BY GIMUSI 25/10/17

Limiti 3

Argomenti: limiti di funzioni di più variabili **Difficoltà:** ⋆⋆⋆⋆

Prerequisiti: tecniche per il calcolo di limiti in un punto per funzioni di più variabili

In ogni riga è assegnata una funzione, di cui si chiede di calcolare liminf e limsup per $(x, y) \to (0, 0)$. Nelle varie colonne, la funzione si intende definita nel suo "naturale dominio" intersecato l'insieme definito dalle relazioni indicate in testa alla colonna stessa.

		a) $(x, y) \in \mathbb{R}^2$		b) $x > 0,\ y > 0$		c) $0 \le x \le y$		d) $x > 0,\ y \le x^2$	
	Funzione	liminf	limsup	liminf	limsup	liminf	limsup	liminf	limsup
1)	$\dfrac{\sin(xy)}{xy}$	1	1	1	1	1	1	1	1
2)	$\dfrac{\sin(xy)}{\sqrt{\lvert x\rvert + 2\lvert y\rvert}}$	0	0	0	0	0	0	0	0
3)	$\dfrac{\sin(x-y)}{x+y}$	$-\infty$	$+\infty$	0	0	0	0	$-\infty$	$+\infty$
4)	$\dfrac{y \sin x}{x^2+y}$	$-\infty$	$+\infty$	0	0	0	0	$-\infty$	0
5)	$\dfrac{\cos x - \cos y}{x+y}$	0	0	0	0	0	0	0	0
6)	$\dfrac{e^x - e^{2y}}{x^2+y^2+x^2y^2}$	0	0	0	0	0	0	0	0
7)	$\dfrac{\sin x - \sin y}{x-y}$	1	1	1	1	1	1	1	1
8)	$\dfrac{\sin x + \sin y}{x+y}$	1	1	1	1	1	1	1	1
9)	$\dfrac{\sin^2 x - \sin^2 y}{x-y}$	0	0	0	0	0	0	0	0
10)	$\dfrac{x-\sqrt{xy}}{x^2-y^2}$	$-\infty$	$+\infty$	0	$+\infty$	0	$+\infty$	0	0
11)	$\displaystyle\int_x^y \frac{e^{-t^2}}{x+y}\,dt$	$-\infty$	$+\infty$	0	0	0	0	$-\infty$	$+\infty$
12)	$\displaystyle\int_x^y \frac{\arctan(t^3)}{x+y}\,dt$	0	0	0	0	0	0	0	0

1/7

1) $\dfrac{\sin(xy)}{xy}$ $\qquad xy = u \to 0$

$$\frac{\sin(xy)}{xy} = \frac{\sin u}{u} \to 1$$

LIMINF = LIMSUP = 1

2) $$\frac{\sin(xy)}{\sqrt{|x|+2|y|}} = \frac{\sin(xy)}{xy}\,\frac{xy}{\sqrt{|x|+2|y|}} = \overset{\to 1}{\frac{\sin(xy)}{xy}}\;\frac{\rho^{3/2}\overset{\to 0}{\cos\theta\sin\theta}}{\underset{\geq m>0}{\sqrt{|\cos\theta|+2|\sin\theta|}}} \to 0$$

LIMINF = LIMSUP = 0

3) $$\frac{\sin(x-y)}{x+y} = \frac{\sin(x-y)}{x-y}\,\frac{x-y}{x+y}$$

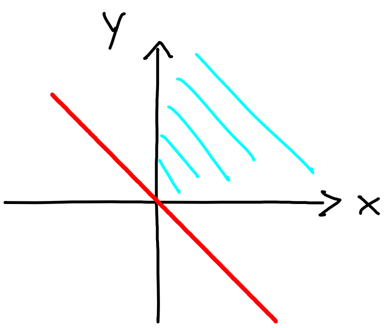
a) $(x,y)\in\mathbb{R}^2$

$$\begin{cases} x=\delta \\ y=\delta^2-\delta \end{cases} \quad \frac{\sin(x-y)}{x-y}\,\frac{x-y}{x+y} = \frac{\sin(x-y)}{x-y}\,\frac{2\delta-\delta^2}{\delta^2} =$$

$$= \overset{\to 1}{\frac{\sin(x-y)}{x-y}}\;\overset{\to \pm\infty}{\frac{2-\delta}{\delta}} \to \pm\infty$$

