{Z}^2\right\}.$$

2. Let us consider the set $\mathbb{R}^2 \setminus [(\mathbb{R}\setminus\mathbb{Q})\times\{0\}]$ (namely the plane minus the irrational points on the x axis), with the topology inherited from the plane.
   (a) Prove that X is a Baire space.
   (b) Prove that X admits a closed subset that is not a Baire space.

3. (Continuity of derivatives)
   (a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function that admits a derivative $f'(x)$ for every $x\in\mathbb{R}$. Prove that the function $f'(x)$ is continuous in at least one point (and actually in a residual set of points).
   (b) Let d be a positive integer, and let $f:\mathbb{R}^d\to\mathbb{R}$ be a function that admits all partial derivatives in every point $x\in\mathbb{R}^d$.
   Prove that the function $f(x)$ is differentiable in at least one point (and actually in a residual set of points).

4. Let $f:(0,+\infty)\to\mathbb{R}$ be a continuous function. Let us assume that there exists $\ell\in\mathbb{R}$ such that

$$\lim_{n\to+\infty} f(nx)=\ell \qquad \forall x\in(0,+\infty).$$

   (a) Prove that

$$\lim_{x\to+\infty} f(x)=\ell.$$

   (b) Discuss the following variants of the problem:
   - the case where $\ell=\pm\infty$,
   - the case where $f(x)$ is not assumed to be continuous,
   - the case where $f(x)$ is assumed to be continuous but ℓ is allowed to depend on x.

5. (Apparently similar to the previous one) Let $f:(0,+\infty)\to\mathbb{R}$ be a uniformly continuous function. Let us assume that there exists $\ell\in\mathbb{R}$ such that

$$\lim_{n\to+\infty} f(x+n)=\ell \qquad \forall x\in(0,+\infty).$$

(a) Prove that

$$\lim_{x \to +\infty} f(x) = \ell.$$

(b) Discuss the following variants of the problem:

- the case where $\ell = \pm\infty$,
- the case where $f(x)$ is assumed to be merely continuous,
- the case where $f(x)$ is assumed to be of class C^∞ but ℓ is allowed to depend on x.

6. (Discontinuity sets of real functions)

(a) Prove that there exists $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous only in $\mathbb{Q}$.

(b) Prove that there do not exist $f : \mathbb{R} \to \mathbb{R}$ which is continuous only in $\mathbb{Q}$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

- f is separately continuous, namely the function $x \to f(x, y)$ is continuous with respect to x for every fixed $y \in \mathbb{R}$, and the function $y \to f(x, y)$ is continuous with respect to y for every fixed $x \in \mathbb{R}$,
- there exists a dense subset $D \subseteq \mathbb{R}^2$ such that $f(x, y) = 0$ for every $(x, y) \in D$.

Prove that $f(x, y) = 0$ for every $(x, y) \in \mathbb{R}^2$.

Minimizing the distance from a point

Subject: ... **Difficulty**: too easy

Prerequisites: ...

1. (Closed and mid-point convex implies comvex)
2. (Distance from a closed set in finite dimension) Let d be a positive integer, and $K \subseteq \mathbb{R}^d$ be a nonempty *closed* subset. Let un consider in $\mathbb{R}^d$ the usual Euclidean norm.
   (a) (Existence of a closest point) Prove that for every $x \in \mathbb{R}^d$ there exist at least one point $z \in K$ such that
   $$\|x - z\| \le \|x - y\| \qquad \forall y \in K.$$
   (b) (non-uniqueness) Give an example in $\mathbb{R}^2$ where the minimizer is not unique.
3. (Manhattan norm in $\mathbb{R}^2$) Let us endow $\mathbb{R}^2$ with the *Manhattan norm* defined by
   $$\|(x, y)\|_1 := |x| + |y| \qquad \forall (x, y) \in \mathbb{R}^2.$$
   (a) Check that this is actually a norm in $\mathbb{R}^2$.
   (b) Prove that for every nonempty *closed* (but not necessarily convex) subset $K \subseteq \mathbb{R}^2$ there exists at least one point $z \in K$ that minimizes the distance from $(3, 4)$.
   (c) Find the set of minimizers (again of the distance from $(3, 4)$) in the case of the following closed and convex subsets:
   - $K_1 := \{(x, y) \in \mathbb{R}^2 : y = 0\}$,
   - $K_2 := \{(x, y) \in \mathbb{R}^2 : x + y = 3\}$,
   - $K_3 := \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 \le 8\}$,
   - $K_4 := \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 \le 1\}$,
   - $K_5 := \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 \le 6\}$.
4. (p-norms in $\mathbb{R}^2$) Let $p \in (1, +\infty)$ be a real number. Let us endow $\mathbb{R}^2$ with the *p-norm* defined by
   $$\|(x, y)\|_p := (|x|^p + |y|^p)^{1/p} \qquad \forall (x, y) \in \mathbb{R}^2.$$
   (a) Check that this is actually a norm in $\mathbb{R}^2$.
   (b) Prove that, for every nonempty *closed* subset $K \subseteq \mathbb{R}^2$ and every $x_0 \in \mathbb{R}^2$, there exists at least one point $z \in K$ that minimizes the distance from x_0.
   (c) Determine whether the minimizers are unique when K is *convex*.
   (d) Discuss the same questions for the ∞-norm defined by
   $$\|(x, y)\|_\infty := \max\{|x|, |y|\} \qquad \forall (x, y) \in \mathbb{R}^2.$$

5. (Distance from a closed set in infinite dimension) Let H be a separable Hilbert space, let $K \subseteq H$ be a nonempty *closed* set (not necessarily convex), and let $x_0 \in H \setminus K$.

   Determine whether it is true that there exists at least one point $z \in K$ such that

$$\|x - z\| \leq \|x - y\| \qquad \forall y \in K.$$

6. (Projection into a ball in normed spaces) Let V be a vector space with norm $\|v\|$, and let B be the closed ball with center in the origin and radius 1. Let us define the *projection* $P_B : V \to B$ as

$$P_B(v) := \begin{cases} v & \text{if } \|v\| \leq 1, \\ \dfrac{v}{\|v\|} & \text{if } \|v\| \geq 1. \end{cases}$$

   (a) Prove that for every $v \in V$ it turns out that

$$\|v - P_B(v)\| \leq \|v - y\| \qquad \forall y \in B,$$

   namely that $P_B(v)$ minimizes the distance from v among all elements of B.

   (b) Give an example where $P_B(v)$ is *not* the *unique* minimizer.

   (c) Prove that

$$\|P_B(v) - P_B(w)\| \leq 2\|v - w\| \qquad \forall (v, w) \in V^2,$$

   with strict inequality when $v \neq w$.

   (d) Give an example where the constant 2 in the previous inequality is optimal.

   Curiosity: it can be shown that, if the dimension of V is greater than or equal than 3, the projection P_B has Lipschitz constant equal to 1 if and only if the norm originates from a scalar product! A proof can be found in the paper

DeFigueiredo, D. G.; Karlovitz, L. A. On the radial projection in normed spaces. *Bull. Amer. Math. Soc.* **73** (1967), no. 3, 364–368.

Parallelogram Law

Subject: ... **Difficulty**: too easy

Prerequisites: ...

1. (There is no "parallelogram inequality") Let V be a normed space.
   Let us assume that either
   $$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \qquad \forall (x,y) \in V^2,$$
   or
   $$\|x+y\|^2 + \|x-y\|^2 \geq 2\|x\|^2 + 2\|y\|^2 \qquad \forall (x,y) \in V^2.$$
   Prove that the norm in V originates from a scalar product.
2. Prove that ℓ^p is a Hilbert spaces if and only if $p = 2$.
3. Characterize all measure spaces $(\mathbb{X}, \mathcal{M}, \mu)$ such that $L^3(\mathbb{X}, \mathcal{M}, \mu)$ is a Hilbert space.

Compact operators

Subject: ... **Difficulty**: too easy

Prerequisites: ...

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1. Let V be a *finite dimensional* normed space.

   Prove that every continuous (but not necessarily linear) operator $f : V \to V$ is compact.

2. (Counterexamples) Let H be a Hilbert space.

   (a) Find an example of a compact operator $f : H \to H$ that is not continuous.

   (b) Find an example of a linear and continuous operator $f : H \to H$ that is not compact.

3. Let H be a Hilbert space, and let $f : H \to H$ be a function such that

$$\langle f(v), w\rangle = \langle f(w), v\rangle \qquad \forall (v, w) \in H^2.$$

   Prove that f is linear.

4. Let $\{e_n\}$ be an orthonormal basis in a separable Hilbert space, and let $\{\lambda_n\}$ be a sequence of real numbers. Let us consider the linear function $A : H \to H$ such that

$$Ae_n = \lambda_n e_n \qquad \forall n \in \mathbb{N}.$$

   (a) Prove that A is well-defined if and only if $\{\lambda_n\}$ is bounded.

   (b) Prove that A is symmetric whenever it is well-defined.

   (c) Prove that A is strong-strong continuous whenever it is well-defined.

   (d) Prove that A is compact if and only if $\lambda_n \to 0$ as $n \to +\infty$.

   (e) Prove that A is weak-strong continuous if and only if it is compact.

5. Let H be a Hilbert space with infinite dimension.

   (a) Prove that there exists a linear, symmetric and compact operator on H that admits infinitely many different eigenvalues and an infinite dimensional kernel.

   (b) Prove that there exists a linear, symmetric and compact operator on H such that

$$\max\left\{\frac{\langle Av, v\rangle}{\|v\|^2} : v \in H \setminus \{0\}\right\}$$

   does not exist.

6. Let $H := L^2((0, 1))$, and let $A : H \to H$ be the operator defined by

$$(Af)(x) := xf(x) \qquad \forall f \in H.$$

(a) Prove that A is linear, symmetric, and Lipschitz continuous.

(b) Determine the exact Lipschitz constant of A.

(c) Prove that A is injective but not surjective.

(d) Prove that A has no eigenvalues.

(e) Prove that A is not compact in at least two ways:

- by deducing it from the failure of the spectral theorem,
- by exhibiting a bounded sequence $\{f_n\}$ such that the images $\{Af_n\}$ do not admit a converging subsequence.

(f) Determine the infimum and the supremum of the Rayleigh quotient in $H \setminus \{0\}$.

7. Let $H := L^2((0,1))$, and let $A : H \to H$ be the operator defined by

$$(Af)(x) := (|7x-2| + |4-7x|) \cdot f(x) \qquad \forall f \in H.$$

(a) Prove that A is linear, symmetric, and Lipschitz continuous.

(b) Determine the exact Lipschitz constant of A.

(c) Determine whether A is injective and/or surjective.

(d) Prove that A has a unique eigenvalue, and that the corresponding eigenspace has infinite dimension.

(e) Determine whether A is compact or not.

Chapter 8

Sobolev spaces

1D Sobolev Spaces

Subject: ... **Difficulty**: too easy

Prerequisites: ...

1. (Piecewise C^1 functions)

   (a) Let $u : (0, 2) \to \mathbb{R}$ be defined as

   $$u(x) := \begin{cases} x^2 & \text{if } x \in (0, 1] \\ (x-2)^3 & \text{if } x \in [1, 2) \end{cases}$$

   Prove that u admits a weak derivative, and find this weak derivative.

