

Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Prova in Itinere di Analisi Matematica II

Pisa, ?? ?? ????

1. Sia

$$V := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 2, z \geq 0\} \cap \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq 2\}.$$

Calcolare l'area della sua superficie.

2. Siano $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^3 = 9, 0 \leq z \leq 2\}$ orientata prendendo in $(2, 2, 1)$ la normale che punta verso le y negative e

$$F(x, y, z) = (x + y^2, -2y + z, x^2 + z).$$

Calcolare il flusso di F attraverso S .

3. Sia

$$D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}.$$

Stabilire se convergono

$$\int_D \frac{x}{x^4 + y^4} dx dy, \quad \int_D \frac{x}{x^2 + y^4} dx dy.$$

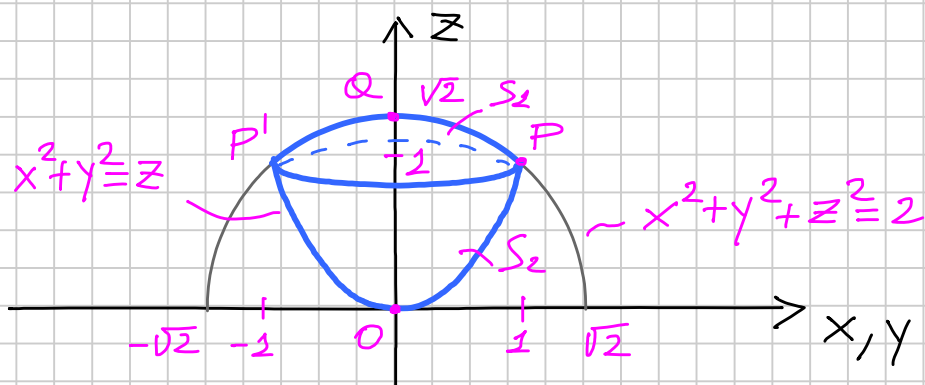
Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato.

Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

1. Sia

$$V := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 2, z \geq 0\} \cap \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq 2\}.$$

Calcolare l'area della sua superficie.



$$P: \begin{cases} x^2 + y^2 + z^2 = 2 \\ x^2 + y^2 = z \end{cases} \leadsto z^2 + z - 2 = 0 \quad (z+2)(z-1) = 0$$

$$P(1, 1) \quad P'(-1, 1)$$

CALOTTA SFERICA "P'QP" (MODO POCO FURBO)

$$\begin{cases} (x, y, \sqrt{2-x^2-y^2}) \\ (x, y) \in \Omega \quad \Omega = \{(x, y) : x^2 + y^2 \leq 1\} \end{cases}$$

$$\begin{cases} f_x = \frac{1}{2} \frac{-2x}{\sqrt{2-x^2-y^2}} = \frac{-x}{\sqrt{2-x^2-y^2}} \\ f_y = \frac{-y}{\sqrt{2-x^2-y^2}} \end{cases}$$

$$A_1 = \int_{\Omega} \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy = \int_{\Omega} \sqrt{\frac{2 - \cancel{x^2} - \cancel{y^2} + \cancel{x^2} + \cancel{y^2}}{2 - x^2 - y^2}} \, dx \, dy =$$

$$= \int_0^1 \int_0^{2\pi} \sqrt{\frac{2}{2-\rho^2}} \, \rho \, d\rho \, d\theta = 2\pi \sqrt{2} \int_0^1 \frac{\rho}{\sqrt{2-\rho^2}} \, d\rho$$

$$= 2\sqrt{2} \pi \left[-\sqrt{2-\rho^2} \right]_0^1 = 2\sqrt{2} \pi (\sqrt{2}-1) = (5-2\sqrt{2})\pi$$

METODO PIÙ RAPIDO \leadsto FORMULE X SUP. DI ROTAZIONE

CALOTTA SFERICA "P'QP"

$$\equiv \text{SUP. DI ROTAZIONE} \quad \varphi(z) = \sqrt{2-z^2}$$

$$A_1 = 2\pi \int_1^{\sqrt{2}} \varphi(z) \sqrt{1 + \dot{\varphi}(z)^2} dz = \quad \dot{\varphi}(z) = \frac{-z}{\sqrt{2-z^2}}$$

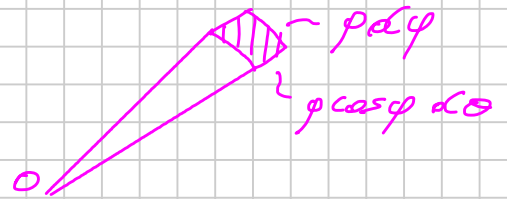
$$= 2\pi \int_1^{\sqrt{2}} \sqrt{2-z^2} \sqrt{1 + \frac{z^2}{2-z^2}} dz =$$

$$= 2\pi \int_1^{\sqrt{2}} \sqrt{2} dz = 2\pi \sqrt{2} (\sqrt{2} - 1) = 4\pi - 2\pi\sqrt{2}$$

