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Linguaggio degli infinitesimi 2 – Parte principale

Argomenti: ordine di infinitesimo e parte principale **Difficoltà:** ⋆⋆⋆

Prerequisiti: limiti notevoli, sviluppi di Taylor

Le funzioni indicate nella seguente tabella sono infinitesime per $x \to 0^+$. Determinare il loro ordine di infinitesimo e la parte principale (scrivere solo la parte principale)

	a) Funzione	P. P.	b) Funzione	P. P.	c) Funzione	P. P.
1)	$\sin x$	x	$\sin x^3$	x^3	$\sin^3 x$	x^3
2)	$\cos x - 1$	$-x^2/2$	$\cos x^2 - 1$	$-x^5/2$	$\cos^2 x - 1$	$-x^2$
3)	$\cos \sqrt{x} - 1$	$-x/2$	$\cos x - e^x$	$-x$	$\cos x - e^{x^2}$	$-3x^2/2$
4)	$\sin x - x$	$-x^3/6$	$\sin x^2 - x^2$	$-x^6/6$	$\sin^2 x - x^2$	$-x^5/3$
5)	$\sinh x - 2x$	$-x$	$\sqrt{x} - \sinh \sqrt{x}$	$-\sqrt{x^3}/6$	$\sinh^{22} x - x^{22}$	$11x^{25}/3$
6)	$\frac{\sin x}{x} - 1$	$-\frac{x^2}{6}$	$(1+x)^{1/x} - e$	$-\frac{ex}{2}$	$\frac{1}{\sin x} - \frac{1}{x}$	$\frac{x}{6}$
7)	$\log x - \log(\sin x)$	$\frac{x^2}{6}$	$2^x - 2^{\sin x}$	$\frac{x^3}{6}\log 2$	$\cos x - \cos(\sin x)$	$-x^5/6$
8)	$\arccos(1-x)$	$\sqrt{2x}$	$\sqrt{e^x} - \sqrt[3]{\cos x}$	$x/2$	$\sin(\sinh x) - \sinh(\sin x)$	$-x^7/45$

Le funzioni indicate nella seguente tabella sono infinitesime per $x \to x_0$. Determinare il loro ordine di infinitesimo e la parte principale (scrivere solo la parte principale)

	a) Funzione	x_0	P. P.	b) Funzione	x_0	P. P.	c) Funzione	x_0	P. P.
9)	$\log x - \log 3$	3	$\frac{x-3}{3}$	$\cos x$	$\pi/2$	$-(x-\frac{\pi}{2})$	$x^2 - x$	1	$x-1$
10)	$2x^3 - 3x^2 + 1$	1	$3(x-1)^2$	$3^{x^2} - 3$	1	$\log 3^6 (x-1)$	$\sin x - \cos x$	$\pi/4$	$\sqrt{2}(x-\frac{\pi}{4})$

Le successioni indicate nella seguente tabella sono infinitesime. Determinare il loro ordine di infinitesimo e la parte principale (scrivere solo la parte principale)

	a) Successione	P. P.	b) Successione	P. P.	c) Successione	P. P.
11)	$\sqrt[n]{2} - 1$	$\frac{\log 2}{n}$	$1 - \cos \frac{1}{n}$	$\frac{1}{2n^2}$	$\sqrt[n]{2} - \cos \frac{n+3}{n^2-5}$	$\frac{\log 2}{n}$
12)	$\sqrt{n+1} - \sqrt{n}$	$\frac{1}{2\sqrt{n}}$	$\sqrt[22]{n+2} - \sqrt[22]{n}$	$\frac{n^{-21/22}}{11}$	$\frac{n^3+5}{n^3+7} - \sqrt[n]{7}$	$-\frac{\log 7}{n}$
13)	$\left(1 + \frac{1}{n^{33}}\right)^{n^{22}}$	1	$\frac{\sqrt[3]{n+1} - \sqrt{3n}}{n}$	$-\frac{\sqrt{3}}{\sqrt{n}}$	$\sqrt[n]{\frac{2n+1}{3n-1}} - \sqrt[n]{\frac{4n+1}{5n-1}}$	$\frac{\log(5/6)}{n}$

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1.a) $\sin x = x + o(x)$

1.b) $\sin x^3 = x^3 + o(x^3)$

1.c) $\sin^3 x = x^3 + o(x^3)$

2.a) $\cos x - 1 = -\frac{x^2}{2} + o(x^2)$

2.b) $\cos x^2 - 1 = -\frac{x^5}{2} + o(x^5)$

2.c) $\cos^2 x - 1 = \left(1 - \frac{x^2}{2} + o(x^2)\right)^2 - 1 = -x^2 + o(x^2)$