   (b) Prove that every piecewise C^1 function admits weak derivative.

2. (Smaller class of test functions) Let u and v be two functions in $L^1((-1, 1))$ such that

   $$\int_{-1}^{1} u(x)\varphi'(x)\,dx = -\int_{-1}^{1} v(x)\varphi(x)\,dx$$

   for every $\varphi \in C_c^\infty((-1, 1))$ such that $v(0) = 3$.

   Can we conclude that v is the weak derivative of u?

3. (Functions in $W^{1,1}$ are not necessarily Hölder continuous) Find a function $u : (-1, 1) \to \mathbb{R}$ such that

   - $u \in W^{1,1}((-1, 1))$,
   - u is not α-Hölder continuous for every $\alpha \in (0, 1]$.

4. (Continuity of functions with weak derivatives)

   (a) Prove that $W^{1,1}_{\text{loc}}((a, b)) \subseteq C^0((a, b))$.

   (b) Prove that $W^{1,1}_{\text{loc}}((a, b))$ contains functions that are not uniformly continuous in (a, b).

5. Find a continuous function $u : [-1, 1] \to \mathbb{R}$ such that $u \notin W^{1,1}((-1, 1))$.

Sobolev Spaces

Subject: ... **Difficulty**: too easy

Prerequisites: ...

1. ($H \neq W$ for $p = +\infty$) Let d and m be positive integers, and let $\Omega \subseteq \mathbb{R}^d$ be an open set.
   (a) Prove that for every $m \geq 1$ the space $W^{m,\infty}(\Omega)$ contains functions that do not belong to $C^m(\Omega)$.
   (b) Prove that for every $m \geq 1$ it turns out that $H^{m,\infty}(\Omega) = C^{m,\infty}(\Omega)$ (the space of the functions of class C^m whose derivatives are bounded up to order m).

Gagliardo-Brascamp-Lieb inequalities

Subject: Integral inequalities **Difficulty**: ⋆⋆⋆⋆

Prerequisites: Hölder inequality, proof of Gagliardo's inequality

1. (Understanding the proof of Gagliardo inequality)
   (a) Write explicitly the inductive step from $d = 2$ to $d = 3$ in the proof of Gagliardo inequality.
   (b) Write explicitly the inductive step from $d = 3$ to $d = 4$ in the proof of Gagliardo inequality.
2. (Equality cases)
   (a) Characterize all equality cases in Gagliardo's inequality with $d = 2$.
   (b) Find at least one triple of nontrivial functions (meaning that none of them vanishes identically) that realizes the equality case in the Gagliardo inequality with $d = 3$.
3. Find all pairs $(p, q) \in [1, +\infty]^2$ of exponents for which it turns out that
$$\int_{\mathbb{R}^2} a(x, y) \cdot b(x)\, dx\, dy \leq \|a\|_{L^p(\mathbb{R}^2)} \cdot \|b\|_{L^q(\mathbb{R})}$$
   for every pair of functions $a \in C_c^\infty(\mathbb{R}^2)$ and $b \in C_c^\infty(\mathbb{R})$.
4. Find all triples $(p, q, r) \in [1, +\infty]^3$ of exponents for which it turns out that
$$\int_{\mathbb{R}^3} a(x, y, z) \cdot b(x, y) \cdot c(y, z)\, dx\, dy\, dz \leq \|a\|_{L^p(\mathbb{R}^3)} \cdot \|b\|_{L^q(\mathbb{R}^2)} \cdot \|c\|_{L^r(\mathbb{R}^2)}$$
   for every triple of functions $a \in C_c^\infty(\mathbb{R}^3)$, $b \in C_c^\infty(\mathbb{R}^2)$, and $c \in C_c^\infty(\mathbb{R}^2)$.
5. Find all triples $(p, q, r) \in [1, +\infty]^3$ of exponents for which it turns out that
$$\int_{\mathbb{R}^3} a(x, y) \cdot b(y, z) \cdot c(x)\, dx\, dy\, dz \leq \|a\|_{L^p(\mathbb{R}^2)} \cdot \|b\|_{L^q(\mathbb{R}^2)} \cdot \|c\|_{L^r(\mathbb{R})}$$
   for every triple of functions $a \in C_c^\infty(\mathbb{R}^2)$, $b \in C_c^\infty(\mathbb{R}^2)$, and $c \in C_c^\infty(\mathbb{R})$.
6. Find all quadruples $(p, q, r, s) \in [1, +\infty]^4$ of exponents for which it turns out that
$$\int_{\mathbb{R}^4} a(x, y) \cdot b(y, z) \cdot c(z, w) \cdot d(w, x)\, dx\, dy\, dz\, dw \leq \|a\|_{L^p(\mathbb{R}^2)} \cdot \|b\|_{L^q(\mathbb{R}^2)} \cdot \|c\|_{L^r(\mathbb{R}^2)} \cdot \|d\|_{L^s(\mathbb{R}^2)}$$
   for every quadruple of functions a, b, c, d in $C_c^\infty(\mathbb{R}^2)$.
7. State and prove an inequality of Gagliando-Brascamp-Lieb type for functions of four variables that are the product of six functions of two variables (all pairs of variables are the argument of one of the six functions).

Extension operators 1

Subject: ... **Difficulty**: too easy

Prerequisites: ...

1. (Half-line to line) In the following points we extend a function $u : (0, +\infty) \to \mathbb{R}$ to the whole real line by defining $u(x)$ in different ways for $x < 0$, and we ask whether this procedure provides a strong m-extension from $(0, +\infty)$ to $\mathbb{R}$.

   (a) Prove that if we set
   $$u(x) := u(-x) \qquad \forall x < 0$$
   we obtain a strong 1-extension.

   (b) Prove that if we set
   $$u(x) := 3u(-x) - 2u(-2x) \qquad \forall x < 0$$
   we obtain a strong 2-extension.

   (c) Find real numbers a and b such that
   $$u(x) := au(-5x) + bu(-2018x) \qquad \forall x < 0$$
   is a strong 2-extension.

   (d) Find real numbers a, b and c such that
   $$u(x) := au(|x|) + bu(4|x| + cx) \qquad \forall x \in \mathbb{R}$$
   is a strong 2-extension.

   (e) Find real numbers a, b and c such that
   $$u(x) := au(-x) + bu(-2x) + u(-3x) \qquad \forall x < 0$$
   is a strong 3-extension.

2. (Starting from intervals) Give an explicit example of

   (a) a strong 1-extension from $(0, 1)$ to $(-1, 1)$,

   (b) a strong 1-extension from $(0, 1)$ to $\mathbb{R}$,

   (c) a strong 3-extension from $(0, 1)$ to $(-1, 1)$,

   (d) a strong 2-extension from $(-1, 0) \cup (0, 1)$ to $(-2, 0) \cup (0, 2)$,

   (e) a strong 2-extension from $(-2, -1) \cup (1, 2)$ to $(-2, 2)$.

3. (Interval and half-lines to the whole line)

   (a) Let $\Omega \subseteq \mathbb{R}$ be a nonempty open *convex* subset. Prove that for every positive integer m there exists a strong m-extension from Ω to $\mathbb{R}$.

   (b) Give an example of a nonempty open subset $\Omega \subseteq \mathbb{R}$ that does not admit a strong 1-extension to $\mathbb{R}$.

Extension operators 2

Subject: ... **Difficulty**: too easy

Prerequisites: ...

1. Let us consider the open set

$$\Omega := \left\{(x,y) \in \mathbb{R}^2 : y > \sin x\right\}.$$

   (a) Prove that the formula

$$Eu(x,y) := u(x, |y - \sin x| + \sin x)$$

   defines an extension of u to the whole $\mathbb{R}^2$, and it represents a strong 1-extension operator from Ω to $\mathbb{R}^2$.

   (b) Find a strong 2-extension operator from Ω to $\mathbb{R}^2$.

   (c) Generalize the result to domains of the form $\Omega := \{(x,y) \in \mathbb{R}^2 : y > f(x)\}$ under suitable assumptions on f.

2. Let us consider the open set

$$\Omega := \left\{(x,y) \in \mathbb{R}^2 : y > x^2\right\}.$$

   (a) Prove that the formula

$$Eu(x,y) := u(x, |y - x^2| + x^2)$$

   defines an extension of u to the whole $\mathbb{R}^2$, but it does not represent a strong 1-extension operator from Ω to $\mathbb{R}^2$.

   (b) Determine if there exists a strong 1-extension operator from Ω to $\mathbb{R}^2$.

3. Let us consider a quadrant $Q := (0, +\infty)^2$ in the plane.

   (a) Find an explicit example of strong 1-extension from the quadrant to $\mathbb{R}^2$.

   (b) Find an explicit example of strong 2-extension from the quadrant to $\mathbb{R}^2$.

4. Let us consider the open set $\Omega := \{(x,y) \in \mathbb{R}^2 : y > |x|\}$.

   (a) Prove that the formula

$$Eu(x,y) := u(x, |y - |x|| + |x|)$$

   defined a strong 1-extension from Ω to $\mathbb{R}^2$.

   (b) Prove that the same formula does not define a strong 2-extension from Ω to $\mathbb{R}^2$.

   (c) Prove that there exists a strong 2-extension from Ω to $\mathbb{R}^2$.

Extension operators 3

Subject: ... **Difficulty**: too easy

Prerequisites: ...

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1. Determine for which of the following open subsets of $\mathbb{R}^2$ there exists a strong 1-extension to the whole $\mathbb{R}^2$.

   (a) The square $(-1,1)^2$.

   (b) The triangle with vertices in $(-1,-1)$, $(0,2)$, $(3,1)$.

   (c) The union of three quadrants $[(-\infty,0)\times\mathbb{R}]\cup[\mathbb{R}\times(-\infty,0)]$.

   (d) The union of two opposite quadrans $(-\infty,0)^2\cup(0,+\infty)^2$.

   (e) The union of two half-planes $\mathbb{R}\times[(-\infty,-1)\cup(1,+\infty)]$.

   (f) The union of two half-planes with the same boundary $\{(x,y)\in\mathbb{R}^2 : y\neq x\}$.

   (g) The strip $(0,1)\times\mathbb{R}$.

   (h) The epigraph $\{(x,y)\in\mathbb{R}^2 : y>|\sin x|\}$.

   (i) The epigraph $\left\{(x,y)\in\mathbb{R}^2 : y>|x|^{1/2}\right\}$.

   (j) The hypograph $\{(x,y)\in\mathbb{R}^2 : y<x^2\}$.

   (k) The hypograph $\left\{(x,y)\in\mathbb{R}^2 : y<|x|^{1/2}\right\}$.

2. Let us consider the open set

$$\Omega := \left\{(x,y)\in\mathbb{R}^2 : x>1,\ 0<y<\frac{e^{-2018x}}{x\log^2 x}\right\},$$

   and the function $u(x,y) := e^x$.

   (a) Prove that $u\in L^p(\Omega)$ if and only if $p\leq 2018$.

   (b) Prove that $u\in W^{m,p}(\Omega)$ if and only if $p\leq 2018$, independently on m.

   (c) Determine if there exists a strong 1-extension operator from Ω to $\mathbb{R}^2$.

