An introduction to Calculus of Variations

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Change log

- *Versione 28 settembre 2014.* Iniziato il progetto. Aggiunti primi esercizi su logica elementare.
- *Versione 5 ottobre 2014.* Aggiunto il test del precorso. Aggiunto quasi tutto il materiale relativo ai preliminari. Iniziati i capitoli saper dire e saper fare.
- *Versione 12 ottobre 2014.* Aggiunto due esercizi in Induzione 1 e 2. Iniziati esercizi sui limiti.
- *Versione 19 aprile 2015.* Aggiunto fino alla continuità uniforme.
- *Versione 17 maggio 2015.* Aggiornati saper dire e saper fare fino a fine corso.
- *Versione settembre 2016.* Aggiunti scritti d'esame. Restyling generale. Restyling generale.
- *Versione dicembre 2017.* Aggiunti scritti d'esame 2017. Nel frattempo erano state aggiunge due schede su Modica–Mortola.
- *Versione settembre 2018.* Aggiunti scritti d'esame 2018.

To do

- finire gli esercizi sui limiti
- scheda di esercizi diversi sullo studio di funzione (dato il grafico, trovare la funzione)
- esercizi sup/inf/max/min basati su studi di funzione
- esercizi teorici sugli integrali
- completare e sistemare le equazioni differenziali (forse spostare altrove il finale delle lineari omogenee)
- equazioni differenziali: scheda sulla non unicità
- scheda sulle funzioni semicontinue.

Contents

Prefazione

[Farsi venire in mente qualcosa da scrivere qui]

Buon lavoro!

Part I

Schede [Nice translation needed]

Modica-Mortola 1 – Minimization problem

1. **The functional**. For every interval $(a, b) \subseteq \mathbb{R}$ and every $\varepsilon > 0$, the Modica-Mortola functional is defined as

$$MM_\varepsilon(u) := \int_a^b \left(\varepsilon \dot{u}^2 + \frac{1}{\varepsilon}(1-u^2)^2 \right) dx.$$

Depending on the context, one can assume that u is of class C^1, C^∞, or H^1, with or without boundary conditions.

2. **An auxiliary function**. Let us consider the function $G : \mathbb{R} \to \mathbb{R}$ defined as

$$G(x) := \int_0^x |1 - s^2| \, ds.$$

3. **The minimization problem**. Let us consider the minimum problem

$$m_\varepsilon := \min \left\{ MM_\varepsilon(u) : u \in C^1([a,b]),\ u(a) = 0,\ u(x) \geq 0 \quad \forall x \in [a,b] \right\}.$$

The following statements hold true.

(a) (Existence of the minimum) For every $\varepsilon > 0$ the minimum exists, and any minimizer $u_\varepsilon(x)$ is an increasing function that satisfies

$$0 \leq u_\varepsilon(x) \leq 1 \qquad \forall x \in [a,b].$$

(b) (Asymptotic behavior of the minimum) It turns out that

$$\lim_{\varepsilon \to 0^+} m_\varepsilon = G(1).$$

(c) (Asymptotic behavior of quasi-minimizers) If $\{u_\varepsilon(x)\}$ is any family of quasi-minimizers, namely such that

$$\lim_{\varepsilon \to 0^+} MM_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0^+} m_\varepsilon,$$

then

$$u_\varepsilon(x) \to 1 \qquad \text{uniformly on compact subsets of } (a,b].$$

(d) (Blow-up of quasi-minimizers) If $\{u_\varepsilon(x)\}$ is any family of quasi-minimizers, then

$$u_\varepsilon(\varepsilon x) \to \tanh(x) \qquad \text{uniformly on compact subsets of } [0, +\infty).$$

4. **The key estimate**. The main point is that the L^1 norm of the derivative of $G(u(x))$ can be estimated in terms of the Modica-Mortola functional. This results from the arithmetic-mean geometric-mean inequality as follows:

$$MM_\varepsilon(u) \geq \int_a^b \left| \dot{u} \cdot (1-u^2) \right| dx = \int_a^b \left| G'(u) \cdot \dot{u} \right| dx \geq \left| G(u(b)) - G(u(a)) \right|.$$

Modica-Mortola 2 – Gamma convergence

1. **Gamma convergence and characterization of the Gamma limit**. The family $MM_\varepsilon(u)$ (extended to $+\infty$ outside the natural domain) admits a Gamma limit $MM_0(u)$ with respect to the metric of $L^1((a,b))$, or any $L^p((a,b))$ with $p \in [1,+\infty)$.

   If $u \in L^1((a,b))$ is any function such that $MM_0(u) < +\infty$, then

   (a) there exists a positive integer n such that
   $$MM_0(u) = 2G(1) \cdot n,$$

   (b) there exist a boolean integer $k \in \{0,1\}$, and a partition $a = x_0 < x_1 < \ldots < x_n = b$ of the interval such that
   $$u(x) = (-1)^{i+k}$$
   for every $i = 1, \ldots, n$ and almost every $x \in (x_{i-1}, x_i)$.

2. **Asymptotic behavior of recovery sequences**. Let $u_\varepsilon \to u$ be any converging family such that
   $$\limsup_{\varepsilon \to 0^+} MM_\varepsilon(u_\varepsilon) \leq MM_0(u) < +\infty,$$
   and let n, k, and the partition of (a,b) be as in the characterization of the Gamma limit.

   Then

   - (asymptotic behavior) for every $i \in \{1, \ldots, n\}$ it turns out that
   $$u_\varepsilon(x) \to (-1)^{k+i} \qquad \text{uniformly on compact subsets of } (x_{i-1}, x_i).$$
   - (blow up) for every $i \in \{1, \ldots, n\}$ there exists $x_{i,\varepsilon} \to x_i$ such that
   $$u_\varepsilon(\varepsilon(x - x_{i,\varepsilon})) \to \tanh x \qquad \text{uniformly on compact subsets of } \mathbb{R}.$$

3. **Transition intervals and key estimates**. An interval $(c,d) \subseteq (a,b)$ is called a *transition interval* if

   - $|u(x)| \leq 1/2$ for every $x \in [c,d]$,
   - $|u(x)| = 1/2$ for every $x \in \{c,d\}$ (namely $u(x) = \pm 1/2$ at the endpoints),
   - $u(c) \cdot u(d) = -1/4$, namely $u(x)$ takes values of opposite sign at the endpoints.

   The key points are that

   (a) the number of transition intervals of u is bounded from above in terms of $MM_\varepsilon(u)$,

   (b) the length of every transition interval is bounded from above by ε times a constant depending on $MM_\varepsilon(u)$.

Part II

Exercises

Approximation results 1

Subject: smooth approximation of given functions **Difficulty**: $\star\star$

Prerequisites: sequences of functions, uniform convergence

1. Let $[c, d] \subseteq (a, b)$ be two intervals. Prove that there exists a function $v : \mathbb{R} \to \mathbb{R}$ of class C^∞ such that
   - $0 \leq v(x) \leq 1$ for every $x \in \mathbb{R}$,
   - $v(x) = 0$ for every $x \notin [a, b]$,
   - $v(x) = 1$ for every $x \in [c, d]$.

2. Let $[c, d] \subseteq (a, b)$ be two intervals. Prove that there exists a sequence of functions $v_n(x)$ in $C_c^\infty((a, b))$ such that
   - $0 \leq v_n(x) \leq 1$ for every $n \in \mathbb{N}$ and every $x \in [a, b]$,
   - $v(x) \to 1$ uniformly in $[c, d]$,
   - $v(x) \to 0$ uniformly on compact subsets of $[a, b] \setminus [c, d]$.

3. Let $v : [a, b] \to \mathbb{R}$ be a continuous function.

   Prove that there exists a sequence $v_n(x)$ in $C_c^\infty((a, b))$ such that
   - there exists a constant $M \in \mathbb{R}$ such that $|v_n(x)| \leq M$ for every $n \in \mathbb{N}$ and every $x \in [a, b]$,
   - $v_n(x) \to v(x)$ uniformly on compact subsets of (a, b).

4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $v : [a, b] \to \mathbb{R}$ be any function (not necessarily continuous). Let M be a real number, let $S \subseteq [a, b]$ be a finite set, and let $v_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions such that

   (i) $|v_n(x)| \leq M$ for every $n \in \mathbb{N}$ and every $x \in [a, b]$,

   (ii) $v_n(x) \to v(x)$ uniformly on compact subsets of $(a, b) \setminus S$.

   Prove that $v(x)$ is Riemann integrable and

$$\lim_{n \to +\infty} \int_a^b f(x) v_n(x)\, dx = \int_a^b f(x) v(x)\, dx.$$

Variations on the Fundamental Lemma

Subject: fundamental lemma in CV **Difficulty**: $\star\star\star$

Prerequisites: fundamental lemma and Du Bois Reymond lemma

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that

$$\int_a^b f(x)v(x)\,dx = 0 \qquad \forall v \in V.$$

Determine which of the following classes V of test functions allow to conclude that $f(x) = 0$ for every $x \in [a, b]$.

   (a) $V = \{v \in C^1([a, b]) : v(a) = 7\}$

   (b) $V = \{v \in C^3([a, b]) : v((a + b)/2) = 7\}$

   (c) $V = \{v \in C^\infty([a, b]) : v'((a + b)/2) = 7\}$

   (d) $V = \left\{v \in C^\infty([a, b]) : \int_a^b v(x)\,dx = 7\right\}$

2. Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function such that

$$\int_0^{2\pi} f(x)v(x)\,dx = 0$$

for every $v \in C_c^\infty((0, 2\pi))$ such that

$$\int_0^{2\pi} v(x)\sin x\,dx = 0.$$

What can we conclude?

3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that

$$\int_a^b f(x)v''(x)\,dx = 0 \qquad \forall v \in C_c^\infty((a, b)).$$

What can we conclude?

4. (Requires Stone-Weierstrass theorem) Let $f : [-1, 2] \to \mathbb{R}$ be a continuous function such that

$$\int_{-1}^2 f(x)x^n\,dx = 0 \qquad \forall n \in \mathbb{N}.$$

   (a) Can we deduce that $f(x) = 0$ for every $x \in [-1, 2]$?

   (b) And if we limit ourselves to powers of x with even exponent?

Boundary conditions 1

Subject: minimization of functionals – indirect methods **Difficulty**: ⋆⋆

Prerequisites: Euler equation, genesis of boundary conditions

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Let us consider, for every $u \in C^1([0,2])$, the following functionals:

$$F_1(u) = \int_0^2 [u'(x)]^2 \, dx, \qquad F_2(u) = \int_0^2 \left([u'(x)]^2 + [u(x)]^2\right) dx,$$

$$F_3(u) = \int_0^2 \left([u'(x)]^2 - 7u(x)\right) dx, \qquad F_4(u) = \int_0^2 \left([u'(x)]^2 + [u(x) - x^3]^2\right) dx.$$

For each of them, in the following table it is required to discuss the minimum problem subject to the extra constraints presented in each row. When the minimum exists, the number of minimizers is required. When the minimum does not exist, a characterization of the infimum is required. It is strongly recommended to work again on these problems in a second step by means of direct methods.

Constraints	$F_1(u)$	$F_2(u)$	$F_3(u)$	$F_4(u)$
$u(0) = 3$ and $u(2) = 8$				
$u(0) = 5$				
no further condition				
$u(0) = u(2)$				
$u(0) = -u(2)$				
$\int_0^2 u(x)\, dx = 8$				
$u(0) = 3$ and $\int_0^2 u(x)\, dx = 8$				
$u(2) = 5u(0)$				
$u(2) = 5 + u(0)$				
$u(2) = u'(0)$				
$u'(0) = u'(2)$				
$u'(0) = 3$				
$u(0) = 3$ and $u'(1) = 3$				
$u''(0)$ exists and $u''(0) = 3$				
$\int_0^1 u(x)\, dx = 1$ and $\int_1^2 u(x)\, dx = 2$				

Boundary conditions 2

Subject: minimization of functionals – indirect methods **Difficulty**: ⋆⋆⋆

Prerequisites: Euler equation, genesis of boundary conditions

Let us consider, for every $u \in C^2([0,\pi])$, the following functionals:

$$F_1(u) = \int_0^\pi \ddot{u}^2\,dx, \qquad F_2(u) = \int_0^\pi \left(\ddot{u}^2 + \dot{u}^2\right)\,dx,$$

$$F_3(u) = \int_0^\pi \left(\ddot{u}^2 + u^2\right)\,dx, \qquad F_4(u) = \int_0^\pi \left(\ddot{u}^2 + \dot{u} + u^2\right)\,dx.$$

For each of them, in the following table it is required to discuss the minimum problem subject to the extra constraints presented in each row. When the minimum exists, the number of minimizers is required. When the minimum does not exist, a characterization of the infimum is required.

Constraints	$F_1(u)$	$F_2(u)$	$F_3(u)$	$F_4(u)$
no further condition				
$u(0) = 1$				
$u'(0) = 1$				
$u''(0) = 1$				
$u(0) = 1$ and $u(\pi) = 2$				
$u'(0) = 1$ and $u'(\pi) = 2$				
$u'(0) = 1$ and $u(\pi) = 2$				
$u(0) = u(\pi)$				
$u'(0) = u'(\pi)$				
$u(0) = u'(\pi)$				
$u(0) = u(\pi)$ and $u'(0) = 2$				
$u(0) = u(\pi)$ and $u'(0) = u'(\pi)$				
$u(0) = u(\pi)$ and $u''(0) = u''(\pi)$				
$\int_0^\pi u(x)\,dx = 3$				
$u(\pi) = u(0) - 44$				

As in the previous exercise sheet, it is strongly recommended to work again on these problems in a second step by means of direct methods.

Minimum problems 1

Subject: minimization of functionals – indirect methods **Difficulty**: $\star\star$

Prerequisites: Euler equation, optimality through convexity

1. Solve the minimum problem for the functional

$$F(u) = \int_0^1 (\dot{u} - u)^2 \, dx$$

subject to each of the following boundary conditions:

(a) $u(0) = 2015$,

(b) $u(0) = u(1) = 2015$.

2. Solve the minimum problem for the functional

$$F(u) = \int_{-2}^{2} (u'(x) - |x|)^4 \, dx$$

subject to each of the following boundary conditions:

(a) $u(2) = 2015$,

(b) $u(-2) = u(2) = 2015$,

(c) $\displaystyle\int_{-2}^{2} u(x)\, dx = 2015$.

3. For every continuous function $f : [a, b] \to \mathbb{R}$, let us consider the minimum problem

$$\min \left\{ \int_a^b \left[\dot{u}^2 + (u - f(x))^2 \right] \, dx : u \in C^1([a, b]) \right\}.$$

(a) Prove that the problem admits a unique solution.

(b) Prove that the solution satisfies

$$\min_{x\in[a,b]} f(x) \leq \min_{x\in[a,b]} u(x) \leq \max_{x\in[a,b]} u(x) \leq \max_{x\in[a,b]} f(x).$$

(c) Prove that the solution satisfies

$$\int_a^b u(x)\, dx = \int_a^b f(x)\, dx.$$

4. Solve the minimum problem

$$\min \left\{ \int_0^3 \left[\dot{v}^2 + (v - u)^2 + (u - x)^2 \right] \, dx : u \in C^1([0, 3]),\ v \in C^1([0, 3]) \right\}.$$

Minimum problems 2

Subject: minimization of functionals – indirect methods **Difficulty**: ⋆ ⋆ ⋆

Prerequisites: Euler equation, optimality through convexity

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1. Let us consider the following functional

$$F(u) := \int_0^2 \left(\dot{u}^2 + u^2\right)\, dx.$$

Discuss existence/uniqueness/regularity for the following minimum problems:

(a) $\min\{u(2) + F(u) : u(0) = 0\}$,
(b) $\min\left\{[u(2)]^3 + F(u) : u(0) = 0\right\}$,
(c) $\min\{u(1) + F(u) : u(0) = 0\}$,
(d) $\min\{u(0) - u(2) + F(u)\}$.

