Basi ortogonali e ortonormali 2

Argomenti: Basi ortogonali e ortonormali **Difficoltà:** $\star\star\star\star$

Prerequisiti: ortogonalizzazione di Gram-Schmidt, cambi di base

Nel seguente esercizio viene assegnato un sottospazio vettoriale W di un certo $\mathbb{R}^n$. Il sottospazio W, a seconda dei casi, viene descritto in maniera parametrica (cioè come Span), o in maniera cartesiana (cioè mediante equazioni). Si chiede di fornire la descrizione parametrica e cartesiana di W e $W^\perp$, di determinare una base ortogonale per W e per $W^\perp$ (costituite da vettori a coordinate intere), e le matrici che rappresentano, rispetto alla base canonica di $\mathbb{R}^n$, la proiezione ortogonale su W e $W^\perp$.

1. Sottospazi di $\mathbb{R}^2$.
   (a) $W = (x, y) \in \mathbb{R}^2 : x + y = 0$,
   (b) $W = (x, y) \in \mathbb{R}^2 : x = 0$,
   (c) $W = \text{Span}((1, 3))$,
   (d) $W = \text{Span}((0, 4))$.
2. Sottospazi di $\mathbb{R}^3$.
   (a) $W = \{(x, y, z) \in \mathbb{R}^3 : x - 2z = 0\}$,
   (b) $W = \{(t, t + s, 2s - 3t) : (s, t) \in \mathbb{R}^2\}$,
   (c) $W = \text{Span}((1, -1, 1))$,
   (d) $W = \text{Span}((1, -1, 1), (1, 2, 3))$,
   (e) $W = \{(x, y, z) \in \mathbb{R}^3 : x - 2z = y + z = 0\}$.
3. Sottospazi di $\mathbb{R}^4$.
   (a) $W = \text{Span}((1, 0, 0, 1), (1, 2, 0, 0))$,
   (b) $W = \{(t + s, 0, t - s, s) : (s, t) \in \mathbb{R}^2\}$,
   (c) $W = \{(x, y, z, w) \in \mathbb{R}^4 : x + y = z - 2w\}$,
   (d) $W = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y = 3y + 4z = 5z + 6w\}$,
   (e) $W = \{(x, y, z, w) \in \mathbb{R}^4 : x = y = z + w = 0\}$.
4. Sottospazi di $\mathbb{R}^5$.
   (a) $W = \text{Span}((1, 1, 0, 0, 0), (1, 0, 0, 0, 1), (-1, 0, 1, 0, 0))$,
   (b) $W = \{(x, y, z, w, v) \in \mathbb{R}^5 : x + v = y + w = 0\}$,
   (c) $W = \{(x, y, z, w, v) \in \mathbb{R}^5 : x + y + z + v = 0\}$,
   (d) $W = \{(x, y, z, w, v) \in \mathbb{R}^5 : z = w = 0\}$,
   (e) $W = \{(a, b, c, b, a) : (a, b, c) \in \mathbb{R}^3\}$.

SOLUZIONI? BY GIMOSI 02/02/15

1. Sottospazi di $\mathbb{R}^2$.

(a) $W = (x, y) \in \mathbb{R}^2 : x + y = 0$,

(b) $W = (x, y) \in \mathbb{R}^2 : x = 0$,

(c) $W = \mathrm{Span}((1, 3))$,

(d) $W = \mathrm{Span}((0, 4))$.

(a) $W: \ x+y=0 \qquad W = \mathrm{SPAN}\{(1,-1)\} \qquad \text{BASE}: \{\overset{w_1}{(1,-1)}\}$

$W^\perp = \mathrm{SPAN}\{(1,1)\} \qquad W^\perp: \ x-y=0 \qquad \text{BASE}: \{\overset{w_2}{(1,1)}\}$

MATRICI DI PROIEZIONE ORT. – MODO 1 (CLASSICO)

$$\text{PROJ}_W: \ \underset{\varepsilon \to e}{\hat{A} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}} \qquad \underset{\varepsilon \to e}{M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}} \ \rightsquigarrow \ \underset{e \to \varepsilon}{M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

$$A = \hat{A} M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\text{PROJ}_{W^\perp}: \ \hat{B} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \rightsquigarrow B = \hat{B} M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

MATRICI DI PROIEZIONE ORT. – MODO 2 (CON MATRICI ORTOGONALI)

$$\text{BASE ORTONORMALE} \ \begin{cases} w_1 \rightsquigarrow 1/\sqrt{2}\,(1,-1) \\ w_2 \rightsquigarrow 1/\sqrt{2}\,(1,1) \end{cases}$$

MAT. ORTOGONALE

$$\text{PROJ}_W: \ \underset{\varepsilon \to e}{\hat{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}} \quad \underset{\varepsilon \to e}{M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}} \ \rightsquigarrow \ \underset{e \to \varepsilon}{M^{-1} = M^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

$$A = \hat{A} M^T = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\text{PROJ}_{W^\perp}: \ \hat{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \rightsquigarrow B = \hat{B} M^{-1} = \hat{B} M^T = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(b) $W: x=0$ $W = \text{SPAN}\{(0,1)\}$ BASE: $\{(0,1)\}$ (w_1)

$W^\perp = \text{SPAN}\{(1,0)\}$ $W^\perp: y=0$ BASE: $\{(1,0)\}$ (w_2)

BASE ORTONORMALE

MODO 1 $\equiv$ MODO 2

$$\text{PROJ}_W: \hat{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow M^{-1} = M^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \hat{A}\, M^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{PROJ}_{W^\perp}: \hat{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \hat{B}\, M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(c) $W = \text{SPAN}\{(1,3)\}$ $W: 3x-y=0$ BASE: $\{(1,3)\}$ (w_1)

$W^\perp = \text{SPAN}\{(3,-1)\}$ $W^\perp: x+3y=0$ BASE: $\{(3,-1)\}$ (w_2)

$$\text{MATRICI CAMBIO BASE: } \underset{\varepsilon \to e}{M = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}} \rightsquigarrow M^{-1} = -\frac{1}{10} \underset{e \to \varepsilon}{\begin{pmatrix} -1 & -3 \\ -3 & 1 \end{pmatrix}}$$