   (d) Determine if there exists a strong 1-extension operator from Ω to each of the following two open sets:

$$\Omega' := \left\{(x,y)\in\mathbb{R}^2 : x>1,\ |y|<\frac{e^{-2018x}}{x\log^2 x}\right\},$$

$$\Omega'' := \left\{(x,y)\in\mathbb{R}^2 : x>1,\ y<\frac{e^{-2018x}}{x\log^2 x}\right\}.$$

Regularity

Subject: ... **Difficulty**: too easy

Prerequisites: ...

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1. For every $\alpha \in (0, 2\pi)$ let $\Omega_\alpha \subseteq \mathbb{R}^2$ be the open set described in polar coordinates by

$$\Omega_\alpha := \{(\rho \cos\theta, \rho \sin\theta) : 0 < \rho < 1, \ 0 < \theta < \alpha\}.$$

   (a) Prove that the function defined in polar coordinates by

$$v(\rho, \theta) := \rho^{3/2} \sin\left(\frac{2}{3}\pi\right)$$

   is harmonic in $\Omega_{3\pi/2}$.

   (b) Prove that $v \in H^1(\Omega)$ but $v \notin H^2(\Omega)$.

   (c) Prove that there exists $f \in C_c^\infty(\mathbb{R}^2)$ such that the solution of Poisson's equation $\Delta u = f$ with homogeneous Dirichlet boundary conditions in $\Omega_{3\pi/2}$ does not belong to $H^2(\Omega_{3\pi/2})$.

   (d) Deduce that the usual regularity result $\Delta u \in L^2 \Rightarrow u \in H^2$ does not hold true in $\Omega_{3\pi/2}$.

   (e) Prove that the same solution belongs to $H^m_{\text{loc}}(\Omega_{3\pi/2})$.

   (f) Extend the result to the case of homogenous Neumann boundary conditions.

   (g) Extend the result to domains Ω_α with $\alpha \in (\pi, 2\pi)$.

2. Let us consider the first quadrant $Q := (0, +\infty)^2$ in the plane.

   (a) Prove that the function defined in polar coordinates by

$$v(\rho, \theta) := \rho^2 \log\rho \cdot \sin(2\theta) * \rho^2 \cdot \theta \cos(\theta)$$

   is harmonic and of class C^∞ in Q.

   (b) Write the same function in cartesian coordinates.

   (c) Prove that the second derivatives of v are unbounded in a neighborhood of the origin.

   (d) Determine for which values of m it turns out that $u \in H^m(\Omega)$, where Ω is the intersection of Q with the unit ball with center in the origin.

3. Let us consider Poisson's equation $\Delta u = f$ in a rectangle of the plane, with homogeneous Dirichlet boundary conditions.

   (a) Prove that the solution satisfies a $L^2 \rightsquigarrow H^2$ regularity result.

   (b) Prove that the solution does not satisfy a $H^1 \rightsquigarrow H^3$ regularity result.

Chapter 9

Calculus of Variations

Generalized Weierstrass Theorem

Subject: Weierstrass theorem **Difficulty**: too easy

Prerequisites: notion of convergence, Weierstrass theorem

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1. Characterize all sets $\mathbb{S}$ for which every function $F : \mathbb{S} \to \mathbb{R}$ attains its minimum on $\mathbb{S}$.
2. Characterize all functions $F : \mathbb{Z} \to \mathbb{R}$ that attain both the minimum and the maximum on every nonempty subset of $\mathbb{Z}$.
3. ("Converse" of Weierstrass theorem) Let $\mathbb{S}$ be a set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function that attains its minimum on $\mathbb{S}$.

   Prove that there exists a notion of convergence in $\mathbb{S}$ with respect to which $\mathbb{S}$ is compact and F is lower semicontinuous.
4. (Easy uniqueness criterion) [This will never converge to a correct stable version!]
   (a) Let $\mathbb{X}$ be a *metric space*, and let $F : \mathbb{X} \to \mathbb{R}$ be a function that admits at least one minimum point in $\mathbb{X}$. Let us assume that every minimizing sequence is convergent. Prove that the minimum point is necessarily *unique*.
   (b) Is the previous statement true if we do not assume the existence of the minimum?
5. Find a continuous function $f : (0, 1] \to \mathbb{R}$ that attains neither the maximum nor the minimum on $(0, 1]$.
6. (Extended range) Extend the notion of lower semicontinuous function and Weierstrass theorem to functions with values
   (a) in $\mathbb{R} \cup \{+\infty\}$
   (b) in the extended real line $\mathbb{R} \cup \{+\infty, -\infty\}$.
7. State and prove the analogous of Weierstrass Theorem (and its generalizations) in the setting of maximum problems.
8. (Weird notions of convergence) Let us consider the set $\mathbb{S} := \mathbb{N}$. For each of the following notions of convergence in $\mathbb{S}$, determine all compact subsets of $\mathbb{S}$, and all lower semicontinuous functions $f : \mathbb{S} \to \mathbb{R}$.

   Determine also if the following property is true: "if $x_n \to x_\infty$, then $x_{2n} \to x_\infty$".
   (a) Only constant sequences are convergent, and for every $k \in \mathbb{N}$ the constant sequence $x_n \equiv k$ converges only to k.
   (b) Only constant sequences are convergent, and for every $k \in \mathbb{N}$ the constant sequence $x_n \equiv k$ converges only to $k + 1$.
   (c) Every sequence converges, and it converges only to 2019.
   (d) Every sequence converges, and it converges to every possible limit.
   (e) Every bounded sequence converges only to 0, and every unbounded sequence converges only to 2019.

Approximation results 1

Subject: smooth approximation of given functions **Difficulty**: too easy

Prerequisites: sequences of functions, uniform convergence

1. Let $[c,d] \subseteq (a,b)$ be two intervals. Prove that there exists a function $v : \mathbb{R} \to \mathbb{R}$ of class C^∞ such that
   - $0 \le v(x) \le 1$ for every $x \in \mathbb{R}$,
   - $v(x) = 0$ for every $x \notin [a,b]$,
   - $v(x) = 1$ for every $x \in [c,d]$.
2. Let $[c,d] \subseteq (a,b)$ be two intervals. Prove that there exists a sequence of functions $v_n(x)$ in $C_c^\infty((a,b))$ such that
   - $0 \le v_n(x) \le 1$ for every $n \in \mathbb{N}$ and every $x \in [a,b]$,
   - $v(x) \to 1$ uniformly in $[c,d]$,
   - $v(x) \to 0$ uniformly on compact subsets of $[a,b] \setminus [c,d]$.
3. Let $v : [a,b] \to \mathbb{R}$ be a continuous function.

   Prove that there exists a sequence $v_n(x)$ in $C_c^\infty((a,b))$ such that
   - there exists a constant $M \in \mathbb{R}$ such that $|v_n(x)| \le M$ for every $n \in \mathbb{N}$ and every $x \in [a,b]$,
   - $v_n(x) \to v(x)$ uniformly on compact subsets of (a,b).
4. Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and let $v : [a,b] \to \mathbb{R}$ be any function (not necessarily continuous). Let M be a real number, let $S \subseteq [a,b]$ be a finite set, and let $v_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions such that

   (i) $|v_n(x)| \le M$ for every $n \in \mathbb{N}$ and every $x \in [a,b]$,

   (ii) $v_n(x) \to v(x)$ uniformly on compact subsets of $(a,b) \setminus S$.

   Prove that $v(x)$ is Riemann integrable and

$$\lim_{n \to +\infty} \int_a^b f(x) v_n(x)\, dx = \int_a^b f(x) v(x)\, dx.$$

Variations on the Fundamental Lemma – Part 1

Subject: Fundamental Lemma in the Calculus of Variations **Difficulty**: too easy

Prerequisites: FLCV, FLCV with zero average, DBR

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1. (Restricted classes of test functions) Let $a < c < b$ be three real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function such that

$$\int_a^b f(x)v(x)\,dx = 0 \qquad \forall v \in V,$$

where V is a suitable class of test functions.

Determine for which of the following choices of V we can conclude that $f(x) = 0$ for every $x \in (a, b)$.

(a) $V = \{v \in C^1([a, b]) : |v(x)| + |v'(x)| \leq 1/7 \quad \forall x \in [a, b]\}$.

(b) $V = \{v \in C^1([a, b]) : v(a) = 7\}$.

(c) $V = \{v \in C^3([a, b]) : v(c) = 7\}$.

(d) $V = \{v \in C^\infty([a, b]) : v'(c) = 7\}$.

(e) $V = C_c^\infty((a, c) \cup (c, b))$.

(f) $V = \left\{v \in C^\infty([a, b]) : \int_a^b v(x)\,dx = 7\right\}$.

(g) $V = \left\{v \in C^\infty([a, b]) : \int_a^c v(x)\,dx = 0\right\}$.

2. Discuss the previous statements in the Lebesgue setting, namely assuming only that $f \in L^1_{\text{loc}}((a, b))$.

3. (Disconnected domain) Let (a, b) and (c, d) be two disjoint intervals of the real line, let $\Omega := (a, b) \cup (c, d)$, and let $f : \Omega \to \mathbb{R}$ be a continuous function.

(a) Let us assume that

$$\int_\Omega f(x)v(x)\,dx = 0$$

for every $v \in C_c^\infty(\Omega)$ with zero average. What can we conclude?

(b) Let us assume that

$$\int_\Omega f(x)v'(x)\,dx = 0$$

for every $v \in C_c^\infty(\Omega)$. What can we conclude?

(c) Generalize to the case $f \in L^1_{\text{loc}}(\Omega)$.

Variations on the Fundamental Lemma – Part 2

Subject: Fundamental Lemma in the Calculus of Variations **Difficulty**: too easy

Prerequisites: FLCV, FLCV with zero average, DBR

1. (Sign conditions in FLCV) Let d be a positive integer, let $\Omega \subset \mathbb{R}^d$ be an open set, and let $f : \Omega \to \mathbb{R}$ be a continuous function.

   (a) Let us assume that
   $$\int_\Omega f(x)v(x)\,dx = 0$$
   for every $v \in C_c^\infty(\Omega)$ such that $v(x) \leq 0$ for every $x \in \Omega$. What can we conclude?

   (b) Let us assume that
   $$\int_\Omega f(x)v(x)\,dx \geq 0$$
   for every $v \in C_c^\infty(\Omega)$ such that $v(x) \geq 0$ for every $x \in \Omega$. What can we conclude?

   (c) Generalize to the case $f \in L^1_{\text{loc}}(\Omega)$.

2. (Sign conditions in DBR) Let $(a, b) \subseteq \mathbb{R}$ be a continuous function.

   (a) Let us assume that
   $$\int_a^b f(x)v'(x)\,dx \geq 0$$
   for every $v \in C_c^\infty((a, b))$ such that $v(x) \geq 0$ for every $x \in (a, b)$. What can we conclude?

   (b) Generalize to the case $f \in L^1_{\text{loc}}((a, b))$.

3. (Higher order derivatives) Let $f : [a, b] \to \mathbb{R}$ be a continuous function.

   (a) Let us assume that
   $$\int_a^b f(x)v''(x)\,dx = 0 \qquad \forall v \in C_c^\infty((a, b)).$$
   What can we conclude?

   (b) Let us assume that
   $$\int_a^b f(x)v''(x)\,dx = 0$$
   for every $v \in C_c^\infty((a, b))$ such that $v(x) \geq 0$ for every $x \in (a, b)$. What can we conclude?