3.a) $\cos\sqrt{x} - 1 = -\frac{x}{2} + o(x)$

3.b) $\cos x - e^x = 1 - 1 - x + o(x) = -x + o(x)$

3.c) $\cos x - e^{x^2} = 1 - \frac{x^2}{2} - 1 - x^2 + o(x^2) = -\frac{3}{2}x^2 + o(x^2)$

5.a) $\sin x - x = x - \frac{x^3}{6} - x + o(x^3) = -\frac{x^3}{6} + o(x^3)$

5.b) $\sin x^2 - x^2 = x^2 - \frac{x^6}{6} - x^2 + o(x^6) = -\frac{x^6}{6} + o(x^6)$

5.c) $\sin^2 x - x^2 = \left(x - \frac{x^3}{6} + o(x^3)\right)^2 - x^2 = -\frac{x^4}{3} + o(x^5)$

5.a) $\sinh x - 2x = x - 2x + o(x) = -x + o(x)$

5.b) $\sqrt{x} - \sinh\sqrt{x} = \cancel{\sqrt{x}} - \cancel{\sqrt{x}} - \frac{(\sqrt{x})^3}{6} + o(x^{3/2})$

5.c) $\sinh^{22} x - x^{22} = \left(x + \frac{x^3}{6} + o(x^3)\right)^{22} - x^{22} = \frac{11}{3} x^{25} + o(x^5)$

6.a) $\frac{\sin x}{x} - 1 = \frac{1}{x}\left(x - \frac{x^3}{6} + o(x^3)\right) - 1 = -\frac{x^2}{6} + o(x^2)$

6.b) $(1+x)^{1/x} - e = e^{\frac{1}{x}\log(1+x)} - e = e^{1 - \frac{x}{2} + o(x)} - e =$

$= e \cdot e^{-\frac{x}{2} + o(x)} - e = e\left(1 - \frac{x}{2} + o(x)\right) - e = -\frac{e}{2}x + o(x)$

6.c) $\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} = \frac{x - x + \frac{x^3}{6} + o(x^3)}{x^2 + o(x^2)} = \frac{\frac{x}{6} + o(x)}{1 + \frac{o(x^2)}{x^2}} =$

$= \left(\frac{x}{6} + o(x)\right)\left(1 + \omega(x)\right)^{-1} = \left(\frac{x}{6} + o(x)\right)\left(1 - \omega(x) + o(\omega(x))\right) =$

$= \frac{x}{6} + o(x)$

7.a) $\log x - \log(\sin x) = -\log \frac{\sin x}{x} = -\log\left(1 - \frac{x^2}{6} + o(x^2)\right) =$

$= -\left(-\frac{x^2}{6} + o(x^2)\right) = \frac{x^2}{6} + o(x^2)$

7.b) $2^x - 2^{\sin x} = 2^x - 2^{x - x^3/6 + o(x^3)} = 1 + x\log 2 - \left(1 + x\log 2 - x^3\frac{\log 2}{6} + o(x^3)\right) =$

$= \frac{\log 2}{6}x^3 + o(x^3)$

7.c) $\cos x - \cos(\sin x) = \cos x - \cos\left(x - \frac{x^3}{6} + o(x^3)\right) =$

$= \cancel{1} - \frac{x^2}{2} + \frac{x^5}{25} + o(x^5) - \cancel{1} + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 +$

$- \frac{1}{25}\left(x - \frac{x^3}{6} + o(x^3)\right)^5 + o(x^5) =$

$= \cancel{-\frac{x^2}{2}} + \cancel{\frac{x^5}{25}} + \cancel{\frac{x^2}{2}} - \frac{x^5}{6} - \cancel{\frac{x^5}{25}} + o(x^5) = -\frac{x^5}{6} + o(x^5)$

8.a) $\arccos(1-x)$

$\left(\arccos(1-x)\right)' = \frac{-1}{\sqrt{1-(1-x)^2}}(-1) = (1 - 1 - x^2 + 2x)^{-1/2} =$

$= (2x)^{-1/2}(1 - x/2)^{-1/2} = (2x)^{-1/2}\left(1 + \frac{1}{5}x + o(x)\right) = (2x)^{-1/2} + o\left(x^{-1/2}\right)$