MODO 1

$$\text{PROJ}_W: \hat{A} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \quad A = \hat{A} M^{-1} = -\frac{1}{10}\begin{pmatrix} -1 & -3 \\ -3 & -9 \end{pmatrix} \quad \text{PROJ}_{W^\perp} = -\frac{1}{10}\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}$$

MODO 2

$$\text{BASE ORTONORMALE} \begin{cases} w_1 \rightsquigarrow 1/\sqrt{10}\,(1,3) \\ w_2 \rightsquigarrow 1/\sqrt{10}\,(3,-1) \end{cases}$$

$$\text{MATRICI DI C.B.} \quad \underset{\varepsilon \to e}{M = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}} \rightsquigarrow M^{-1} = M^T = \frac{1}{10} \underset{e \to \varepsilon}{\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}}$$

$$\text{PROJ}_W: \hat{A} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \quad A = \hat{A} M^T = \frac{1}{10}\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \text{PROJ}_{W^\perp} = \frac{1}{10}\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}$$

(d) $W = \text{SPAN}\{0, 5\} = \text{SPAN}\{(0,1)\} \equiv$ P.TO (b)

2. Sottospazi di $\mathbb{R}^3$.

(a) $W = \{(x, y, z) \in \mathbb{R}^3 : x - 2z = 0\}$,

(b) $W = \{(t, t + s, 2s - 3t) : (s, t) \in \mathbb{R}^2\}$,

(c) $W = \mathrm{Span}((1, -1, 1))$,

(d) $W = \mathrm{Span}((1, -1, 1), (1, 2, 3))$,

(e) $W = \{(x, y, z) \in \mathbb{R}^3 : x - 2z = y + z = 0\}$.

(a) $W: x - 2z = 0 \quad W = \mathrm{SPAN}\{\overset{w_1}{(0,1,0)}, \overset{w_2}{(2,0,1)}\} \quad \langle w_1, w_2 \rangle = 0 \quad$ BASE: $\{w_1, w_2\}$

$W^{\perp} = \mathrm{SPAN}\{\overset{w_3 \,\equiv\, \perp \text{ PIANO}}{(1,0,-2)}\} \quad W^{\perp}: \begin{cases} 2x + z = 0 \\ y = 0 \end{cases} \quad$ BASE: $\{w_3\}$

MODO 1

M DI C. BASE: $M = \underset{\mathcal{E} \to e}{\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}} \rightsquigarrow M^{-1} = \underset{e \to \mathcal{E}}{\frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}}$

PROJ, W: $\hat{A} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A = \hat{A} M^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

PROJ, $W^{\perp}$: $\hat{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad B = \hat{B} M^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

MODO 2 $\quad w_1 \rightsquigarrow (0,1,0) \quad w_2 \rightsquigarrow 1/\sqrt{5}\,(2,0,1) \quad w_3 \rightsquigarrow 1/\sqrt{5}\,(1,0,-2) \quad$ BASE ORTON. DI $\mathbb{R}^3$

M DI C. BASE: $M = \underset{\mathcal{E} \to e}{\frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 2 & 1 \\ \sqrt{5} & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}} \rightsquigarrow M^{-1} = M^{\delta} = \underset{e \to \mathcal{E}}{\frac{1}{\sqrt{5}}\begin{pmatrix} 0 & \sqrt{5} & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}}$

PROJ, W: $\hat{A} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 2 & 0 \\ \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A = \hat{A} M^{\delta} = \frac{1}{5}\begin{pmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

PROJ, $W^{\perp}$: $\hat{B} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad B = \hat{B} M^{\delta} = \frac{1}{5}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

(b) $W: (\delta, \delta+\varepsilon, 2\varepsilon-3\delta) = \text{SPAN}\{ \overset{w_1}{(1,1,-3)}, \overset{\hat{w}_2}{(0,1,2)} \}$ $\langle w_1, \hat{w}_2 \rangle = -5$ $\|v_1\|^2 = 11$

$ax+by+cz=0$ $\begin{cases} a+b-3c=0 \\ b+2c=0 \end{cases}$ $\begin{cases} a=5c \\ b=-2c \end{cases}$ $5x-2y+z=0$

$$w_2^* = \hat{w}_2 - \frac{\langle \hat{w}_2, w_1 \rangle}{\|w_1\|^2} w_1 = \hat{w}_2 + \frac{5}{11} w_1 = \left(\frac{5}{11}, 1+\frac{5}{11}, 2-\frac{15}{11}\right) = \left(\frac{5}{11}, \frac{16}{11}, \frac{7}{11}\right)$$

$w_2 = (5,16,7) \perp w_1$ BASE: $\{w_1, w_2\}$

$W^\perp = \text{SPAN}\{ \overset{w_3 \perp \text{PIANO}}{(5,-2,1)} \}$ $W^\perp: \begin{cases} x+y-3z=0 & \perp w_1 \\ y+2z=0 & \perp w_2 \end{cases}$ BASE: $\{w_3\}$

MODO 1

MATRICI DI C.B. $M = \begin{array}{c} \begin{array}{ccc} w_1 & \hat{w}_2 & w_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 5 \\ 1 & 1 & -2 \\ -3 & 2 & 1 \end{pmatrix} \end{array} \rightsquigarrow M^{-1} = \frac{1}{30} \begin{pmatrix} 5 & 10 & -5 \\ 5 & 16 & 7 \\ 5 & -2 & 1 \end{pmatrix}$

PROJ$_W$: $\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 2 & 0 \end{pmatrix}$ $A = \hat{A} M^{-1} = \frac{1}{30} \begin{pmatrix} 5 & 10 & -5 \\ 10 & 26 & 2 \\ -5 & 2 & 29 \end{pmatrix}$

PROJ$_{W^\perp}$: $\hat{B} = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ $B = \hat{B} M^{-1} = \frac{1}{30} \begin{pmatrix} 25 & -10 & 5 \\ -10 & 5 & -2 \\ 5 & -2 & 1 \end{pmatrix}$

MODO 2 $w_1 \rightsquigarrow 1/\sqrt{11}\,(1,1,-3)$ $w_2 \rightsquigarrow \frac{1}{\sqrt{330}}(5,16,7)$ $w_3 \rightsquigarrow 1/\sqrt{30}\,(5,-2,1)$