LIMINF $= -\infty$ $\qquad$ LIMSUP $= +\infty$

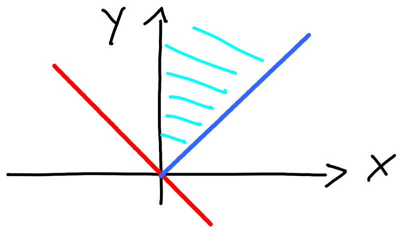
b) $x>0$ $y>0$



$$0 \leq \frac{|x-y|}{x+y} \leq \frac{\max\{x,y\}}{\underset{>0}{x+y}} \to 0 \qquad \overset{\to 1}{\frac{\sin(x-y)}{x-y}}\;\overset{\to 0}{\frac{x-y}{x+y}} \to 0$$

LIMINF = LIMSUP = 0

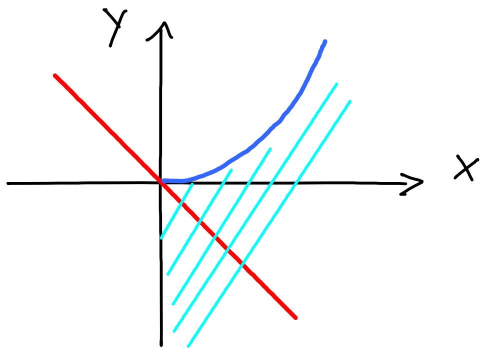
c) $0 \leq x \leq y$



$$0 \leq \frac{|x-y|}{x+y} \leq \frac{y}{\underset{>0}{x+y}} \to 0 \qquad \overset{\to 1}{\frac{\sin(x-y)}{x-y}} \; \overset{\to 0}{\frac{x-y}{x+y}} \to 0$$

LIMINF = LIMSUP = 0

d) $x > 0 \quad y \leq x^2$



$$\begin{cases} x = \delta \\ y = \delta^2 - \delta \end{cases} \quad \frac{\sin(x-y)}{x-y} \, \frac{x-y}{x+y} = \frac{\sin(x-y)}{x-y} \, \frac{2\delta - \delta^2}{\delta^2} =$$

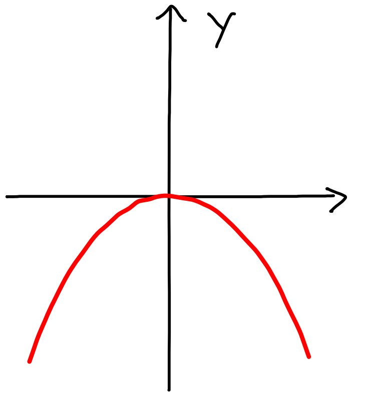
$$= \overset{\to 1}{\frac{\sin(x-y)}{x-y}} \; \overset{\to +\infty}{\frac{2-\delta}{\delta}} \to +\infty$$

$$\begin{cases} x = \delta \\ y = -\delta^3 - \delta \end{cases} \quad \frac{\sin(x-y)}{x-y} \, \frac{x-y}{x+y} = \frac{\sin(x-y)}{x-y} \, \frac{2\delta + \delta^3}{-\delta^3} =$$

$$= \overset{\to 1}{\frac{\sin(x-y)}{x-y}} \; \overset{\to -\infty}{\frac{2+\delta^2}{-\delta^2}} \to -\infty$$

LIMINF = $-\infty$ LIMSUP = $+\infty$

4) $\dfrac{y\sin x}{x^2+y} = \dfrac{\sin x}{x}\,\dfrac{xy}{x^2+y}$ $\quad y \neq -x^2$



a) $(x,y)\in\mathbb{R}^2$

$$\begin{cases} x=\delta \\ y=-\delta^2+\delta^4 \end{cases}$$

$$\frac{\sin x}{x}\,\frac{xy}{x^2+y} = \frac{\sin x}{x}\,\frac{-\delta^3+\delta^5}{\delta^4} = \underbrace{\frac{\sin x}{x}}_{\to 1}\,\underbrace{\frac{-1+\delta^2}{\delta}}_{\to \pm\infty} \to \pm\infty$$

