BY GIMUSI 31/10/15

Serie 4

Argomenti: convergenza di serie a termini di segno qualunque **Difficoltà:** ⋆⋆⋆

Prerequisiti: criterio di Leibnitz, assoluta CONVERGENZA

Stabilire se le seguenti serie numeriche sono assolutamente convergenti (AC), oppure convergenti ma non assolutamente convergenti (C), oppure non convergenti (N).

	a) Serie	AC/C/N	b) Serie	AC/C/N	c) Serie	AC/C/N
1)	$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$	C	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	C	$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	AC
2)	$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$	N	$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$	C	$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+1}$	AC
3)	$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$	AC	$\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}$	AC (≥0)	$\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$	N
4)	$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$	C	$\sum_{n=4}^{\infty} \frac{(-1)^n}{\log(\log n)}$	C	$\sum_{n=4}^{\infty} \frac{(-1)^n}{n \log^2 n}$	AC
5)	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt[3]{n}}$	C	$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-\sqrt[3]{n}}$	C	$\sum_{n=1}^{\infty} \frac{\sqrt{n}\cos(n!)}{7+n^2}$	AC

Stesse domande della tabella precedente, avendo cura di giustificare nei dettagli i passaggi (tenendo presente che non esistono criteri del confronto asintotico per serie a segno alterno).

	a) Serie	AC/C/N	b) Serie	AC/C/N
6)	$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+3n-\sin n}{n^4-\arctan n^2}$	AC	$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+3n-\sin n}{n^3-\arctan n^2}$	C
7)	$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+3n-\sin n}{n^2-\arctan n^2}$	N	$\sum_{n=1}^{\infty} (-1)^n \frac{3n+\cos(\pi n)}{n}$	N
8)	$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{\pi n}{2}\right)$	C	$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{\pi n}{3}\right)$	C
9)	$\sum_{n=1}^{\infty} (-1)^n \sinh\left(\frac{n^2+\sin^2(n!)}{n+\sin(n!)}\right)$	N	$\sum_{n=1}^{\infty} (-1)^n \sinh\left(\frac{n+\sin(n!)}{n^2+\sin^2(n!)}\right)$	C

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1.a) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ CONVERGE (PER LEIBNITZ)

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{1}{n} \qquad \begin{cases} \alpha_n \geq 0 \quad \forall n \\ \alpha_{n+1} \leq \alpha_n \\ \alpha_n \to 0 \end{cases}$$

$\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} \frac{1}{n}$ NON CONVERGE ASSOLUTAMENTE

1.b) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ CONVERGE (PER LEIBNITZ)

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{1}{\sqrt{n}} \qquad \begin{cases} \alpha_n \geq 0 \quad \forall n \\ \alpha_{n+1} \leq \alpha_n \\ \alpha_n \to 0 \end{cases}$$

$\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$ NON CONVERGE ASSOLUTAMENTE

1.c) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$ CONVERGE ASSOLUTAMENTE

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{1}{n^2} \qquad \begin{cases} \alpha_n \geq 0 \quad \forall n \\ \alpha_{n+1} \leq \alpha_n \\ \alpha_n \to 0 \end{cases}$$

$\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} \frac{1}{n^2}$ CONVERGE ASSOLUTAMENTE

2.a) $\sum_{n=1}^{+\infty} (-1)^n \frac{n}{n+1}$ NON CONVERGE (MANCANZA COND. NEC.)

$$a_n \to \begin{cases} 1 & n \text{ PARI} \\ -1 & n \text{ DISPARI} \end{cases}$$

2.b) $\sum_{n=1}^{+\infty} (-1)^n \frac{n}{n^2+1}$ CONVERGE (PER LEIBNITZ)

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{n}{n^2+1} \qquad \begin{cases} \alpha_n \geq 0 \quad \forall n \\ \alpha_{n+1} \leq \alpha_n \quad ? \\ \alpha_n \to 0 \end{cases}$$

$$\alpha_{n+1} \leq \alpha_n \qquad \frac{n+1}{(n+1)^2+1} \overset{?}{\leq} \frac{n}{n^2+1} \qquad \cancel{n^3}+n^2+n+1 \leq \cancel{n^3}+2n^2+2n$$

$$n^2+n-1 \geq 0 \qquad \forall n \geq 1$$

$$\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} \frac{n}{n^2+1}$$ NON CONVERGE ASSOLUTAMENTE

$$|a_n| = \frac{n}{n^2+1} \geq \frac{n+1}{n^2+2n+1} = \frac{1}{n+1} = b_n \qquad \sum_{n=1}^{+\infty} b_n = +\infty$$

