Sviluppi di Taylor 2

Argomenti: sviluppi di Taylor con centro qualunque **Difficoltà:** ⋆⋆⋆

Prerequisiti: sviluppi di Taylor, operazioni algebriche con i polinomi di Taylor

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In ogni riga delle seguenti tabelle sono indicati una funzione $f(x)$ ed un numero reale x_0. Si chiede di determinare il polinomio di Taylor di $f(x)$ di ordine 3 con centro in x_0, cioè l'unico polinomio $P_3(h)$ di grado minore o uguale a 3 tale che $f(x_0+h) = P_3(h) + o(h^3)$ per $h \to 0$.

	$f(x)$	x_0	a) $P_3(h)$	$f(x)$	x_0	b) $P_3(h)$
1)	e^x	2	$e^2+e^2h+\frac{e^2}{2}h^2+\frac{e^2}{6}h^3$	$\log x$	5	$\log 5+\sum_1^3\frac{(-1)^{n+1}}{n!}\left(\frac{h}{5}\right)^k$
2)	x^3-x	1	$2h+3h^2+h^3$	x^5	-1	$-1+5h-10h^2+10h^3$
3)	$\sqrt{2+x}$	0	$\sqrt2+\frac{\sqrt2}{4}h-\frac{\sqrt2}{32}h^2+\frac{\sqrt2}{128}h^3$	$(5+x)^{-1}$	0	$\frac15-\frac{h}{25}+\frac{h^2}{125}-\frac{h^3}{625}$
4)	$\dfrac{x+5}{x^2+4}$	0	$\frac54+\frac h4-\frac{5}{16}h^2-\frac{h^3}{16}$	$\dfrac{x+5}{x^2+4}$	3	$\frac{8}{13}-\frac{35}{13^2}h+\frac{106}{13^3}h^2-\frac{161}{13^4}h^3$
5)	$\sin x$	π	$-h+h^3/6$	$\cos x$	5π	$-1+h^2/2$
6)	$\sin x$	$\pi/6$	$\frac12+\frac{\sqrt3}{2}h-\frac{h^2}{4}-\frac{\sqrt3}{12}h^3$	$\arctan x$	1	$\frac{\pi}{4}+\frac h2-\frac{h^2}{4}+\frac{h^3}{12}$
7)	$e^{\cos x}$	0	$e-e/2\,h^2$	$\sin(\cos x)$	0	$\sin 1-\frac{\cos 1}{2}h^2$
8)	$\arccos x$	0	$\frac{\pi}{2}-h-\frac16h^3$	$\log(3+\sin x)$	0	$\log 3+\frac h3-\frac{h^2}{18}-\frac{7}{162}h^3$

In ogni riga della seguente tabella sono indicati una funzione $f(x)$, un punto x_0 ed un intero positivo n. Si chiede di determinare il polinomio di Taylor $P_n(h)$ di ordine n di $f(x)$ con centro in x_0, ed il più grande intero k per cui vale lo sviluppo $f(x_0+h) = P_n(h) + o(h^k)$ per $h \to 0$.

	$f(x)$	x_0	n	$P_n(x)$	k
9)	$\sin(x+x^4)\cdot\cos(\log(1+x))$	0	4	$h-\frac23h^3+\frac32h^4$	4
10)	$\sqrt{x}\arctan(x\sqrt{x})$	0	8	$h^2-\frac13h^5+\frac15h^8$	10
11)	$\sqrt[3]{\cos x}$	0	3	$1-h^2/6$	3
12)	$\log(1+\sin^2(e^x-1))$	0	4	$h^2+h^3-h^4/4$	4
13)	$\sqrt{x}-\sqrt[3]{2x}$	2	3	$\sqrt2-\sqrt[3]{4}+\left(\frac{\sqrt2}{4}-\frac{\sqrt[3]{4}}{6}\right)h+\left(\frac{\sqrt[3]{4}}{36}-\frac{\sqrt2}{32}\right)h^2+\left(\frac{\sqrt2}{128}-\frac{5\sqrt[3]{4}}{648}\right)h^3$	3
14)	$x^3 3^{-x}$	1	3	$\frac13+\left(1-\frac{\log 3}{3}\right)h+\left(1-\log 3+\frac{\log^2 3}{6}\right)h^2+\left(\frac13-\log 3+\frac{\log^2 3}{2}-\frac{\log^3 3}{18}\right)h^3$	3
15)	$\log x-\log(4-x)$	2	3	$h+h^3/12$	4
16)	$\arctan(e^x)$	0	3	$\pi/4+h/2-h^3/12$	4
17)	$\sqrt{\dfrac{e^x+1}{\cos x+2}}$	0	3	$\sqrt{\frac23}+\sqrt{\frac23}\,\frac h4+\frac{17}{96}\sqrt{\frac23}\,h^2+\frac{5}{128}\sqrt{\frac23}\,h^3$	3