2. Let us consider the minimum problem

$$\min\left\{\int_0^1 (1 + u^2)\dot{u}^2\, dx : u(0) = 1,\ u(1) = \alpha\right\}.$$

(a) [This point seems to require the direct method] Prove that for every $\alpha \in \mathbb{R}$ the problem admits at least a solution.
(b) Prove that every minimizer is monotone.
(c) Prove that for every $\alpha \in \mathbb{R}$ the solution is unique.
(d) Discuss convexity/concavity of the solution.

3. Let us consider the minimum problem

$$\min\left\{\int_0^1 \left(\dot{u}^4 + u\right)\, dx : u(0) = 0,\ u(1) = \alpha\right\},$$

where α is a real parameter.

(a) Prove that the problem admits a unique solution for every $\alpha \in \mathbb{R}$.
(b) Discuss monotonicity and regularity of the solution.

4. Let us consider the minimum problem

$$\min\left\{\int_0^1 \left(e^{u'(x)} + u^4(x)\right)\, dx : u(0) = u(1) = \alpha\right\},$$

where α is a real parameter.

(a) Compute explicitly the solution in the case $\alpha = 0$.
(b) [Uhm, existence is not so clear] Prove that for every $\alpha \in \mathbb{R}$ the problem admits a unique solution, and this solution is strictly convex when $\alpha > 0$ and strictly concave when $\alpha < 0$.

Minimum problems 3

Subject: minimization of functionals – direct methods **Difficulty**: ⋆⋆⋆⋆

Prerequisites: direct methods in H^1, optimality through convexity

1. Determine which of the following functionals attains the minimum in the class of all functions $u \in C^1([0,1])$ such that $u(0) = 1$:

$$F_1(u) = \int_0^1 \left(\dot{u}^2 + \arctan(u^2)\right)\,dx, \qquad F_2(u) = \int_0^1 \left(u^2 + \arctan(\dot{u}^2)\right)\,dx,$$

$$F_3(u) = \int_0^1 \arctan\left(\dot{u}^2 + u^2\right)\,dx, \qquad F_4(u) = \int_0^1 \left(\dot{u}^2 - \arctan(u^2)\right)\,dx.$$

2. Let us consider the functionals

$$F(u) = \int_0^\pi \left(\dot{u}^2 + \sin x \cdot u^4\right)\,dx, \qquad G(u) = \int_0^\pi \left(\dot{u}^2 + \cos x \cdot u^4\right)\,dx.$$

   Discuss existence/uniqueness/regularity for the minimization of $F(u)$ and $G(u)$ subject to the boundary conditions $u(0) = u(\pi) = 4$.

3. Discuss existence/uniqueness/regularity for the minimum problem

$$\min\left\{\int_0^1 \left(e^{x^2} \cdot \dot{u}^2 + e^{u^4}\right)\,dx : \int_0^1 u(x)\,dx = 2015\right\}.$$

4. Discuss existence/uniqueness/regularity for the minimum problem

$$\min\left\{\int_0^{\pi/4} \left(\cos x \cdot \dot{u}^2 + \sin x \cdot u^4 - \tan x \cdot u\right)\,dx : u \in C^1([0,\pi/4])\right\}.$$

5. Determine which of the following functionals attains the minimum in the class of all function $u \in C^1([0,7])$ such that $u(0) = u(7) = 0$:

$$F(u) = \int_0^7 \left(\sqrt{1+\dot{u}^4} - \sqrt{1+u^2}\right)\,dx, \qquad G(u) = \int_0^7 \left(\sqrt{1+\dot{u}^2} - \sqrt{1+u^4}\right)\,dx.$$

6. Let us consider the following minimum problem

$$\min\left\{\int_0^1 \left[\frac{\dot{u}}{u^2+1}\right]^2 dx : u(0) = 0,\ u(1) = 1\right\}.$$

   (a) Solve the Euler equation associated to the problem.

   (b) Prove that in the minimization process it is enough to consider nondecreasing functions.

   (c) Prove that the solution of the Euler equation is actually the unique global minimizer.

Boundary value problems 1

Subject: minimum problems vs BVP **Difficulty**: ⋆⋆⋆⋆

Prerequisites: direct methods in H^1, Euler equation, genesis of boundary conditions

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In each row of the following table a boundary value problem is presented. It is required to determine a variational problem for which the given BVP is the Euler equation, and then to discuss existence/uniqueness/regularity of the solution.

Equation	Boundary conditions	Variational problem
$u'' = \sinh u$	$u(0) = 0$ $u(1) = 2015$	
$u'' = e^u$	$u(0) = 2015$ $u'(0) = 0$	
$u'' = \dfrac{\arctan(u+x)}{x^2+1}$	$u'(0) = 0$ $u'(2015) = 0$	
$u'' = \lvert\sin x\rvert \cdot \lvert\cos u\rvert$	$u(0) = 0$ $u(2015) = 7$	
$u'' = \dfrac{\cos x}{u}$	$u(2) = 1/20$ $u(3) = 2015$	
$u'' = x^2 u^5 - \sin x$	$u(-1) = u(1)$ $u'(-1) = u'(1)$	
$u'' = u^3 \cdot \arctan x$	$u'(1) = 1$ $u(2015) = 1$	
$u'' = \arctan(x^2 u)$	$u'(1) = 2$ $u'(2015) = 3$	
$u^{IV} = e^{-u}$	$u(0) = u(2015) = 7$ $u'(0) = u''(0) = 0$	
$u^{IV} = x^3 - \log u$	$u(0) = u'(2105) = 3$ $u'(0) = u''(2015) = 0$	
$u^{IV} = (u')^2 u'' - u^5$	$u(0) = u'(0) = 3$ $u(4) = u''(4) = 0$	

[Aggiungere una scheda]

Relaxation 1

Subject: relaxation of functions **Difficulty**: $\star\star\star$

Prerequisites: definition of relaxation and lower semicontinuous envelope

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1. Compute the relaxation of the following functions defined in $\mathbb{R}$:

$$f_1(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}, \end{cases} \qquad f_2(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

$$f_3(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap (-\infty, 0], \\ 0 & \text{otherwise}, \end{cases} \qquad f_4(x) = \begin{cases} x^2 - 3 & \text{if } x \in \mathbb{Q}, \\ 3 - x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

2. State precisely, and then prove or disprove the following facts.
   (a) The relaxation of a bounded function is a bounded function.
   (b) The relaxation of a continuous function is a continuous function.
   (c) The relaxation of an injective function is an injective function.
   (d) The relaxation of a surjective function is a surjective function.
   (e) The relaxation of a nondecreasing function is a nondecreasing function.
   (f) The relaxation of a periodic function is a periodic function.
   (g) The relaxation of a convex function is a convex function.

3. State precisely, and then prove or disprove the following facts.
   (a) If $F(x) \geq G(x)$ for every x, then $\overline{F}(x) \geq \overline{G}(x)$ for every x.
   (b) The set $\overline{F}(x) \leq M$ is the closure of the set $F(x) \leq M$.
   (c) If $H(x) := \max\{F(x), G(x)\}$, then $\overline{H}(x) = \max\{\overline{F}(x), \overline{G}(x)\}$.
   (d) If $H(x) := \min\{F(x), G(x)\}$, then $\overline{H}(x) = \min\{\overline{F}(x), \overline{G}(x)\}$.

4. (Relaxation of the sum of two functions) Let $\mathbb{X}$ be a metric space, let $F : \mathbb{X} \to \mathbb{R}$ and $G : \mathbb{X} \to \mathbb{R}$, and let $H(x) := F(x) + G(x)$.
   (a) Prove that
   $$\overline{H}(x) \geq \overline{F}(x) + \overline{G}(x) \qquad \forall x \in \mathbb{X}.$$
   (b) Prove that equality holds true for $x_0 \in \mathbb{X}$ whenever $G(x)$ is continuous in x_0.
   (c) Prove that strict inequality can occur in x_0 even if $G(x)$ is lower semicontinuous in x_0.
   (d) What about the case where both $F(x)$ and $G(x)$ are lower semicontinuous in x_0?

5. Complete and prove the following statement.

   Let $< \ldots >$. Then the epigraph of $\overline{F}(x)$ is $< \ldots >$ of the epigraph of $F(x)$.

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ whose relaxation is $-\infty$ for every $x \in \mathbb{R}$.

Gamma convergence 1

Subject: Gamma convergence – One dimensional examples **Difficulty**: ⋆⋆⋆

Prerequisites: definitions of Gamma-limit, Gamma-liminf and Gamma-limsup

1. In each row of the following table a sequence of real functions is given. Determine whether the sequence is equicoercive or not, and then compute the pointwise limit and the Gamma-limit.

Sequence	Equicoercive?	Pointwise limit	Gamma-limit
e^{-nx}			
e^{-nx^2}			
$n^3e^{-nx^2}$			
$n!e^{-nx}$			
e^{nx^2}			
$\arctan(nx)$			
$\|3x\|^n$			
$\|x\|^{-n}$			
$x^2 + n\cos^2 x$			
$\arctan^2(nx-1)$			
$(n^2x - n^2 + 3n)^2$			
$(n^2x - n^2 + 3n)^3$			
$nx \cdot e^{-n^2x^2}$			
$(x/n) \cdot e^{-(x/n)^2}$			
$x \cdot e^{-(x/n)^2}$			
$n \sin x \cdot e^{-n^2x^2}$			

2. Compute Gamma-liminf and Gamma-limsup of the following sequences of real functions.

Sequence	Gamma-liminf	Gamma-limsup
x^n		
$(-1)^n x$		
$\sin(nx)$		
$\arctan(x^2 + (-1)^n nx)$		
$\cos(n \sin(nx))$		

Gamma convergence 2

Subject: Gamma convergence – Simple properties **Difficulty**: ⋆⋆⋆

Prerequisites: definitions of Gamma-limit, Gamma-liminf and Gamma-limsup

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1. State precisely, and then prove or disprove the following facts.
   (a) The Gamma-limit of bounded functions is a bounded function.
   (b) The Gamma-limit of continuous functions is a continuous function.
   (c) The Gamma-limit of injective functions is an injective function.
   (d) The Gamma-limit of surjective functions is a surjective function.
   (e) The Gamma-limit of nondecreasing functions is a nondecreasing function.
   (f) The Gamma-limit of periodic functions is a periodic function.
   (g) The Gamma-limit of convex functions is a convex function.
   (h) The Gamma-limit of Lipschitz continuous functions, with equi-bounded Lipschitz constants, is a Lipschitz continuous function.
2. Same as before, with Gamma-liminf and Gamma-limsup.
3. (Gamma-limit vs uniform limit) State precisely, and then prove the following facts.
   (a) Uniform convergence implies Gamma-convergence.
   (b) Uniform convergence on compact subsets implies Gamma-convergence.
4. (Stability under continuous and lower semicontinuous perturbations) Let $\mathbb{X}$ be a metric space, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functions.
   (a) Prove that for every *continuous* function $g : \mathbb{X} \to \mathbb{R}$ it turns out that
   $$\Gamma\!-\liminf_{n\to+\infty} [g(x) + f_n(x)] = g(x) + \Gamma\!-\liminf_{n\to+\infty} f_n(x) \qquad \forall x \in \mathbb{X},$$
   $$\Gamma\!-\limsup_{n\to+\infty} [g(x) + f_n(x)] = g(x) + \Gamma\!-\limsup_{n\to+\infty} f_n(x) \qquad \forall x \in \mathbb{X}.$$
   (b) Prove that the previous statements are not necessarily true if $g(x)$ is just lower semicontinuous.
   (c) Prove that the first equality holds true for every *lower semicontinuous* function $g(x)$ provided that the Gamma-liminf of $f_n(x)$ coincides with the pointwise liminf of $f_n(x)$.
   (d) Prove that the second equality holds true for every *lower semicontinuous* function $g(x)$ provided that the Gamma-limsup of $f_n(x)$ coincides with the pointwise limsup of $f_n(x)$.
   (e) Deduce that the equality
   $$\Gamma\!-\lim_{n\to+\infty} [g(x) + f_n(x)] = g(x) + \Gamma\!-\lim_{n\to+\infty} f_n(x) \qquad \forall x \in \mathbb{X}$$
   holds true when either $g(x)$ is continuous, or $g(x)$ is lower semicontinuous and the Gamma-limit of $f_n(x)$ coincides with the pointwise limit.

Gamma convergence 3

Subject: Gamma convergence and relaxation **Difficulty**: $\star\star\star\star$

Prerequisites: relaxation, Gamma-liminf and Gamma-limsup

1. (Gamma-limit vs pointwise limit)
   (a) State precisely and prove the following inequalities:
   $$\Gamma\text{–}\liminf_{n\to+\infty} f_n(x) \le \overline{\liminf_{n\to+\infty} f_n(x)}, \qquad \Gamma\text{–}\limsup_{n\to+\infty} f_n(x) \le \overline{\limsup_{n\to+\infty} f_n(x)}.$$
   (b) Show that strict inequalities can occur.
2. (Gamma-limits of subsequences) Let $\mathbb{X}$ be a metric space, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functions.

   Prove that for every increasing sequences n_k of natural numbers it turns out that
   $$\Gamma\text{–}\liminf_{n\to+\infty} f_n(x) \le \Gamma\text{–}\liminf_{k\to+\infty} f_{n_k}(x) \le \Gamma\text{–}\limsup_{k\to+\infty} f_{n_k}(x) \le \Gamma\text{–}\limsup_{n\to+\infty} f_n(x) \qquad \forall x \in \mathbb{X}.$$
3. (Gamma-limits of sums) Let $\mathbb{X}$ be a metric space, and let $f_n : \mathbb{X} \to \mathbb{R}$ and $g_n : \mathbb{X} \to \mathbb{R}$ be two sequences of functions.
   (a) Prove that
   $$\Gamma\text{–}\liminf_{n\to+\infty} [f_n(x) + g_n(x)] \ge \Gamma\text{–}\liminf_{n\to+\infty} f_n(x) + \Gamma\text{–}\liminf_{n\to+\infty} g_n(x) \qquad \forall x \in \mathbb{X}.$$
   (b) Prove that strict inequalities can occur, even if the sequences $f_n(x)$ and $g_n(x)$ admit a Gamma-limit.
   (c) Prove that equalities hold true whenever the sequences $g_n(x)$ converges *uniformly* on compact subsets of $\mathbb{X}$ to a *continuous* function.
   (d) Prove that there is no relation between the Gamma-limsup of $f_n(x) + g_n(x)$ and the Gamma-limsup of $f_n(x)$ and $g_n(x)$.
4. (Gamma-limit of monotone sequences) Let $\mathbb{X}$ be a metric space, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functions.
   (a) Let us assume that
   $$f_{n+1}(x) \le f_n(x) \qquad \forall x \in \mathbb{X}, \ \forall n \in \mathbb{N}.$$
   Prove that the Gamma-limit is the relaxation of the pointwise limit, namely
   $$\Gamma\text{–}\lim_{n\to+\infty} f_n(x) = \overline{\lim_{n\to+\infty} f_n(x)}.$$
   (b) Prove or disprove the analogous statement in the case where
   $$f_{n+1}(x) \ge f_n(x) \qquad \forall x \in \mathbb{X}, \ \forall n \in \mathbb{N}.$$
5. (Gamma-limits vs relaxation) State precisely and then prove the following relations
   $$\Gamma\text{–}\liminf_{n\to+\infty} f_n(x) = \Gamma\text{–}\liminf_{n\to+\infty} \overline{f_n}(x), \qquad \Gamma\text{–}\limsup_{n\to+\infty} f_n(x) = \Gamma\text{–}\limsup_{n\to+\infty} \overline{f_n}(x).$$

Gamma convergence 4

Subject: minimum problems with parameters **Difficulty**: ⋆⋆⋆⋆

Prerequisites: Gamma convergence, convergence of minima and minimizers

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1. Discuss the equicoerciveness and compute the Gamma-limit of the following sequences of functions defined in $\mathbb{R}^2$:

$$x^2 + y^4 + n(x^2 - y^2), \qquad x + \arctan(nxy), \qquad x^2 + n(y - \sin(nx))^2,$$
$$x^{2n} - y^{4n}, \qquad (nx+1)^{2n} - (ny-1)^{2n} \qquad (nx+1)^{2n} - (n^2y-1)^{2n},$$
$$\left[x^{2n} - y^{4n}\right]^2, \qquad \left[(nx+1)^{2n} - (ny-1)^{2n}\right]^2, \qquad \left[(nx+1)^{2n} - (n^2y-1)^{2n}\right]^2.$$

2. Let x_n be any sequence of real numbers with

$$x_n \in \operatorname{argmin}\left\{x^2 + e^x + n\sin x : x \in \mathbb{R}\right\}.$$

Compute the limit of x_n as $n \to +\infty$.