LIMINF $= -\infty$ $\quad$ LIMSUP $= +\infty$

b) $x>0\quad y>0 \qquad x=u\quad y=v^2$

$$\frac{\sin x}{x}\,\frac{xy}{x^2+y} = \frac{\sin u}{u}\,\frac{uv^2}{u^2+v^2} = \underbrace{\frac{\sin u}{u}}_{\to 1}\,\rho\,\underbrace{(\cos\theta\sin^2\theta)}_{\to 0} \to 0$$

LIMINF $=$ LIMSUP $= 0$

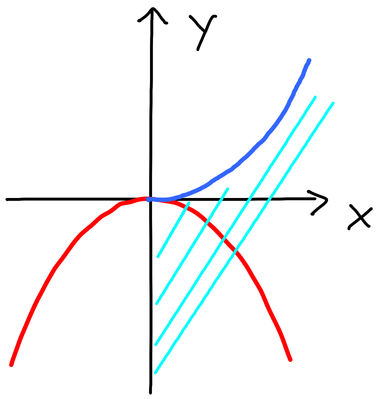
c) $0\le x\le y$

$$0 \le \frac{\sin x}{x}\,\frac{xy}{x^2+y} \le 1\cdot\frac{xy}{x^2+x} = \frac{y}{x+1} \to 0$$

LIMINF $=$ LIMSUP $= 0$

d) $x>0\quad y\le x^2$

$$\begin{cases} x=\delta \quad \delta\to 0^+ \\ y=-\delta^2+\delta^4 \end{cases} \qquad \frac{\sin x}{x}\,\frac{xy}{x^2+y} \xrightarrow{a)} -\infty$$



$$\begin{cases} x>0 \quad y\le 0 \leadsto f(x,y)\le 0 \\ x>0 \quad 0<y\le x^2 \leadsto 0\le \dfrac{\sin x}{x}\,\dfrac{xy}{x^2+y} \le 1\cdot\dfrac{xy}{2y} \to 0 \end{cases}$$

LIMINF $= -\infty$ $\quad$ LIMSUP $= 0$

5) $$\frac{\cos x-\cos y}{x+y}=\frac{-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}{x+y}=$$

$$=-\overset{\to 1}{\frac{\sin\left(\frac{x+y}{2}\right)}{\frac{x+y}{2}}}\;\overset{\to 0}{\sin\left(\frac{x-y}{2}\right)}\to 0$$ LIMINF = LIMSUP = 0

6) $$\frac{e^x-e^{2y}}{x^2+y^2+x^2y^2}=\frac{e^x-1}{x^2+y^2+x^2y^2}-\frac{e^{2y}-1}{x^2+y^2+x^2y^2}=$$

$$=\overset{\to 1}{\frac{e^x-1}{x}}\;\overset{\to 0}{\frac{x}{x^2+y^2+x^2y^2}}-\overset{\to 1}{\frac{e^{2y}-1}{2y}}\;\overset{\to 0}{\frac{2y}{x^2+y^2+x^2y^2}}\to 0$$

7) $$\frac{\sin x-\sin y}{x-y}=\frac{2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)}{x-y}=$$

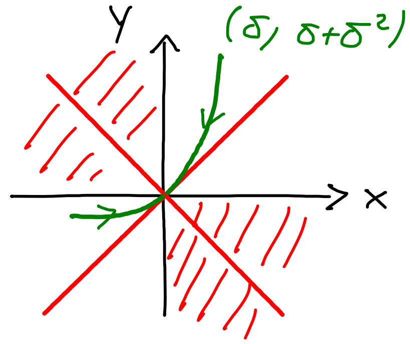
$$=\overset{\to 1}{\frac{\sin\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}}}\;\overset{\to 1}{\cos\left(\frac{x+y}{2}\right)}\to 1$$ LIMINF = LIMSUP = 1

8) $$\frac{\sin x+\sin y}{x+y}=\frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{x+y}=$$

$$=\overset{\to 1}{\frac{\sin\left(\frac{x+y}{2}\right)}{\frac{x+y}{2}}}\;\overset{\to 1}{\cos\left(\frac{x-y}{2}\right)}\to 1$$ LIMINF = LIMSUP = 1

9) $$\frac{\sin^2 x-\sin^2 y}{x-y}=\overset{\to 1}{\frac{\sin x-\sin y}{x-y}}\;\overset{\to 0}{(\sin x+\sin y)}\to 0$$