VERIFICA IN COORD. POLARI

$$A_1 = \int_0^{2\pi} \int_{\pi/5}^{\pi/2} \rho^2 \cos \varphi d\theta d\varphi =$$

$$= 4\pi \left[\sin \varphi \right]_{\pi/5}^{\pi/2} = 4\pi (1 - \sqrt{2}/2) = (4 - 2\sqrt{2})\pi$$



CALOTTA "P'OP" \equiv SUP. DI ROTAZIONE $\varphi(z) = \sqrt{z}$

$$A_2 = 2\pi \int_0^1 \varphi(z) \sqrt{1 + \dot{\varphi}(z)^2} dz = \quad \dot{\varphi}(z) = \frac{1}{2\sqrt{z}}$$

$$= 2\pi \int_0^1 \frac{1}{\sqrt{z}} \sqrt{\frac{1 + \frac{1}{4z}}{4z}} dz =$$

$$= 2\pi \int_0^1 \sqrt{\frac{1}{4} + z} dz = \quad z + \frac{1}{4} = \mu$$

$$= 2\pi \int_{1/4}^{5/4} \sqrt{\mu} d\mu = 2\pi \left[\frac{2}{3} \mu^{3/2} \right]_{1/4}^{5/4} = \frac{4}{3} \pi \left(\frac{5\sqrt{5}}{8} - \frac{1}{8} \right) =$$

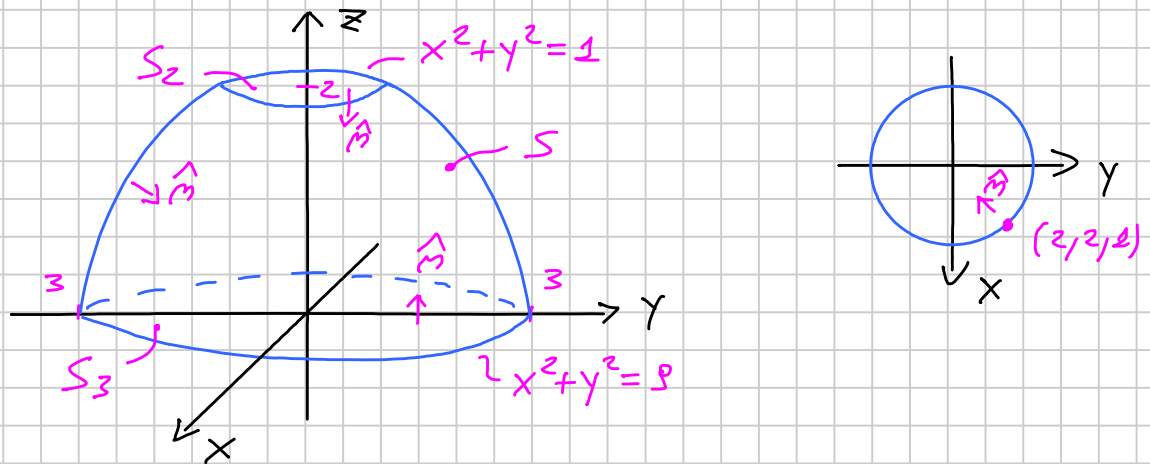
$$= \left(\frac{5\sqrt{5}}{6} - \frac{1}{6} \right) \pi$$

$$\leadsto A = A_1 + A_2 = \left(\frac{5\sqrt{5}}{6} - 2\sqrt{2} + \frac{23}{6} \right) \pi$$

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$$F(x, y, z) = (x + y^2, -2y + z, x^2 + z).$$

Calcolare il flusso di F attraverso S .



$$\int_S \langle \bar{F}, \hat{n} \rangle d\sigma = \int_{S \cup S_2 \cup S_3} \langle \bar{F}, \hat{n} \rangle d\sigma - \int_{S_2 \cup S_3} \langle \bar{F}, \hat{n} \rangle d\sigma$$

$$\Omega = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9, 0 \leq z \leq 2\}$$

SEGNO MENO XCHÉ \hat{n} PUNTA VERSO L'INTERNO

$$\int_{S \cup S_2 \cup S_3} \langle \bar{F}, \hat{n} \rangle d\sigma = - \int_{\Omega} \operatorname{div} \bar{F} dx dy dz = - \int_{\Omega} (1 - 2 + 1) dx dy dz = 0$$

$$\leadsto \int_S \langle \bar{F}, \hat{n} \rangle d\sigma = - \int_{S_2} \langle \bar{F}, \hat{n} \rangle d\sigma - \int_{S_3} \langle \bar{F}, \hat{n} \rangle d\sigma$$

$$\int_{S_2} \langle \bar{F}, \hat{n} \rangle d\sigma = \int_{S_2} x^2 d\sigma = \int_0^{2\pi} \int_0^{\pi} \rho^2 \cos^2 \theta \rho d\rho d\theta =$$

$$= \int_0^{2\pi} \cos^2 \theta d\theta \int_0^{\pi} \rho^3 d\rho = \left[\theta/2 + \frac{\sin 2\theta}{4} \right]_0^{2\pi} \left[\rho^4/4 \right]_0^{\pi} =$$

$$= \frac{81}{5} \pi$$

$$\begin{aligned} \int_{S_3} \langle \bar{F}, \hat{n} \rangle d\sigma &= \int_{S_3} (-x^2 - 2) d\sigma = - \int_{S_3} x^2 d\sigma - 2 \int_{S_3} d\sigma = \\ &= - \int_0^{2\pi} \cos^2 \theta d\theta \int_0^1 \rho^3 d\rho - 2\pi = - \frac{\pi}{5} - 2\pi = - \frac{9}{5} \pi \end{aligned}$$