$\arccos(1-x) = (2x)^{1/2} + o(\sqrt{x})$

8.b) $\sqrt{e^x} - \sqrt[3]{\cos x}$

$$\sqrt{e^x} = (1 + x + o(x))^{1/2} = 1 + \frac{1}{2}x + o(x)$$

$$\sqrt[3]{\cos x} = \left(1 - x^2/2 + o(x^2)\right)^{1/3} = 1 - \frac{1}{6}x^2 + o(x^2)$$

$$\sqrt{e^x} - \sqrt[3]{\cos x} = \frac{1}{2}x + o(x)$$

8.c) $\sin(\sinh x) - \sinh(\sin x)$

$$\sin(\sinh x) = \sin\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{7!} + o(x^7)\right) =$$

$$= x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{7!} - \frac{1}{6}\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{7!}\right)^3 +$$

$$+ \frac{1}{120}\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{7!}\right)^5 - \frac{1}{7!}\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{7!}\right)^7 + o(x^7) =$$

$$= x + \cancel{\frac{x^3}{6}} + \frac{x^5}{120} + \cancel{\frac{x^7}{7!}} - \cancel{\frac{x^3}{6}} - \frac{x^5}{12} - \frac{x^7}{72} - \frac{x^7}{240} + \frac{x^5}{120} + \frac{x^7}{144} - \cancel{\frac{x^7}{7!}} + o(x^7) =$$

$$= x - \frac{x^5}{15} - \frac{x^7}{30} + o(x^7)$$

$$\frac{-10-3+5}{720} = \frac{-8}{72} = -\frac{1}{30}$$

$$\sinh(\sin x) = \sinh\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + o(x^7)\right) =$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \frac{1}{6}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}\right)^3 +$$

$$+ \frac{1}{120}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}\right)^5 + \frac{1}{7!}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}\right)^7 + o(x^7)$$

$$= x - \cancel{\frac{x^3}{6}} + \frac{x^5}{120} - \cancel{\frac{x^7}{7!}} + \cancel{\frac{x^3}{6}} - \frac{x^5}{12} + \frac{x^7}{72} + \frac{x^7}{240} + \frac{x^5}{120} - \frac{x^7}{144} + \cancel{\frac{x^7}{7!}} + o(x^7)$$

$$= x - \frac{x^5}{15} + \frac{x^7}{30} + o(x^7)$$

$$\sin(\sinh x) - \sinh(\sin x) = -\frac{x^7}{45} + o(x^7)$$

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9.a) $\log x - \log 3 \qquad x_0 = 3 \qquad x = x_0 + h$

$$\log(3+h) - \log 3 = \log(1 + h/3) = \frac{h}{3} + o(h) =$$

$$= \frac{x-3}{3} + o(x-3)$$

9.b) $\cos x \qquad x_0 = \pi/2 \qquad x = x_0 + h$

$$\cos(\pi/2 + h) = -\sin h = -h + o(h) = -(x - \pi/2) + o(x - \pi/2)$$

9.c) $x^2 - x \qquad x_0 = 1 \qquad x = x_0 + h$

$$x^2 - x = (h+1)^2 - (h+1) = (h+1)(h) = h + o(h) = (x-1) + o(x-1)$$

10.a) $2x^3 - 3x^2 + 1 \qquad x_0 = 1 \qquad x = x_0 + h$

$$2x^3 - 3x^2 + 1 = 2(1+h)^3 - 3(1+h)^2 + 1 = \cancel{2} + 6h^2 + \cancel{6h} + 2h^3 +$$

$$\cancel{-3} - \cancel{6h} - 3h^2 + \cancel{1} = 3h^2 + o(h^2) = 3(x-1)^2 + o((x-1)^2)$$

10.b) $3^{x^2} - 3 \qquad x_0 = 1 \qquad x = x_0 + h$

$$3^{x^2} = 3^{(1+h)^2} = 3^{1+2h+h^2} = e^{(1+2h+h^2)\log 3} =$$

$$= 3 \cdot e^{2\log 3 \cdot h + o(h)} = 3(1 + 2\log 3 \cdot h + o(h)) =$$

$$= 3 + 6\log 3 \cdot h + o(h)$$

$$3^{x^2} - 3 = 3^{(1+h)^2} - 3 = 6\log 3 \cdot h + o(h) = 6\log 3\,(x-1) + o(x-1)$$

10.c) $\sin x - \cos x \qquad x_0 = \pi/4 \qquad x = x_0 + h$

$$\begin{cases} \sin x = \sin(\pi/4 + h) = \sin(\pi/4)\cdot\cos h + \sin(h)\cdot\cos(\pi/4) \\ \cos x = \cos(\pi/4 + h) = \cos(\pi/4)\cdot\cos(h) - \sin(\pi/4)\cdot\sin(h) \end{cases}$$

$$\sin x - \cos x = \sqrt{2}\sin h = \sqrt{2}\,h + o(h) = \sqrt{2}(x - \pi/4) + o(x - \pi/4)$$