MATRICI DI C.B. $M = \frac{1}{\sqrt{330}} \begin{pmatrix} \sqrt{30} & 5 & 5\sqrt{11} \\ \sqrt{30} & 16 & -2\sqrt{11} \\ -3\sqrt{30} & 7 & \sqrt{11} \end{pmatrix} \rightsquigarrow M^{-1} = M^T = \frac{1}{\sqrt{330}} \begin{pmatrix} \sqrt{30} & \sqrt{30} & -3\sqrt{30} \\ 5 & 16 & 7 \\ 5\sqrt{11} & -2\sqrt{11} & \sqrt{11} \end{pmatrix}$

PROJ$_W$: $\hat{A} = \frac{1}{\sqrt{330}} \begin{pmatrix} \sqrt{30} & 5 & 0 \\ \sqrt{30} & 16 & 0 \\ -3\sqrt{30} & 7 & 0 \end{pmatrix}$ $A = \hat{A} M^T = \frac{1}{30} \begin{pmatrix} 5 & 10 & -5 \\ 10 & 26 & 2 \\ -5 & 2 & 29 \end{pmatrix}$

PROJ$_{W^\perp}$: $\hat{B} = \frac{1}{\sqrt{330}} \begin{pmatrix} 0 & 0 & 5\sqrt{11} \\ 0 & 0 & -2\sqrt{11} \\ 0 & 0 & \sqrt{11} \end{pmatrix}$ $B = \hat{B} M^T = \frac{1}{30} \begin{pmatrix} 25 & -10 & 5 \\ -10 & 5 & -2 \\ 5 & -2 & 1 \end{pmatrix}$

(c) $W = \text{SPAN}\{\overset{w_1}{(1,-1,1)}\}$ $\quad W: \begin{cases} x+y=0 \\ y+z=0 \end{cases}$ $\quad$ BASE: $\{w_1\}$

$$W^\perp = \{ \overset{n \equiv w_1}{x-y+z=0} \} = \text{SPAN}\{ \overset{w_2}{(1,1,0)}, \overset{\hat{w}_3}{(1,0,-1)} \} \quad \langle w_2, \hat{w}_3 \rangle = 1 \quad \|w_2\|^2 = 2$$

$$w_3^* = \hat{w}_3 - \frac{1}{2} w_2 = \left(1-\frac{1}{2}, -\frac{1}{2}, -1\right) = \left(\frac{1}{2}, -\frac{1}{2}, -1\right) \quad w_3 = \overset{\perp w_2}{(1,-1,-2)} \quad \text{BASE: } \{w_2, w_3\}$$

MODO 1

M DI C.BASE:
$$M = \underset{\mathcal{E} \to e}{\begin{array}{c} \begin{array}{ccc} w_1 & w_2 & \hat{w}_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \end{array}} \rightsquigarrow \underset{e \to \mathcal{E}}{M^{-1}} = \frac{1}{3} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & -2 \end{pmatrix}$$

PROJ_W:
$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A = \hat{A} M^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$\text{PROJ}_{W^\perp}$:
$$\hat{B} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \hat{B} M^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

MODO 2 $\quad w_1 \rightsquigarrow 1/\sqrt{3}(1,-1,1) \quad w_2 \rightsquigarrow 1/\sqrt{2}(1,1,0) \quad w_3 \rightsquigarrow 1/\sqrt{6}(1,-1,-2)$

$$M = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ -\sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & -2 \end{pmatrix} \qquad M^{-1} = M^T = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & 0 \\ 1 & -1 & -2 \end{pmatrix}$$

PROJ_W:
$$\hat{A} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix} \qquad A = \hat{A} M^T = \frac{1}{3} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$\text{PROJ}_{W^\perp}$:
$$\hat{B} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & \sqrt{3} & 1 \\ 0 & \sqrt{3} & -1 \\ 0 & 0 & -2 \end{pmatrix} \qquad B = \hat{B} M^T = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

(d) $W = SPAN\{\overset{w_1}{(1,-1,1)}, \overset{\hat{w}_2}{(1,2,3)}\}$ $\quad n = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-5,-2,3)$ $\quad W = \{5x+2y-3z=0\}$

$\langle \hat{w}_2, w_1 \rangle = 2 \quad \|w_1\|^2 = 3 \quad w_2^* = \hat{w}_2 - \frac{2}{3} w_1 = \left(1-\frac{2}{3}, 2+\frac{2}{3}, 3-\frac{2}{3}\right) = \left(\frac{1}{3}, \frac{8}{3}, \frac{7}{3}\right)$

$w_2 = (1,8,7) \perp w_1 \quad$ BASE: $\{w_1, w_2\}$

$W^\perp = SPAN\{\overset{w_3}{(5,2,-3)}\}$ $\quad W^\perp: \begin{cases} x-y+z=0 \\ x+2+3z=0 \end{cases}$ $\quad$ BASE: $\{w_3\}$

MODO 1

M DI C.BASE: $M = \begin{array}{c} \begin{matrix} w_1 & \hat{w}_2 & w_3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 5 \\ -1 & 2 & 2 \\ 1 & 3 & -3 \end{pmatrix} \end{array}$ $\varepsilon \to e$ $\quad \rightsquigarrow M^{-1} = \frac{1}{38} \begin{pmatrix} 12 & -18 & 8 \\ 1 & 8 & 7 \\ 5 & 2 & -3 \end{pmatrix}$ $e \to \varepsilon$

PROJ, W: $\hat{A} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix} \quad A = \hat{A} M^{-1} = \frac{1}{38} \begin{pmatrix} 13 & -10 & 15 \\ -10 & 34 & 6 \\ 15 & 6 & 29 \end{pmatrix}$

PROJ, $W^\perp$: $\hat{B} = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{pmatrix} \quad B = \hat{B} M^{-1} = \frac{1}{38} \begin{pmatrix} 25 & 10 & -15 \\ 10 & 4 & -6 \\ -15 & -6 & 9 \end{pmatrix}$

MODO 2 $\quad w_1 \rightsquigarrow 1/\sqrt{3}\,(1,-1,1) \quad w_2 \rightsquigarrow \frac{1}{\sqrt{114}}(1,8,7) \quad w_3 \rightsquigarrow 1/\sqrt{38}\,(5,2,-3)$