3. Let us consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \sin x + n^2 \cos x + n(x-1)^{2n}.$$

(a) Prove that $f_n(x)$ has a minimum in $\mathbb{R}$ for every $n \geq 1$.

(b) Prove that every sequence x_n of minimizers has a finite limit as $n \to +\infty$.

(c) Prove that for n large enough the minimizer is unique.

4. Let us consider the parametric minimization problem

$$I_n := \min\left\{\int_0^1 \left(n\dot{u}^2 + \sin u\right)\,dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = 3\right\}.$$

(a) Prove that I_n is well defined for every positive integer n, namely the minimun exists.

(b) Prove that $I_n \to +\infty$, and determine its principal part as $n \to +\infty$.

Gamma convergence 5

Subject: convergence of minima and minimizers **Difficulty**: ⋆⋆⋆⋆

Prerequisites: Gamma convergence, convergence of minima and minimizers

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1. (Convergence of minima – Truths and urban legends) Let $\mathbb{X}$ be a metric space, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functions. Let us set

$$F^-_\infty(x) := \Gamma\!-\liminf_{n\to+\infty} f_n(x), \qquad F^+_\infty(x) := \Gamma\!-\limsup_{n\to+\infty} f_n(x).$$

   (a) Prove that

$$\liminf_{n\to+\infty}\left(\inf_{x\in\mathbb{X}} f_n(x)\right) \le \inf_{x\in\mathbb{X}} F^-_\infty(x), \qquad \limsup_{n\to+\infty}\left(\inf_{x\in\mathbb{X}} f_n(x)\right) \le \inf_{x\in\mathbb{X}} F^+_\infty(x).$$

   (b) Prove that strict inequalities can occurr, even if the following three conditions are satisfied:

   - $|f_n(x)| \le 1$ for every $n \in \mathbb{N}$ and every $x \in \mathbb{X}$,
   - all inf's are actually min's,
   - the Gamma-limit of $f_n(x)$ exists.

   (c) Let us assume that the sequence $f_n(x)$ is equicoercive, and let $K \subseteq \mathbb{X}$ be the compact set such that

$$\inf_{x\in\mathbb{X}} f_n(x) = \inf_{x\in K} f_n(x) \qquad \forall n \in \mathbb{N}.$$

   Prove that in this case it turns out that

$$\min_{x\in\mathbb{X}} F^-_\infty(x) = \min_{x\in K} F^-_\infty(x) = \liminf_{n\to+\infty}\left(\inf_{x\in\mathbb{X}} f_n(x)\right),$$

$$\min_{x\in\mathbb{X}} F^+_\infty(x) = \min_{x\in K} F^+_\infty(x) = \limsup_{n\to+\infty}\left(\inf_{x\in\mathbb{X}} f_n(x)\right).$$

2. (Convergence of minimizers – Truths and urban legends) Let $\mathbb{X}$, $f_n(x)$, $F^-_\infty(x)$ and $F^+_\infty(x)$ be as in the previous exercise.

   Prove or disprove the following statements, both as stated, and with "quasi-minimizers" instead of "minimizers".

   (a) If a sequence of minimizers converges to some x_∞, then x_∞ is a minimum point for $F^-_\infty(x)$ and/or $F^+_\infty(x)$.

   (b) If a sequence of minimizers has a *subsequence* converging to some x_∞, then x_∞ is a minimum point for $F^-_\infty(x)$ and/or $F^+_\infty(x)$.

   (c) If the Gamma-limit of $f_n(x)$ exists, and a sequence of minimizers has a *subsequence* converging to some x_∞, then x_∞ is a minimum point for the Gamma-limit.

   (d) Let us assume that the Gamma-limit of $f_n(x)$ exists and its minimum exists. Then every minimum point of the Gamma-limit is the limit of a sequence of minimizers.

   (e) Let us assume that the sequence $f_n(x)$ is equicoercive. Then every sequence of minimizers admits a converging subsequence.

Gamma convergence n

Subject: Gamma convergence **Difficulty**: ⋆⋆⋆⋆⋆

Prerequisites: everything concerning Gamma convergence

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1. (A ... "user friendly" definition) Let $\mathbb{X}$ be a metric space, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functions. Let $B_r(x)$ denote the closed ball with center in x and radius r.

   (a) Decrypt and prove the following statement. For every $x \in \mathbb{X}$ it turns out that

   $$\Gamma^{-}-\liminf_{n\to+\infty} f_n(x) = \lim_{r\to 0^+} \liminf_{n\to+\infty} \inf_{y\in B_r(x)} f_n(y) = \sup_{r>0} \sup_{k\geq 0} \inf_{n\geq k} \inf\{f_n(y) : y \in B_r(x)\}.$$

   (b) Provide an analogous characterization of the Gamma-limsup.

   (c) What happens if we take open balls?

2. (Gamma-convergence and epigraphs) Let $\mathbb{X}$ be a metric space, and let $f_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functions.

   (a) Prove that the epigraph of the Gamma-liminf is the ω-limit of the ephigraphs, namely

   $$\operatorname{epi}\left(\Gamma-\liminf_{n\to+\infty} f_n(x)\right) = \bigcap_{n\geq 0} \operatorname{Clos}\left(\bigcup_{k\geq n} \operatorname{epi}(f_k(x))\right).$$

   (b) Use this result in order to obtain an alternatove proof that

   $$\Gamma-\liminf_{n\to+\infty} f_n(x) = \Gamma-\liminf_{n\to+\infty} \overline{f_n}(x).$$

   (c) (???) Find an analogous characterization of the epigraph of the Gamma-limsup.

3. Is there any relation between $\Gamma^{-}-\limsup$ and $\Gamma^{+}-\liminf$?

4. Find a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that

   $$\Gamma^{-}-\liminf_{n\to+\infty} f_n(x), \qquad \Gamma^{-}-\limsup_{n\to+\infty} f_n(x), \qquad \Gamma^{+}-\liminf_{n\to+\infty} f_n(x), \qquad \Gamma^{+}-\limsup_{n\to+\infty} f_n(x)$$

   are four distint numbers for every $x \in \mathbb{R}$.

5. Gamma-compactness?

6. Metrization of Gamma-convergence?

Quadratic functionals

Subject: sign of quadratic functionals **Difficulty**: $\star\star$

Prerequisites: Legendre and Jacobi conditions, conjugate points

1. Determine for which values of the parameter $\ell > 0$ it turns out that
$$\int_0^\ell (\cos x \cdot \dot{u}^2 + u^2)\, dx \geq 0$$
for every $u \in C^1([0, \ell])$ such that $u(0) = u(\ell) = 0$.

2. For every $\ell > 0$ let us set
$$I_\ell := \inf \left\{ \int_0^\ell (\dot{u}^2 - 3u^2)\, dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$
   (a) Find $\ell_0 := \sup\{\ell > 0 : I_\ell = 0\}$.
   (b) Determine I_{ℓ_0}.

3. For every $\ell > 0$ let us set
$$I_\ell := \inf \left\{ \int_0^\ell (\dot{u}^2 - x^2 u^2)\, dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$
Prove that there exists $\ell_0 > 0$ such that $I_\ell = 0$ if and only if $\ell \leq \ell_0$.

4. Let us consider the two functionals
$$F(u) = \int_0^{10} (\dot{u}^2 + u^2 + 2a\, u\dot{u})\, dx, \qquad G(u) = \int_0^{10} (\dot{u}^2 + u^2 + 2ax\, u\dot{u})\, dx.$$
Determine the values of the parameter a for which $F(u) \geq 0$ and $G(u) \geq 0$ for every $u \in C^1([0, 10])$ such that $u(0) = u(10) = 0$.

Local Minima 1

Subject: sign of quadratic functionals **Difficulty**: ⋆⋆⋆

Prerequisites: Legendre and Jacobi conditions, conjugate points

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1. Let us consider the minimum problem

$$\min\left\{\int_0^1 \dot{u}^3\,dx : u \in C^1([0,1]),\ u(0)=0,\ u(1)=1\right\},$$

   and the function $u_0(x) = x$ for every $x \in [0,1]$.

   (a) Prove that $u_0(x)$ is the unique minimum point if we restrict ourselves to competitors with $u'(x) \geq 0$ for every $x \in [0,1]$.

   (b) Prove that $u_0(x)$ is the unique minimum point if we restrict ourselves to competitors with $u'(x) \geq -2$ for every $x \in [0,1]$.

   (c) Prove that, for every $\varepsilon > 0$, it turns out that $u_0(x)$ is *not* a minimum point if we restrict ourselves to competitors with $u'(x) \geq -2-\varepsilon$ for every $x \in [0,1]$.

   (d) Find the infimum if we restrict ourselves to competitors with $u'(x) \geq -4$ for every $x \in [0,1]$.

2. (a) Prove that there exists $a \in (0,\pi)$ such that the minimum problem

$$\min\left\{\int_0^1 \cos(u'(x))\,dx : u \in C^1([0,1]),\ u(0)=0,\ u(1)=a,\ u'(x) \geq 0\ \forall x \in [0,1]\right\}$$

   admits a solution.

   (b) Find all values $a \in \mathbb{R}$ for which the minimum problem

$$\min\left\{\int_0^1 \cos(u'(x))\,dx : u \in C^1([0,1]),\ u(0)=0,\ u(1)=a,\ u'(x) \geq -4\ \forall x \in [0,1]\right\}$$

   admits a solution.

3. Let us consider the minimization problem for the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 - \varepsilon \dot{u}^4 + u^2\right)\,dx$$

   in $C^1([0,1])$ subject to the boundary conditions $u(0) = 0$ and $u(1) = 2$.

   (a) Prove that for every $\varepsilon > 0$ the problem does not admit strong local minima.

   (b) Prove that when $\varepsilon > 0$ is small enough there exists at least a weak local minimum.

Local Minima 2

Subject: weak/strong local minima **Difficulty**: ⋆⋆⋆

Prerequisites: Legendre and Jacobi conditions, conjugate points, Weierstrass condition

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In each row of the following table a function $\psi(x, u, \dot{u})$ is presented. It is required to consider the minimum problem for the functional

$$F(u) = \int_0^\ell \psi(x, u, \dot{u})\, dx$$

in the class of all $u \in C^1([0, \ell])$ such that $u(0) = u(\ell) = 0$, and to determine the values of the parameter $\ell > 0$ for which the infimum is a real number, the minimum exists, and the function $u_0(x) \equiv 0$ is a weak local minimum, a strong local minimum, and a global minimum.

$\psi(x, u, \dot{u})$	$\inf \in \mathbb{R}$	min exists	u_0 WLM	u_0 SLM	u_0 GM
$\dot{u}^2 - u^2$					
$\dot{u}^2 - u^4$					
$\dot{u}^2 + u^4$					
$\dot{u}^2 - u^2 + u^4$					
$\dot{u}^2 - u^2 - u^4$					
$\dot{u}^2 + u^2 - u^4$					
$\dot{u}^4 - u^2$					
$\dot{u}^4 - \dot{u}^2 + u^2$					
$-\dot{u}^4 + \dot{u}^2 + u^2$					
$\dot{u}^4 - \dot{u}^2 - u^2$					
$-\dot{u}^4 + \dot{u}^2 - u^2$					
$\dot{u}^2 - \sin(u^2)$					
$\dot{u}^2 - \arctan^2 u$					
$\dot{u}^2 + \cos u$					
$\dot{u}^2 - \sinh(u^2)$					
$\sinh(\dot{u}^2) - \sin(u^2)$					
$\sin(\dot{u}^2) - \sinh(u^2)$					
$\cos(\dot{u}^2) + u^2$					
$\dot{u}^2 - u^2 + xu^4$					

Minimum problems 4

Subject: **Difficulty**: $\star\star\star\star$

Prerequisites:

1. Let us consider the minimum problem

$$\min\left\{\int_0^\ell \left[\cos x \cdot \dot{u}^2 + \sin(x^2 u)\right]\, dx : u \in C^1([0,\ell]),\ u(0) = 1,\ u(\ell) = 2015\right\},$$

   where ℓ is a positive parameter

   (a) Prove that the problem admits a solution for every $\ell < \pi/2$, and every minimum point is of class C^∞.

   (b) Prove that the problem does not admit a solution for every $\ell > \pi/2$.

   (c) Discuss the case $\ell = \pi/2$.

2. Compute, as a function of $\ell > 0$, the

$$\inf\left\{\int_0^\ell \left[(1+u^2)\dot{u}^2 - u^2\right]\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

3. Discuss existence of solutions to the minimum problem

$$\min\left\{\int_0^\ell \left[\sqrt{1+\dot{u}^4} - 7u^2 + 4e^x u\right]\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 2015\right\}$$

   depending on the parameter $\ell > 0$.

Gamma convergence n – Linearization effects

Subject: **Difficulty**: ⋆⋆⋆⋆

Prerequisites:

1. The families of functionals $F_\varepsilon(u)$, $G_\varepsilon(u)$ and $H_\varepsilon(u)$ are defined as
$$F_\varepsilon(u) = \int_0^1 \frac{\sinh(\varepsilon \dot{u}^2)}{\varepsilon}\,dx, \qquad G_\varepsilon(u) = \int_0^1 \frac{\sin(\varepsilon \dot{u}^2)}{\varepsilon}\,dx, \qquad H_\varepsilon(u) = \int_0^1 \frac{\sqrt{1+\varepsilon\dot{u}^2}-1}{\varepsilon}\,dx$$
if $u \in C^1([0,1])$, and as $+\infty$ elsewhere.
Compute the Gamma-limit as $\varepsilon \to 0^+$ in $C^0([0,1])$ and in $L^2((0,1))$.

2. Let us consider the minimum problem
$$m_\varepsilon := \min\left\{\int_0^1 \left[\sinh(\dot{u}^2) + \varepsilon x \sin u\right]\,dx : u \in C^1([0,1]),\ u(0) = u(1) = 0\right\}.$$
   (a) Prove that m_ε is well defined for every $\varepsilon > 0$.
   (b) Compute the principal part of m_ε as $\varepsilon \to 0^+$.
   (c) Prove that the minimizer is unique when ε is small enough.