10) $\frac{x-\sqrt{xy}}{x^2-y^2} = \frac{\cos\theta - \sqrt{\cos\theta \sin\theta}}{\rho\,(\cos^2\theta - \sin^2\theta)}$ $\quad x \neq \pm y \quad xy \geq 0$



a) $(x,y) \in \mathbb{R}^2$

$$f(\delta, \delta+\delta^2) = \frac{\delta - \sqrt{\delta^2+\delta^3}}{\cancel{\delta^2} - \cancel{\delta^2} - 2\delta^3 - \delta^4} =$$

$$= \frac{1-\sqrt{1+\delta}}{-2\delta^2-\delta^3} = \frac{-\frac{\delta}{2} + o(\delta)}{-2\delta^2-\delta^3} = \frac{1+o(1)}{4\delta+2\delta^2} \to \pm\infty$$

LIMINF $= -\infty$ $\quad$ LIMSUP $= +\infty$

b) $x>0 \quad y>0$

$$\begin{cases} x>y \rightsquigarrow \frac{x-\sqrt{xy}}{x^2-y^2} \geq \frac{x-\sqrt{x^2}}{x^2-y^2} = 0 \\ x<y \rightsquigarrow \frac{x-\sqrt{xy}}{x^2-y^2} \geq \frac{x-\sqrt{y^2}}{x^2-y^2} = \frac{1}{x+y} \geq 0 \end{cases} \rightsquigarrow f(x,y) \geq 0$$

$$\begin{cases} f(\delta, \delta+\delta^2) \xrightarrow{a)} +\infty \\ f(0,\delta) = 0 \end{cases}$$

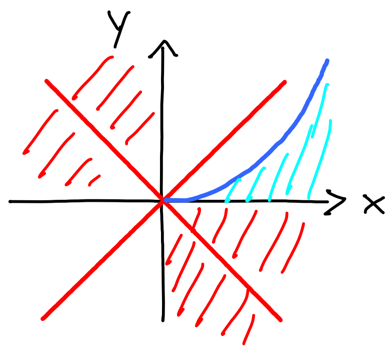
LIMINF $= 0$ $\quad$ LIMSUP $= +\infty$

c) $0 \leq x \leq y$

$$\begin{cases} f(\delta, \delta+\delta^2) \xrightarrow{a)} +\infty \\ f(0,\delta) = 0 \end{cases}$$

LIMINF $= 0$ $\quad$ LIMSUP $= +\infty$

d) $x>0 \quad 0 \leq y \leq x^2 \quad 0 \leq xy < x^3$



$$0 \leq \frac{x-\sqrt{xy}}{x^2-y^2} = \sqrt{x}\,\frac{\sqrt{x}-\sqrt{y}}{(x-y)(x+y)} =$$

$$= \sqrt{x}\,\frac{\cancel{\sqrt{x}-\sqrt{y}}}{\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})(x+y)} \to 0 \qquad x \neq y$$

LIMINF $=$ LIMSUP $= 0$

11) $\displaystyle\int_x^y \frac{e^{-\sigma^2}}{x+y}\,d\sigma = \underbrace{\frac{G(y)-G(x)}{y-x}}_{\to 1 = G'(0)} \;\frac{y-x}{y+x}$ vol. 3) $\qquad G(x)=\displaystyle\int_0^x e^{-\sigma^2}\,d\sigma$

a) $(x,y)\in\mathbb{R}^2 \rightsquigarrow$ LIMINF $=-\infty$ LIMSUP $=+\infty$

b) $x>0\ \ y>0 \rightsquigarrow$ LIMINF = LIMSUP = 0

c) $0\le x\le y \rightsquigarrow$ LIMINF = LIMSUP = 0

d) $x>0\ \ y\le x^2 \rightsquigarrow$ LIMINF $=-\infty$ LIMSUP $=+\infty$

12) $\displaystyle\int_x^y \frac{\arctan(\sigma^3)}{x+y}\,d\sigma \simeq \frac{1}{5}\,\frac{y^5-x^5}{x+y} =$

$$= \frac{1}{5}\,\frac{(y^2-x^2)(y^2+x^2)}{x+y} = \frac{1}{5}\,\frac{(y-x)\cancel{(y+x)}(y^2+x^2)}{\cancel{x+y}} \to 0$$

LIMINF = LIMSUP = 0