$M = \frac{1}{\sqrt{114}} \begin{pmatrix} \sqrt{38} & 1 & 5\sqrt{3} \\ -\sqrt{38} & 8 & 2\sqrt{3} \\ \sqrt{38} & 7 & -3\sqrt{3} \end{pmatrix} \rightsquigarrow M^{-1} = M^T = \frac{1}{\sqrt{114}} \begin{pmatrix} \sqrt{38} & -\sqrt{38} & \sqrt{38} \\ 1 & 8 & 7 \\ 5\sqrt{3} & 2\sqrt{3} & -3\sqrt{3} \end{pmatrix}$

PROJ, W: $\hat{A} = \frac{1}{\sqrt{114}} \begin{pmatrix} \sqrt{38} & 1 & 0 \\ -\sqrt{38} & 8 & 0 \\ \sqrt{38} & 7 & 0 \end{pmatrix} \quad A = \hat{A} M^T = \frac{1}{38} \begin{pmatrix} 13 & -10 & 15 \\ -10 & 34 & 6 \\ 15 & 6 & 29 \end{pmatrix}$

PROJ, $W^\perp$: $\hat{B} = \frac{1}{\sqrt{114}} \begin{pmatrix} 0 & 0 & 5\sqrt{3} \\ 0 & 0 & 2\sqrt{3} \\ 0 & 0 & -3\sqrt{3} \end{pmatrix} \quad B = \hat{B} M^T = \frac{1}{38} \begin{pmatrix} 25 & 10 & -15 \\ 10 & 4 & -6 \\ -15 & -6 & 9 \end{pmatrix}$

(e) $W: \begin{cases} x-2z=0 \\ y+z=0 \end{cases}$ $\quad W = SPAN\{(\overset{}{2}, \overset{w_1}{-1}, 1)\}$ $\quad$ BASE: $\{w_1\}$

$$W^\perp = \{ \overset{m \equiv w_1}{2x-y+z=0} \} = SPAN\{\overset{w_2}{(1,2,0)}, \overset{\hat{w}_3}{(1,0,-2)}\} \quad \langle \hat{w}_3, w_2 \rangle = 1 \quad \|w_2\|^2 = 5$$

$$w_3^* = \hat{w}_3 - \frac{1}{5} w_2 = \left(1-\frac{1}{5}, -\frac{2}{5}, -2\right) = \left(\frac{4}{5}, \frac{-2}{5}, -2\right) \quad \overset{\perp w_2}{w_3 = (4,-2,-10)} \quad \text{BASE: } \{w_2, w_3\}$$

MODO 1

$$\text{M DI C.BASE: } \underset{\varepsilon \to e}{M} = \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & \hat{w}_3 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \end{array} \rightsquigarrow \underset{e \to \varepsilon}{M^{-1}} = \frac{1}{12}\begin{pmatrix} 4 & -2 & 2 \\ 2 & 5 & 1 \\ 2 & -1 & -5 \end{pmatrix}$$

$$PROJ_W: \hat{A} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A = \hat{A} M^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

$$PROJ_{W^\perp}: \hat{B} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad B = \hat{B} M^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & 1 \\ -2 & 1 & 5 \end{pmatrix}$$

MODO 2 $\quad w_1 \rightsquigarrow 1/\sqrt{6}\,(2,-1,1) \quad w_2 \rightsquigarrow 1/\sqrt{5}\,(1,2,0) \quad w_3 \rightsquigarrow \frac{1}{2\sqrt{30}}(4,-2,-10)$

$$M = \frac{1}{2\sqrt{30}}\begin{pmatrix} 4\sqrt{5} & 2\sqrt{6} & 4 \\ -2\sqrt{5} & 4\sqrt{6} & -2 \\ 2\sqrt{5} & 0 & -10 \end{pmatrix} \rightsquigarrow M^{-1} = M^T = \frac{1}{2\sqrt{30}}\begin{pmatrix} 4\sqrt{5} & -2\sqrt{5} & 2\sqrt{5} \\ 2\sqrt{6} & 4\sqrt{6} & 0 \\ 4 & -2 & -10 \end{pmatrix}$$

$$PROJ_W: \hat{A} = \frac{1}{2\sqrt{30}}\begin{pmatrix} 4\sqrt{5} & 0 & 0 \\ -2\sqrt{5} & 0 & 0 \\ 2\sqrt{5} & 0 & 0 \end{pmatrix} \rightsquigarrow A = \hat{A} M^T = \frac{1}{6}\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

$$PROJ_{W^\perp}: \hat{B} = \frac{1}{2\sqrt{30}}\begin{pmatrix} 0 & 2\sqrt{6} & 4 \\ 0 & 4\sqrt{6} & -2 \\ 0 & 0 & -10 \end{pmatrix} \rightsquigarrow B = \hat{B} M^T = \frac{1}{6}\begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & 1 \\ -2 & 1 & 5 \end{pmatrix}$$

3. Sottospazi di $\mathbb{R}^4$.

(a) $W = \mathrm{Span}((1,0,0,1),(1,2,0,0))$,

(b) $W = \{(t+s, 0, t-s, s) : (s,t) \in \mathbb{R}^2\}$,

(c) $W = \{(x,y,z,w) \in \mathbb{R}^4 : x+y = z-2w\}$,

(d) $W = \{(x,y,z,w) \in \mathbb{R}^4 : x+2y = 3y+4z = 5z+6w\}$,

(e) $W = \{(x,y,z,w) \in \mathbb{R}^4 : x = y = z+w = 0\}$.