1.a) $e^x \qquad x_0 = 2 \qquad x = x_0 + h$

$$e^{2+h} = e^2 \cdot e^h = e^2\left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + o(h^3)\right) =$$

$$= e^2 + e^2 h + \frac{e^2}{2} h^2 + \frac{e^2}{6} h^3 + o(h^3)$$

1.b) $\log x \qquad x_0 = 5 \qquad x = x_0 + h$

$$\log(5+h) = \log 5 + \log(1 + h/5) =$$

$$= \log 5 + \frac{h}{5} - \frac{1}{2}\left(\frac{h}{5}\right)^2 + \frac{1}{3}\left(\frac{h}{5}\right)^3 - \frac{1}{4}\left(\frac{h}{5}\right)^4 + \frac{1}{5}\left(\frac{h}{5}\right)^5 + o(h^5)$$

2.a) $x^3 - x \qquad x_0 = 1 \qquad x = x_0 + h$

$$(1+h)^3 - (1+h) = \cancel{1} + 3h + 3h^2 + h^3 - \cancel{1} - h = 2h + 3h^2 + h^3$$

2.b) $x^5 \qquad x_0 = -1 \qquad x = x_0 + h$

$$(-1+h)^5 = -1 + 5h - 10h^2 + 10h^3 + o(h^3)$$

3.a) $\sqrt{2+x} \qquad x_0 = 0 \qquad x = h$

$$\sqrt{2+h} = \sqrt{2}\sqrt{1 + h/2} =$$

$$= \sqrt{2}\left(1 + \frac{1}{2}\,\frac{h}{2} + \frac{1}{2!}\,\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{h}{2}\right)^2 + \frac{1}{3!}\left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)\left(\frac{h}{2}\right)^3 + o(h^3)\right) =$$

$$= \sqrt{2}\left(1 + \frac{h}{4} - \frac{h^2}{32} + \frac{h^3}{128} + o(h^3)\right) =$$

$$= \sqrt{2} + \frac{\sqrt{2}}{4} h - \frac{\sqrt{2}}{32} h^2 + \frac{\sqrt{2}}{128} h^3 + o(h^3)$$

3.b) $(5+x)^{-1} \qquad x_0 = 0 \qquad x = h$

$$(5+h)^{-1} = 5^{-1}(1 + h/5)^{-1} =$$

$$= 5^{-1}\left(1 - 1\,\frac{h}{5} + \frac{(-1)(-2)}{2!}\left(\frac{h}{5}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{h}{5}\right)^3 + o(h^3)\right) =$$

$$= \frac{1}{5} - \frac{h}{25} + \frac{h^2}{125} - \frac{h^3}{625} + o(h^3)$$

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2$$

5.a) $\dfrac{x+5}{x^2+4}$ $\quad x_0 = 0 \quad x = \delta$

$$(\delta^2+4)^{-1} = \frac{1}{4}\left(1+\frac{\delta^2}{4}\right)^{-1} = \frac{1}{4}\left(1-\frac{\delta^2}{4}+o(\delta^3)\right)$$

$$\frac{\delta+5}{\delta^2+4} = (\delta+5)\frac{1}{4}\left(1-\frac{\delta^2}{4}+o(\delta^3)\right) = \frac{5}{4}+\frac{\delta}{4}-\frac{5}{16}\delta^2-\frac{\delta^3}{16}+o(\delta^3)$$

