

3. (a) Determinare per quali esponenti α esiste una costante reale c tale che

$$\log(1+x) \leq c(x - \sin x)^\alpha \quad \forall x \geq 0.$$

(b) Determinare per quali esponenti α esiste una costante reale c tale che

$$e^{-1/x^2} \leq cx^\alpha \quad \forall x > 0.$$

(a)

$$\log(1+x) \leq c(x - \sin x)^\alpha \Rightarrow \frac{1}{c} \leq \frac{(x - \sin x)^\alpha}{\log(1+x)} \quad x \neq 0$$

$$\text{CONSIDERIAMO: } f(x) = \frac{(x - \sin x)^\alpha}{\log(1+x)} \quad \text{CONTINUA } (x > 0)$$

COMPORTAMENTO AI LIMITI:

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{(x - \sin x)^\alpha}{\log(1+x)} &= \lim_{x \rightarrow 0^+} \frac{(x - x + x^3/6 + o(x^3))^\alpha}{x + o(x)} = \\ &= \lim_{x \rightarrow 0^+} \frac{x^{3\alpha} \left(\frac{1}{6} + \frac{o(x^3)}{x^3} \right)^\alpha}{x \left(1 + \frac{o(x)}{x} \right)} = \begin{cases} 0 & 3\alpha > 1 & \alpha > 1/3 \\ 1/6^\alpha & 3\alpha = 1 & \alpha = 1/3 \\ +\infty & 3\alpha < 1 & \alpha < 1/3 \end{cases} \end{aligned}$$

$$\lim_{x \rightarrow +\infty} \frac{(x - \sin x)^\alpha}{\log(1+x)} = \frac{x^\alpha \left(1 - \frac{\sin x}{x} \right)^\alpha}{\log(1+x)} = \begin{cases} +\infty & \alpha > 0 \\ 0 & \alpha \leq 0 \end{cases}$$

$\Rightarrow 0 < \alpha \leq 1/3$ $f(x)$ È LIMITATA INFERIORMENTE

$$\Rightarrow \exists c \in \mathbb{R} \text{ s.c. } \frac{1}{c} \leq \frac{(x - \sin x)^\alpha}{\log(1+x)} \quad \forall x > 0$$

$$x=0 \quad \log(1+x) = 0 \leq c \cdot (0-0)^\alpha = 0 \quad \forall c \in \mathbb{R}$$

(b)

$$e^{-1/x^2} \leq c x^2 \quad \forall x > 0 \Leftrightarrow c \geq \frac{e^{-1/x^2}}{x^2}$$

CONSIDERIAMO: $f(x) = \frac{e^{-1/x^2}}{x^2}$ CONTINUA ($x > 0$)

COMPORTAMENTO AI LIMITI:

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x^2} = \lim_{x \rightarrow 0^+} \frac{1}{x^2 e^{1/x^2}} = \lim_{y \rightarrow +\infty} \frac{y^2}{e^{y^2}} = 0 \quad \forall \epsilon$$

$$\lim_{x \rightarrow +\infty} \frac{e^{-1/x^2}}{x^2} = \lim_{x \rightarrow +\infty} \frac{1}{x^2 e^{1/x^2}} = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha > 0 \\ +\infty & \alpha < 0 \end{cases}$$

$\leadsto \alpha \geq 0$ $f(x)$ È LIMITATA SUPERIORMENTE

$$\Rightarrow \exists c \in \mathbb{R} \text{ s.c. } c \geq \frac{e^{-1/x^2}}{x^2} \quad \forall x > 0$$