   (c) Generalize the first two points to functions in $L^1_{\text{loc}}((a, b))$.

   (d) Generalize the first point to higher order derivatives.

Variations on the Fundamental Lemma – Part 3

Subject: Fundamental Lemma in the Calculus of Variations **Difficulty**: too easy

Prerequisites: FLCV, FLCV with zero average, DBR

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1. (Weighted zero average) Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function such that
$$\int_0^{2\pi} f(x)v(x)\,dx = 0$$
for every $v \in C_c^\infty((0, 2\pi))$ such that
$$\int_0^{2\pi} v(x)\sin x\,dx = 0.$$
What can we conclude?

2. (Requires Stone-Weierstrass theorem) Let $f : [-1, 2] \to \mathbb{R}$ be a continuous function such that
$$\int_{-1}^{2} f(x)x^n\,dx = 0 \qquad \forall n \in \mathbb{N}.$$
   (a) Can we deduce that $f(x) = 0$ for every $x \in [-1, 2]$?
   (b) And if we limit ourselves to powers of x with even exponent?

3. Ci sarà una versione vettoriale del DBR?

First Variation

Subject: ?? **Difficulty**: too easy

Prerequisites: First Variation along lines, Gateaux derivatives, DLM

1. Let $\mathbb{S}$ be a set, and let $F : \mathbb{S} \to \mathbb{R}$ be a function. Let us assume that there exists $x_0 \in \mathbb{S}$ such that $\delta F(x_0, \gamma) = 0$ for every curve γ through x_0.

   What can we conclude?

2. (a) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ for which $(0, 0)$ is a directional local minimum point with respect to all directions $v \in \mathbb{R}^2$, but it is not even a local minimum point.

   (b) Is it possible to find the previous example of class C^∞?

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = |x^2 + y^2|^{1/2019}$.

   Determine whether there exists an *injective* curve γ through $x_0 = (0, 0)$ such that $\delta F(x_0, \gamma) = 0$.

4. (Inner/horizontal variation) Let $[a, b] \subseteq \mathbb{R}$ be an interval, let $L : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian of class C^2, and let $F(u)$ be the corresponding integral functional.

   Let $u : [a, b] \to \mathbb{R}$ be a function of class C^2, and let $v \in C_c^\infty((a, b))$.

   (a) Prove that there exists $t_0 > 0$ such that

   $$x + tv(x) \in [a, b] \qquad \forall x \in [a, b], \quad \forall t \in (-t_0, t_0).$$

   (b) Prove that the expression $[\gamma(t)](x) := u(x + tv(x))$ defines a curve through u in the space $C^2([a, b])$, and that all the elements in the support of this curve do have the same boundary data of u.

   (c) Prove that

   $$\delta F(u, v) = \int_a^b \left\{ -\frac{d}{dx} L_p(x, u(x), u'(x)) + L_s(x, u(x), u'(x)) \right\} \cdot u'(x) \cdot v(x)\, dx.$$

   (d) Motivate the presence of $u'(x)$ in the previous relation.

   (e) Compare the Euler-Lagrange equations generated by the inner (horizontal) and the outer (vertical) variation.

5. (Lagrangian independent of p) Determine

   $$\inf \left\{ \int_0^\pi (u - \sin x)^2\, dx : u \in C^1([-1, 1]),\ u(0) = u(\pi) = 1 \right\}$$

   and

   $$\inf \left\{ \int_0^\pi (u - \sin x)^2\, dx : u \in C^1([-1, 1]),\ u(\pi/2) = 0 \right\}.$$

6. (Lagrangian independent of s) Determine

$$\inf\left\{\int_0^\pi (\dot{u}-\sin x)^2\,dx : u \in C^1([-1,1]),\ u(0)=u(\pi)=1\right\}$$

and

$$\inf\left\{\int_0^\pi (\dot{u}-\cos x)^2\,dx : u \in C^1([-1,1]),\ u(0)=u(\pi)=1\right\}.$$

7. (Generalization of Erdmann's equation) Let $L : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a Lagrangian, and let $u : [a,b]\to\mathbb{R}$ be a solution to the corresponding Euler-Lagrange equation.

   Prove that, under suitable regularity assumption on L and u (to be made precise), there exists a constant $c \in \mathbb{R}$ such that

$$L_p(x,u(x),u'(x))\cdot u'(x) - L(x,u(x),u'(x)) = c - \int_a^x L_x(t,u(t),u'(t))\,dt \qquad \forall x \in [a,b].$$

8. (Violation of uniqueness?) Let us assume that, for some autonomous Lagrangian, the Euler-Lagrange equation has a non-constant solution $u_0(x)$. If we interpret Beltrami identity as a differential equation, then $u_0(x)$ is a solution to this equation. On the other hand, also all constant functions are solution to the same equation.

   Does this violate the classical uniqueness result for solutions to first order differential equations?

9. (Open problem! I have no solution) Is it possible to deduce Beltrami identity from Euler-Lagrange equation assuming only that the Lagrangian $L(x,s,p)$ and the solution $u(x)$ are of class C^1?

Boundary conditions 1

Subject: minimization of functionals – indirect methods **Difficulty**: too easy

Prerequisites: Euler equation, genesis of boundary conditions

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Let us consider, for every $u \in C^1([0,2])$, the following functionals:

$$F_1(u) = \int_0^2 [u'(x)]^2 \, dx, \qquad F_2(u) = \int_0^2 \left([u'(x)]^2 + [u(x)]^2\right) dx,$$

$$F_3(u) = \int_0^2 \left([u'(x)]^2 - 7u(x)\right) dx, \qquad F_4(u) = \int_0^2 \left([u'(x)]^2 + [u(x) - x^3]^2\right) dx.$$

For each of them, in the following table it is required to discuss the minimum problem subject to the extra constraints presented in each row. When the minimum exists, the number of minimizers is required. When the minimum does not exist, a characterization of the infimum is required. It is strongly recommended to work again on these problems in a second step by means of direct methods.

Constraints	$F_1(u)$	$F_2(u)$	$F_3(u)$	$F_4(u)$
$u(0) = 3$ and $u(2) = 8$				
$u(0) = 5$				
no further condition				
$u(0) = u(2)$				
$u(0) = -u(2)$				
$\int_0^2 u(x)\, dx = 8$				
$u(0) = 3$ and $\int_0^2 u(x)\, dx = 8$				
$u(2) = 5u(0)$				
$u(2) = 5 + u(0)$				
$u(2) = u'(0)$				
$u'(0) = u'(2)$				
$u'(0) = 3$				
$u(0) = 3$ and $u'(1) = 3$				
$u''(0)$ exists and $u''(0) = 3$				
$\int_0^1 u(x)\, dx = 1$ and $\int_1^2 u(x)\, dx = 2$				

Boundary conditions 2

Subject: minimization of functionals – indirect methods **Difficulty**: too easy

Prerequisites: Euler equation, genesis of boundary conditions

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Let us consider, for every $u \in C^2([0,\pi])$, the following functionals:

$$F_1(u) = \int_0^\pi \ddot{u}^2 \, dx, \qquad F_2(u) = \int_0^\pi \left(\ddot{u}^2 + \dot{u}^2\right) dx,$$

$$F_3(u) = \int_0^\pi \left(\ddot{u}^2 + u^2\right) dx, \qquad F_4(u) = \int_0^\pi \left(\ddot{u}^2 + \dot{u} + u^2\right) dx.$$

For each of them, in the following table it is required to discuss the minimum problem subject to the extra constraints presented in each row. When the minimum exists, the number of minimizers is required. When the minimum does not exist, a characterization of the infimum is required.

Constraints	$F_1(u)$	$F_2(u)$	$F_3(u)$	$F_4(u)$
no further condition				
$u(0) = 1$				
$u'(0) = 1$				
$u''(0) = 1$				
$u(0) = 1$ and $u(\pi) = 2$				
$u'(0) = 1$ and $u'(\pi) = 2$				
$u'(0) = 1$ and $u(\pi) = 2$				
$u(0) = u(\pi)$				
$u'(0) = u'(\pi)$				
$u(0) = u'(\pi)$				
$u(0) = u(\pi)$ and $u'(0) = 2$				
$u(0) = u(\pi)$ and $u'(0) = u'(\pi)$				
$u(0) = u(\pi)$ and $u''(0) = u''(\pi)$				
$\int_0^\pi u(x)\, dx = 3$				
$u(\pi) = u(0) - 44$				

As in the previous exercise sheet, it is strongly recommended to work again on these problems in a second step by means of direct methods.

Minimum problems 1

Subject: minimization of functionals – indirect methods **Difficulty**: too easy

Prerequisites: Euler equation, optimality through convexity

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1. Solve the minimum problem for the functional

$$F(u) = \int_0^1 (\dot{u} - u)^2 \, dx$$

subject to each of the following boundary conditions:

   (a) $u(0) = 2015$,

   (b) $u(0) = u(1) = 2015$.

2. Solve the minimum problem for the functional

$$F(u) = \int_{-2}^2 (u'(x) - |x|)^4 \, dx$$

subject to each of the following boundary conditions:

   (a) $u(2) = 2015$,

   (b) $u(-2) = u(2) = 2015$,

   (c) $\displaystyle\int_{-2}^2 u(x)\, dx = 2015$.

3. For every continuous function $f : [a, b] \to \mathbb{R}$, let us consider the minimum problem

$$\min \left\{ \int_a^b \left[\dot{u}^2 + (u - f(x))^2 \right] \, dx : u \in C^1([a, b]) \right\}.$$

   (a) Prove that the problem admits a unique solution.

   (b) Prove that the solution satisfies

$$\min_{x\in[a,b]} f(x) \le \min_{x\in[a,b]} u(x) \le \max_{x\in[a,b]} u(x) \le \max_{x\in[a,b]} f(x).$$

   (c) Prove that the solution satisfies

$$\int_a^b u(x)\, dx = \int_a^b f(x)\, dx.$$

4. Solve the minimum problem

$$\min \left\{ \int_0^3 \left[\dot{v}^2 + (v - u)^2 + (u - x)^2 \right] \, dx : u \in C^1([0, 3]), \ v \in C^1([0, 3]) \right\}.$$

5. Determine for which values of the real parameter λ the

$$\inf \left\{ \int_{-1}^{1} \left[\dot{u}^2 + (u-x)^2 \right] \, dx : u(0) = \lambda, \ u \in C^1([-1,1]) \right\}$$

is actually a minimum (please note that the condition is given in a point that is *not* one of the endpoints of the interval).

Minimum problems 2

Subject: minimization of functionals – indirect methods **Difficulty**: too easy

Prerequisites: Euler equation, optimality through convexity

1. Let us consider the following functional

$$F(u) := \int_0^2 \left(\dot{u}^2 + u^2\right) dx.$$

Discuss existence/uniqueness/regularity for the following minimum problems:

(a) $\min \{u(2) + F(u) : u(0) = 0\}$,

(b) $\min \left\{[u(2)]^3 + F(u) : u(0) = 0\right\}$,

(c) $\min \{u(1) + F(u) : u(0) = 0\}$,

(d) $\min \{u(0) - u(2) + F(u)\}$.