(a) $W = \mathrm{SPAN}\{\overset{V_1}{(1,0,0,1)}, \overset{\hat{V}_2}{(1,2,0,0)}\}$

$$(x,y,z,w) = a w_1 + b w_2 \Rightarrow \begin{cases} x = a+b & z=0 \\ y = 2b & w = a \end{cases} \quad \begin{cases} z=0 \\ x = w + y/2 \end{cases} \quad W: \begin{cases} z=0 \\ 2x-y-2w=0 \end{cases}$$

$$\langle \hat{w}_2, w_1 \rangle = 1 \quad \|w_1\|^2 = 2 \quad w_2^* = \hat{w}_2 - \frac{1}{2} w_1 = \left(1-\frac{1}{2}, 2, 0, -\frac{1}{2}\right) = \left(\frac{1}{2}, 2, 0, -\frac{1}{2}\right)$$

$V_2 = (1,4,0,-1) \perp V_1$ BASE: $\{V_1, V_2\}$

$$W^\perp: \overset{\perp V_1, V_2}{\begin{cases} x+w=0 \\ x+2y=0 \end{cases}} = \mathrm{SPAN}\{\overset{V_3}{(2,-1,0,-2)}, \overset{V_4}{(0,0,1,0)}\} \quad \langle V_3, V_4 \rangle = 0 \quad \text{BASE: } \{V_3, V_4\}$$

MODO 2 $\quad V_1 \rightsquigarrow 1/\sqrt{2}\,(1,0,0,1) \quad V_2 \rightsquigarrow 1/3\sqrt{2}\,(1,4,0,-1)$ B. ORTONORMALE

$V_3 \rightsquigarrow 1/3\,(2,-1,0,-2) \quad V_4 \rightsquigarrow (0,0,1,0)$

$$\text{MATRICI DI C.B.} \quad M = \frac{1}{3\sqrt{2}} \begin{array}{c} \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{pmatrix} 3 & 1 & 2\sqrt{2} & 0 \\ 0 & 4 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 3\sqrt{2} \\ 3 & -1 & -2\sqrt{2} & 0 \end{pmatrix} \end{array} \rightsquigarrow M^{-1} = M^T = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & 0 & 0 & 3 \\ 1 & 4 & 0 & -1 \\ 2\sqrt{2} & -\sqrt{2} & 0 & -2\sqrt{2} \\ 0 & 0 & 3\sqrt{2} & 0 \end{pmatrix}$$

$$\mathrm{PROJ}_W: \quad \hat{A} = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \hat{A} M^{-1} = \frac{1}{9} \begin{pmatrix} 5 & 2 & 0 & 4 \\ 2 & 8 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 4 & -2 & 0 & 5 \end{pmatrix}$$

$$\mathrm{PROJ}_{W^\perp}: \quad \hat{B} = \frac{1}{3\sqrt{2}} \begin{pmatrix} 0 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 3\sqrt{2} \\ 0 & 0 & -2\sqrt{2} & 0 \end{pmatrix} \rightsquigarrow \hat{B} M^{-1} = \frac{1}{9} \begin{pmatrix} 4 & -2 & 0 & -4 \\ -2 & 1 & 0 & 2 \\ 0 & 0 & 9 & 0 \\ -4 & 2 & 0 & 4 \end{pmatrix}$$

(b) $W: \{(\delta+s, 0, \delta-s, s)\} = \text{SPAN}\{\overset{v_1}{(1,0,1,0)}, \overset{v_2}{(1,0,-1,1)}\}$ $\langle w_1, w_2\rangle = 0$

$$(x,y,z,w) = a\,w_1 + b\,w_2 \Rightarrow \begin{cases} x = a+b & z = a-b \\ y = 0 & w = b \end{cases} \quad \begin{cases} x - z - 2w = 0 \\ y = 0 \end{cases} \quad \text{BASE}\{v_1, v_2\}$$

$$W^{\perp} = \text{SPAN}\{\overset{v_3}{(1,0,-1,-2)}, \overset{v_4}{(0,1,0,0)}\} \quad W: \begin{cases} x + z = 0 \\ x - z + w = 0 \end{cases} \quad \langle w_3, w_4\rangle = 0 \quad \text{BASE}\{v_3, v_4\}$$

MODO 2 $\quad v_1 \rightsquigarrow 1/\sqrt{2}\,(1,0,1,0) \quad v_2 \rightsquigarrow 1/\sqrt{3}\,(1,0,-1,1)$

$\quad v_3 \rightsquigarrow 1/\sqrt{6}\,(1,0,-1,-2) \quad v_4 \rightsquigarrow (0,1,0,0)$

$$\text{MATRICI DI C.B.} \quad M = \frac{1}{\sqrt{6}} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 & \sqrt{6} \\ \sqrt{3} & -\sqrt{2} & -1 & 0 \\ 0 & \sqrt{2} & -2 & 0 \end{pmatrix} \end{array} \rightsquigarrow M^{-1} = M^{T} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & -1 & -2 \\ 0 & \sqrt{6} & 0 & 0 \end{pmatrix}$$

$$\text{PROJ}_W: \quad \hat{A} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{3} & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix} \qquad A = \hat{A} M^{-1} = \frac{1}{6} \begin{pmatrix} 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & -2 \\ 2 & 0 & -2 & 2 \end{pmatrix}$$

$$\text{PROJ}_{W^{\perp}}: \quad \hat{B} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \qquad B = \hat{B} M^{-1} = \frac{1}{6} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 6 & 0 & 0 \\ -1 & 0 & 1 & 2 \\ -2 & 0 & 2 & 5 \end{pmatrix}$$

(c) $W: x+y=z-2w = \mathrm{SPAN}\{\overset{V_1}{(1,-1,0,0)},\overset{V_2}{(0,0,2,1)},\overset{V_3}{(3,3,1,-2)}\}$

$\langle w_1,w_2\rangle=0 \quad \langle w_1,w_3\rangle=0 \quad \langle w_2,w_3\rangle=0 \qquad \mathrm{BASE}: \{V_1,V_2,V_3\}$

$$W^{\perp}\begin{cases}\perp V_1 & x-y=0\\ \perp V_2 & 2z+w=0\\ \perp V_3 & 3x+3y+z-2w=0\end{cases} \qquad \begin{cases}a=b\\ d=-2c\\ 6a+5c=0\end{cases} \qquad \begin{cases}b=-5c/6\\ d=-2c\\ a=-5c/6\end{cases} = \mathrm{SPAN}\{\overset{V_5}{(5,5,-6,12)}\}$$

$\|V_5\|^2=230$

MODO 2 $\quad V_1 \rightsquigarrow 1/\sqrt{2}\,(1,-1,0,0) \quad V_2 \rightsquigarrow 1/\sqrt{5}\,(0,0,2,1)$