3. Let us consider the minimum problem
$$m_{\varepsilon,k} := \min\left\{\int_0^1 \left[\cosh(\dot{u}) + \arctan(u^k)\right]\,dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = \varepsilon\right\}.$$
   (a) Prove that $m_{\varepsilon,k}$ is well defined for every $\varepsilon > 0$ and every positive integer k.
   (b) Discuss, as k varies in the set of positive integers, the asymptotic behavior of minima and minimizers as $\varepsilon \to 0^+$.

4. Discuss existence/uniqueness/regularity and asymptotic behavior as $\varepsilon \to 0^+$ for the following minimum problems
$$\min\left\{\int_0^1 \left[\cosh(\dot{u}^4) + \varepsilon u^2\right]\,dx : u \in C^1([0,1]),\ u(0) = u(1) = 1\right\},$$
$$\min\left\{\int_0^1 \left[\dot{u}^2 + \sin(\dot{u}) + \sinh(u^2)\right]\,dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = \varepsilon\right\}.$$

5. Let us consider the minimization problem
$$I_\varepsilon = \min\left\{\int_0^1 \left[\dot{u}^4 + \sinh(\dot{u}^2)\right]\,dx : u \in C^1([0,1]),\ u(0) = 0,\ \int_0^1 u^2\,dx = \varepsilon\right\}.$$
Determine two positive constants $\alpha < \beta$ for which there exists $a \neq 0$ and $b \neq 0$ such that
$$I_\varepsilon = a\varepsilon^\alpha + b\varepsilon^\beta + o(\varepsilon^\beta) \qquad \text{as } \varepsilon \to 0^+.$$

Modica-Mortola 1 – Minimization problems

Subject: asymptotic behavior of minimum problems **Difficulty**: ⋆⋆⋆

Prerequisites: Modica-Mortola functionals, basic AM–GM trick

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1. Prove all the results stated in the sheet ...
2. Discuss the same problem without the restriction that u is nonnegative.
3. (Different boundary conditions) Discuss the asymptotic behavior of minima and minimizers to the Modica-Mortola functional with boundary condition(s)
   (a) $u(a) = A$ with $A \in (0, 1)$,
   (b) $u(a) = A$ with $A > 1$,
   (c) $u(a) = A$ and $u(b) = B$ with A and B in $(0, 1)$,
   (d) $u(a) = A$ and $u(b) = B$ with A and B in $(-1, 1)$,
   (e) $u(a) = A$ and $u(b) = B$ with $A < -1$ and $B > 1$.
4. (Integral constraint) Discuss the asymptotic behavior of minima and minimizers to the Modica-Mortola functional in the class of all $u \in C^1([a, b])$ such that
$$\int_a^b u(x)\, dx = m(b - a),$$
   where $m \in (-1, 1)$ is a parameter.
5. (Different potential)
   (a) Discuss the asymptotic behavior of minima and minimizers to the functionals
$$\int_a^b \left(\varepsilon \dot{u}^2 + \frac{1}{\varepsilon}(1 - \cos u) \right) dx.$$
   (b) Consider the functional where the potential $(1 - u^2)^2$ in the Modica-Mortola functional is replaced by $W(u)$, where $W : \mathbb{R} \to \mathbb{R}$ is a *double-well potential*, namely a continuous nonnegative function such that $W(s) = 0$ if and only if $|s| = 1$.
6. (Different exponent)
   (a) Discuss the asymptotic behavior of minima and minimizers to the functionals
$$\int_a^b \left(\varepsilon \dot{u}^4 + \frac{1}{\varepsilon}(1 - u^2)^2 \right) dx.$$
   (b) More generally, consider the functional where the quadratic term $\dot{u}^2$ in the Modica-Mortola functional is replaced by any power $|\dot{u}|^p$ with exponent $p > 1$.
   (c) What happens when $p = 1$?

Modica-Mortola 2 – Asymptotic profile

Subject: minimum problem in unbounded domains **Difficulty**: ⋆⋆⋆

Prerequisites: direct method, Gamma convergence, basic AM–GM trick

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1. (Minimum problem in a half-line) Let us consider the minimum problem

$$\min\left\{\int_0^{+\infty}\left[\dot v^2+(1-v^2)^2\right]\,dx : v\in C^1([0,+\infty)),\ v(0)=0\right\}.$$

   (a) Prove that in the minimization process we can limit ourselves to functions v with

$$\sup_{x\ge 0}|v(x)|\ge 1.$$

   (b) Prove that the minimum is $G(1)$.

   (c) Prove that there are exactly two minimizers, namely $v(x)=\pm\tanh x$.

2. (Minimum problem with different boundary conditions) State and prove analogous results for the minimum problem

   (a) in the half-line $x\ge 0$ with boundary condition $v(0)=A$, where A is a given real number,

   (b) in the the whole real line $\mathbb{R}$, with limit conditions

$$\lim_{x\to-\infty} v(x)=-1, \qquad \lim_{x\to+\infty} v(x)=1.$$

3. (Minimization over increasing intervals)

   Let us consider, for every positive integer n, the minimization problem

$$M_n := \min\left\{\int_0^{n}\left[\dot v^2+(1-v^2)^2\right]\,dx : v\in C^1([0,n]),\ v(0)=0\right\}.$$

   (a) Prove that M_n is well-defined for every n,

   (b) Prove that M_n to $G(1)$.

   (c) If $u_n(x)$ is any sequence of minimizers, prove that $u_n(x)\to\tanh x$ uniformly on compact subsets of $[0,+\infty)$.

   (d) Interpret the result in terms of Gamma convergence.

4. (Generalizations) State and prove analogous results for Lagrangians such as

$$|\dot u|^p+W(u),$$

   where $p>1$ and W is a double-well potential.

Modica-Mortola 3 – Gamma convergence

Subject: Modica-Mortola functional **Difficulty**: ⋆⋆⋆⋆

Prerequisites: Gamma convergence, basic AM–GM trick, transition intervals

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1. Prove all the results stated in the sheet ...
2. (More general exponents and potentials) State and prove analogous results (characterization of the Gamma limit and asymptotic behavior of recovery sequences) for the family of functionals
$$\int_a^b \left(\varepsilon|\dot{u}|^p + \frac{1}{\varepsilon}W(u)\right) dx,$$
where $p > 1$ and W is a double-well potential.
3. (More zeroes of the potential) State and prove analogous results (characterization of the Gamma limit and asymptotic behavior of recovery sequences) for the family of functionals
$$\int_a^b \left(\varepsilon\dot{u}^2 + \frac{1}{\varepsilon}(1 - \cos u)\right) dx.$$
4. (Modica-Mortola on derivatives) State and prove analogous results (characterization of the Gamma limit and asymptotic behavior of recovery sequences) for the family of functionals
$$\int_a^b \left(\varepsilon\ddot{u}^2 + \frac{1}{\varepsilon}(1 - \dot{u}^2)^2\right) dx.$$
5. (Modica-Mortola with restrictions) Let us consider the family of functionals that coincide with the Modica-Mortola functionals if u is in some class V, and is equal to $+\infty$ if $u \notin V$. Compute the Gamma limit of the family in the case where
   (a) (Boundary data) V is the set of functions satisfying the boundary conditions $u(a) = A$ and $u(b) = B$ (A and B are given real numbers),
   (b) (Integral condition) V is the set of functions whose integral in $[a, b]$ vanishes.
6. [Achtung: this is more a research project rather than an exercise!]

   (Higher order Modica-Mortola) Develop a theory (asymptotic of the minimization problem, characterization of the Gamma limit and recovery sequences) for the family of functionals
$$\int_a^b \left(\varepsilon\ddot{u}^2 + \frac{1}{\varepsilon}(1 - u^2)^2\right) dx.$$

Minimum problems 10

Subject: asymptotic behavior of minimum problems **Difficulty**: ⋆⋆⋆⋆

Prerequisites: direct method, Gamma convergence, Modica-Mortola

1. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min \left\{ \int_0^1 \left(n\dot{u}^2 + u^2 - \arctan u \right) dx : u(0) = 0,\ u(1) = 2017 \right\}.$$
   (a) Prove that M_n is well-defined for every n.
   (b) Determine the principal part of M_n as $n \to +\infty$.
2. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min \left\{ \int_0^1 \left(n\dot{u}^2 + u^2 - \arctan u \right) dx : u(0) = u(1) = 0 \right\}.$$
   (a) Prove that M_n is well-defined and negative for every n.
   (b) Determine the principal part of M_n as $n \to +\infty$.
3. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min \left\{ \int_0^1 \left(n\dot{u}^2 + u^2 - \arctan u \right) dx : u(0) = u(1) = 2017 \right\}.$$
   (a) Prove that M_n is well-defined and negative for every n.
   (b) Determine the principal part of M_n as $n \to +\infty$.
4. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min \left\{ \int_0^1 \left(\dot{u}^2 + nu^2 - \arctan u \right) dx : u(0) = u(1) = 0 \right\}.$$
   (a) Prove that M_n is well-defined and negative for every n.
   (b) Determine the principal part of M_n as $n \to +\infty$.
5. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min \left\{ \int_0^1 \left(\dot{u}^2 + nu^2 - \arctan u \right) dx : u(0) = u(1) = 2017 \right\}.$$
   (a) Prove that M_n is well-defined for every n.
   (b) Determine the principal part of M_n as $n \to +\infty$.

Minimum problems 11

Subject: asymptotic behavior of minimum problems **Difficulty**: ⋆⋆⋆⋆

Prerequisites: direct method, Gamma convergence, Modica-Mortola

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1. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min\left\{\int_0^1 \left(\dot{u}^2 + u^2 + n\arctan u\right)\,dx : u(0) = 0,\ u(1) = 2017\right\}.$$
   (a) Prove that M_n is well-defined for every n.
   (b) Determine the principal part of M_n as $n \to +\infty$.

2. Let us consider, for every $\varepsilon > 0$, the minimum problem
$$M_n := \min\left\{\int_0^1 \left(\dot{u}^2 + u^2 + \varepsilon\arctan u\right)\,dx : u(0) = u(1) = 0\right\}.$$
   (a) Prove that M_ε is well-defined for every $\varepsilon > 0$.
   (b) Determine the principal part of M_ε as $\varepsilon \to 0^+$.

3. Let us consider, for every $\lambda > 0$, the minimum problem
$$M_\lambda := \min\left\{\int_0^1 \left(\dot{u}^2 + \lambda u^4 + u^3\right)\,dx : u(0) = u(1) = 0\right\}.$$
   (a) Prove that M_λ is well-defined for every $\lambda > 0$.
   (b) Prove that the function $u(x) \equiv 0$ is a SLM for every $\lambda > 0$.
   (c) Prove that the function $u(x) \equiv 0$ is a GM when λ is large enough.
   (d) Prove that the function $u(x) \equiv 0$ is not a WLM when λ is small enough.
   (e) Determine the principal part of M_λ as $\lambda \to 0^+$.

4. Let us consider, for every $\lambda > 0$, the minimum problem
$$M_\lambda := \min\left\{\int_0^1 \left(\dot{u}^2 + \lambda u^4 + u^3\right)\,dx : u(0) = u(1) = 2017\right\}.$$
   (a) Prove that M_λ is well-defined for every $\lambda > 0$.
   (b) Determine the principal part of M_λ as $\lambda \to 0^+$.

5. Let us consider, for every $\lambda > 0$, the minimum problem
$$M_\lambda := \min\left\{\int_0^1 \left(\dot{u}^2 + u^4 + \lambda u^3\right)\,dx : u(0) = u(1) = 0\right\}.$$
   (a) Prove that M_λ is well-defined for every $\lambda > 0$.
   (b) Determine the principal part of M_λ as $\lambda \to 0^+$.
   (c) Determine the principal part of M_λ as $\lambda \to +\infty$.

Minimum problems 12

Subject: asymptotic behavior of minimum problems **Difficulty**: ⋆⋆⋆⋆

Prerequisites: direct method, Gamma convergence, Modica-Mortola

1. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min\left\{\int_0^1 \left(\dot{u}^2 + n\sinh^2 u\right)\,dx : u(0) = 3,\ u(1) = 0\right\}.$$
   (a) Prove that M_n is well-defined for every n, and the minimizer u_n is always unique and monotone.
   (b) Determine the principal part of M_n as $n \to +\infty$.
   (c) Prove that u_n converges to 0 uniformly on compact subsets of $(0, 3]$.
   (d) Determine the order of vanishing of the sequence x_n such that $u_n(x_n) = 2$.
2. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min\left\{\int_0^{2017} \left(\dot{u}^4 + n\cos u\right)\,dx : u(0) = 0,\ u(2017) = 2\pi\right\}.$$
   (a) Prove that M_n is well-defined for every n.
   (b) Determine the first two terms in the asymptotic expansion of M_n.
   (c) Describe the asymptotic behavior of any sequence of minimizers.
3. Let us consider, for every positive integer n, the minimum problem
$$M_n := \min\left\{\int_0^3 \left(\dot{u}^2 + n\cosh u\right)\,dx : u(0) = 1,\ u(3) = 2\pi\right\}.$$
   (a) Prove that M_n is well-defined for every n.
   (b) Determine the first two terms in the asymptotic expansion of M_n.
   (c) Describe the asymptotic behavior of any sequence of minimizers.
4. Prove that the following sequences are well-defined and compute their leading terms:
$$K_n := \min\left\{\int_0^1 \left(\dot{u}^8 + nu^5\right)\,dx : u(0) = 0,\ u(1) = 0\right\},$$
$$L_n := \min\left\{\int_0^1 \left(\dot{u}^8 + nu^5\right)\,dx : u(0) = 1,\ u(1) = 2\right\},$$
$$M_n := \min\left\{\int_0^1 \left(\dot{u}^8 + nu^{12}\right)\,dx : u(0) = 1,\ u(1) = 2\right\}.$$

Minimum problems 13

Subject: asymptotic behavior of minimum problems **Difficulty**: ⋆⋆⋆⋆

Prerequisites: direct method, Gamma convergence, Modica-Mortola

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1. Let us consider, for every positive integer n, the minimum problem

$$M_n := \min \left\{ \int_0^1 \left(\dot{u}^2 + n \arctan u \right) \, dx : u(0) = u(1) = 0 \right\}.$$

   (a) Prove that M_n is well-defined for every n.

   (b) Determine the first two terms in the asymptotic expansion of M_n.

   (c) [I have no idea] If u_n is any sequence of minimizers, determine the order of infinity of the sequence

$$\min_{x \in [0,1]} u_n(x).$$

Minimum problems $+\infty$

Subject: **Difficulty**: ???

Prerequisites:

[Mixed problems to be re-organized]

1. Let us consider the minimum problem

$$\min\left\{\int_0^\ell \left(\cosh(\dot{u}) - u^k\right)\,dx : u(0) = 0,\ u(\ell) = 2015\right\}$$

where ℓ is a positive real number and k is a positive integer.

   (a) Prove that the minimum problem admits a solution for every choice of the parameters.

   (b) Prove that every minimizer is of class C^∞.

2. Let us consider the minimum problem

$$\min\left\{\int_0^\ell \left(\cosh(\dot{u}) - u^k\right)\,dx : u(0) = u(\ell) = 0\right\}$$

where ℓ is a positive real number and k is a positive integer.

Prove that the function $u_0(x) \equiv 0$

   (a) is a SLM for every $k > 2$,

   (b) is not a global minimum if k is large enough.