2. Let us consider the minimum problem

$$\min \left\{ \int_0^1 (1 + u^2)\dot{u}^2 \, dx : u(0) = 1, \; u(1) = \alpha \right\}.$$

(a) [This point seems to require the direct method; uhm, with a clever variable change] Prove that for every $\alpha \in \mathbb{R}$ the problem admits at least a solution.

(b) Prove that every minimizer is monotone.

(c) Prove that for every $\alpha \in \mathbb{R}$ the solution is unique.

(d) Discuss convexity/concavity of the solution.

3. Let us consider the minimum problem

$$\min \left\{ \int_0^1 \left(\dot{u}^4 + u\right) dx : u(0) = 0, \; u(1) = \alpha \right\},$$

where α is a real parameter.

(a) Prove that the problem admits a unique solution for every $\alpha \in \mathbb{R}$.

(b) Discuss monotonicity and regularity of the solution.

4. Let us consider the minimum problem

$$\min \left\{ \int_0^1 \left(e^{u'(x)} + u^4(x)\right) dx : u(0) = u(1) = \alpha \right\},$$

where α is a real parameter.

(a) Compute explicitly the solution in the case $\alpha = 0$.

(b) [Uhm, existence is not so clear] Prove that for every $\alpha \in \mathbb{R}$ the problem admits a unique solution, and this solution is strictly convex when $\alpha > 0$ and strictly concave when $\alpha < 0$.

Minimum problems 3

Subject: minimization of functionals – direct methods **Difficulty**: too easy

Prerequisites: direct methods in H^1, optimality through convexity

1. Determine which of the following functionals attains the minimum in the class of all functions $u \in C^1([0,1])$ such that $u(0) = 1$:

$$F_1(u) = \int_0^1 \left(\dot{u}^2 + \arctan(u^2)\right) dx, \qquad F_2(u) = \int_0^1 \left(u^2 + \arctan(\dot{u}^2)\right) dx,$$

$$F_3(u) = \int_0^1 \arctan\left(\dot{u}^2 + u^2\right) dx, \qquad F_4(u) = \int_0^1 \left(\dot{u}^2 - \arctan(u^2)\right) dx.$$

2. Let us consider the functionals

$$F(u) = \int_0^\pi \left(\dot{u}^2 + \sin x \cdot u^4\right) dx, \qquad G(u) = \int_0^\pi \left(\dot{u}^2 + \cos x \cdot u^4\right) dx.$$

Discuss existence/uniqueness/regularity for the minimization of $F(u)$ and $G(u)$ subject to the boundary conditions $u(0) = u(\pi) = 4$.

3. Discuss existence/uniqueness/regularity for the minimum problem

$$\min\left\{\int_0^1 \left(e^{x^2} \cdot \dot{u}^2 + e^{u^4}\right) dx : \int_0^1 u(x)\, dx = 2015\right\}.$$

4. Discuss existence/uniqueness/regularity for the minimum problem

$$\min\left\{\int_0^{\pi/4} \left(\cos x \cdot \dot{u}^2 + \sin x \cdot u^4 - \tan x \cdot u\right) dx : u \in C^1([0,\pi/4])\right\}.$$

5. Determine which of the following functionals attains the minimum in the class of all function $u \in C^1([0,7])$ such that $u(0) = u(7) = 0$:

$$F(u) = \int_0^7 \left(\sqrt{1+\dot{u}^4} - \sqrt{1+u^2}\right) dx, \qquad G(u) = \int_0^7 \left(\sqrt{1+\dot{u}^2} - \sqrt{1+u^4}\right) dx.$$

6. Let us consider the following minimum problem

$$\min\left\{\int_0^1 \left[\frac{\dot{u}}{u^2+1}\right]^2 dx : u(0) = 0,\ u(1) = 1\right\}.$$

   (a) Solve the Euler equation associated to the problem.

   (b) Prove that in the minimization process it is enough to consider nondecreasing functions.

   (c) Prove that the solution of the Euler equation is actually the unique global minimizer.

ELE for multiple integrals

Subject: **Difficulty**: too easy

Prerequisites:

[Bozza per il futuro]

1. For each of the following Lagrangians, compute the corresponding Euler-Lagrange equation (in expanded form).

Lagrangian	Euler-Lagrange equation
$u_x^2 + u_y^2$	
$u_x^4 + u_y^4$	
$(u_x^2 + u_y^2)^2$	
$u_x^2 + u_y^2 - 2u_x u_y$	
$u_x^2 u_y^2$	
$\log(u_x^2 + u_y^2)$	
$x^2 u_x^2 + y^2 u_y^2$	
$y^2 u_x^2 + x^2 u_y^2$	
$u^4 u_x^2 + u^2 u_y^4$	
$x^2 u^4 u_x^6$	
$x^4 u^4 u_y^6$	

2. Let p be a positive real number. Compute the Euler-Lagrange equation corresponding to the following Lagrangian, known as p-Laplacian (it is the p-norm of the gradient):

$$(u_x^2 + u_y^2)^{p/2}.$$

3. Let Ω be the unit ball with center in the origin of the plane. Let us consider the functional

$$F(u) = \int_\Omega (u_x^2 + u_y)^2 \, dx \, dy.$$

   (a) Let us set $u_n(x, y) := |x^2 - y^2|^{1/n}$. Compute the limit of $F(u_n)$ as $n \to +\infty$.

   (b) Determine the infimum of $F(u)$ among all functions $u \in C^\infty(\Omega)$ such that $u(0, 0) = 1$.

Boundary value problems 1

Subject: minimum problems vs BVP **Difficulty**: too easy

Prerequisites: direct methods in H^1, Euler equation, genesis of boundary conditions

In each row of the following table a boundary value problem is presented. It is required to determine a variational problem for which the given BVP is the Euler equation, and then to discuss existence/uniqueness/regularity of the solution.

Equation	Boundary conditions	Variational problem
$u'' = \sinh u$	$u(0) = 0$ $u(1) = 2015$	
$u'' = e^u$	$u(0) = 2015$ $u'(0) = 0$	
$u'' = \dfrac{\arctan(u+x)}{x^2+1}$	$u'(0) = 0$ $u'(2015) = 0$	
$u'' = \lvert\sin x\rvert \cdot \lvert\cos u\rvert$	$u(0) = 0$ $u(2015) = 7$	
$u'' = \dfrac{\cos x}{u}$	$u(2) = 1/20$ $u(3) = 2015$	
$u'' = x^2u^5 - \sin x$	$u(-1) = u(1)$ $u'(-1) = u'(1)$	
$u'' = u^3 \cdot \arctan x$	$u'(1) = 1$ $u(2015) = 1$	
$u'' = \arctan(x^2u)$	$u'(1) = 2$ $u'(2015) = 3$	
$u^{IV} = e^{-u}$	$u(0) = u(2015) = 7$ $u'(0) = u''(0) = 0$	
$u^{IV} = x^3 - \log u$	$u(0) = u'(2105) = 3$ $u'(0) = u''(2015) = 0$	
$u^{IV} = (u')^2u'' - u^5$	$u(0) = u'(0) = 3$ $u(4) = u''(4) = 0$	

[Aggiungere una scheda]

Weak lower semicontinuity

Subject: Continuity of integral functionals **Difficulty**: $\star\star\star$

Prerequisites: Weak convergence, strong and convex implies weak

1. Let $a < b$ be two real numbers, and let $F : L^2((a,b)) \to \mathbb{R}$ be the functional defined by

$$F(v) := \int_a^b (7 + \sin x) \cdot v(x)^2 \, dx \qquad \forall v \in L^2((a,b)).$$

   (a) Prove that F is *strongly continuous* and convex in $L^2((a,b))$.

   (b) Prove that F is *weakly lower semicontinuous* in $L^2((a,b))$.

2. Let $a < b$ be two real numbers, and let $F : L^2((a,b)) \to \mathbb{R} \cup \{+\infty\}$ (note that now the value $+\infty$ is allowed) be the functional defined by

$$F(v) := \int_a^b (7 + \sin x) \cdot v(x)^4 \, dx \qquad \forall v \in L^2((a,b)).$$

   (a) Prove that F is *strongly lower semicontinuous* and convex in $L^2((a,b))$.

   (b) Prove that F is *weakly lower semicontinuous* in $L^2((a,b))$.

3. Generalize the results of the previous exercise to functionals of the form

$$F(v) := \int_a^b L(x, v(x)) \, dx \qquad \forall v \in L^2((a,b)),$$

   where the Lagrangian $L : (a,b) \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

   - (measurability with respect to x) for every $p \in \mathbb{R}$, the function $x \to L(x,p)$ is measurable in (a,b),
   - (convexity with respect to p) for every $x \in (a,b)$, the function $p \to L(x,p)$ is convex in $\mathbb{R}$,
   - (boundedness from below) there exists $A \in \mathbb{R}$ such that

$$L(x,p) \geq -A \qquad \forall (x,p) \in (a,b) \times \mathbb{R}.$$

4. Generalize the results of the previous exercise to functionals of the form

$$F(v_1, \ldots, v_k) := \int_a^b L(x, v_1(x), \ldots, v_k(x)) \, dx \qquad \forall v \in \left[L^2((a,b))\right]^k,$$

   where k is a positive integer, and $L : (a,b) \times \mathbb{R}^k \to \mathbb{R}$.

Part III

Exam preparation

Chapter 10

Saper dire

[Nice translation needed]

Updated at the end of the classes of the course 2018/19.

10.1 Normed and Banach Spaces

B-1 Definition of norm, normed space and Banach space. Continuity of the norm.

B-2 Normal convergence implies convergence for series in Banach spaces.

B-3 Finite dimensional subspaces are closed.

B-4 Characterizations of linear continuous operators between normed spaces.

B-5 The space of linear continuous functions between normed spaces: definition of the norm and discussion of completeness.

B-6 Analytic form of Hahn-Banach theorem.

B-7 Topological dual space, aligned functional (existence/uniqueness), dual characterization of the norm and applications.

B-8 Definition of weak and weak* convergence.

B-9 Lower semicontinuity of the norm with respect to weak and weak* convergence.

B-10 Weak* compactness of balls.

B-11 Bidual of a normed space and canonical embedding of a space into its bidual.

B-12 Reflexive spaces and weak compactness of balls.

B-13 Gauge of a convex set.

B-14 Geometric forms of Hahn-Banach theorem.

B-15 Relations between convexity, strong/weak convergence, strong/weak semicontinuity (in normed spaces).

B-16 The dual of ℓ^p is $\ell^{p'}$ if $p < +\infty$.

B-17 The dual of the sequence spaces c_{00}, c_0, c is ℓ^1 (in which sense?).

B-18 The dual of L^p is $L^{p'}$ if $p < +\infty$.

B-19 The dual of ℓ^∞ is not ℓ^1, and the dual of L^∞ is not L^1.

B-20 Weak/weak* compactness of balls in L^p spaces: statements, proofs, counterexamples.

B-21 Separability of a normed space vs separability of the dual.

B-22 Baire spaces: equivalent definitions, examples, counterexamples.

B-23 Complete metric spaces are Baire spaces.

B-24 Open subsets of Baire spaces are Baire spaces. Closed subsets of Baire spaces are not Baire spaces.

B-25 The set of discontinuity points of a function is a F_σ.

B-26 Weakly convergent sequences are bounded.

B-27 The algebraic basis of a Banach space is either finite or uncountable.

B-28 Banach-Steinhaus theorem (two statements).

