(c) Detta c_n la più grande costante reale per cui vale la (1), calcolare il limite $n\sqrt[n]{c_n}$

$$f(x)=\frac{\arctan x + x^n}{2^x}$$

STUDIO IL MAX PER $n\to+\infty$ CON $x>1$

$$f(x)=\frac{\arctan x + x^n}{2^x}=\frac{x^n}{2^x}\left(\frac{\arctan x}{x^n}+1\right)$$

PER $n\to+\infty$ $\quad f_{MAX}\simeq g_{MAX}$ CON $g(x)=\frac{x^n}{2^x}$

$g(0)=0 \quad \lim_{x\to+\infty} g(x)=0 \Rightarrow$ x W. $\exists\ g_{MAX}$

$$g'(x)=\frac{nx^{n-1}\cdot\cancel{2^x}-\cancel{2^x}\lg 2\, x^n}{(2^x)^{\cancel{2}}}=0 \qquad nx^{n-1}-\lg 2\cdot x^n=0$$

$$x^n(n-\lg 2\cdot x)=0 \Rightarrow \begin{cases} x=0 \\ x=n/\lg 2 \end{cases}$$

PUNTO DI MAX PER g

PERTANTO:

$$f_{MAX}=f(x_n)=\frac{\arctan x_n + x_n^n}{2^{x_n}} \qquad x_n\sim n/\lg 2 \quad n\to+\infty$$

$$c_n=\frac{1}{f_{MAX}}=\frac{2^{x_n}}{\arctan x_n + x_n^n}$$

$$a_n=n\sqrt[n]{c_n}=\sqrt[n]{b_n} \qquad b_n=n^n\cdot c_n$$

$$\frac{b_{n+1}}{b_n}=\frac{(n+1)^{(n+1)}}{n^n}\cdot\frac{2^{x_{n+1}}}{2^{x_n}}\cdot\frac{\arctan x_n + x_n^n}{\arctan x_{n+1} + x_{n+1}^{n+1}}=$$

$$= \frac{2^{x_{n+1}}}{2^{x_n}} \frac{(n+1)^{(n+1)}}{n^n} \frac{x_n^n}{x_{n+1}^{n+1}} \frac{\frac{\arctan x_n}{x_n^n}+1}{\frac{\arctan x_{n+1}}{x_{n+1}^{n+1}}+1}$$

$$\frac{2^{x_{n+1}}}{2^{x_n}} \sim \frac{2^{(n+1)/\log 2}}{2^{n/\log 2}} = 2^{1/\log 2} = 2^{\log_2 e} = e$$

$$\frac{(n+1)^{(n+1)}}{n^n} \frac{x_n^n}{x_{n+1}^{n+1}} \sim \frac{(n+1)^{(n+1)}}{n^n} \frac{\cancel{n^n}}{\cancel{(n+1)^{(n+1)}}} \frac{(\log 2)^{\cancel{n}+1}}{\cancel{(\log 2)^n}} = \log 2$$

$$\frac{b_{n+1}}{b_n} = \underbrace{\frac{2^{x_{n+1}}}{2^{x_n}}}_{\to e} \underbrace{\frac{(n+1)^{(n+1)}}{n^n} \frac{x_n^n}{x_{n+1}^{n+1}}}_{\to \log 2} \underbrace{\frac{\frac{\arctan x_n}{x_n^n}+1}{\frac{\arctan x_{n+1}}{x_{n+1}^{n+1}}+1}}_{\to 1} \to e \log 2$$

PER IL CRITERIO RAPPORTO $\to$ RADICE

$$a_n = n\sqrt[n]{c_n} = \sqrt[n]{b_n} \to e \log 2$$