5.b) $\dfrac{x+5}{x^2+4}$ $\quad x_0 = 3 \quad x = x_0+\delta$

$$\frac{x+5}{x^2+4} = \frac{8+\delta}{\delta^2+6\delta+13}$$

$$(\delta^2+6\delta+13)^{-1} = \frac{1}{13}\left(1+\frac{6}{13}\delta+\frac{\delta^2}{13}\right)^{-1} =$$

$$= \frac{1}{13}\left(1-\frac{6}{13}\delta-\frac{\delta^2}{13}+\frac{(-1)(-2)}{2!}\left(\frac{6}{13}\delta+\frac{\delta^2}{13}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{6}{13}\delta+\frac{\delta^2}{13}\right)^3+o(\delta^3)\right) =$$

$12 \cdot 13 = 156 \qquad 216$

$$= \frac{1}{13}\left(1-\frac{6}{13}\delta-\frac{\delta^2}{13}+\frac{6^2}{13^2}\delta^2+\frac{12}{13^{\cancel{2}\,3}}\delta^3-\frac{\cancel{6^3}}{13^3}\delta^3+o(\delta^3)\right)$$

$$= \frac{1}{13}-\frac{6}{13^2}\delta+\frac{23}{13^3}\delta^2-\frac{60}{13^4}\delta^3+o(\delta^3)$$

$$\frac{x+5}{x^2+4} = \frac{8+\delta}{\delta^2+6\delta+13} = (8+\delta)(\delta^2+6\delta+13)^{-1} =$$

$$= \frac{8}{13}-\frac{48}{13^2}\delta+\frac{184}{13^3}\delta^2-\frac{480}{13^4}\delta^3+\frac{\delta}{13}-\frac{6}{13^2}\delta^2+\frac{23}{13^3}\delta^3+o(\delta^3) =$$

$23 \cdot 13 = 299$

$$= \frac{8}{13}+\left(\frac{1}{13}-\frac{48}{13^2}\right)\delta+\left(\frac{184}{13^3}-\frac{6}{13^2}\right)\delta^2+\left(\frac{23}{13^{\cancel{3}\,4}}-\frac{480}{13^4}\right)\delta^3+o(\delta^3) =$$

$$= \frac{8}{13}-\frac{35}{13^2}\delta+\frac{106}{13^3}\delta^2-\frac{181}{13^4}\delta^3+o(\delta^3)$$

5.a) $\sin x \quad x_0 = \pi \quad x = x_0 + h$

$$\sin(x) = \sin(\pi + h) = -\sin h = -h + \frac{h^3}{6} + o(h^3)$$

5.b) $\cos x \quad x_0 = 3\pi \quad x = x_0 + h$

$$\cos x = \cos(3\pi + h) = -\cos h = -1 + \frac{h^2}{2} + o(h^3)$$

6.a) $\sin x \quad x_0 = \pi/6 \quad x = x_0 + h$

$$\sin x = \sin(\pi/6 + h) = \sin \pi/6 \cdot \cos h + \cos \pi/6 \cdot \sin h = \frac{1}{2}\cos h + \frac{\sqrt{3}}{2}\sin h =$$

$$= \frac{1}{2}\left(1 - \frac{h^2}{2}\right) + \frac{\sqrt{3}}{2}\left(h - \frac{h^3}{6}\right) + o(h^3) = \frac{1}{2} + \frac{\sqrt{3}}{2}h - \frac{h^2}{4} - \frac{\sqrt{3}}{12}h^3 + o(h^3)$$

6.b) $\arctan x \quad x_0 = 1 \quad x = x_0 + h$

$$\arctan(1) = \frac{\pi}{4} \qquad (\arctan x)'_{x=1} = \left(\frac{1}{1+x^2}\right)_{x=1} = \frac{1}{2}$$

$$(\arctan x)''_{x=1} = \left(\frac{-2x}{(1+x^2)^2}\right)_{x=1} = -\frac{1}{2}$$

$$(\mathrm{artan}\, x)'''_{x=1} = \left(\frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4}\right)_{x=1} = \frac{1}{2}$$

$$\arctan(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + o(h^n) =$$

$$= \frac{\pi}{4} + \frac{h}{2} - \frac{h^2}{4} + \frac{h^3}{12} + o(h^3)$$

7.a) $e^{\cos x} \quad x_0 = 0 \quad x = h$

$$e^{\cos h} = e^{1 - h^2/2 + o(h^3)} = e \cdot e^{-h^2/2 + o(h^3)} = e - \frac{e}{2}h^2 + o(h^3)$$