3. Let us consider the minimum problem

$$\min\left\{\int_0^3 \left(\dot{u}^4 - u^2\right)\,dx : u(0) = u(3) = 0\right\}.$$

   (a) Determine whether the function $u_0(x) \equiv 0$ is a WLM/SLM.

   (b) Prove that the minimum exists and it is a negative number.

   (c) Prove that the minimum is equal to

$$-\int_0^3 u_0^2(x)\,dx,$$

   where $u_0(x)$ is any minimizer.

   (d) Prove that all minimum points are of class C^1 but not of class C^2.

   (e) Find the largest $\alpha \in (0, 1)$ such that all minimum points are of class $C^{1,\alpha}$.

4. Let us consider the minimum problem

$$\min\left\{\int_0^\ell \left(\dot{u}^{22}-|u|^q\right)\,dx : u(0)=u(\ell)=0\right\},$$

where ℓ and q are positive parameters.

Prove that

(a) for $q<22$ the minimum exists and is is a negative number,

(b) for $q>22$ the minimum does not exist,

(c) for $q=22$ there exists $\ell_0>0$ such that

- when $\ell\leq\ell_0$ the minimum exists and it is equal to zero,
- when $\ell>\ell_0$ the minimum does not exist.

5. Discuss existence of solutions to the minimum problem

$$\min\left\{\int_0^\ell \left(\dot{u}^{22}-|u|^q\right)\,dx : u(0)=0,\ u(\ell)=2015\right\},$$

depending on the positive parameters ℓ and q.

6. Let us set

$$M_\varepsilon=\min\left\{\int_0^{2015} \left(\dot{u}^2+\arctan u\right)\,dx : u(0)=0,\ u(2015)=\varepsilon\right\}.$$

(a) Prove that M_ε is well defined for every $\varepsilon>0$.

(b) Compute the principal part of M_ε as $\varepsilon\to 0^+$.

7. Let us set

$$M_n=\min\left\{\int_0^{10\pi} \left(\dot{u}^2+n\cosh u\right)\,dx : u(0)=u(10\pi)=2015,\ u(x)\geq\sin x\ \forall x\in[0,10\pi]\right\}.$$

(a) Prove that M_n is well defined for every $n\in\mathbb{N}$.

(b) Compute the principal part of M_n as $n\to+\infty$.

(c) Describe the asymptotic behavior of any sequence of minimizers.

8. Let us set

$$M_n=\min\left\{\int_0^1 \left(\dot{u}^2+nu^{2015}+e^{-nu}\right)\,dx : u(0)=u(1)=2015\right\}.$$

(a) Prove that M_n is well defined for every $n\in\mathbb{N}$.

(b) Compute the principal part of M_n as $n\to+\infty$.

(c) Describe the asymptotic behavior of any sequence of minimizers.

9. (Lack of coerciveness)

   (a) Discuss existence of solutions to the minimum problem

   $$\min\left\{\int_0^1 \left(x^2\dot{u}^2 + \cos u\right)\,dx : u(0) = 0\right\}.$$

   (b) Discuss existence of solutions to the minimum problem

   $$\min\left\{\int_0^1 \left(|x|^p\dot{u}^2 + \cos u\right)\,dx : u(0) = 0\right\}$$

   depending on the positive parameter p.

   (c) Discuss existence of solutions to the minimum problem

   $$\min\left\{\int_0^1 \left(|x|^{22}|\dot{u}|^{1+p} + \cos u\right)\,dx : u(0) = 0\right\}$$

   depending on the positive parameter p.

10. Let us consider, for every $\lambda > 0$, the minimum problem

    $$\min\left\{\int_0^1 \left(x^2\dot{u}^2 + \lambda u^2\right)\,dx : u(1) = 1\right\}.$$

    (a) Prove that the minimum problem admits a unique solution for every $\lambda > 0$.

    (b) Prove that the solution lies in $C^0([0,1]) \cap C^\infty((0,1])$.

    (c) Determine for which values $\lambda > 0$ the solution lies in $C^1([0,1])$.

11. Let us consider the minimum problem

    $$\min\left\{\int_0^1 \left(\sin^2 x \cdot \dot{u}^2 + u^2 - xu\right)\,dx : u(0) = \lambda,\ u(1) = 2015\right\}.$$

    (a) Prove that the minimum problem admits a solution for just one real value of λ.

    (b) Prove that the infimum is a real numer and does not depend on λ.

12. Let us set

    $$M_\varepsilon = \min\left\{\int_\varepsilon^1 \left(x^2\dot{u}^2 + 10u^2\right)\,dx : u(\varepsilon) = 2015\right\}.$$

    (a) Prove that M_ε is well defined for every $\varepsilon > 0$.

    (b) Compute the principal part of M_ε as $\varepsilon \to 0^+$.

13. Let α and β be two positive constants, and let $a : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function such that

    $$a(x) = \begin{cases} \alpha & \text{if } x \in [0, 1/2), \\ \beta & \text{if } x \in [1/2, 1). \end{cases}$$

(a) Compute the Gamma-limit of the following functionals

$$\int_0^2 a(nx)u^2\,dx, \qquad \int_0^2 a(nx)u^4\,dx, \qquad \int_0^2 a(nx)\left(u^2+u^4\right)\,dx.$$

(b) Compute the Gamma-limit of the following functionals (well, do not compute explicitly the last one, but just think about it)

$$\int_0^2 a(nx)\dot{u}^2\,dx, \qquad \int_0^2 a(nx)\dot{u}^4\,dx, \qquad \int_0^2 a(nx)\left(\dot{u}^2+\dot{u}^4\right)\,dx.$$

(c) Compute the Gamma-limit of the following functionals

$$\int_0^2 a(nx^2)u^2\,dx, \qquad \int_0^2 a(nx^2)\dot{u}^2\,dx, \qquad \int_0^2 \left[a(nx)\dot{u}^4+a(n^2x)u^2\right]\,dx.$$

(d) (Double scales) Compute the Gamma-limit of the following functionals

$$\int_0^2 \left[a(nx)\dot{u}^4+a(n^2x)u^2\right]\,dx, \qquad \int_0^2 \left[a(nx)+a(n^2x)\right]u^2\,dx,$$

$$\int_0^2 a(nx)\cdot a(n^2x)\cdot \dot{u}^2\,dx, \qquad \int_0^2 \left[a(nx)+a(n^2x)\right]\dot{u}^2\,dx.$$

14. Let us consider the minimum problem

$$\min\left\{\int_0^{10\pi}\left(\dot{u}^2+\lambda u^3\right)\,dx : u(0)=u(10\pi)=2015,\ u(x)\ge \sin x\ \forall x\in[0,10\pi]\right\}.$$

(a) Prove that the minimum problem has a solution of class C^1 for every $\lambda>0$.

(b) Prove that there exists $\lambda_0>0$ such that minimizers do not touch the obstacle for every $\lambda<\lambda_0$.

(c) Find the principal part of the minimum as $\lambda\to+\infty$.

(d) Describe the asymptotic behavior of minimizers as $\lambda\to+\infty$.

(e) Prove that there exists $\lambda_1>\lambda_0$ such that the "contact region" of minimizers is connected for every $\lambda>\lambda_1$ (the contact region is defined as the set of all $x\in[0,10\pi]$ where the minimizer coincides with the obstacle).

15. Let us consider the minimum problem

$$\min\left\{\int_0^{10\pi}\left(\dot{u}^4+\lambda u^2\right)\,dx : u(0)=u(10\pi)=0,\ u(x)\ge \sin x\ \forall x\in[0,10\pi]\right\}.$$

(a) Prove that the minimum problem has a solution of class C^1 for every $\lambda>0$, and that the minimizer is always unique.

(b) Find the principal part of the minimum as $\lambda\to+\infty$.

(c) Describe the asymptotic behavior of minimizers as $\lambda\to+\infty$.

16. (Homogenized obstacle)

    (a) Prove that the sequence

    $$\min \left\{ \int_0^1 \left(\sqrt{1 + \dot{u}^4} + \arctan u \right) dx : u(x) \geq \sin(nx) \ \forall x \in [0,1] \right\}$$

    is well-defined, and determine its principal part.

    (b) Determine, for every real number $a > 0$, the principal part of the sequence

    $$\min \left\{ \int_0^1 \left(\sqrt{1 + \dot{u}^4} + n^a \arctan u \right) dx : u(x) \geq \sin(nx) \ \forall x \in [0,1] \right\}.$$

17. Solve the minimum problem

    $$\min \left\{ \int_0^\ell \dot{u}^2 \, dx : u(0) = 0, \ \int_0^\ell u^2 \, dx = 1 \right\},$$

    where $\ell > 0$ is real parameter.

18. Solve the minimum problem

    $$\min \left\{ \int_0^1 \dot{u}^2 \, dx : \int_0^1 u^2 \, dx = \int_0^1 u^3 \, dx = 2015 \right\}.$$

    (a) Prove that the class of competitors is nonempty, namely there exists at least one function that satisfies the constraints.

    (b) Prove that the minimum problem admits a solution.

    (c) Prove that all minimizers are of class C^∞.

    (d) Prove that minimizers are neither affine functions nor polynomials of degree 2.

19. Discuss existence of solutions to the following minimum problems

    $$\min \left\{ \int_0^1 \left(\dot{u}^2 - u^{20} \right) dx : \int_0^1 \dot{u}^4 \, dx = 1, \ u(0) = 3 \right\},$$

    $$\min \left\{ \int_0^1 \left(\dot{u}^6 - u^6 \right) dx : \int_0^1 \left(\dot{u}^4 + u^2 \right) dx = 1 \right\}.$$

20. Determine which of the following problems admit solutions:

    $$\min \left\{ \int_0^1 \left(\dot{u}^2 - u^4 \right) dx : \int_0^1 u^6 \, dx = 1 \right\}, \qquad \min \left\{ \int_0^1 \left(\dot{u}^2 - u^4 \right) dx : \int_0^1 u^{2015} \, dx = 1 \right\},$$

    $$\min \left\{ \int_0^1 \left(\dot{u}^2 + u^4 \right) dx : \int_0^1 u^{2015} \, dx = 1 \right\}, \qquad \min \left\{ \int_0^1 \left(\dot{u}^2 - u^6 \right) dx : \int_0^1 u^4 \, dx = 1 \right\}$$

    $$\min \left\{ \int_0^1 \left(u^{20} - \dot{u}^2 \right) dx : \int_0^1 \dot{u}^4 \, dx = 1 \right\}, \qquad \min \left\{ \int_0^1 \left(u^{20} - \dot{u}^4 \right) dx : \int_0^1 \dot{u}^2 \, dx = 1 \right\}.$$

21. Let us set
$$M_n = \min\left\{ \int_0^{\pi} \left(\dot{u}^2 + e^{n\cos x \cdot u}\right)\, dx : \int_0^{\pi} u\, dx = 1 \right\}.$$
   (a) Prove that M_n is well defined for every $n \in \mathbb{N}$.
   (b) Compute the principal part of M_n as $n \to +\infty$.

Part III

Exam preparation

Chapter 1

Saper dire

[Nice translation needed]

1.1 Indirect method

1.1.1 First variation

- I-1 Fundamental lemma in the calculus of variations: statement, possible proofs, extension to different classes of test functions.
- I-2 Du Bois Reymond lemma: statement, possible proofs, extension to different classes of test functions.
- I-3 Inner/outer (horizontal/vertical) variations for an integral functional.
- I-4 Different forms of the Euler-Lagrange equation, under suitable assumptions on the Lagrangian.
- I-5 Optimality through convexity: statement(s) and proof.
- I-6 Optimality through auxiliary functional: statement, proof, examples.
- I-7 First variation for integral functionals involving multiple integrals: Euler-Lagrange equation in divergence form, Laplacian, Neumann conditions in more space dimensions.
- I-8 Lagrange multipliers for constrained minimization problems: statement and proof.
- I-9 Sufficient condition which guarantees that an extremal to a variational problem with constraint is actually a minimizer.

1.1.2 Quadratic functionals

- I-10 Jacobi equation and its interpretation as an Euler-Lagrange equation. Definition of conjugate points.
- I-11 Riccati equation associated to a quadratic functional. Connections between Riccati equations and solutions with constant sign to linear second order equations.
- I-12 Necessary conditions for a quadratic functional to be nonnegative (Lagrange and Jacobi conditions): statement and proof.
- I-13 Sufficient conditions for a quadratic functional to be nonnegative (reinforced Lagrange and Jacobi conditions): statement and proof.

1.1.3 Second variation

- I-14 Second variation of an integral functional: definition and computation.
- I-15 Different notions of minimum point (DLM, WLM, SLM, GM): definitions, implications, counterexamples.
- I-16 Complete picture of necessary and/or sufficient conditions for an extremal to be WLM or SLM: definitions and statements.

I-17 Necessary conditions for an extremal to be a DLM: statement and proof.

I-18 Sufficient conditions for an extremal to be a WLM: statement and proof.

1.1.4 Calibrations and Weierstrass fields

I-19 Calibration through null Lagrangians: definitions, statements, proofs.

I-20 Calibration through the value function: definition, statement, proof.

I-21 Interpretation in terms of calibrations of the minimality through convexity.

I-22 Interpretation in terms of calibrations of the last part of the proof of the sufficient condition for quadratic functionals to be nonnegative.

I-23 Definition of Weierstrass excess function and connection with convexity properties of the Lagrangian.

I-24 Necessary condition for an extremal to be a SLM (Weierstrass condition): statement and proof.

I-25 Definition of Weierstrass field of extremals and its slope function. Rewriting of the Euler-Lagrange equation in terms of the slope function.

I-26 Weierstrass sufficient conditions for an extremal to be a SLM: definitions, statement, and road map of the proof.

I-27 Weierstrass representation formula: statement and proof *à la Hilbert* (via calibrations).

I-28 Weierstrass representation formula: statement and proof *à la Weierstrass* (in the case of a field of extremals originating from a single point).

I-29 Imbedding theorem: definitions, statement, and sketch of the proof.

1.2 Direct method

1.2.1 General framework

D-1 Compactness, semicontinuity and Weierstrass theorem with respect to a notion of convergence: definitions, statement, proof.

1.2.2 Weak convergence in Hilbert spaces

D-2 Continuity of the norm in a Hilbert space with respect to the strong convergence. Lack of compactness (with respect to the strong convergence) of balls in Hilbert spaces of infinite dimension.

D-3 Hilbert bases in Hilbert spaces and corresponding components of vectors: definitions and main properties.

- D-4 Weak convergence is separable Hilbert spaces: definition and basic properties (behavior with respect to the operations and the scalar product).
- D-5 Can we test the weak convergence just on a dense subset? Statement, proof, counterexamples.
- D-6 Semicontinuity of the norm with respect to weak convergence: statement, proof, counterexamples to continuity.
- D-7 Compactness of balls in separable Hilbert spaces with respect to weak convergence: statement and proof.

1.2.3 Sobolev spaces in dimension one

- D-8 Fundamental lemma in the calculus of variations and Du Bois Reymond lemma for Lebesgue functions: statements and possible proofs.
- D-9 Weak derivative and Sobolev spaces: definition (W) vs definition (H).
- D-10 Uniqueness of the weak derivative.
- D-11 Sobolev functions are the antiderivatives of their weak derivatives. Regularity of functions with continuous weak derivative.
- D-12 Hölder regularity of functions in Sobolev spaces: statements and proofs.
- D-13 Equivalence of the two definitions of Sobolev spaces ($H = W$): statement and proof.
- D-14 Compactness theorems in Sobolev spaces under suitable boundedness assumptions on functions and their derivatives: statements and proofs.
- D-15 Continuity and lower semicontinuity theorems for integral functionals in Sobolev spaces: statements and proofs.
- D-16 Weak convergence in L^p spaces.
- D-17 Compactness and semicontinuity results for integral functionals with p-growth assumptions: statements and proofs in special cases ($p \geq 2$).