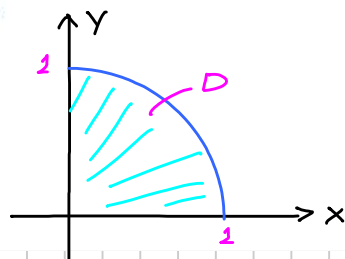
$$\begin{aligned} \Rightarrow \int_S \langle \bar{F}, \hat{n} \rangle d\sigma &= - \int_{S_2} \langle \bar{F}, \hat{n} \rangle d\sigma - \int_{S_3} \langle \bar{F}, \hat{n} \rangle d\sigma = \\ &= - \frac{81}{5} \pi + \frac{9}{5} \pi = - 18\pi \end{aligned}$$

3. Sia

$$D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}.$$

Stabilire se convergono

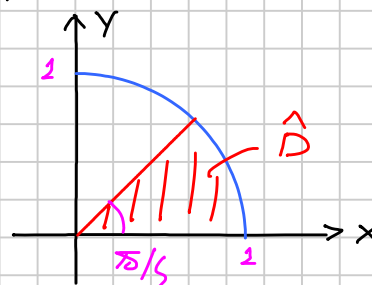
$$\int_D \frac{x}{x^4 + y^4} dx dy, \quad \int_D \frac{x}{x^2 + y^4} dx dy.$$



$$\begin{aligned} \int_D \frac{x}{x^4 + y^4} dx dy &= \int_D \frac{\rho \cos \theta}{\rho^4 (\cos^4 \theta + \sin^4 \theta)} \rho d\rho d\theta = \\ &= \int_D \frac{1}{\rho^2} \frac{\cos \theta}{\cos^4 \theta + \sin^4 \theta} d\rho d\theta \geq \int_{\hat{D}} \frac{1}{\rho^2} \frac{\cos \theta}{\cos^4 \theta + \sin^4 \theta} d\rho d\theta \geq \end{aligned}$$

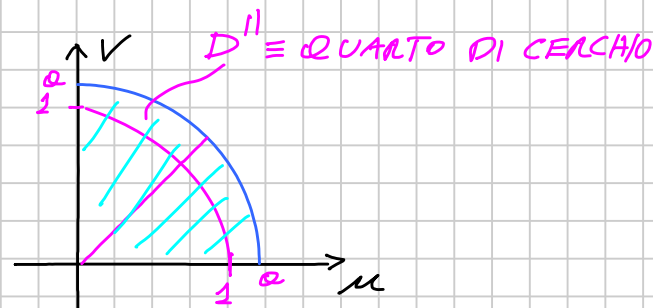
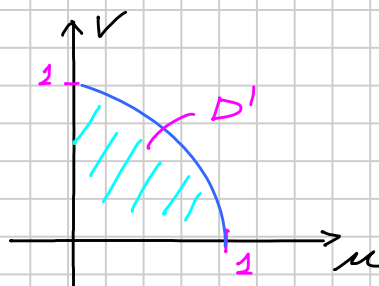
$$0 < m \leq \cos^4 \theta + \sin^4 \theta \leq M$$

$$\geq \int_{\hat{D}} \frac{1}{\rho^2} \frac{\sqrt{2}/2}{M} d\rho d\theta = +\infty$$



$$\int_D \frac{x}{x^2 + y^4} dx dy = \begin{cases} x = \mu^2 \\ y = \nu \end{cases} \quad \begin{vmatrix} 2\mu & 0 \\ 0 & 1 \end{vmatrix} = 2\mu$$

$$= \int_{D'} \frac{\mu^2}{\mu^2 + \nu^4} \cdot 2\mu d\mu d\nu = \int_{D'} \frac{2\mu^3}{\mu^2 + \nu^4} d\mu d\nu =$$



$$\leq \int_{D''} \frac{2 \rho^3 \cos^3 \theta}{\rho^2 (\cos^4 \theta + \sin^4 \theta)} \rho d\rho d\theta = \int_{D''} \frac{2 \cos^3 \theta}{\cos^4 \theta + \sin^4 \theta} d\rho d\theta < +\infty$$

INT. PROPRIO

$$\leadsto \begin{cases} \int_D \frac{x}{x^4 + y^4} dx dy & \text{DIVERGE} \\ \int_D \frac{x}{x^2 + y^4} dx dy & \text{CONVERGE} \end{cases}$$