$V_3 \rightsquigarrow 1/\sqrt{23}\,(3,3,1,-2) \quad V_5 \rightsquigarrow 1/\sqrt{230}\,(5,5,-6,12)$

$$\frac{1}{\sqrt{230}}\begin{array}{c}\begin{matrix}V_1 & V_2 & V_3 & V_5\end{matrix}\\ \begin{pmatrix}\sqrt{115} & 0 & 3\sqrt{10} & 5\\ -\sqrt{115} & 0 & 3\sqrt{10} & 5\\ 0 & 2\sqrt{56} & \sqrt{10} & -6\\ 0 & \sqrt{56} & -2\sqrt{10} & 12\end{pmatrix}\end{array} \rightsquigarrow M^{-1}=\frac{1}{\sqrt{230}}\begin{pmatrix}\sqrt{115} & -\sqrt{115} & 0 & 0\\ 0 & 0 & 2\sqrt{56} & \sqrt{56}\\ 3\sqrt{10} & 3\sqrt{10} & \sqrt{10} & -2\sqrt{10}\\ 5 & 5 & -6 & 12\end{pmatrix}$$

$$\mathrm{PROJ}_W: \hat{A}=\frac{1}{\sqrt{230}}\begin{pmatrix}\sqrt{115} & 0 & 3\sqrt{10} & 0\\ -\sqrt{115} & 0 & 3\sqrt{10} & 0\\ 0 & 2\sqrt{56} & \sqrt{10} & 0\\ 0 & \sqrt{56} & -2\sqrt{10} & 0\end{pmatrix} \rightsquigarrow A=\hat{A}M^{-1}=\frac{1}{230}\begin{pmatrix}205 & -25 & 30 & -60\\ -25 & 205 & 30 & -60\\ 30 & 30 & 195 & 72\\ -60 & -60 & 72 & 86\end{pmatrix}$$

$$\mathrm{PROJ}_{W^{\perp}}: \hat{B}=\frac{1}{\sqrt{230}}\begin{pmatrix}0 & 0 & 0 & 5\\ 0 & 0 & 0 & 5\\ 0 & 0 & 0 & -6\\ 0 & 0 & 0 & 12\end{pmatrix} \rightsquigarrow B=\hat{B}M^{-1}=\frac{1}{230}\begin{pmatrix}25 & 25 & -30 & 60\\ 25 & 25 & -30 & 60\\ -30 & -30 & 36 & -72\\ 60 & 60 & -72 & 155\end{pmatrix}$$

(d)

$W: \begin{cases} x-y-4z=0 \\ x+2y-5z-6w=0 \end{cases}$ $\quad z=1, w=0 \begin{cases} x-y=4 \\ x+2y=5 \end{cases} \begin{cases} 3y=1 \\ x=4+\frac{1}{3}=\frac{13}{3} \end{cases}$

$\hat{v}_2=(13,1,3,0)$ $\quad z=0, w=1 \begin{cases} x=y \\ 3y=6 \end{cases} \begin{cases} x=2 \\ y=2 \end{cases}$ $\quad v_1=(2,2,0,1)$

$\langle \hat{v}_2, v_1 \rangle = 28 \quad \|v_1\|^2=9 \quad v_2^* = \hat{v}_2 - \frac{28}{9} v_1 = \left(13-\frac{56}{9}, 1-\frac{56}{9}, 3, -\frac{28}{9}\right) =$

$= \left(\frac{61}{9}, \frac{-47}{9}, 3, \frac{-28}{9}\right)$ $\quad v_2=(61,-47,27,-28) \perp v_1 \quad$ BASE: $\{v_1, v_2\}$

$W = \text{SPAN}\{v_1, \hat{v}_2\}$

$W^\perp = \text{SPAN}\{\underset{v_3}{(1,-1,-4,0)}, \underset{\hat{v}_4}{(1,2,-5,-6)}\}$ $\quad \begin{cases} 2x+2y+w=0 \\ 13x+y+3z=0 \end{cases}$

$\langle \hat{v}_4, v_3 \rangle = 19 \quad \|v_3\|^2=18 \quad v_4^* = \hat{v}_4 - \frac{19}{8}\, 3 = \left(1-\frac{19}{18}, 2+\frac{19}{18}, -5+\frac{76}{18}, -6\right) =$

$= \left(-\frac{1}{18}, \frac{55}{18}, \frac{-14}{18}, -6\right)$ $\quad v_4=(-1,55,-14,-108) \quad W^\perp = \text{SPAN}\{v_3, \hat{v}_4\} \quad$ BASE: $\{v_3, v_4\}$

MODO 1

MATRICI DI C.B. $\quad M = \begin{array}{c} \begin{array}{cccc} v_1 & \hat{v}_2 & v_3 & \hat{v}_4 \end{array} \\ \begin{pmatrix} 2 & 13 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 0 & 3 & -4 & -5 \\ 1 & 0 & 0 & -6 \end{pmatrix} \end{array} \rightsquigarrow M^{-1} = \frac{1}{827} \begin{pmatrix} -6 & 330 & -84 & 179 \\ 61 & -47 & 27 & -28 \\ 47 & -104 & -169 & 114 \\ -1 & 55 & -14 & -108 \end{pmatrix}$

PROJ_W: $\quad \hat{A} = \begin{pmatrix} 2 & 13 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A = \hat{A} M^{-1} = \frac{1}{827} \begin{pmatrix} 781 & 49 & 183 & -6 \\ 49 & 613 & -141 & 330 \\ 183 & -141 & 81 & -84 \\ -6 & 330 & -84 & 179 \end{pmatrix}$

$\text{PROJ}_{W^\perp}$: $\quad \hat{B} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & -6 \end{pmatrix} \qquad B = \hat{B} M^{-1} = \frac{1}{827} \begin{pmatrix} 46 & -49 & -183 & 6 \\ -49 & 214 & 141 & -330 \\ -183 & 141 & 746 & 84 \\ 6 & -330 & 84 & 648 \end{pmatrix}$

(e)

$$W: \begin{cases} x = 0 \\ y = 0 \\ z + w = 0 \end{cases} = \mathrm{SPAN}\{\overset{v_1}{(0,0,1,-1)}\} \qquad \mathrm{BASE}: \{v_1\}$$

$$W^{\perp} = \mathrm{SPAN}\{\overset{v_2}{(1,0,0,0)}, \overset{v_3}{(0,1,0,0)}, \overset{v_4}{(0,0,1,1)}\} \qquad W^{\perp}: z - w = 0$$

$$\langle w_2, w_3 \rangle = 0 \quad \langle w_3, w_2 \rangle = 0 \quad \langle w_3, w_4 \rangle = 0 \qquad \mathrm{BASE}: \{v_2, v_3, v_4\}$$