1.3 Relaxation

- R-1 Relaxation vs lower semicontinuous envelope: definitions, statement, proof.
- R-2 Infimum of a functional vs infimum of the relaxation: statements, proofs, examples.
- R-3 Stability of the relaxation under continuous perturbations: statement, proof, counterexamples in case of semicontinuous perturbations.
- R-4 Subsets which are dense in energy and their role in the computation of the relaxed functional.

- R-5 Characterization of Sobolev spaces as the set where suitable relaxed functionals are finite: statements and proofs.
- R-6 Relaxation of integral functionals with Lagrangian depending only on the derivative: statements, proofs, counterexamples showing the importance of growth conditions.

1.4 Gamma convergence

- G-1 Basic definitions (Gamma limit, Gamma limsup and Gamma liminf). Gamma convergence vs pointwise and/or uniform convergence.
- G-2 Stability of Gamma convergence with respect to continuous and/or lower semicontinuous perturbations: statements, proofs, counterexamples.
- G-3 The “inf” in the definition of Gamma liminf/limsup is actually a “min”.
- G-4 Lower semicontinuity of Gamma liminf and Gamma limsup.
- G-5 Convergence of minima and minimizers, with or without equicoerciveness assumptions: definitions, statements, counterexamples.
- G-6 The Gamma liminf/limsup of a sequence of functionals coincides with the Gamma liminf/limsup of the sequence of relaxed functionals.
- G-7 Proof of the method of Lagrange multipliers in finite dimension through penalization of the constraint, and interpretation in terms of Gamma convergence.

1.5 Classical examples

- C-1 Geodesics in Euclidean spaces: description, discussion of uniqueness, calibrations.
- C-2 Geodesics in the cylinder and in the sphere: description, uniqueness, calibrations.
- C-3 Transversality conditions for point-to-curve problems: definition, statement, proof.
- C-4 Transversality condition and regularity result for the minimizer of an obstacle problem in a contact point: definition, statements, proof.
- C-5 Discrete-to-continuum problems: piecewise constant and piecewise affine setting, basic examples, Euler-Lagrange equation in discrete setting, convergence results.
- C-6 Characterization of Sobolev spaces as the set where the Gamma limit of suitable functionals defined in a discrete setting is finite.
- C-7 The brachistochrone problem: model, existence/uniqueness of solutions to the Euler-Lagrange equation, cycloids, minimality through Weierstrass fields and through convexity (after suitable variable change).

- C-8 Graph of minimal length which spans a given area: model, existence/uniqueness of solutions to the Euler-Lagrange equation with Lagrange multiplier, minimality through convexity.
- C-9 Minimal surfaces of revolution: model, existence/uniqueness of solutions to the Euler-Lagrange equation, discussion of a special symmetric case.
- C-10 Heavy chain problem: parametric vs non parametric approach, Euler-Lagrange equation with Lagrange multipliers, discussion of a special symmetric case, approximation through Gamma convergence.
- C-11 Homogenization of periodic coefficients, both in front of the function and in front of the derivative.
- C-12 Problems of Modica-Mortola type: asymptotic behavior of minima and minimizers in a model example.

Chapter 2

Know how

[Know how]

2.1 Indirect method

2.1.1 First variation

- Recognizing when the set where a functional is defined/finite is a vector space or an affine space, and identifying the space of admissible variations.

- Computing the Gateaux derivative of a functional along a given curve or in a given direction.

- Being aware of the different classes of test functions that can be involved in the fundamental lemma in the calculus of variations and in the DBR lemma.

- Computing the Euler-Lagrange equation for an integral functional, even in the case of functionals depending on more unknowns and/or involving higher order derivatives.

- Being aware of the different forms of the Euler-Lagrange equation (integral forms, differential form, DBR form, Erdmann form) and of the assumptions on the Lagrangian and on the extremal that are required for each of them.

- Computing the boundary conditions that originate in the computation of an Euler-Lagrange equation (in particular Dirichlet, Neumann and periodic boundary conditions). Being aware of which choice of test functions gives rise to the different boundary conditions.

- Concluding that a solution of the Euler-Lagrange equation is a global minimum point under suitable convexity assumptions on the Lagrangian.

- Concluding that a solution of the Euler-Lagrange equation is a global minimum point through a suitable auxiliary functional.

- Computing the Euler-Lagrange equation and the associated boundary conditions for integral functionals involving multiple integrals.

- Reverse engineering: producing, when possible, a functional whose Euler-Lagrange equation is a given ordinary differential equation or a given partial differential equation.

- Computing the Euler-Lagrange equation with Lagrange multipliers for a minimization problem with integral constraints (keeping into account that one has to check first that the first variation of the constraint does not vanish identically).

- Being aware that there is a the simple sufficient condition which guarantees that a solution in the Lagrange multipliers setting is actually a minimum.

2.1.2 Quadratic functionals

- Exploiting necessary and/or sufficient conditions in order to establish whether a quadratic functional is nonnegative for all functions vanishing at the endpoints.

- Computing the Jacobi equation and conjugate points.

- Being aware that the length of the interval of integration might be relevant when deciding the signature of a quadratic functional.

2.1.3 Second variation

- Computing the second variation of an integral functional
- Exploiting the theory of quadratic functionals (Legendre condition, reinforced Legendre condition, conjugate points) in order to decide whether a given extremal is a WLM for a given functional or not.

2.1.4 Calibrations and Weierstrass fields

- Computing the value function in simple cases, and deducing a calibration for a given extremal.
- Being aware that a calibration through the value function is actually also a calibration for a curve-to-curve problem (the curves are the two levels of the value function corresponding to the initial and final condition).
- Exploiting the Weierstrass condition and the theory of Weierstrass fields in order to decide whether a given extremal is a SLM for a given functional or not.
- Exploiting the imbedding theorem in order to decide whether a given extremal lies in a Weierstrass field or not.

2.2 Direct method

- Providing examples of sequences in Hilbert spaces which converge or do not converge strongly and/or weakly.
- Deciding whether a sequence of functions in a Lebesgue space is strongly/weakly convergent.
- Deducing pointwise estimates on a function from *integral* estimates on its (weak) derivative and boundary conditions or integral estimates on the function.
- Knowing when it is possible (and when it is not possible) to deduce the compactness of a sublevel of a functional with respect to a suitable notion of convergence.
- Deciding which pointwise or integral constraints are stable by a given notion of convergence.
- Road map of the direct method: weak formulation, compactness, semicontinuity, regularity.
- Deriving the Euler-Lagrange equation for a weak solution to a variational problem, focussing in particular on the assumptions needed and on the interpretation of all derivatives which appear in the equation.

- Being familiar with the properties of the Lagrangian (in particular convexity and growth conditions) which yield compactness of sublevels and/or lower semicontinuity of the functional (specifying always carefully the notion of convergence in use).
- Proving regularity of a weak solution to a variational problem, keeping in mind that usually this requires both an *initial step* and a *bootstrap argument*.
- Having clear the variational approach to existence/uniqueness/regularity results for some differential equations (this requires interpreting the differential problem as the Euler-Lagrange equation of a suitable minimization problem).

2.3 Relaxation

- Knowing the characterization of lower semicontinuity in terms of sequences, sublevels, epigraphs.
- Road map for proving that $G(x)$ is the relaxation of $F(x)$: lower semicontinuity of $G(x)$, inequality $G(x) \leq F(x)$, existence of a recovery sequence for every x in a subset which is dense in energy.
- Exploiting the stability of the relaxation under continuous perturbations in order to reduce the computation of a relaxation to the relevant terms.
- Exploiting the relaxed functional in order to determine the infimum when a variational problem does not admit the minimum.
- Being aware that relaxation is the fundamental tool in order to extend a functional to a wider setting.
- Computing the relaxation of an integral functional, always keeping into account the pathologies which can originate from a lack of suitable growth conditions.

2.4 Gamma convergence

- Computing a Gamma limit through the liminf and the limsup inequality.
- Exploiting a subset which is dense in energy in order to simplify the verification of the limsup inequality when computing a Gamma limit.
- Exploiting the stability of Gamma convergence under continuous or lower semicontinuous (when true) perturbations in order to reduce the computation of a Gamma limit to the relevant terms.
- Exploiting Gamma convergence in order to describe the asymptotic behavior of minima and minimizers for variational problems depending on parameters.
- Computing the Gamma limit of sequences of integral functionals in terms of the limit of the Lagrangians.

- Computing the Gamma limit of integral functionals in the discrete-to-continuum framework.

- Guessing the right rescaling of a sequence of functionals in order to obtain a nontrivial Gamma limit.

2.5 Classical examples

- Geodesics in Euclidean spaces or surfaces: writing functionals and Euler-Lagrange equations, finding extremals and calibrations.

- Approximating obstacle problems through Gamma convergence.

- Applying the direct method to obstacle problems, taking care of the stability of the constraints when passing to the limit.

- Finding contact points in obstacle problems by exploiting transversality conditions.

- Finding the asymptotic behavior of minima and minimizers for variational problems with small parameters that produce linearization effects.

- Discrete-to-continuum limit: approximating variational problems in a continuum setting through variational problems defined just for piecewise affine or piecewise constant functions.

- Computing Euler-Lagrange equations for variational problems in the discrete setting.

- Computing the Gamma limit of functionals with homogenized coefficients, having clear the role of the cell problem in the case of a periodic coefficient in front of the derivative.

- Recognizing the Modica-Mortola pattern when it appears in a sequence of variational problems, and keeping in mind the basic AM–GM trick.

Chapter 3

Exam papers

[Spiegare il significato ovvio di questo capitolo]

[In italian for the time being. To be hopefully translated in a near future.]

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 25 Dicembre 2015

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1. Consideriamo il problema

$$\min\left\{\int_0^1 (\ddot{u}^2 + u^2 - x^2 u)\,dx : u(0) = 0\right\}.$$

   (a) Scrivere l'equazione di Eulero e le condizioni al bordo associate al problema.

   (b) Dimostrare che l'equazione di Eulero ha soluzione unica.

   (c) Stabilire se il problema di minimo ha soluzione.

2. Discutere esistenza, unicità e regolarità per il problema

$$\ddot{u} = \frac{xu^3}{1+\dot{u}^2}, \qquad u(0) = 0, \quad u(1) = 2016.$$

3. Consideriamo, per ogni numero reale $\ell > 0$, il problema

$$\min\left\{\int_0^\ell (\dot{u}^2 - u^2 + 7xu)\,dx : u(0) = 0,\ u(\ell) = 2016\right\}.$$

   (a) Scrivere l'equazione di Eulero associata al problema.

   (b) Studiare, al variare del parametro ℓ, l'unicità della soluzione dell'equazione di Eulero.

   (c) Stabilire per quali valori di ℓ il problema di minimo ha soluzione.

4. Per ogni $\varepsilon > 0$ definiamo

$$I_\varepsilon = \inf\left\{\int_0^1 \left[\varepsilon\dot{u}^4 - \dot{u}^2 - u^2\right]\,dx : u(0) = u(1) = 0\right\}.$$

   (a) Dimostrare che $I_\varepsilon \in (-\infty, 0)$ per ogni $\varepsilon > 0$.

   (b) Dimostrare che $I_\varepsilon \to -\infty$ per $\varepsilon \to 0^+$.

   (c) Calcolare l'ordine di infinito di I_ε per $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 12 Gennaio 2016

1. Studiare il problema di minimo

$$\min\left\{\int_0^1 (\dot{u}^2 + u^2 - x^2 u)\,dx : u(0) = 0\right\}.$$

2. Discutere esistenza, unicità e regolarità per il problema

$$\ddot{u} = \frac{u^5}{1+x^3} + x^5, \qquad u(0) = 2016, \quad u'(2016) = 0.$$

3. Consideriamo, per ogni numero reale $\ell > 0$, il problema

$$\min\left\{\int_0^\ell (\dot{u}^2 - 7\arctan^2 u)\,dx : u(0) = u(\ell) = 0\right\}.$$

   (a) Stabilire per quali valori di ℓ il problema ha soluzione.

   (b) Stabilire per quali valori di ℓ il minimo (esiste ed) è negativo.

4. (a) Determinare per quali valori del parametro reale λ il problema

$$\min\left\{\int_0^1 (\dot{u}^4 - 2\dot{u}^2)\,dx : u(0) = 0,\ u(1) = \lambda\right\}$$

   ammette soluzione.

   (b) Determinare per quali valori reali di λ la funzione $u_0(x) = \lambda x$ è un punto di minimo locale (WLM) per il problema precedente.

   (c) Calcolare

$$\inf\left\{\int_0^3 \left[\dot{u}^4 - 2\dot{u}^2 + (u + e^{-x})^2\right]\,dx : u \in C^1([0,3])\right\}.$$

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 02 Febbraio 2016

1. Consideriamo il funzionale

$$F(u) = \int_0^2 \left[\dot{u}^2 + (u - x^2)^2\right] dx.$$

   (a) Studiare il problema di minimo per $F(u)$ con le condizioni al bordo $u(0) = u(2)$.

   (b) Studiare il problema di minimo per $F(u)$ con le condizioni al bordo $u'(0) = u'(2)$.

2. Discutere esistenza, unicità e regolarità per il problema

$$\ddot{u} = 2^{x+u}, \qquad u(2) = 2, \quad u(4) = 4.$$

3. Consideriamo, per ogni numero reale $\ell > 0$, il funzionale

$$F(u) = \int_0^\ell \left[\dot{u}^2 - \sinh(u^2)\right] dx.$$

   Stabilire per quali valori di ℓ valgono le seguenti proprietà.

   (a) La funzione $u_0(x) \equiv 0$ è un minimo locale forte (SLM), ovviamente tra quelle che hanno lo stesso dato al bordo.

   (b) Esiste il minimo di $F(u)$ con le condizioni al bordo $u(0) = u(\ell) = 0$.

   (c) Esiste il minimo di $F(u)$ con le condizioni al bordo $u(0) = u(\ell) = 2016$.

4. (a) Dimostrare che per ogni $\varepsilon > 0$ esiste

$$m_\varepsilon = \min\left\{\int_0^1 \left[\sinh(\dot{u}^2) + \sin^4 u\right] dx : u(0) = 0,\ u(1) = \varepsilon\right\}.$$

   (b) Determinare la parte principale di m_ε per $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 23 Febbraio 2016

1. Studiare il problema di minimo

$$\min\left\{\int_0^1 (\dot{u}^2 - 7\dot{u} + x^3 u)\,dx : u(0) = 0\right\}.$$

2. Discutere esistenza, unicità e regolarità per il problema

$$\ddot{u} = \frac{u^3}{\cos x}, \qquad u(0) = u(1) = 2016.$$

3. Consideriamo, per ogni numero reale $\ell > 0$, il problema di minimo

$$\min\left\{\int_0^\ell \left(\dot{u}^2 - \sin^2 u\right)\,dx : u(0) = u(\ell) = 0\right\}.$$

   (a) Determinare per quali valori di ℓ il problema di minimo ha soluzione.

   (b) Determinare per quali valori di ℓ il valore del minimo (esiste ed) è negativo.

   (c) Stabilire se esistono valori di ℓ per cui il valore del minimo è esattamente -2016.