B-29 Existence of "many" continuous and periodic functions whose Fourier series does not converge in "many" points.

B-30 The pointwise limit of continuous functions is continuous in "many" points.

B-31 The derivative of a derivable function is continuous in "many" points.

B-32 Existence of nowhere differentiable continuous functions.

B-33 Characterization of open mappings through quantitative solvers.

B-34 Open mapping theorem.

B-35 Continuity of the inverse of a linear continuous function.

B-36 Equivalence of norms in Banach spaces and closed graph theorem.

B-37 Existence of a linear quantitative solver vs existence of a topological complement.

B-38 Nonlinear projection onto a compact convex set.

B-39 Approximation of compact operators in normed spaces by operators with finite dimensional range.

B-40 Schauder fixed point theorem.

B-41 Proof of Peano theorem for ordinary differential equations by means of Schauder fixed point theorem.

10.2 Hilbert Spaces

H-1 Definition of scalar product and Hilbert space. Basic properties: Cauchy-Schwarz inequality, continuity of the norm and of the scalar product.

H-2 Direct proof that ℓ^2 is a Hilbert space.

H-3 Example of a non-separable Hilbert space.

H-4 Orthonormal bases in Hilbert spaces and corresponding components of vectors. Representations of vectors, norms, and scalar products in terms of components.

H-5 Existence of orthonormal bases in separable Hilbert spaces.

- H-6 Weak convergence in separable Hilbert spaces: definition and basic properties. Comparison with strong convergence in finite/infinite dimension.
- H-7 Weak convergence and convergence of components with respect to an orthonormal basis: statement, proof, counterexamples.
- H-8 Weak lower semicontinuity of the norm in a Hilbert space.
- H-9 Lack of compactness (with respect to the strong convergence) of balls in Hilbert spaces of infinite dimension.
- H-10 Compactness of balls in separable Hilbert spaces with respect to weak convergence: statement and proof.
- H-11 Parallelogram identity and characterization of norms originating from a scalar product (Riesz-Fréchet-von Neumann theorem).
- H-12 Projection into a closed convex set: existence, uniqueness, 1-Lipschitz continuity, characterization.
- H-13 Projection into a closed subspace: existence, uniqueness, characterization, linearity, orthogonal space and direct sums.
- H-14 Separation of convex sets in Hilbert spaces.
- H-15 Relations between convexity, strong/weak convergence, strong/weak semicontinuity (in Hilbert spaces).
- H-16 Dual of a Hilbert space (proof via orthonormal system and proof via projection into a closed set).
- H-17 Powers of unbounded positive diagonal operators and their domains.
- H-18 Inverse of the second derivative (with different boundary conditions) as an unbounded operator: eigenvalues and eigenfunctions.
- H-19 Inverse of the Laplacian (with different boundary conditions) as an unbounded operator.
- H-20 Sobolev spaces vs domains of powers of the Laplacian vs convergence of Fourier series.
- H-21 In a square of the plane, the trace of a function in H^1 belongs to $H^{1/2}$.
- H-22 Compact operators: definition, relations with strong/weak continuity, examples and counterexamples.
- H-23 Rayleigh quotient and variational characterization of eigenvalues and eigenvectors.
- H-24 Spectral theorem for bounded compact symmetric operators in separable Hilbert spaces.
- H-25 The limit (in a suitable sense) of compact operators is a compact operator.
- H-26 Approximation of compact operators in Hilbert spaces by operators with finite dimensional range.

10.3 Sobolev Spaces

- S-1 All Sobolev functions in intervals of the real line are antiderivatives of their weak derivatives.
- S-2 Hölder regularity of Sobolev functions in intervals of the real line.
- S-3 W-weak derivatives in full generality: definition, basic properties (uniqueness, stability under subsequences, linearity, compatibility with the classical notion).
- S-4 Stability of weak derivatives with respect to weak convergence.
- S-5 W-weak derivatives commute with convolutions.
- S-6 H-weak derivatives in full generality: definition and basic properties.
- S-7 Equivalence between W-weak derivatives and H-weak derivatives.
- S-8 Sobolev spaces in full generality: definition W and definition H. Statement of different forms of approximation theorems.
- S-9 Low-cost approximation theorem in Sobolev spaces: statement and proof.
- S-10 Full approximation theorem in Sobolev spaces (H=W, by Meyers and Serrin): statement and proof.
- S-11 Deluxe approximation theorem in Sobolev spaces: statement, proof, counterexamples.
- S-12 Product of a Sobolev and a smooth function. Product of two Sobolev functions.
- S-13 External composition of a Sobolev function with a smooth function. Absolute value and truncation of Sobolev functions.
- S-14 Embedding theorems for Sobolev functions: statements under different assumptions (different orders of derivation, different assumptions on the open set).
- S-15 Scaling argument showing that the exponents in Sobolev embeddings are the only possible ones.
- S-16 Examples showing the optimality of Sobolev embedding results.
- S-17 Gagliardo's inequality.
- S-18 Proof of Sobolev embedding in the case $p < d$.
- S-19 Proof of Sobolev embedding in the case $p = d$.
- S-20 Proof of Sobolev/Morrey embedding in the case $p > d$.
- S-21 General definitions of extension operators.
- S-22 Extension theorems for Sobolev functions on cylinders.

S-23 Extension theorems for Sobolev functions on suitable smooth domains.

S-24 Partitions of the unity (case A: covering of an open set).

S-25 Partitions of the unity (case B: covering of the closure of an open set with compact boundary).

S-26 Characterization of relatively compact subsets in metric spaces.

S-27 Characterization of relatively compact subsets in L^p spaces (L^p version of Ascoli-Arzelà theorem).

S-28 Compact embedding theorems for Sobolev spaces.

S-29 Traces of Sobolev functions in a half-space: existence and further summability.

S-30 Traces of Sobolev functions in smooth domains: existence and further summability.

S-31 Continuity of the trace.

S-32 Internal composition of Sobolev functions: Sobolev spaces of diffeomorphic domains are isomorphic.

S-33 Spaces $W_0^{1,p}$: the case of $\mathbb{R}^d$ and $\mathbb{R}^d$ minus one point.

S-34 Spaces $W_0^{1,p}$ in general domains: definition and embedding results.

S-35 Spaces $W_0^{1,p}$ in smooth domains: characterization through extension by zero and trace at the boundary.

S-36 Inequalities à la Poincaré-Sobolev-Wirtinger.

S-37 Elliptic equation in divergence form: existence theory via direct method and via representation of the dual of a Hilbert space (à la Lax-Milgram).

S-38 Regularity for elliptic PDEs in divergence form: a priori estimates in the whole space and in a half-space.

S-39 Characterization of Sobolev spaces through difference quotients.

S-40 Regularity for elliptic PDEs in divergence form: rigorous proof via difference quotients in the whole space and in a half-space.

S-41 Interior regularity for elliptic PDEs in divergence form in general domains.

S-42 Boundary regularity for elliptic PDEs in divergence form in smooth domains.

10.4 Indirect methods in the Calculus of Variations

- I-1 First Variation of a functional along a curve. First Variation of a functional along a direction (Gateaux derivative). Directional local minima. Necessary conditions for minimality.
- I-2 Fundamental Lemma in the Calculus of Variations (classic and Lebesgue setting): statement, possible proofs, extension to different classes of test functions.
- I-3 Fundamental Lemma in the Calculus of Variations in the case of test functions with zero average (classic and Lebesgue setting): statement, possible proofs. Du Bois-Reymond lemma.
- I-4 Inner/outer (horizontal/vertical) variations for an integral functional.
- I-5 Different forms of the Euler-Lagrange equation, under suitable assumptions on the Lagrangian.
- I-6 Optimality through convexity: statement(s) and proof.
- I-7 Optimality through auxiliary functional: statement, proof, examples.
- I-8 First variation for integral functionals involving multiple integrals: Euler-Lagrange equation in divergence form, Laplacian, Neumann conditions in more space dimensions.

10.5 Direct methods in the Calculus of Variations

- D-1 Compactness, semicontinuity and Weierstrass theorem with respect to a notion of convergence: definitions, statement, proof.
- D-2 Coercivity and variants of Weierstrass theorem.
- D-3 Weak convergence in L^p spaces: definition and basic properties.
- D-4 Continuity results (with respect to strong convergence) for integral functionals depending on the function (and not on the derivative) in Sobolev spaces, under continuity/semicontinuity assumptions on the Lagrangian.
- D-5 Semicontinuity results (with respect to weak convergence) for integral functionals depending on the derivative in Sobolev spaces, under convexity assumptions on the Lagrangian.
- D-6 Compactness results for integral functionals under suitable growth assumptions on the Lagrangian.
- D-7 Example of non-uniqueness for the Dirichlet problem for a second order equation.

10.6 Macro topics

In this final section we collect an unordered list of "macro topics". Each of these topics involves definitions, statements, proofs, examples, counterexamples, applications coming from different parts of the program. Of course, the intersection between different macro topics is often nonempty.

It is strongly recommended to spend some time on each of the following topics, trying to collect and organize the relevant facts.

1. Sobolev spaces in dimension one.
2. Weak derivatives.
3. Approximation results for Sobolev spaces.
4. Embedding theorems for Sobolev spaces.
5. Extension results for Sobolev spaces.
6. Compact embedding theorems for Sobolev spaces.
7. Traces of Sobolev functions.
8. Internal and boundary regularity for elliptic PDEs in divergence form.
9. Orthonormal bases in Hilbert spaces.
10. Parallelogram identity and projection onto a closed set.
11. Spectral theorem.
12. Approximation of compact operators.
13. Laplacian and its powers as unbounded operators.
14. Analytic form of Hahn-Banach theorem.
15. Geometric forms of Hahn-Banach theorem.
16. Strong vs weak closedness and lower semicontinuity.
17. Weak convergence.
18. Dual spaces.
19. Baire spaces.
20. Quantitative solvers.
21. Schauder fixed point theorem.
22. Lower semicontinuity and compactness for integral functionals.
23. First variation of functionals.
24. Partitions of the unity.