2.c) $\sum_{n=1}^{+\infty} (-1)^n \frac{n}{n^3+1}$ CONVERGE ASSOLUTAMENTE

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{n}{n^3+1} \qquad \begin{cases} \alpha_n \geq 0 \quad \forall n \\ \alpha_{n+1} \leq \alpha_n \quad ? \\ \alpha_n \to 0 \end{cases}$$

$$\alpha_{n+1} \leq \alpha_n \qquad \frac{n+1}{(n+1)^3+1} \overset{?}{\leq} \frac{n}{n^3+1} \qquad \cancel{n^5}+n^3+n+1 \leq \cancel{n^5}+3n^3+3n^2+2n$$

$$2n^3+3n^2+n-1 \geq 0 \qquad \forall n \geq 1$$

$$\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} \frac{n}{n^3+1}$$ CONVERGE ASSOLUTAMENTE

$$|a_n| = \frac{n}{n^3+1} \leq \frac{n+1}{n^3+3n^2+3n+1} = \frac{1}{(n+1)^2} = b_n \qquad \sum_{n=1}^{+\infty} b_n < +\infty$$

3.a) $\displaystyle\sum_{n=1}^{+\infty} \frac{\cos n}{n^2}$ CONVERGE ASSOLUTAMENTE

$$|a_n| = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2} = b_n \qquad \sum_{n=1}^{+\infty} b_n < +\infty$$

3.b) $\displaystyle\sum_{n=1}^{+\infty} \frac{2+\sin n}{n^2}$ CONVERGE ASSOLUTAMENTE

$$0 \leq \frac{2+\sin n}{n^2} \leq \frac{3}{n^2} \qquad \sum_{n=1}^{+\infty} \frac{3}{n^2} < +\infty$$

3.c) $\displaystyle\sum_{n=1}^{+\infty} \frac{2+\sin n}{n}$ NON CONVERGE

$$0 \leq \frac{1}{n} \leq \frac{2+\sin n}{n} \qquad \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

5.a) $\displaystyle\sum_{n=2}^{+\infty} \frac{(-1)^n}{\log n}$ CONVERGE (PER LEIBNITZ)

$$|a_n| = \frac{1}{\log n} \geq \frac{1}{n} \qquad \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$ NON CONVERGE ASSOLUTAMENTE

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{1}{\log n} \quad \begin{cases} \alpha_n \geq 0 \quad \forall n \\ \alpha_{n+1} \leq \alpha_n \\ \alpha_n \to 0 \end{cases}$$

5.b) $\displaystyle\sum_{n=5}^{+\infty} \frac{(-1)^n}{\log \log n}$ CONVERGE (PER LEIBNITZ)

$$|a_n| = \frac{1}{\log \log n} \geq \frac{1}{n} \qquad \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$ NON CONVERGE ASSOLUTAMENTE

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{1}{\log \log n} \quad \begin{cases} \alpha_n \geq 0 \quad \forall n \geq 3 \\ \alpha_{n+1} \leq \alpha_n \\ \alpha_n \to 0 \end{cases}$$

5.c) $\displaystyle\sum_{n=4}^{+\infty} \frac{(-1)^n}{n \log^2 n}$ CONVERGE ASSOLUTAMENTE

$$|a_n| = \frac{1}{n \log^2 n} \to \frac{2^n}{2^n n^2} = \frac{1}{n^2} = b_n \qquad \sum_{n=1}^{+\infty} b_n < +\infty$$

CRITERIO DI CONDENSAZIONE DI CAUCHY

$$\begin{cases} a_n \geq 0 \\ a_{n+1} \leq a_n \\ a_n \to 0 \end{cases} \qquad \sum_{n=0}^{\infty} a_n < +\infty \iff \sum_{n=0}^{\infty} 2^n a_{2^n} < +\infty$$

5.a) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n} + \sqrt[3]{n}}$ CONVERGE (PER LEIBNITZ)

$|a_n| = \dfrac{1}{\sqrt{n} + \sqrt[3]{n}}$ DIVERGE PER C.A. CON $\dfrac{1}{\sqrt{n}}$