4. (a) Dimostrare che per ogni intero positivo n esiste

$$M_n = \min\left\{\int_0^1 \left(\dot{u}^2 + u\sin u\right)\,dx : u(0) = 2016,\ \int_0^1 u^2\,dx = n\right\}.$$

   (b) Calcolare il limite di M_n per $n \to +\infty$.

   (c) Calcolare, al variare del parametro a, il

$$\lim_{n\to+\infty} \frac{M_n}{n^a}.$$

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 28 Giugno 2016

1. Consideriamo il funzionale
$$F(u) = \int_0^1 (\ddot{u}^2 - 7x\dot{u} - 10u)\, dx.$$
Studiare il problema di minimo per $F(u)$ con ciascuna delle seguenti condizioni al bordo:
   (a) $u(0) = u(1) = 0$,
   (b) $u'(0) = u'(1) = 5$.

2. Discutere esistenza, unicità e regolarità per il problema
$$\ddot{u} = u^5 - x^{55}, \qquad u'(0) = 555, \quad u(1) = 0.$$

3. Consideriamo, per ogni numero reale $\ell > 0$, il problema di minimo
$$\min \left\{ \int_0^\ell \left(\dot{u}^2 - u \sin u + u^6\right) dx : u(0) = u(\ell) = 0 \right\}.$$
   (a) Determinare per quali valori di ℓ il problema di minimo ha soluzione.
   (b) Determinare per quali valori di ℓ il valore del minimo (esiste ed) è negativo.

4. (a) Dimostrare che per ogni intero $n \geq 2$ esiste
$$M_n = \min \left\{ \int_0^1 \left(|\dot{u}|^n + (u - x)^{22} + e^{-nu}\right) dx : u(0) = u(1) = 0 \right\}.$$
   (b) Calcolare il limite di M_n per $n \to +\infty$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 22 Luglio 2016

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1. Consideriamo il funzionale
$$F(u) = \int_0^1 (\dot{u} - u)^2 \, dx.$$
Studiare il problema di minimo per $F(u)$ con ciascuna delle seguenti condizioni al bordo:
   (a) $u(1) = 3$,
   (b) $u(0) = u(1) = 3$.

2. Discutere esistenza, unicità e regolarità per il problema
$$\ddot{u} = \frac{x}{x+1} \cdot \frac{u}{u+1}, \qquad u(0) = u(2016) = 3.$$

3. Consideriamo, per ogni numero reale k, il problema di minimo
$$\inf \left\{ \int_0^3 \left(\dot{u}^2 - k \sin(u^2) + u^6 \right) \, dx : u(0) = u(3) = 0 \right\}.$$
   (a) Determinare per quali valori di $k \in \mathbb{R}$ l'estremo inferiore è in realtà un minimo.
   (b) Determinare per quali valori $k \geq 0$ l'estremo inferiore è negativo.
   (c) Determinare se esistono valori $k < 0$ per cui l'estremo inferiore è negativo.

4. (a) Dimostrare che per ogni $\varepsilon > 0$ esiste
$$I_\varepsilon = \min \left\{ \int_0^1 \left(\dot{u}^2 - u^4 + \varepsilon u^6 \right) \, dx : \int_0^1 u^2 \, dx = \varepsilon \right\}.$$
   (b) Calcolare l'ordine di infinitesimo e la parte principale di I_ε per $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 06 Settembre 2016

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1. Consideriamo il funzionale

$$F(u) = \int_0^1 \left[(\dot{u} - x)^2 + (u - x)^2 \right] dx.$$

   Studiare il problema di minimo per $F(u)$ con ciascuna delle seguenti condizioni al bordo:

   (a) nessuna condizione,

   (b) $u'(1) = 0$.

2. Discutere esistenza, unicità e regolarità per il problema

$$\ddot{u} = \frac{u(u^2 + 1)}{\dot{u}^2 + 1}, \qquad u'(0) = u'(1) = 1.$$

3. Consideriamo, per ogni numero reale k, il problema di minimo

$$\min \left\{ \int_0^3 \left(\dot{u}^4 - k \sin(u^2) + u^6 \right) dx : u(0) = u(3) = 0 \right\}.$$

   (a) Determinare per quali valori di $k \in \mathbb{R}$ il problema di minimo ha soluzione.

   (b) Determinare per quali valori $k \geq 0$ il valore del minimo (esiste ed) è negativo.

4. Consideriamo, per ogni numero reale $\ell > 0$, il problema di minimo

$$\min \left\{ \int_0^\ell \left[\sin(\dot{u}^2) + \cos u \right] dx : u(0) = u(\ell) = 0 \right\}.$$

   (a) Determinare per quali $\ell > 0$ la funzione $u_0(x) \equiv 0$ è un punto di minimo locale debole (WLM).

   (b) Determinare per quali $\ell > 0$ la funzione $u_0(x) \equiv 0$ è un punto di minimo locale forte (SLM).

   (c) Determinare per quali $\ell > 0$ il problema di minimo ha soluzione.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 10 January 2017

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1. Let us consider the functional
$$F(u) = \int_0^{\pi} \left(\dot{u}^2 - u\cos x\right)\,dx.$$
   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi) = u(0)$.
   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi) = 2u(0)$.

2. Let us consider the boundary value problem
$$\ddot{u} = \frac{\sinh u - 3}{3 + \cosh \dot{u}}, \qquad u(0) = u(2017) = 3.$$
   (a) Discuss existence, uniqueness, regularity of the solution.
   (b) Prove that the solution satisfies $1 < u(x) \le 3$ for every $x \in [0, 2017]$.

3. Let us consider, for every $\ell > 0$, the problem
$$\inf\left\{\int_0^{\ell} \left(\dot{u}^2 + \arctan u - u^2\right)\,dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$
   (a) Determine for which values of ℓ the infimum is negative.
   (b) Determine for which values of ℓ the infimum is actually a minimum.

4. Let us set
$$m_{\varepsilon} = \min\left\{\int_0^1 \left(\varepsilon\ddot{u}^2 + \cos\dot{u} + \cos u\right)\,dx : u \in C^2([0,1]),\ u(0) = u'(0) = 1\right\}.$$
   (a) Prove that m_{ε} is well-defined (namely the minimum exists) for every positive ε.
   (b) Compute the limit of m_{ε} as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 24 February 2017

1. Discuss the minimum problem
$$\min\left\{\int_0^\ell \left(\dot{u}^2 + u\sin x\right)\,dx : u(0) = u(\ell)\right\}$$
depending on the parameter $\ell > 0$.

2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem
$$\ddot{u} = \frac{u-3}{3+\cos\dot{u}}, \qquad u(0) = 2017, \quad u'(2017) = 0.$$

3. Let us consider, for every $\ell > 0$, the problem
$$\inf\left\{\int_0^\ell \left(\sin(\dot{u}^2) - \sinh(u^2)\right)\,dx : u(0) = u(\ell) = 0\right\}.$$
   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.
   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.
   (c) Compute the infimum as a function of ℓ.

4. Let us set
$$m_\varepsilon = \inf\left\{\int_0^1 \left(\varepsilon\dot{u}^4 - \dot{u}^2 + u^2\right)\,dx : u(0) = u(1) = 2017\right\},$$
where ε is a positive real parameter.
   (a) Determine for which values of ε it turns out that $m_\varepsilon \in \mathbb{R}$.
   (b) Determine for which values of ε the infimum is actually a minimum.
   (c) Determine the leading term of m_ε as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 05 June 2017

1. Discuss the minimum problem
$$\min \left\{ \int_0^1 \left[(\dot{u}+x)^2 + (u+x)^2 \right] dx : u'(1) = \alpha \right\}$$
depending on the real parameter α.

2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem
$$\ddot{u} = \frac{u^3}{u^2+1} \cdot |\cos x|, \qquad u'(0) = 1, \quad u(1) = 0.$$

3. Let us consider, for every $\ell > 0$, the problem
$$\min \left\{ \int_0^\ell \left[\sinh(\dot{u}^2) - \sin(u^2) + u^4 \right] dx : u(0) = u(\ell) = 0 \right\}.$$
   (a) Determine for which values of ℓ the minimum exists.
   (b) Determine for which values of ℓ the minimum (exists and) is negative.
   (c) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

4. Let us set
$$m_\varepsilon = \inf \left\{ \int_0^1 \left(\varepsilon \dot{u}^2 + \arctan \dot{u} + \arctan u \right) dx : u(0) = 3, \ u(1) = 2017 \right\},$$
where ε is a positive real parameter.
   (a) Determine if there exist values of ε for which the infimum is actually a minimum.
   (b) Determine if there exist values of ε for which the infimum is not a minimum.
   (c) Compute the limit of m_ε as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 26 June 2017

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1. Let us consider the functional

$$F(u) = \int_0^{\pi} \left[\ddot{u}^2 + \dot{u}^2 + \sin x \cdot u \right] \, dx.$$

Discuss the minimum problem for $F(u)$ subject to each of the following boundary conditions:

(a) $u(0) = \pi$,

(b) $u'(0) = 2017$.

2. Let us consider the boundary value problem

$$\ddot{u} = \arctan x \cdot \arctan u, \qquad u(0) = \pi, \quad u(\pi) = 0.$$

(a) Discuss existence, uniqueness, and regularity of the solution.

(b) Discuss the monotonicity of the solution.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^{\ell} \left[\arctan(\dot{u}^2) + \sin(u^2) \right] \, dx : u(0) = u(\ell) = 0 \right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine the infimum as a function of ℓ.

4. For every positive integer n, let us set

$$M_n = \inf \left\{ \int_0^1 \left(2017\dot{u}^2 + nu^6 - n^2 u^2 \right) \, dx : u(0) = u(1) = 0 \right\}.$$

(a) Determine for which values of n the infimum is actually a minimum.

(b) Determine for which values of n the infimum is negative.

(c) Determine the leading term of M_n as $n \to +\infty$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 27 July 2017

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1. Discuss the minimum problem

$$\min\left\{\int_0^1 \left(\dot{u}-x^2\right)^2 dx : u(0)=u(1)\right\}.$$

2. Discuss existence, uniqueness, and regularity of the solution to the boundary value problem

$$\ddot{u}=\frac{u^3-x^3}{1+\dot{u}^2}, \qquad u(0)=1, \quad \dot{u}(1)=3.$$

3. Let us consider, for every $\ell>0$, the problem

$$\inf\left\{\int_0^\ell \left(-\cos(\dot{u})+\cos(u)\right) dx : u(0)=u(\ell)=0\right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a strong local minimum.

   (c) Determine the infimum as a function of ℓ.

4. Let us consider, for every $\varepsilon>0$, the problem

$$m_\varepsilon=\min\left\{\int_0^1 \left(\sinh(\dot{u}^2)+u^6\right) dx : u(0)=u(1)=0,\ \int_0^1 u^4\,dx=\varepsilon\right\}.$$

   (a) Prove that the minimum exists for every $\varepsilon>0$.

   (b) Determine all real numbers α for which

$$\lim_{\varepsilon\to 0^+}\frac{m_\varepsilon}{\varepsilon^\alpha}=0.$$

Università di Pisa - Corso di Laurea in Matematica

Exam paper of “Elementi di Calcolo delle Variazioni”

Pisa, 23 September 2017

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1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\dot{u}^2 + u^2 - x^2\dot{u}\right)\, dx.$$

Discuss the minimum problem for $F(u)$ subject to each of the following boundary conditions:

(a) $u(1) = 2u(-1)$,

(b) $u(0) = 0$.

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{u + u^3}{1 + \dot{u}^4}, \qquad u(0) = 1, \quad u(1) = \lambda.$$

(a) Discuss existence, uniqueness, and regularity of the solution.

(b) Determine the values of λ for which the solution is convex.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^{\ell} \left[(1 + u^2)\dot{u}^2 - 10\sin^2(u) + \cos x \cdot u^6\right]\, dx : u(0) = u(\ell) = 0 \right\}.$$

(a) Determine for which values of ℓ the infimum is actually a minimum.

(b) Determine for which values of ℓ the minimum exists and is negative.

4. Let us consider, for every $\varepsilon > 0$, the problem

$$m_\varepsilon = \inf \left\{ \int_0^2 \left(\varepsilon\dot{u}^4 - \dot{u}^2 + \varepsilon^2 u^4\right)\, dx : u(0) = u(2) = 2 \right\}.$$

(a) Determine for which values $\varepsilon > 0$ the infimum is a real number.

(b) Determine for which values $\varepsilon > 0$ the infimum is actually a minimum.

(c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 22 January 2018

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1. Let us consider the functional

$$F(u)=\int_0^{\pi}\left(\dot{u}^2-u\cos x\right)\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi)=u(0)$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(\pi)=2u(0)$.

2. Let us consider the boundary value problem

$$\ddot{u}=\frac{\sinh u-3}{3+\cosh\dot{u}},\qquad u(0)=u(2017)=3.$$

   (a) Discuss existence, uniqueness, regularity of the solution.

   (b) Prove that the solution satisfies $1<u(x)\leq 3$ for every $x\in[0,2017]$.

3. Let us consider, for every $\ell>0$, the problem

$$\inf\left\{\int_0^{\ell}\left(\dot{u}^2+\arctan u-u^2\right)\,dx : u\in C^1([0,\ell]),\ u(0)=u(\ell)=0\right\}.$$

   (a) Determine for which values of ℓ the infimum is negative (possibly $-\infty$).

   (b) Determine for which values of ℓ the infimum is actually a minimum.

4. Let us set

$$m_{\varepsilon}=\min\left\{\int_0^{1}\left(\varepsilon\ddot{u}^2+\cos\dot{u}+\cos u\right)\,dx : u\in C^2([0,1]),\ u(0)=u'(0)=1\right\}.$$

   (a) Prove that m_{ε} is well-defined (namely the minimum exists) for every positive ε.

   (b) Compute the limit of m_{ε} as $\varepsilon\to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 19 February 2018

1. Let us consider the functional

$$F(u) = \int_0^1 \left\{ \left(\dot{u} - x^2 \right)^2 + u \right\} dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = u(0) + 3$.

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{7+u}{7+\sin \dot{u}}, \qquad \dot{u}(0) = 7, \quad u(7) = 0.$$

   Discuss existence, uniqueness, regularity of the solution.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left(\arctan(\dot{u}^2) - \sin(u^2) \right) dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Compute the infimum as a function of ℓ.

4. Let us set

$$m_\varepsilon = \min \left\{ \int_0^1 \left(\dot{u}^2 + \dot{u}^6 + \varepsilon \sin u \right) dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = \varepsilon \right\}.$$

   (a) Prove that m_ε is well-defined (namely the minimum exists) for every positive ε.

   (b) Determine for which positive values of ε all minimum points are convex.

   (c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 08 June 2018

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1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\dot{u}^2 + xu + u^2\right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_{-1}^{1} u(x)\,dx = 7$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u(0) = 7$.

2. Let us consider, for any value of the real parameter a, the boundary value problem

$$\ddot{u} = (x+7)(u+7), \qquad u(-7) = u(7) = a.$$

   (a) Discuss existence, uniqueness and regularity of the solution.

   (b) Determine the values of a for which the solution is convex.

3. Let us set, for every $\ell > 0$,

$$I(\ell) := \inf\left\{\int_0^{\ell} \left(\cos x \cdot \dot{u}^2 + x^2 \cdot \cos u\right) dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

   (a) Determine the value of $I(2)$.

   (b) Determine whether $I(3/2)$ is actually a minimum.

   (c) Determine the value of $I(1)$.