11.a) $\sqrt[n]{2}-1 = e^{\log 2/n} - 1 = \log 2 \, \frac{1}{n} + o\left(\frac{1}{n}\right)$

11.b) $1 - \cos\frac{1}{n} = \cancel{1} - \cancel{1} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)$

11.c) $\sqrt[n]{2} - \cos\frac{n+3}{n^2-5} = \cancel{1} + \frac{\log 2}{n} - \cancel{1} + \frac{1}{2n^2} + o\left(\frac{1}{n}\right) = \frac{\log 2}{n} + o\left(\frac{1}{n}\right)$

12.a) $\sqrt{n+1} - \sqrt{n} = \sqrt{n}\left((1+1/n)^{1/2} - 1\right) = \sqrt{n}\left(\cancel{1} + \frac{1}{2n} + o\left(\frac{1}{n}\right) - \cancel{1}\right) =$

$= \frac{1}{2\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$

12.b) $\sqrt[22]{n+2} - \sqrt[22]{n} = \sqrt[22]{n}\left((1+2/n)^{1/22} - 1\right) = \sqrt[22]{n}\left(1 + \frac{1}{11n} + o\left(\frac{1}{n}\right) - 1\right) =$

$= \frac{1}{11 n^{21/22}} + o\left(\frac{1}{n^{21/22}}\right)$

12.c) $\frac{n^3+5}{n^3+7} - \sqrt[n]{7} = (n^3+5)n^{-3}(1+7/n^3)^{-1} - e^{\frac{\log 7}{n}} =$

$= \left(1 + \frac{5}{n^3}\right)\left(1 - \frac{7}{n^3} + o\left(\frac{1}{n^3}\right)\right) - 1 - \frac{\log 7}{n} + o\left(\frac{1}{n}\right) =$

$= \cancel{1} - \frac{2}{n^3} + o\left(\frac{1}{n^3}\right) - \cancel{1} - \frac{\log 7}{n} + o\left(\frac{1}{n}\right) = -\frac{\log 7}{n} + o\left(\frac{1}{n}\right)$

13.a) $\left(1 + \frac{1}{n^{33}}\right)^{n^{22}} = e^{n^{22}\log\left(1+\frac{1}{n^{33}}\right)} = e^{n^{22}\left(\frac{1}{n^{33}} + o\left(\frac{1}{n^{33}}\right)\right)} =$

$= e^{\frac{1}{n^{11}} + o\left(\frac{1}{n^{11}}\right)} = 1 + \frac{1}{n^{11}} + o\left(\frac{1}{n^{11}}\right)$

13.b) $\frac{\sqrt[3]{n+1} - \sqrt{3n}}{n} = n^{-2/3}\left(1 + \frac{1}{n}\right)^{1/3} - \sqrt{3}\, n^{-1/2}) =$

$= n^{-2/3}\left(1 + \frac{1}{3n} + o\left(\frac{1}{n}\right)\right) - \sqrt{3}\, n^{-1/2} = -\sqrt{3}\, n^{-1/2} + o(n^{-1/2})$

13.c) $\sqrt[n]{\dfrac{2n+1}{3n-1}} - \sqrt[n]{\dfrac{5n+1}{5n-1}}$

$$\sqrt[n]{\frac{2n+1}{3n-1}} = e^{\frac{\log(2n+1)-\log(3n-1)}{n}}$$

$$\frac{\log(2n+1)-\log(3n-1)}{n} = \frac{1}{n}\left(\log(1+1/2n)-\log(1-1/3n)+\log(2/3)\right) =$$

$$= \frac{1}{n}\left(\frac{1}{2n}+\frac{1}{3n}+o\left(\frac{1}{n}\right)+\log(2/3)\right) = \frac{\log(2/3)}{n}+o\left(\frac{1}{n}\right)$$

$$\sqrt[n]{\frac{5n+1}{5n-1}} = \frac{\log(5/5)}{n}+o\left(\frac{1}{n}\right)$$

$$\sqrt[n]{\frac{2n+1}{3n-1}} - \sqrt[n]{\frac{5n+1}{5n-1}} = \frac{\log(2/3)}{n} - \frac{\log(5/5)}{n} + o\left(\frac{1}{n}\right) =$$

$$= \frac{\log(3/6)}{n}+o\left(\frac{1}{n}\right)$$