Chapter 11

Know how

[Know how]

11.1 Normed and Banach spaces

- Proving that given functions are norms or scalar products. Proving/disproving that a given structure in a normed/Banach/Hilbert space.
- Deciding whether a given operator is linear and/or continuous, and in case being able to compute its norm as an operator.
- Computing the aligned functional(s) of a given vector in a normed space.
- Computing the norm of elements in the topological dual.
- Providing a direct proof that the classical sequence spaces are Banach spaces.
- [to be continued]

11.2 Hilbert spaces

- Computing the projection of a given point onto a given convex set.
- In an interval, being aware of the properties as operators of (minus) the second derivative and its inverse with different boundary conditions, and in particular:
  - computing eigenvalues and eigenvectors,
  - exploiting symmetry and compactness properties,
  - computing the domain of powers of these operators.
- In an open set, being aware of the properties as operators of (minus) the Laplacian and its inverse with different boundary conditions.
- [to be continued]

11.3 Sobolev spaces

- Given an open set, deciding which results hold true for Sobolev functions in that set, and in particular
  - continuous embedding theorems,
  - compact embedding theorems,
  - extension and approximation theorems,
  - existence of the trace,
  - inequalities à la Poincaré-Sobolev-Wirtinger,
  - existence/regularity for the Dirichlet Laplacian.
- [to be continued]

11.4 Indirect method in the Calculus of Variations

- Recognizing when the set where a functional is defined/finite is a vector space or an affine space, and identifying the space of admissible variations.
- Computing the first variation of a functional along a given curve or the Gateaux derivative in a given direction.
- Being aware of the different classes of test functions that can be involved in the fundamental lemma in the calculus of variations and in the Du Bois-Reymond lemma.
- Computing the Euler-Lagrange equation for an integral functional, even in the case of functionals depending on more unknowns and/or involving higher order derivatives.
- Being aware of the different forms of the Euler-Lagrange equation (integral forms, differential form, DBR form, Erdmann form), and of the assumptions on the Lagrangian and on the extremal that are required for each of them.
- Computing the boundary conditions that originate in the computation of an Euler-Lagrange equation (in particular Dirichlet, Neumann and periodic boundary conditions). Being aware of which choice of test functions gives rise to the different boundary conditions.
- Concluding that a solution of the Euler-Lagrange equation is a global minimum point under suitable convexity assumptions on the Lagrangian.
- Concluding that a solution of the Euler-Lagrange equation is a global minimum point through a suitable auxiliary functional.
- Computing the Euler-Lagrange equation and the associated boundary conditions for integral functionals involving multiple integrals.
- Reverse engineering: producing, when possible, a functional whose Euler-Lagrange equation is a given ordinary differential equation or a given partial differential equation.
- Being able to exploit truncation arguments in order to prove qualitative properties of minimum points.

11.5 Direct method in the Calculus of Variations

- Providing examples of sequences in Hilbert spaces, and in particular in L^2, that converge or do not converge strongly and/or weakly.
- Deciding whether a sequence of functions in a L^p space is strongly/weakly convergent.
- Deducing pointwise estimates on a function from *integral* estimates on its (weak) derivative and boundary conditions or integral estimates on the function.
- Knowing when it is possible (and when it is not possible) to deduce the compactness of a sublevel of a functional with respect to a suitable notion of convergence.

- Deciding which pointwise or integral constraints are stable by a given notion of convergence.
- Setting the road map of the direct method: weak formulation, compactness, semicontinuity, regularity.
- Deriving the Euler-Lagrange equation for a weak solution to a variational problem, focussing in particular on the assumptions needed and on the interpretation of all derivatives which appear in the equation.
- Being familiar with the properties of the Lagrangian (in particular convexity and growth conditions) which yield compactness of sublevels and/or lower semicontinuity of the functional (specifying always carefully the notion of convergence in use).
- Proving regularity of a weak solution to a variational problem, keeping in mind that usually this requires both an *initial step* and a *bootstrap argument*.
- Having clear the variational approach to existence/uniqueness/regularity results for some differential equations (this requires interpreting the differential problem as the Euler-Lagrange equation of a suitable minimization problem).

Chapter 12

Exam papers

[Spiegare il significato ovvio di questo capitolo]

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 25 December 2018

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1. Let Ω be the ball in $\mathbb{R}^3$ with center in $(5,4,3)$ and radius 2. Let us consider the problem

$$\min\left\{\int_\Omega \left(x^{25}\,u_x^2+y^{12}\,u_y^2+z^{2018}\,u_z^2+u^{8102}\right)dx\,dy\,dz : u\equiv 1 \text{ on } \partial\Omega\right\}.$$

   (a) Determine the Euler-Lagrange equation for the problem.

   (b) Discuss existence, uniqueness, and regularity of the solution.

2. Let us consider the function $F:\ell^{25}\to\ell^{12}$ defined by

$$F(x_1,\ldots,x_n,\ldots):=(x_1^{2018},\ldots,x_n^{2018},\ldots).$$

   Determine if this function is linear, continuous, surjective, compact.

3. Let us consider the open set $\Omega:=(-1,1)^3\subseteq\mathbb{R}^3$. For every positive real number p, let us consider the problem

$$\inf\left\{\int_\Omega \left(|u_x|^{25}+|u_y|^{12}+|u_z|^{2018}-|u|^p\right):\ u\in C_c^\infty(\Omega)\right\}.$$

   Determine whether the infimum is a real number in each of the following special cases:

   (a) $p=11$,

   (b) $p=2000$,

   (c) $p=3000$.

4. Let us consider a function $f\in W^{25,12}(\mathbb{R}^{2018})$.

   (a) Determine for which values of p we can conclude that $f_{x_1x_2x_3x_4x_5}$, namely the partial derivative of f with respect to the first five variables, belongs to $L^p(\mathbb{R}^{2018})$.

   (b) Let us set $g(x_1,\ldots,x_{2017}):=f(x_1,\ldots,x_{2017},x_1+\ldots+x_{2017})$.
   Determine for which values of q we can conclude that $g\in L^q(\mathbb{R}^{2017})$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 01 January 2019

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1. (a) Let Ω be the unit ball in $\mathbb{R}^2$ with center in the origin. Determine whether

$$\inf \left\{ \int_\Omega \sinh u \, dx\, dy : u \in C_c^\infty(\Omega), \ \int_\Omega \left(u_x^6 + u_y^4\right) dx\, dy \leq 3 \right\}$$

is a real number.

(b) Let Ω be the unit ball in $\mathbb{R}^{2019}$ with center in the origin. Determine whether

$$\inf \left\{ \int_\Omega \sinh u \, dx : u \in C_c^\infty(\Omega), \ \int_\Omega |\nabla u|^2 \, dx \leq 3 \right\}$$

is a real number.

2. The "new year norm" on $\mathbb{R}^2$ is defined by

$$\|(x, y)\| := 2018|x| + 2019|y|.$$

(a) Determine all aligned functionals of $(0, 1)$ with respect to this norm.

(b) Determine all aligned functionals of $(-2, 3)$ with respect to this norm.

(c) Determine the norm of the identity as an operator from $\mathbb{R}^2$ with the "new year norm" to $\mathbb{R}^2$ with the Euclidean norm.

3. For every function $f : (0, 1) \to \mathbb{R}$, and every real number $a > 0$, let us set

$$[T_a f](x) := f(x^a) \qquad \forall x \in (-1, 1).$$

(a) Determine all values of a for which T_a defines a continuous operator from $L^{2019}((0, 1))$ to $L^1((0, 1))$, and in these cases determine the norm of the operator.

(b) Determine all values of a for which T_a defines a continuous operator from $L^{2018}((0, 1))$ to $L^{2019}((0, 1))$, and in these cases determine the norm of the operator.

4. Let us consider the open set

$$\Omega := \left\{ (x, y) \in \mathbb{R}^2 : 0 < x < 1, \ 0 < y < x^{2019} \right\}.$$

(a) Determine if there exists a positive integer m such that

$$\inf \left\{ \int_\Omega \left(|u_x|^{2018} + |u_y|^{2019} + u^m\right) dx : u \in C_c^\infty(\Omega), \ \int_\Omega u(x)^2 \, dx \leq 1 \right\} = -\infty.$$

(b) Determine if there exists a 1-extender from Ω to $\mathbb{R}^2$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 15 January 2019

1. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u} = \frac{1 + u^3 + x^2}{1 + \dot{u}^2}, \qquad u(0) = u'(3) = 3.$$

2. Let V denote the set of sequences $\{x_n\}_{n \geq 1}$ of real numbers such that

$$\sum_{n=1}^{\infty} n|x_n| < +\infty,$$

with norm defined by the series above.

   (a) Characterize the dual space of V.

   (b) Determine all aligned functionals of the sequence with $x_1 = 9$, $x_2 = 8$, $x_3 = 7$, and $x_n = 0$ for every $n \geq 4$.

   (c) Determine all aligned functionals of the sequence with $x_n = (-1)^n \cdot n^{-4}$.

(A aligned functional of a vector v is a linear 1-Lipschitz functional f such that $f(v) = \|v\|$)

3. Let $\Omega = (-1, 1)^2$ be a square in the plane.

   (a) Determine whether

$$\sup \left\{ \int_\Omega u_{xy}^2 \, dx \, dy : u \in C_c^2(\Omega), \ \int_\Omega u_{xx}^2 \, dx \, dy \leq 7, \ \int_\Omega u_{yy}^2 \, dx \, dy \leq 8 \right\}$$

   is finite or infinite.

   (b) Determine whether

$$\sup \left\{ \int_\Omega u_{yy}^2 \, dx \, dy : u \in C_c^2(\Omega), \ \int_\Omega u_{xx}^2 \, dx \, dy \leq 7, \ \int_\Omega u_{xy}^2 \, dx \, dy \leq 8 \right\}$$

   is finite or infinite.

4. Let us consider the open set $\Omega = (0, \pi)^2$. Determine if there exists a constant C such that

$$\int_\Omega \sin(xy) \cdot u^2 \, dx \, dy \leq C \int_\Omega \left(e^{xy} \cdot u_x^2 + u_y^2 + \cos x \cdot u_x \cdot u_y \right) \, dx \, dy$$

for every $u \in C_c^1(\Omega)$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 02 February 2019

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1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 - 3u\dot{u} + xu\right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u(1) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

2. Let us consider the square $Q := (0, \pi)^2$ in the plane.

   Find all exponents $p \geq 1$ for which there exists a constant C_p such that

$$\int_0^\pi [f(t, \sin t)]^2 \, dt \leq C_p \left\{ \int_Q \|\nabla f(x,y)\|^p \, dx \, dy \right\}^{2/p} \qquad \forall f \in C_c^\infty(Q).$$

3. Let $\Omega \subseteq \mathbb{R}^2$ be the unit ball with center in $(4, 5)$. For every real number λ, let us set

$$I(\lambda) = \inf \left\{ \int_\Omega \left(\arctan y \cdot u_x^2 + \arctan x \cdot u_y^2 - \lambda \frac{u^4}{1+u^2} \right) dx \, dy : u \in C_c^\infty(\Omega) \right\}.$$

   (a) Determine whether there exists $\lambda > 0$ such that $I(\lambda)$ is a real number.

   (b) Determine whether there exists $\lambda > 0$ such that $I(\lambda) = -\infty$.

4. For every sequence $\{x_n\}_{n \geq 1}$, let us set

$$T(x_1, x_2, x_3, \ldots, x_n, \ldots) = \left(\frac{x_1}{\sqrt{1}}, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{3}}, \ldots, \frac{x_n}{\sqrt{n}}, \ldots \right).$$

   (a) Determine whether the restriction of T defines a continuous operator for each of the following choices of the sequence space:

$$\ell^2 \to \ell^2, \qquad \ell^2 \to \ell^1, \qquad \ell^3 \to \ell^2.$$

   When the answer is positive, determine the norm of the operator.

   (b) Determine for which values of the exponent $p \geq 1$ the restriction of T defines a continuous operator $\ell^p \to \ell^1$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 23 February 2019

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1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\ddot{u}^2 + \dot{u}^2\right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) = u'(0) = 1$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Let us consider the boundary value problem

$$u''(x) = \frac{1 + e^{u(x)}}{1 + e^{u'(x)}}, \qquad u(0) = 3, \quad u(3) = 0.$$

   (a) Discuss existence, uniqueness and regularity of the solution.

   (b) Prove that $u'(0) < -1$.