4. Let us set

$$m_\varepsilon := \inf\left\{\int_0^1 \left(\varepsilon\dot{u}^6 - \dot{u}^2 + \sin u\right) dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = 0\right\}.$$

   (a) Determine for which positive values of ε it turns out that m_ε is finite.

   (b) Determine for which positive values of ε it turns out that m_ε is actually a minimum.

   (c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 29 June 2018

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1. Let us consider the two functionals

$$F_1(u) = \int_0^2 \left\{ (\dot{u} - 2x)^2 + (u - x^2)^2 \right\} dx, \qquad F_2(u) = \int_0^2 \left\{ (\dot{u} - 2x) + (u - x^2)^2 \right\} dx.$$

   (a) Discuss the minimum problem for $F_1(u)$ subject to the boundary condition $u(0) = u(2)$.

   (b) Discuss the minimum problem for $F_2(u)$ subject to the boundary condition $u(0) = u(2)$.

2. Let us consider the boundary value problem

$$\ddot{u} = \frac{x^7 u^9}{\dot{u}^8 + 10}, \qquad u'(0) = 11, \quad u(11) = 0.$$

   Discuss existence, uniqueness and regularity of the solution.

3. Let us consider, for every $\lambda > 0$, the problem

$$m_\lambda := \inf \left\{ \int_0^4 \left(\dot{u}^2 - \lambda u \arctan u + \sin^4 u \right) dx : u \in C^1([0,4]), \ u(0) = u(4) = 0 \right\}.$$

   (a) Determine for which values of λ the infimum is actually a minimum.

   (b) Determine for which values of λ the infimum is negative.

   (c) Determine the leading term of m_λ as $\lambda \to +\infty$.

4. Let us consider, for every value of the real parameter a, the minimum problem

$$\min \left\{ \int_0^1 u'(x) \cdot e^{u'(x)} \, dx : u \in C^1([0,1]), \ u(0) = 0, \ u(1) = a \right\}.$$

   (a) Determine for which values of a the function $u(x) = ax$ is a weak local minimum.

   (b) Determine for which values of a the function $u(x) = ax$ is a strong local minimum.

   (c) Determine for which values of a the minimum exists.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 20 July 2018

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1. Let us consider the functional

$$F(u) = \int_0^3 \left(\ddot{u}^2 + 3u\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u'(0) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$\ddot{u} = \sinh(x + u), \qquad u'(0) = 0, \quad u'(1) = 7.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left\{\tanh(\dot{u}^2) + \arctan(u^3 - u^2)\right\} dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Compute the infimum as a function of ℓ.

4. For every real number $\ell > 0$, let us set

$$I(\ell) := \inf \left\{ \int_0^\ell \left(\dot{u}^6 - \dot{u}^2 + u^2 - u^4\right) dx : u \in C^1([0, \ell]),\ u(0) = 0,\ u(\ell) = 0 \right\}.$$

   (a) Determine for which positive values of ℓ it turns out that $I(\ell)$ is negative.

   (b) Determine for which positive values of ℓ it turns out that $I(\ell)$ is finite.

   (c) Compute the leading term of $I(\ell)$ as $\ell \to +\infty$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 18 September 2018

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1. Let us consider the functional

$$F(u) = \int_0^2 \left(\dot{u}^2 + (u - x^2)^2\right)\,dx.$$

   Discuss the minimum problem for $F(u)$ with each of the following sets of boundary conditions:

   (a) $u(0) = 0$ and $u(2) = 4$.

   (b) $u(0) = 0$ and $u'(2) = 4$.

2. Discuss existence, uniqueness and regularity of solutions to equation

$$\left(\dot{u}^2 + e^{\dot{u}}\right) \cdot \ddot{u} = x^2 + e^u$$

   with each of the following sets of boundary conditions:

   (a) $u(-1) = u(1) = 0$,

   (b) $u'(-1) = u'(1) = 0$.

3. Let us consider, for every $\ell > 0$ and $\mu \in \mathbb{R}$, the problem

$$\min\left\{\int_0^\ell \left(\dot{u}^2 + \sin u - u^2\right)\,dx : u \in C^1([0,\ell]),\ u(0) = 0,\ u(\ell) = \mu\right\}.$$

   (a) Discuss the existence of the minimum in the case $\ell = \mu = 2018$.

   (b) Discuss the existence of the minimum in the case $\ell = 3$ and $\mu = 0$.

   (c) Discuss the existence of the minimum in the case $\ell = 3$ and $\mu = 2018$.

4. Let us set

$$m_\varepsilon := \inf\left\{\int_0^1 \left(\frac{\dot{u}^6}{\varepsilon^2} - \frac{\dot{u}^2}{\varepsilon^6} + \arctan u\right)\,dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = 0\right\}.$$

   (a) Determine for which positive values of ε it turns out that m_ε is finite.

   (b) Determine for which positive values of ε it turns out that m_ε is actually a minimum.

   (c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 15 January 2019

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1. Let us consider the functional

$$F(u)=\int_0^{\pi}\left[(\dot{u}-x^2)^2+\sin x\cdot u\right]\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(0)=0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u'(\pi)=\pi$.

2. Discuss existence, uniqueness, regularity of the solution to the boundary value problem

$$\ddot{u}=\frac{1+u^3+x^2}{1+\dot{u}^2}, \qquad u(0)=u'(3)=3.$$

3. Let us consider, for every $\ell>0$, the problem

$$\inf\left\{\int_0^{\ell}\left(\sqrt{1+\dot{u}^2}-\sqrt{1+u^4}\right)\,dx : u\in C^1([0,\ell]),\ u(0)=u(\ell)=0\right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a strong local minimum.

   (c) Determine for which values of ℓ the infimum is a real number.

4. Let us consider the functional

$$F(u)=\int_0^1\{\sin\dot{u}+\sin(u-\sin x)\}\,dx.$$

   (a) Compute the infimum of $F(u)$ in the class $C_c^{\infty}((0,1))$.

   (b) Is is true that any minimizing sequence for the previous point converges in some sense to a continuous function?

Università di Pisa - Corso di Laurea in Matematica

Exam paper of “Elementi di Calcolo delle Variazioni”

Pisa, 02 February 2019

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1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 - 3u\dot{u} + xu \right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u(1) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

2. Let us consider ordinary differential equation

$$\ddot{u} = \frac{(7 + \sin x)u}{7 + \sin \dot{u}}.$$

   (a) Prove that the equation admits a 2π-periodic solution.

   (b) Prove that every 4π-periodic solution is actually 2π-periodic.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left(\sqrt{1 + \dot{u}^4} - \sqrt{1 + u^2} \right) dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (b) Determine for which values of ℓ the infimum is a real number.

4. Let us set

$$m_\varepsilon = \inf \left\{ \int_0^1 \left(\varepsilon \dot{u}^2 + \dot{u}^4 - \sin(u^2) + u^4 \right) dx : u \in C^1([0, 1]),\ u(0) = 0,\ u(1) = \varepsilon \right\},$$

   and

$$M_\varepsilon = \inf \left\{ \int_0^1 \left(\varepsilon \dot{u}^2 + \dot{u}^4 - \sin(u^{22}) + u^4 \right) dx : u \in C^1([0, 1]),\ u(0) = 0,\ u(1) = \varepsilon \right\}.$$

   For every real number $\alpha > 0$, compute the following limits:

$$\lim_{\varepsilon \to 0^+} \frac{m_\varepsilon}{\varepsilon^\alpha}, \qquad \lim_{\varepsilon \to 0^+} \frac{M_\varepsilon}{\varepsilon^\alpha}.$$

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 23 February 2019

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1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\ddot{u}^2 + \dot{u}^2\right) dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) = u'(0) = 1$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Let us consider the boundary value problem

$$u''(x) = \frac{1 + e^{u(x)}}{1 + e^{u'(x)}}, \qquad u(0) = 3, \quad u(3) = 0.$$

   (a) Discuss existence, uniqueness and regularity of the solution.

   (b) Prove that $u'(0) < -1$.

3. Let us set, for every $\ell > 0$,

$$I(\ell) := \inf\left\{\int_0^{\ell} \left(\dot{u}^2 - x\sin^2 u\right) dx : u \in C_c^\infty((0,\ell))\right\}.$$

   (a) Determine whether there exist positive values of ℓ such that $I(\ell) = 0$.

   (b) Determine whether there exist positive values of ℓ such that $I(\ell) < 0$.

4. Let us set, for every $\varepsilon > 0$,

$$m_\varepsilon := \inf\left\{\int_0^{2\pi} \left(\varepsilon\ddot{u}^2 + \sin\dot{u} + \cos u\right) dx : u \in C^2([0,1]),\ u(0) = 0,\ u(2\pi) = 2\pi\right\}.$$

   (a) Determine for which positive values of ε it turns out that m_ε is a minimum.

   (b) Compute the limit of m_ε as $\varepsilon \to 0^+$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 11 June 2019

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1. Let us consider the functional

$$F(u) = \int_0^{\pi} \left(\dot{u}^2 - u \sin x\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_0^{\pi} u(x)\, dx = 0$.

   (b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u'' = u^7 - x^7, \qquad u(0) = 7, \quad u'(7) = 7.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^{\ell} \left\{\dot{u}^2 + \dot{u}^5 - \sin(u^2) + u^5\right\} dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Determine for which values of ℓ the infimum is actually a minimum.

4. Let us set, for every $\varepsilon > 0$,

$$m_{\varepsilon} := \inf \left\{ \int_0^{4} \left(\dot{u}^2 - u \sin(u^2)\right) dx : u \in C^1([0, 4]),\ u(0) = 0,\ u(4) = \varepsilon \right\}.$$

   (a) Determine for which values of ε the infimum is actually a minimum.

   (b) Compute the leading term of m_{ε} as $\varepsilon \to 0^+$.

   (c) Compute the leading term of m_{ε} as $\varepsilon \to +\infty$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 05 July 2019

1. Let us consider the functional
$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u} + x^3 u\right) dx.$$
   (a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) + u(1) = 3$.
   (b) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) - u(1) = 3$.

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem
$$u'' = u^3 + \sin u + |x - 1|^{3/2}, \qquad u(0) = u(2) = 5.$$

3. Let us set, for every $\lambda > 0$,
$$I(\lambda) := \inf \left\{ \int_0^4 \left(\dot{u}^4 + \dot{u}^2 - \lambda \sin(\dot{u}^2) + x u^2\right) dx : u \in C^1([0,4]),\ u(0) = u(4) = 0 \right\}.$$
   (a) Determine for which values of λ it turns out that $I(\lambda)$ is a real number.
   (b) Determine for which values of λ it turns out that $I(\lambda)$ is actually a minimum.
   (c) Determine the limit of $I(\lambda)$ as $\lambda \to +\infty$.

4. For every real number $m > 0$, let us set
$$J(m) := \inf \left\{ \int_0^1 \left(u^{19} + \arctan(u^2)\right) dx : u \in C^1([0,1]),\ u(0) = 0,\ \int_0^1 |\dot{u}|^7 \, dx \leq m \right\}.$$
   (a) Determine for which values of m it turns out that $J(m)$ is a real number.
   (b) Determine whether there exists $m > 0$ such that $J(m) = 0$.
   (c) Determine for which real values of α it turns out that
$$\lim_{m \to +\infty} \frac{J(m)}{m^\alpha} = 0.$$

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 03 September 2019

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1. Determine whether the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u}u + u^2 + u\right)\, dx$$

has the minimum in the class $C^1([0,1])$.

2. Let us consider the boundary value problem

$$u'' = -\frac{x^3}{u^3}, \qquad u(0) = 1, \quad u(2) = 3.$$

(a) Discuss existence, uniqueness and regularity of positive solutions.

(b) Determine the minimum of the solution in the interval $[0,2]$.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \arctan\left(\dot{u}^2 - u^2\right)\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

(c) Determine the infimum as a function of ℓ.

4. For every real number $m > 0$, let us set

$$J(m) := \inf\left\{\int_0^1 \left(u^{19} - \sin(u^2)\right)\, dx : u \in C_c^1((0,1)),\ \int_0^1 |\dot{u}|^7\, dx \le m\right\}.$$

(a) Determine whether there exists $m > 0$ such that $J(m) = 0$.

(b) Determine for which real values of α it turns out that

$$\lim_{m\to 0^+} \frac{J(m)}{m^\alpha} = 0.$$

(c) Determine for which real values of β it turns out that

$$\lim_{m\to +\infty} \frac{J(m)}{m^\beta} = 0.$$

Università di Pisa - Corso di Laurea in Matematica

Exam paper of “Elementi di Calcolo delle Variazioni”

Pisa, 11 January 2020

1. Determine for which values of the real parameter a the problem
$$\min \left\{ \int_{-\pi}^{\pi} \left\{ (\dot{u} - \cos x)^2 + (u - \sin x)^2 \right\} dx : u \in C^1([-\pi, \pi]),\ u(0) = a \right\}$$
admits a solution (note that the condition is given in the midpoint of the interval).

2. Discuss existence, uniqueness, and regularity of functions $u : \mathbb{R} \to \mathbb{R}$ that are *periodic* and satisfy
$$u'' = u^3 + \sin^2 x \qquad \forall x \in \mathbb{R}.$$

3. Let us consider, for every $\ell > 0$, the problem
$$\inf \left\{ \int_0^\ell \left[\sin(\dot{u}^2) + \cos(u) - \arctan(u^4) \right] dx : u \in C^1([0, \ell]),\ u(0) = u(\ell) = 0 \right\}.$$
   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.
   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.
   (c) Determine the infimum as a function of ℓ.

4. For every real number $\ell > 0$, and every real number α, let us set
$$I(\alpha, \ell) := \inf \left\{ \int_0^\ell \left(\dot{u}^2 - u^2 \right) dx : u \in C^1([0, \ell]),\ \int_0^\ell u(x)\, dx = \alpha \right\}.$$
   (a) Determine whether there exists $\ell > 0$ such that $I(0, \ell) = 0$.
   (b) Determine whether there exists $\ell > 0$ such that $I(0, \ell) = -\infty$.
   (c) Determine whether there exists $\ell > 0$ such that $I(2020, \ell) = -\infty$.

Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 21 February 2020

1. Let us consider the functionals

$$F(u) = u(0) + \int_0^1 \left(\dot{u}^2 + u\right)\, dx, \qquad G(u) = [u(0)]^3 + \int_0^1 \left(\dot{u}^2 + u\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = 3$.

   (b) Discuss the minimum problem for $G(u)$ with boundary condition $u(1) = 3$.

2. Discuss existence, uniqueness and regularity of solutions to the boundary value problem

$$u'' = -1 + \sqrt{u}, \qquad u(0) = 1/2, \quad \dot{u}(2020) = 1.$$

3. Let us consider, for every $\ell > 0$, the problem

$$\inf \left\{ \int_0^\ell \left[\sin(\dot{u}^2) - \cos(u) - \arctan(u^4)\right] dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0 \right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a weak local minimum.

   (b) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (c) Determine for which values of ℓ the infimum is actually a minimum.

4. For every real number $\ell > 0$, let us set

$$m(\ell) := \min \left\{ \int_0^\ell \left(\dot{u}^4 - u^2\right) dx : u \in C^1([0,\ell]),\ \int_0^\ell u(x)\, dx = 2020 \right\}.$$

   (a) Prove that $m(\ell)$ is well-defined and negative for every $\ell > 0$.

   (b) Prove that $m(\ell) \to -\infty$ as $\ell \to +\infty$.

   (c) Determine all real numbers α such that

$$\lim_{\ell \to +\infty} \frac{m(\ell)}{\ell^\alpha} \in (-\infty, 0).$$