7.b) $\sin(\cos x) \quad x_0 = 0 \quad x = h$

$$\cos h = 1 - \frac{h^2}{2} + o(h^3) \qquad \sin(\cos h) = \sin\left(1 - \frac{h^2}{2} + o(h^3)\right) =$$

$$= \sin 1 \cdot \cos\left(-\frac{h^2}{2} + o(h^3)\right) + \cos 1 \cdot \sin\left(-\frac{h^2}{2} + o(h^3)\right) =$$

$$= \sin 1 \cdot 1 + \cos 1 \cdot \left(-\frac{h^2}{2} + o(h^3)\right) = \sin 1 - \frac{\cos 1}{2}h^2 + o(h^3)$$

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8.a) ARCCOSX $x_0=0$ $x=h$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{-1/2} = -\left(1-\frac{1}{2}(-x^2)+o(x^2)\right) =$$

$$= -1-\frac{1}{2}x^2+o(x^2)$$

$$\arccos h = c - h - \frac{1}{6}h^3 + o(h^3) \qquad \arccos(0)=\frac{\pi}{2} \Rightarrow c=\frac{\pi}{2}$$

$$\arccos h = \frac{\pi}{2} - h - \frac{1}{6}h^3 + o(h^3)$$

8.b) $\log(3+\sin x)$ $x_0=0$ $x=h$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\log(3+\sin h) = \log 3 + \log(1+\sin h/3) =$$

$$= \log 3 + \log\left(1+\frac{h}{3}-\frac{h^3}{18}+o(h^3)\right) =$$

$$= \log 3 + \frac{h}{3} - \frac{h^3}{18} - \frac{1}{2}\left(\frac{h}{3}-\frac{h^3}{18}\right)^2 + \frac{1}{3}\left(\frac{h}{3}-\frac{h^3}{18}\right)^3 + o(h^3) =$$

$$= \log 3 + \frac{h}{3} - \frac{h^3}{18} - \frac{h^2}{18} + \frac{h^3}{81} + o(h^3) =$$

$$= \log 3 + \frac{h}{3} - \frac{h^2}{18} - \frac{7}{162}h^3 + o(h^3)$$

8) $\sin(x+x^5)\cdot\cos(\log(1+x))$ $x_0=0$ $n=5$ $h=x$

$$\sin(x+x^5) = x+x^5-\frac{1}{6}(x+x^5)^3+o(x^5) = x-\frac{1}{6}x^3+x^5+o(x^5)$$

$$\log(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)$$

$$\cos(\log(1+x)) = \cos\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)\right) =$$

$$= 1 - \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)^2 + \frac{1}{24}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)^4 + o(x^4) =$$

$$= 1 - \frac{1}{2}\left(x^2 + \frac{1}{4}x^4 - x^3 + \frac{2}{3}x^4\right) + \frac{1}{24}x^4 + o(x^4) =$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{-3-8+1}{24}x^4 + o(x^4) =$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + o(x^4)$$

$$\sin(\chi + \chi^4) \cdot \cos(\log(1+\chi)) =$$

$$= \left(\chi - \frac{1}{6}\chi^3 + \chi^4\right)\left(1 - \frac{1}{2}\chi^2 + \frac{1}{2}\chi^3 - \frac{5}{12}\chi^4\right) + o(\chi^4) =$$

$$= \chi - \frac{1}{2}\chi^3 + \frac{1}{2}\chi^4 - \frac{1}{6}\chi^3 + \chi^4 + o(\chi^4) =$$

$$= \chi - \frac{2}{3}\chi^3 + \frac{3}{2}\chi^4 + o(\chi^4)$$

10) $\sqrt{x}\,\arctan(x\sqrt{x}) \quad x_0 = 0 \quad m = 8 \quad \chi = x$

$$\arctan(x\sqrt{x}) = x\sqrt{x} - \frac{1}{3}(x\sqrt{x})^3 + \frac{1}{5}(x\sqrt{x})^5 + o((x\sqrt{x})^5)$$

$$\sqrt{\chi}\,\arctan(\chi\sqrt{\chi}) = \chi^2 - \frac{1}{3}\chi^5 + \frac{1}{5}\chi^8 + o(\chi^8)$$