3. Let Ω be a ball in $\mathbb{R}^3$. For every positive integer m, let us set

$$\sup \left\{ \int_\Omega |u - \arctan(xyz)|^m \, dx\, dy\, dz : u \in C^1(\Omega), \ \int_\Omega (u^2 + 2u_x^2 + 3u_y^2 + 4u_z^2) \, dx\, dy\, dz \leq 5 \right\}.$$

   (a) Prove that in the case $m = 3$ the supremum is actually a maximum, at least in the larger class $H^1(\Omega)$.

   (b) Determine all positive integers m such that the supremum is a real number.

4. For every $f : (0, 5) \to \mathbb{R}$, let us set

$$[Tf](x) := \sin(f(x)).$$

   Determine whether the restriction of T defines

   (a) a *continuous* mapping $L^8((0, 5)) \to L^1((0, 5))$,

   (b) a *continuous* mapping $L^1((0, 5)) \to L^8((0, 5))$,

   (c) a *compact* mapping $H^8((0, 5)) \to L^{2019}((0, 5))$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 11 June 2019

1. Let us consider the functional

$$F(u) = \int_0^\pi \left(\dot{u}^2 - u \sin x\right)\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_0^\pi u(x)\,dx = 0$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u'' = u^7 - x^7, \qquad u(0) = 7, \quad u'(7) = 7.$$

3. Let us consider, for every real number $\ell > 0$, the square $Q_\ell := (0, \ell) \times (0, \ell)$.
   Determine for which values of ℓ there exist two constants A_ℓ and B_ℓ such that

$$\int_{Q_\ell} |u(x,y)|^{2019}\,dx\,dy \le A_\ell \left| \int_{Q_\ell} \left(\cos y \cdot u_x(x,y)^2 + \cos x \cdot u_y(x,y)^2\right)\,dx\,dy \right|^{B_\ell}$$

   for every $u \in C^1_c(Q_\ell)$.

4. For every $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^2) \qquad \forall x \in (0,1).$$

   Determine for which real numbers $p \ge 1$ the restriction of T defines

   (a) a *continuous* operator $L^p((0,1)) \to L^2((0,1))$,

   (b) a *continuous* operator $L^2((0,1)) \to L^p((0,1))$,

   (c) a *compact* operator $H^1((0,1)) \to L^p((0,1))$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 05 July 2019

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1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u} + x^3 u\right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) + u(1) = 3$.

   (b) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) - u(1) = 3$.

2. For every $f \in L^2((0,1))$, let us consider the Dirichlet problem

$$u'' = u^3 + \sin u + f(x), \qquad u(0) = u(1) = 0.$$

   (a) Prove that the problem admits a unique solution.

   (b) Discuss the regularity of this solution.

   (c) Let $S : L^2((0,1)) \to L^2((0,1))$ be the operator that associates to each function f the corresponding solution u. Determine whether S is a compact operator.

3. Let d be a positive integer, and let B_d denote the unit ball in $\mathbb{R}^d$ with center in the origin. For every real number $m > 0$, let us set

$$I_d(m) := \inf \left\{ \int_{B_d} \left(u^{19} + \arctan(u^2)\right) dx : u \in C^1_c(B_d),\ \int_{B_d} \|\nabla u(x)\|^7 \, dx \leq m \right\}.$$

   (a) In dimension $d = 3$, determine whether there exists $m > 0$ such that $I_3(m) = 0$.

   (b) Determine for which values of d it turns out that $I_d(m)$ is a real number for every $m > 0$.

4. For every $f : (1, +\infty) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^4) \qquad \forall x \in (1, +\infty).$$

   Determine for which real numbers $p \geq 1$ the restriction of T defines

   (a) a *continuous* operator $L^p((1,+\infty)) \to L^2((1,+\infty))$,

   (b) a *continuous* operator $L^2((1,+\infty)) \to L^p((1,+\infty))$,

   (c) a *compact* operator $H^1((1,+\infty)) \to L^p((1,+\infty))$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 03 September 2019

1. Determine whether the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u}u + u^2 + u\right)\,dx$$

has the minimum in the class $C^1([0,1])$.

2. For every positive integer d, let us consider the following three inequalities:

$$\int_{\mathbb{R}^d} u(x)^{32}\,dx \le K_d, \qquad u(0) \le K_d, \qquad \|\nabla u(0)\| \le K_d,$$

For each of them, determine the values of d for which there exists a constant K_d that makes it true for every $u \in C_c^\infty(\mathbb{R}^d)$ whose norm in $W^{20,19}(\mathbb{R}^d)$ is less than or equal to 1.

3. Let us consider the square $\Omega = (0,1)^2$, and for every real number $\varepsilon > 0$ let us set

$$I(\varepsilon) := \inf\left\{\int_\Omega \left(u_x^2 + u_y^2 + u^7\right)\,dx\,dy : u \in C_c^1(\Omega),\ \int_\Omega \left(u_x^2 + 5u_y^2\right)\,dx\,dy \le \varepsilon\right\}.$$

   (a) Prove that $I(\varepsilon)$ is a real number for every $\varepsilon > 0$.

   (b) Determine whether there exists $\varepsilon > 0$ such that $I(\varepsilon) = 0$.

   (c) Find the limit of $I(\varepsilon)$ as $\varepsilon \to +\infty$.

4. For every $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^2) \qquad \forall x \in (0,1).$$

Determine whether the restriction of T defines

   (a) a *continuous* operator $H^1((0,1)) \to L^4((0,1))$,

   (b) a *continuous* operator $W^{1,4}((0,1)) \to W^{1,4}((0,1))$,

   (c) a *compact* operator $H^1((0,1)) \to H^1((0,1))$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 11 January 2020

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1. Determine for which values of the real parameter a the problem

$$\min \left\{ \int_{-\pi}^{\pi} \left\{ (\dot{u} - \cos x)^2 + (u - \sin x)^2 \right\} dx : u \in C^1([-\pi, \pi]),\ u(0) = a \right\}$$

   admits a solution (note that the condition is given in the midpoint of the interval).

2. Discuss existence, uniqueness, and regularity of functions $u : \mathbb{R} \to \mathbb{R}$ that are *periodic* and satisfy

$$u'' = u^3 + \sin^2 x \qquad \forall x \in \mathbb{R}.$$

3. For every positive real numbers R, c, and α, let us set

$$I(R, c, \alpha) := \inf \left\{ \int_{B_R} \left(|\nabla u|^2 - cu^2 \right) dx : u \in C^\infty(B_R) \cap H^1(B_R),\ \int_{B_R} u\, dx = \alpha \right\},$$

   where B_R denotes the open ball in $\mathbb{R}^3$ with center in the origin and radius R.

   (a) Determine whether there exists $c > 0$ such that $I(1, c, 0) = 0$.

   (b) Determine whether there exists $c > 0$ such that $I(1, c, 0) = -\infty$.

   (c) Determine whether there exists $R > 0$ such that $I(R, 1, 2020) = -\infty$.

4. For every measurable function $f : [0, 1] \to \mathbb{R}$, let us set

$$[Tf](x) = \int_0^{\sin x} \sin(f(t))\, dt \qquad \forall x \in [0, 1].$$

   Determine whether the restriction of T defines

   (a) a continuous mapping $L^2((0, 1)) \to L^{2020}((0, 1))$,

   (b) a compact mapping $L^{2020}((0, 1)) \to L^2((0, 1))$,

   (c) a compact mapping $C^0([0, 1]) \to C^1([0, 1])$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 31 January 2020

1. Let us consider the functionals

$$F(u) = u(0) + \int_0^1 \left(\dot{u}^2 + u^2\right)\, dx, \qquad G(u) = [u(0)]^3 + \int_0^1 \left(\dot{u}^2 + u^2\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = 3$.

   (b) Discuss the minimum problem for $G(u)$ with boundary condition $u(1) = 3$.

2. Let a be a positive real number, and let us consider the boundary value problem

$$u'' = \log u, \qquad u(0) = u(2020) = a.$$

   (a) Discuss existence, uniqueness and regularity of solutions.

   (b) Determine the values of a for which solutions are less than 1 for every $x \in [0, 2020]$.

3. Let B denote an open ball in $\mathbb{R}^3$. For every real number $p > 1$, let us set

$$S(p) := \sup\left\{\int_B u^5\, dx : u \in C^\infty(B),\ \int_B |\nabla u|^p\, dx = \int_B u\, dx = 5\right\}.$$

   (a) Determine whether there exists $p_0 < 2$ such that $S(p)$ is a real number for every $p \geq p_0$.

   (b) Determine whether there exists a real number $p > 1$ such that $S(p) = +\infty$.

   (c) Determine whether there exists a real number M such that $S(p) \leq M$ for every $p \geq 2$.

4. For every sequence $\{x_n\}$ of real numbers, let us set

$$C(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots\right).$$

   In other words, $C(\{x_n\})$ is the sequence $\{y_n\}$ with

$$y_n := \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \forall n \geq 1.$$

   Determine whether the restriction of C defines

   (a) a bounded operator $\ell^1 \to \ell^1$,

   (b) a bounded operator $\ell^1 \to \ell^2$,

   (c) a bounded operator $c \to c$ (as usual c denotes the space of sequences with a finite limit),

   (d) a compact operator $\ell^\infty \to \ell^\infty$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 21 February 2020

1. Prove that there exists a sequence of functions $u_n : [0,1] \to \mathbb{R}$ of class C^∞ such that
   - $\{u_n\}$ is an orthonormal basis of $L^2((0,1))$,
   - $u_n(0) = u_n(1) = 0$ for every positive integer n,
   - for every positive integer n, there exists a negative real number λ_n such that

   $$(\cos x \cdot u_n'(x))' = \lambda_n u_n(x) \qquad \forall x \in [0,1].$$

2. Discuss existence, uniqueness and regularity of solutions to the boundary value problem

   $$u'' = -1 + \sqrt{u}, \qquad u(0) = 1/2, \quad \dot{u}(2020) = 1.$$

3. For every positive real number R, let B_R denote the ball in $\mathbb{R}^3$ with center in the origin and radius R. For every real number $p > 1$, and every real number $r \in (0,1)$, let us set

   $$I(p,r) := \inf \left\{ \int_{B_1 \setminus B_r} \left(|\nabla u|^p + u^{2020} \right) dx : u \in C^\infty(B_1),\ u(x) = 1 \text{ for every } x \in B_r \right\}.$$

   (a) Prove that $I(p,r) > 0$ for every $p > 1$ and every $r \in (0,1)$,

   (b) Prove that for every $p > 1$ there exists

   $$\ell(p) := \lim_{r \to 0^+} I(p,r).$$

   (c) Determine the values of $p > 1$ such that $\ell(p) = 0$.

4. For every measurable function $f : [0,1] \to \mathbb{R}$, let us define $Tf : [0,1] \to \mathbb{R}$ as

   $$[Tf](x) = \int_0^x t f(t)\, dt \qquad \forall x \in [0,1].$$

   Determine whether the restriction of T defines

   (a) a bounded operator $C^0([0,1]) \to C^0([0,1])$ (in case, compute the norm of the operator),

   (b) a bounded operator $L^2((0,1)) \to L^\infty((0,1))$ (in case, compute the norm of the operator),

   (c) a compact operator $L^2((0,1)) \to C^0([0,1])$

   (d) an open mapping $L^2((0,1)) \to L^{2020}((0,1))$.