MODO 2 $\quad v_1 \rightsquigarrow 1/\sqrt{2}(0,0,1,-1) \quad v_2 \rightsquigarrow (1,0,0,0) \quad v_3 \rightsquigarrow (0,1,0,0) \quad v_4 \rightsquigarrow 1/\sqrt{2}(0,0,1,1)$

MATRICI DI C.B.

$$M = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \end{array} \rightsquigarrow M^{-1} = M^{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

PROJ,W:

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow A = \hat{A} M^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

PROJ,W$^{\perp}$:

$$\hat{B} = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow B = \hat{B} M^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

4. Sottospazi di $\mathbb{R}^5$.

(a) $W = \mathrm{Span}((1,1,0,0,0),(1,0,0,0,1),(-1,0,1,0,0))$,

(b) $W = \{(x,y,z,w,v) \in \mathbb{R}^5 : x+v = y+w = 0\}$,

(c) $W = \{(x,y,z,w,v) \in \mathbb{R}^5 : x+y+z+v = 0\}$,

(d) $W = \{(x,y,z,w,v) \in \mathbb{R}^5 : z = w = 0\}$,

(e) $W = \{(a,b,c,b,a) : (a,b,c) \in \mathbb{R}^3\}$.

(a) $W = SPAN\{\underset{u_1}{(1,1,0,0,0)}, \underset{\hat u_2}{(1,0,0,0,1)}, \underset{\hat u_3}{(-1,0,1,0,0)}\}$

$(x,y,z,w,v) = a w_1 + b\hat w_2 + c\hat w_3 \Rightarrow \begin{cases} x = a+b-c \quad y = a \\ z = c \quad w=0 \quad v=b \end{cases}$ $\quad W: \begin{cases} x-y+z-v=0 \\ w=0 \end{cases}$

$\langle \hat u_2, u_1\rangle = 1 \quad \|u_1\|^2 = 2 \quad u_2^* = \hat u_2 - \frac{1}{2}u_1 = \left(1-\frac12, -\frac12, 0, 0, 1\right) = \left(\frac12, -\frac12, 0, 0, 1\right)$

$u_2 = (1,-1,0,0,2) \quad \langle \hat u_3, u_1\rangle = -1 \quad \langle \hat u_3, u_2\rangle = -1 \quad \|u_2\|^2 = 6$

$u_3^* = \hat u_3 + \frac16 u_2 + \frac12 u_1 = \left(-1+\frac16+\frac12, -\frac16+\frac12, 1, 0, \frac26\right) = \left(-\frac13, \frac13, 1, 0, \frac13\right)$ $\quad u_3 = (-1,1,3,0,1)$ BASE: $\{u_1,u_2,u_3\}$

$W^\perp = SPAN\{\underset{u_4}{(1,-1,1,0,-1)}, \underset{u_5}{(0,0,0,1,0)}\}$ $\quad W^\perp: x+y=0, x+v=0, x-z=0$

$\langle u_4, u_5\rangle = 0$ BASE: $\{u_4, u_5\}$

MODO 2 $\quad u_1 \rightsquigarrow 1/\sqrt{2}(1,1,0,0,0) \quad u_2 \rightsquigarrow \frac{1}{\sqrt6}(1,-1,0,0,2) \quad u_3\ \frac{1}{2\sqrt3}(-1,1,3,0,1) \quad u_4 \rightsquigarrow \frac12(1,-1,1,0,-1) \quad u_5 \equiv e_4$

$$M = \frac{1}{2\sqrt6}\begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{pmatrix} 2\sqrt3 & 2 & -\sqrt2 & \sqrt6 & 0 \\ 2\sqrt3 & -2 & \sqrt2 & -\sqrt6 & 0 \\ 0 & 0 & 3\sqrt2 & \sqrt6 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt6 \\ 0 & 4 & \sqrt2 & -\sqrt6 & 0 \end{pmatrix} \end{array} \rightsquigarrow M^{-1} = M^T = \frac{1}{2\sqrt6}\begin{pmatrix} 2\sqrt3 & 2\sqrt3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 4 \\ -\sqrt2 & \sqrt2 & 3\sqrt2 & 0 & \sqrt2 \\ \sqrt6 & -\sqrt6 & \sqrt6 & 0 & -\sqrt6 \\ 0 & 0 & 0 & 2\sqrt6 & 0 \end{pmatrix}$$

$PROJ_W$

$$\hat A = \frac{1}{2\sqrt6}\begin{pmatrix} 2\sqrt3 & 2 & -\sqrt2 & 0 & 0 \\ 2\sqrt3 & -2 & \sqrt2 & 0 & 0 \\ 0 & 0 & 3\sqrt2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & \sqrt2 & 0 & 0 \end{pmatrix} \rightsquigarrow \hat A M^{-1} = \frac14 \begin{pmatrix} 3 & 1 & -1 & 0 & 1 \\ 1 & 3 & 1 & 0 & -1 \\ -1 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 3 \end{pmatrix}$$

$PROJ_{W^\perp}$

$$\hat B = \frac{1}{2\sqrt6}\begin{pmatrix} 0 & 0 & 0 & \sqrt6 & 0 \\ 0 & 0 & 0 & -\sqrt6 & 0 \\ 0 & 0 & 0 & \sqrt6 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt6 \\ 0 & 0 & 0 & -\sqrt6 & 0 \end{pmatrix} \rightsquigarrow \hat B M^{-1} = \frac14 \begin{pmatrix} 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 4 & 0 \\ -1 & 1 & -1 & 0 & 1 \end{pmatrix}$$

(e) $W: \begin{cases} x+v=0 \\ y+w=0 \end{cases} = SPAN\{\overset{u_1}{(1,0,0,0,-1)}, \overset{u_2}{(0,1,0,-1,0)}, \overset{u_3}{(0,0,1,0,0)}\}$

$\langle u_1,u_2\rangle = \langle u_1,u_3\rangle = \langle u_2,u_3\rangle = 0$ BASE: $\{u_1,u_2,u_3\}$

$W^\perp = SPAN\{\overset{u_4}{(1,0,0,0,1)}, \overset{u_5}{(0,1,0,1,0)}\}$ $W^\perp: \begin{cases} x-v=0 \quad z=0 \\ y-w=0 \end{cases}$