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{1}{\sqrt{n} + \sqrt[3]{n}} \qquad \begin{cases} \alpha_n \geq 0 \quad \forall n \geq 1 \\ \alpha_{n+1} \leq \alpha_n \\ \alpha_n \to 0 \end{cases}$$

5.b) $\displaystyle\sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt{n} - \sqrt[3]{n}}$ CONVERGE (PER LEIBNITZ)

$$|a_n| = \frac{1}{\sqrt{n} - \sqrt[3]{n}} \geq \frac{1}{\sqrt{n}} = b_n \qquad \sum_{n=1}^{+\infty} b_n = +\infty$$

$$a_n = (-1)^n \alpha_n \qquad \alpha_n = \frac{1}{\sqrt{n} - \sqrt[3]{n}} \qquad \begin{cases} \alpha_n \geq 0 \quad \forall n \geq 2 \\ \alpha_{n+1} \leq \alpha_n \quad ? \\ \alpha_n \to 0 \end{cases}$$

$$\alpha_{n+1} \leq \alpha_n \qquad \sqrt{n+1} - \sqrt[3]{n+1} \geq \sqrt{n} - \sqrt[3]{n} \geq 0$$

$$\sqrt{n+1} - \sqrt{n} \geq \sqrt[3]{n+1} - \sqrt[3]{n} \geq 0$$

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n+1}-\sqrt[3]{n}} \geq 1 \qquad \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n+1}-\sqrt[3]{n}} = \frac{\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2+n}+\sqrt[3]{n^2}}{\sqrt{n+1}+\sqrt{n}} =$$

$$= \frac{\sqrt[3]{n^2}}{\sqrt{n}} \frac{\sqrt[3]{(1+1/n)^2}+\sqrt[3]{1+1/n}+1}{\sqrt{1+1/n}+1} \to +\infty$$

5.c) $\displaystyle\sum_{n=1}^{+\infty} \frac{\sqrt{n}\cos(n!)}{7+n^2}$ CONVERGE ASSOLUTAMENTE

$$|a_n| = \frac{\sqrt{n}\,|\cos(n!)|}{7+n^2} \leq \frac{\sqrt{n}}{7+n^2}$$ CONVERGE PER C.A. CON $\frac{1}{\sqrt{n^3}}$

6.a) $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n-\sin n}{n^4-\arctan n^2}$ CONVERGE ASSOLUTAMENTE

$$|a_n| = \frac{n^2+3n-\sin n}{n^4-\arctan n^2} \sim \frac{1}{n^2} \qquad \sum_{n=1}^{+\infty} |a_n| < +\infty \quad \text{PER C.A.}$$

$$b_n = \frac{1}{n^2} \qquad \sum_{n=1}^{+\infty} b_n < +\infty \qquad \frac{|a_n|}{b_n} = \frac{n^4+3n^3-\sin n \cdot n^2}{n^4-\arctan n^2} \to 1$$

6.b) $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n-\sin n}{n^3-\arctan n^2}$ CONVERGE

$$|a_n| = \frac{n^2+3n-\sin n}{n^3-\arctan n^2} \sim \frac{1}{n}$$ NON CONVERGE ASSOLUTAMENTE

$$\sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n-\sin n}{n^3-\arctan n^2} = \overbrace{\sum_{n=1}^{+\infty} (-1)^n \left(\frac{n^2+3n-\sin n}{n^3-\arctan n^2} - \frac{1}{n}\right)}^{\text{CONVERGE}} + \underbrace{\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}}_{\text{CONVERGE}}$$

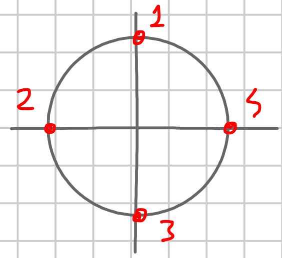
$$\sum_{n=1}^{+\infty} (-1)^n \left(\frac{n^2+3n-\sin n}{n^3-\arctan n^2} - \frac{1}{n}\right) = \sum_{n=1}^{+\infty} (-1)^n \left(\frac{\cancel{n^3}+3n^2-n\sin n-\cancel{n^3}+\arctan n^2}{n^4-n\arctan n^2}\right) =$$

$$\sum_{n=1}^{+\infty} (-1)^n \left(\frac{3n^2-n\sin n+\arctan n^2}{n^4-n\arctan n^2}\right)$$ A.C. × CONFR. ASINT. CON $b_n = \frac{1}{n^2}$