11) $\sqrt[3]{\cos x} \quad x_0 = 0 \quad m = 3 \quad \chi = x$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3)$$

$$\sqrt[3]{\cos \chi} = \left(1 - \frac{\chi^2}{2} + o(\chi^3)\right)^{1/3} = 1 + \frac{1}{3}\left(-\frac{\chi^2}{2}\right) + o(\chi^3) = 1 - \frac{1}{6}\chi^2 + o(\chi^3)$$

12) $\log(1 + \sin^2(e^x - 1)) \quad x_0 = 0 \quad m = 4 \quad \chi = x$

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$$

$$\sin(x) = x - \frac{x^3}{6} + o(x^4) \qquad \sin^2 x = x^2 - \frac{x^4}{3} + o(x^5)$$

$$\sin^2(e^x-1) = \left(x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\right)^2 - \frac{1}{3}\left(x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\right)^4 + o(x^4) =$$

$$= x^2+\frac{x^4}{4}+x^3+\cancel{\frac{x^4}{3}}-\cancel{\frac{1}{3}x^4}+o(x^4) = x^2+x^3+\frac{x^4}{4}+o(x^4)$$

$$\log(1+\sin^2(e^s-1)) = \log\left(1+s^2+s^3+\frac{s^4}{4}+o(s^4)\right) =$$

$$= s^2+s^3+\frac{s^4}{4}-\frac{1}{2}\left(s^2+s^3+\frac{s^4}{4}\right)^2+o(s^4) =$$

$$= s^2+s^3+\frac{s^4}{4}-\frac{1}{2}s^4+o(s^4) = s^2+s^3-\frac{s^4}{4}+o(s^4)$$

13) $\sqrt{x}-\sqrt[3]{2x}$ $\quad x_0=2 \quad n=3 \quad s=x-2$

$$\sqrt{x}-\sqrt[3]{2x} = \sqrt{2+s}-\sqrt[3]{4+2s} = \sqrt{2}\,(1+s/2)^{1/2}-\sqrt[3]{4}\,(1+s/2)^{1/3} =$$

$$= \sqrt{2}\left(1+\frac{1}{2}\frac{s}{2}+\frac{1}{2}\frac{1}{2}\left(-\frac{1}{2}\right)\frac{s^2}{4}+\frac{1}{\cancel{6}\,2}\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{\cancel{3}}{2}\right)\frac{s^3}{8}+o(s^3)\right)+$$

$$-\sqrt[3]{4}\left(1+\frac{1}{3}\frac{s}{2}+\frac{1}{\cancel{2}}\frac{1}{3}\left(-\frac{\cancel{2}}{3}\right)\frac{s^2}{4}+\frac{1}{\cancel{6}_{3}}\frac{1}{3}\left(-\frac{\cancel{2}}{3}\right)\left(-\frac{5}{3}\right)\frac{s^3}{8}+o(s^3)\right) =$$

$$= \sqrt{2}\left(1+\frac{s}{4}-\frac{s^2}{32}+\frac{s^3}{128}+o(s^3)\right)-\sqrt[3]{4}\left(1+\frac{s}{6}-\frac{s^2}{36}+\frac{5}{648}s^3+o(s^3)\right) =$$

$$= \sqrt{2}-\sqrt[3]{4}+\left(\frac{\sqrt{2}}{4}-\frac{\sqrt[3]{4}}{6}\right)s+\left(\frac{\sqrt[3]{4}}{36}-\frac{\sqrt{2}}{32}\right)s^2+\left(\frac{\sqrt{2}}{128}-\frac{5\sqrt[3]{4}}{648}\right)s^3+o(s^3)$$

14) $x^3\,3^{-x}$ $\quad x_0=1 \quad n=3 \quad s=x-1$

$$x^3=(1+s)^3=1+3s+3s^2+s^3$$

$$3^{-1-s}=\frac{1}{3}\cdot 3^{-s}=\frac{1}{3}e^{-s\log 3}=\frac{1}{3}\left(1-s\log 3+\frac{1}{2}s^2\log^2 3-\frac{1}{6}s^3\log^3 3+o(s^3)\right)=$$

$$=\frac{1}{3}-\frac{\log 3}{3}s+\frac{\log^2 3}{6}s^2-\frac{\log^3 3}{18}s^3+o(s^3)$$