$\langle u_4,u_5\rangle = 0$ BASE: $\{u_4,u_5\}$

MODO 2 $u_1 \rightsquigarrow 1/\sqrt{2}(1,0,0,0,-1)$ $u_2 \rightsquigarrow 1/\sqrt{2}(0,1,0,-1,0)$ $u_3 \rightsquigarrow e_3$ $u_4 \rightsquigarrow 1/\sqrt{2}(1,0,0,0,1)$ $u_5 \rightsquigarrow 1/\sqrt{2}(0,1,0,1,0)$

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow M^{-1} = M^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

PROJ W

$$\hat{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \hat{A}M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

PROJ $W^\perp$

$$\hat{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \hat{B}M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(C) $W: x+y+z+v=0$

$$= \text{SPAN}\{\overset{\mu_1}{(1,-1,0,0,0)}, \overset{\mu_2}{(1,1,-2,0,0)}, \overset{\mu_3}{(1,1,1,0,-3)}, \overset{\mu_4}{(0,0,0,1,0)}\}$$

$\langle \mu_i,\mu_j\rangle = 0 \ \forall i\neq j \ 1,4 \implies$ BASE: $\{\mu_1,\mu_2,\mu_3,\mu_4\}$

$W^\perp = \text{SPAN}\{\overset{\mu_5}{(1,1,1,0,1)}\}$ $\quad W^\perp: \begin{cases} x-y=0 \quad x+y-2z=0 \\ x+y+z-3v=0 \quad w=0 \end{cases}$ $\quad$ BASE: $\{\mu_5\}$

MODO 2 $\mu_1 \rightsquigarrow 1/\sqrt{2}(1,-1,0,0,0)$ $\mu_2 \rightsquigarrow 1/\sqrt{6}(1,1,-2,0,0)$ $\mu_3 \rightsquigarrow \frac{1}{2\sqrt{3}}(1,1,1,0,-3)$ $\mu_4 \rightsquigarrow e_4$ $\mu_5 \rightsquigarrow \frac{1}{5}(1,1,1,0,1)$

$$M = \frac{1}{5\sqrt{6}}\begin{pmatrix} 5\sqrt{3} & 5 & 2\sqrt{2} & 0 & \sqrt{6} \\ -5\sqrt{3} & 5 & 2\sqrt{2} & 0 & \sqrt{6} \\ 0 & -10 & 2\sqrt{2} & 0 & \sqrt{6} \\ 0 & 0 & 0 & 5\sqrt{6} & 0 \\ 0 & 0 & 6\sqrt{2} & 0 & \sqrt{6} \end{pmatrix} \rightsquigarrow M^{-1} = M^T = \frac{1}{5\sqrt{6}}\begin{pmatrix} 5\sqrt{3} & -5\sqrt{3} & 0 & 0 & 0 \\ 5 & 5 & -10 & 0 & 0 \\ 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 0 & 6\sqrt{2} \\ 0 & 0 & 0 & 5\sqrt{6} & 0 \\ \sqrt{6} & \sqrt{6} & \sqrt{6} & 0 & \sqrt{6} \end{pmatrix}$$

PROJ,W $\quad \hat{A} = \frac{1}{5\sqrt{6}}\begin{pmatrix} 5\sqrt{3} & 5 & 2\sqrt{2} & 0 & 0 \\ -5\sqrt{3} & 5 & 2\sqrt{2} & 0 & 0 \\ 0 & -10 & 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 5\sqrt{6} & 0 \\ 0 & 0 & 6\sqrt{2} & 0 & 0 \end{pmatrix} \rightsquigarrow \hat{A}M^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 5 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix}$

PROJ,$W^\perp$ $\quad \hat{B} = \frac{1}{5\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} \end{pmatrix} \rightsquigarrow \hat{B}M^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$

(a) $W: \begin{cases} z=0 \\ w=0 \end{cases} = \text{SPAN}\{\overset{u_1 \equiv e_1}{(1,0,0,0,0)}, \overset{u_2 \equiv e_2}{(0,1,0,0,0)}, \overset{u_3 \equiv e_5}{(0,0,0,0,1)}\}$

$\langle u_1, u_2 \rangle = \langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle = 0$ BASE: $\{u_1, u_2, u_3\}$

$W^\perp = \text{SPAN}\{\overset{u_4 \equiv e_3}{(0,0,1,0,0)}, \overset{u_5 \equiv e_4}{(0,0,0,1,0)}\}$ $W^\perp: x=y=v=0$ BASE $\{u_4, u_5\}$

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$$M = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{\text{MAT. ORTOGONALE}} \rightsquigarrow M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad M^{-1} \equiv M^T$$

PROJ, W

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \hat{A}M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

PROJ, $W^\perp$

$$\hat{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \hat{B}M^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(e) $W = SPAN\{\overset{u_1}{(1,0,0,0,1)}, \overset{u_2}{(0,1,0,1,0)}, \overset{u_3}{(0,0,1,0,0)}\}$ $W: x-v = y-w = 0$

$\langle u_1,u_2\rangle = \langle u_1,u_3\rangle = \langle u_2,u_3\rangle = 0$ BASE: $\{u_1,u_2,u_3\}$

$W^\perp : \begin{cases} x+v=0 \quad z=0 \\ y+w=0 \end{cases}$ $W^\perp = SPAN\{\overset{u_4}{(1,0,0,0,-1)}, \overset{u_5}{(0,1,0,-1,0)}\}$

$\langle u_4,u_5\rangle = 0$ BASE: $\{u_4,u_5\}$

MODO 2 $\quad u_1 \rightsquigarrow 1/\sqrt{2}(1,0,0,0,1)$ $u_2 \rightsquigarrow (0,1,0,1,0)$ $u_3 \rightsquigarrow e_3$ $u_4 \rightsquigarrow 1/\sqrt{2}(1,0,0,0,-1)$ $u_5 \rightsquigarrow 1/\sqrt{2}(0,1,0,-1,0)$

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix} \rightsquigarrow M^{-1} = M^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$PROJ_W$

$$\hat{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \hat{A}M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$PROJ_{W^\perp}$

$$\hat{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \rightsquigarrow \hat{B}M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$