7.a) $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n-\sin n}{n^2-\arctan n^2}$ NON CONVERGE $(a_n \not\to 0)$

7.b) $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \frac{3n+\cos(\pi n)}{n}$ NON CONVERGE $(a_n \not\to 0)$

8.a) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n} \sin\left(\frac{\pi n}{2}\right)$ CONVERGE

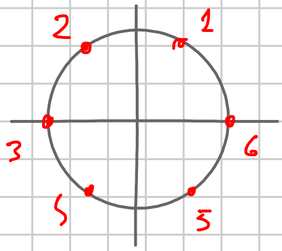


$$\sum_{n=1}^{+\infty} \frac{1}{n} \sin\left(\frac{\pi n}{2}\right) = \sum_{k=1}^{+\infty} \frac{1}{2k-1} \cdot (-1)^{k+1}$$

CONVERGE PER LEIBNITZ (MA NON A.C.)

$$\alpha_k = \frac{1}{2k-1} \begin{cases} \alpha_k \geq 0 \\ \alpha_{k+1} \leq \alpha_k & \left(\dfrac{1}{2k+1} \leq \dfrac{1}{2k-1}\right) \\ \alpha_k \to 0 \end{cases}$$

8.b) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n} \sin\left(\frac{\pi n}{3}\right)$ CONVERGE



$$\sum_{n=1}^{+\infty} \frac{1}{n} \sin\left(\frac{\pi n}{3}\right) = \underbrace{\sum_{k=1}^{+\infty} \frac{1}{3k-2}\, \frac{\sqrt{3}}{2} (-1)^{k+1}}_{\downarrow} + \underbrace{\sum_{k=1}^{+\infty} \frac{1}{3k-1}\, \frac{\sqrt{3}}{2} (-1)^{k+1}}_{\downarrow}$$

CONVERGONO PER LEIBNITZ

9.a) $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \sinh\left(\frac{n^2+\sin^2(n!)}{n+\sin(n!)}\right)$ NON CONVERGE $(a_n \not\to 0)$

9.b) $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \sinh\left(\frac{n+\sin(n!)}{n^2+\sin^2(n!)}\right)$ CONVERGE

PONIAMO $\displaystyle\omega(n) = \frac{n+\sin(n!)}{n^2+\sin^2(n!)} \to 0^+$

$$\sum_{n=1}^{+\infty} (-1)^n \sinh\left(\frac{n+\sin(n!)}{n^2+\sin^2(n!)}\right) = \overbrace{\sum_{n=1}^{+\infty} (-1)^n \left[\sinh(\omega(n)) - \omega(n)\right]}^{\text{CONVERGE}} + \overbrace{\sum_{n=1}^{+\infty} (-1)^n \omega(n)}^{\text{CONVERGE}}$$

$$\sum_{n=1}^{+\infty}(-1)^n\left[\sinh(\omega(n))-\omega(n)\right] \quad \text{A.C. PER CONFR. ASINT. CON } \omega(n)^3$$

$$\frac{|\sinh(\omega(n))-\omega(n)|}{\omega(n)^3}=\frac{|\cancel{\omega(n)}+\omega(n)^3/6-\cancel{\omega(n)}+o(\omega(n))|}{\omega(n)^3}\to 1/6$$

$$\sum_{n=1}^{+\infty}(-1)^n\omega(n)=\underbrace{\sum_{n=1}^{+\infty}(-1)^n\left(\omega(n)-\frac{1}{n}\right)}^{\text{CONVERGE}}+\overbrace{\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}}^{\text{CONVERGE}}$$

$$\sum_{n=1}^{+\infty}(-1)^n\left(\omega(n)-\frac{1}{n}\right) \quad \text{A.C. PER CONFR. ASINT. CON } 1/n^2$$

$$\frac{|\omega(n)-1/n|}{1/n^2}=n^2\left(\frac{n+\sin(n!)}{n^2+\sin^2(n!)}-\frac{1}{n}\right)=$$

$$=n^2\left(\frac{\cancel{n^2}+n\sin(n!)-\cancel{n^2}-\sin(n!)}{n^3+n\sin^2(n!)}\right)\to 1$$