$$x^3 3^{-x} = (1+s)^3 \cdot 3^{-1-s} =$$

$$=(1+3s+3s^2+s^3)\left(\frac{1}{3}-\frac{\log 3}{3}s+\frac{\log^2 3}{6}s^2-\frac{\log^3 3}{18}s^3+o(s^3)\right)=$$

$$=\frac{1}{3}-\frac{\log 3}{3}s+\frac{\log^2 3}{6}s^2-\frac{\log^3 3}{18}s^3+s-\log 3\, s^2+\frac{\log^2 3}{2}s^3+$$

$$+s^2-\log 3\, s^3+\frac{s^3}{3}+o(s^3)=$$

$$=\frac{1}{3}+\left(1-\frac{\log 3}{3}\right)s+\left(1-\log 3+\frac{\log^2 3}{6}\right)s^2+\left(\frac{1}{3}-\log 3+\frac{\log^2 3}{2}-\frac{\log^3 3}{18}\right)s^3+o(s^3)$$

15) $\log x - \log(5-x) \qquad x_0=2 \qquad n=3 \qquad s=x-2$

$$\log x-\log(5-x) = \log\frac{x}{5-x}$$

$$\frac{x}{5-x} = (2+s)(5-2-s)^{-1} = (2+s)(2-s)^{-1}$$

$$(2-s)^{-1} = \frac{1}{2}\left(1-\frac{s}{2}\right)^{-1} = \frac{1}{2}\left(1+\frac{s}{2}+\frac{1}{2}(-1)(-2)\frac{s^2}{5}+\frac{1}{6}(-1)(-2)(-3)\frac{-s^3}{8}+o(s^3)\right)=$$

$$=\frac{1}{2}+\frac{s}{5}+\frac{s^2}{8}+\frac{s^3}{16}+o(s^3)$$

$$\frac{x}{5-x} = (2+s)(2-s)^{-1} = (2+s)\left(\frac{1}{2}+\frac{s}{5}+\frac{s^2}{8}+\frac{s^3}{16}+o(s^3)\right)=$$

$$=1+\frac{s}{2}+\frac{s^2}{5}+\frac{s^3}{8}+\frac{s}{2}+\frac{s^2}{5}+\frac{s^3}{8}+o(s^3)=$$

$$=1+s+\frac{s^2}{2}+\frac{s^3}{5}+o(s^3)$$

$$\log x-\log(5-x) = \log\frac{x}{5-x} = \log\left(1+s+\frac{s^2}{2}+\frac{s^3}{5}+o(s^3)\right)=$$

$$=s+\frac{s^2}{2}+\frac{s^3}{5}-\frac{1}{2}\left(s+\frac{s^2}{2}\right)^2+\frac{1}{3}s^3+o(s^3)=$$

$$=s+\cancel{\frac{s^2}{2}}+\frac{7}{12}s^3-\cancel{\frac{1}{2}}s^2-\frac{1}{2}s^3+o(s^3)=s+\frac{s^3}{12}+o(s^3)$$

16) $\arctan(e^x) \qquad x_0=0 \qquad n=3 \qquad z=x$

$$(\arctan(e^x))' = \frac{e^x}{1+e^{2x}} = e^x(1+e^{2x})^{-1}$$

$$e^x = e^z = 1+z+\frac{z^2}{2}+\frac{z^3}{6}+o(z^3)$$

$$e^{2x} = e^{2z} = 1+2z+2z^2+\frac{4}{3}z^3+o(z^3)$$

$$(1+e^{2x})^{-1} = (1+e^{2z})^{-1} = \left(2+2z+2z^2+\frac{4}{3}z^3+o(z^3)\right)^{-1} =$$

$$= \frac{1}{2}\left(1+z+z^2+\frac{2}{3}z^3+o(z^3)\right)^{-1} =$$

$$= \frac{1}{2}\left(1-z-z^2-\frac{2}{3}z^3+\frac{1}{2}(-1)(-2)\left(z+z^2\right)^2+\frac{1}{6}(-1)(-2)(-3)\,z^3+o(z^3)\right) =$$

$$= \frac{1}{2}-\frac{z}{2}-\cancel{\frac{z^2}{2}}-\frac{z^3}{3}+\cancel{\frac{z^2}{2}}+z^3-\frac{z^3}{2}+o(z^3) = \frac{1}{2}-\frac{z}{2}+\frac{z^3}{6}+o(z^3)$$

$$e^x(1+e^{2x})^{-1} = e^z(1+e^{2z})^{-1} =$$

$$= \left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+o(z^3)\right)\left(\frac{1}{2}-\frac{z}{2}+\frac{z^3}{6}+o(z^3)\right) =$$

$$= \frac{1}{2}-\cancel{\frac{z}{2}}+\frac{z^3}{6}+\cancel{\frac{z}{2}}-\frac{z^2}{2}+\frac{z^2}{4}-\frac{z^3}{4}+\frac{z^3}{12}+o(z^3) =$$

$$= \frac{1}{2}-\frac{z^2}{4}+o(z^3)$$

$$\arctan(e^x) = \arctan(e^z) = c+\frac{z}{2}-\frac{z^3}{12}+o(z^4)$$

$$\arctan(1) = \pi/4 \rightsquigarrow c = \pi/4$$

$$\arctan(e^x) = \arctan(e^z) = \frac{\pi}{4}+\frac{z}{2}-\frac{z^3}{12}+o(z^4)$$

17) $\sqrt{\dfrac{e^x+1}{\cos x+2}} \qquad x_0=0 \qquad n=3 \qquad x=\delta$

$$e^x+1 = e^\delta+1 = 2+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}+o(\delta^3)$$

$$\cos x+2 = \cos\delta+2 = 3-\frac{\delta^2}{2}+o(\delta^3)$$

$$(\cos x+2)^{-1} = (\cos\delta+2)^{-1} = \frac{1}{3}\left(1-\frac{\delta^2}{6}+o(\delta^3)\right)^{-1} =$$

$$= \frac{1}{3}\left(1+\frac{\delta^2}{6}+o(\delta^3)\right)$$

$$\frac{e^x+1}{\cos x+2} = (e^\delta+1)(\cos\delta+2)^{-1} =$$

$$= \left(2+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}+o(\delta^3)\right)\cdot\frac{1}{3}\left(1+\frac{\delta^2}{6}+o(\delta^3)\right) =$$

$$= \frac{2}{3}+\frac{\delta}{3}+\frac{\delta^2}{6}+\frac{\delta^3}{18}+\frac{\delta^2}{9}+\frac{\delta^3}{18}+o(\delta^3) =$$

$$= \frac{2}{3}+\frac{\delta}{3}+\frac{5}{18}\delta^2+\frac{\delta^3}{9}+o(\delta^3)$$

$$\sqrt{\frac{e^x+1}{\cos x+2}} = \left(\frac{2}{3}+\frac{\delta}{3}+\frac{5}{18}\delta^2+\frac{\delta^3}{9}+o(\delta^3)\right)^{1/2} =$$

$$= \left(\frac{2}{3}\right)^{1/2}\left(1+\frac{\delta}{2}+\frac{5}{12}\delta^2+\frac{\delta^3}{6}+o(\delta^3)\right)^{1/2} =$$

$$= \left(\frac{2}{3}\right)^{1/2}\left(1+\frac{\delta}{4}+\frac{5}{24}\delta^2+\frac{\delta^3}{12}+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{\delta}{2}+\frac{5}{12}\delta^2\right)^2+\frac{1}{6}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{\delta^3}{8}+o(\delta^3)\right) =$$

$$= \left(\frac{2}{3}\right)^{1/2}\left(1+\frac{\delta}{4}+\frac{5}{24}\delta^2+\frac{\delta^3}{12}-\frac{\delta^2}{32}-\frac{5}{96}\delta^3+\frac{\delta^3}{128}+o(\delta^3)\right) =$$

$$= \left(\frac{2}{3}\right)^{1/2}\left(1+\frac{\delta}{4}+\frac{20-3}{96}\delta^2+\frac{\overset{5\ \cancel{15}}{32-20+3}}{\cancel{384}\ 128}\delta^3+o(\delta^3)\right) =$$

$$= \sqrt{\frac{2}{3}}+\sqrt{\frac{2}{3}}\,\frac{\delta}{4}+\frac{17}{96}\sqrt{\frac{2}{3}}\,\delta^2+\frac{5}{128}\sqrt{\frac{2}{3}}\,\delta^3+o(\delta^3)$$

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